

SI 1: Proof Writing and Induction

1 Contra-positive

A Definition: The contra-positive is a sort of “negative” form of a statement “if A then B ”. That is, given a statement “if A then B ”, then that immediately implies the contra-positive, or “if not B then not A ”. This can be useful when presented with some statement which seems very difficult to prove directly, but whose contra-positive is easier to prove. Proving the contra-positive of a statement is equivalent to proving the original statement.

1.1 Find the contra-positive form for the following:

1. If it is raining, my car is wet.
2. If $n \neq m$, then $n + p \neq p + m$.
3. If $n = m$, and $m = p$, then $n = p$.
4. If a student is not sleep deprived during finals, they are not taking Math 250.

2 Definitions

Importance of Definitions: As many of you may have already noticed, definitions are very important in proof writing. Even for something as simple and intuitive as \leq , we cannot reason about it unless we have made a precise definition for it.

2.1 Writing Definitions

5. In this course, the symbol $|$ is used to signify *divides*. For example, $2|4$ means “2 divides 4” and is a true statement. Write a precise definition for $|$.

2.2 Using Definitions

6. **Prove:** For all $m, n, p \in \mathbb{N}$, if $m \not\leq n$, then either $m \not\leq p$ or $p \not\leq n$

3 Basic Proof Writing

For the next two problems, you may use the following axioms:

1. **Progression:** To take an n^{th} step, we must begin at some step s such that $s < n$ and $s \geq 0$, and end at the n^{th} step.
2. **Counting:** For all m, n in \mathbb{N} such that $m < p$, if we begin counting at m , and count up to p , we have counted on the interval (m, p)
3. **Skipping:** For all m, n, p in \mathbb{N} such that $m < n < p$, to count from m to p without counting to n , means we have skipped at least one number n .
4. **Axiom of induction:** It is impossible to remove a natural number without violating the vocabulary of \mathbb{N} .

3.1 Lemma 1

Prove: When walking from any point A to any point B, it is impossible to take a second step without first having taken a first step.

3.2 Proof of the Linearity of taking 3 steps

Prove by Induction: For all n in \mathbb{N} , it is impossible to take an $n+1^{\text{th}}$ step without having taken an n^{th} step.

4 Some More Difficult Proofs

4.1 Factorization of numbers greater than 1

Prove: Every natural number $n > 1$ is either prime or a product of primes

4.2 Towers of Hanoi

Prove: For a tower of n disks, it takes $2^n - 1$ moves to solve the problem of the Towers of Hanoi.

The problem of the Towers of Hanoi: There are n disks stacked on one of 3 pegs, ordered from smallest disk to largest (smallest disk on the top). The task is to move all the disks to another peg following this set of rules:

1. Only one disk may be moved at a time
2. No disk may be placed on a peg with a smaller disk underneath it