SI 1: Proof Writing and Induction

1 Contrapositive

A Definition: The contrapositive is a sort of “negative” form of a statment “if A then B”. That is, given a statment “if A then B”, then that immediately implies the contrapositive, or “if not B then not A”. This can be useful when presented with some statement which seems very difficult to prove directly, but whose contrapositive is easier to prove. Proving the contrapositive of a statment is equivalent to proving the original statment.

1.1 Find the contrapositive form for the following:

1. If it is raining, my car is wet.
2. If $n \neq m$, then $n + p \neq p + m$.
3. If $n = m$, and $m = p$, then $n = p$.
4. If a student is not sleep deprived during finals, they are not taking Math 250.

2 Definitions

Importance of Definitions: As many of you may have already noticed, definitions are very important in proof writing. Even for something as simple and intuitive as $\leq$, we cannot reason about it unless we have made a precise definition for it.

2.1 Writing Definitions

Write a definition for $\leq$ in $\mathbb{N}$. Hint: what exactly does $m \leq n$ mean $\forall n, m \in \mathbb{N}$?

2.2 Using Definitions

Prove: $\forall m, n, p \in \mathbb{N}$, if $m \nleq n$, then either $m \nleq p$ or $p \nleq n$

3 Basic Proof Writing

For the next two problems, you may use the following axioms:

1. **Progression:** To take an n^{th} step, we must begin at some step s such that $s < n$ and $s \geq 0$, and end at the n^{th} step
2. **Counting:** $\forall m, n \in \mathbb{N}$ such that $m < p$, if we begin at counting at m, and count up to p, we have counted on the interval (m, p)
3. **Skipping:** $\forall m, n, p \in \mathbb{N}$ such that $m < n < p$, to count from m to p without counting to n, means we have skipped at least one number n.
4. **Linearity:** In a real-world progression of things, it is impossible to skip an item in the progression.

3.1 Lemma 1

Prove: When walking from any point A to any point B, it is impossible to take a second step without first having taken a first step.

3.2 Proof of the Linearity of taking 3 steps

Prove by Induction: $\forall n \in \mathbb{N}$, it is impossible to take an $n + 1^{th}$ step without having taken an n^{th} step.

4 Some More Difficult Proofs

4.1 Factorization of numbers greater than 1

Prove: Every natural number $n > 1$ is either prime or a product of primes

4.2 Towers of Hanoi

Prove: For a tower of n disks, it takes $2^n - 1$ moves to solve the problem of the Towers of Hanoi.

The problem of the Towers of Hanoi: There are n disks stacked on one of 3 pegs, ordered from smallest disk to largest (smallest disk on the top). The task is to move all the disks to another peg following this set of rules:

1. Only one disk may be moved at a time
2. No disk may be placed on a peg with a smaller disk underneath it