Problem 2)

a.) We say that a time-series Y_t is weakly stationary if:

1.) $\mu_t = const.$ for all t

2.) $Cov(Y_n, Y_m)$ depends only on $|n-m|$

The simplest example of a stationary process is $Y_t = 1$

$\mu_t = 1$

$cov(Y_t, Y_m) = 0$

b.) An ARMA (P,q) process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \dots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \dots - \theta_q e_{t-q}$$

is stationary only if:

1.) $|\phi_p| < 1$

2.) $\sum_{i=1}^{p} \phi_i < 1$

It is stationary if and only if the equation

$$1 - \phi_1 x - \phi_2 x^2 - \dots - \phi_p x^p = 0$$

Has solutions $x_i \in \mathbb{C}$ only outside the unit disk. Likewise, it is invertible only if:

1.) $|\theta_p| < 1$

2.) $\sum_{i=1}^{q} \theta_i < 1$

And, invertible if and only if the equation:

$$1-\theta_1 x - \theta_2 x^2 - \dots - \theta_q x^q = 0$$

has solutions $x_i \in \mathbb{C}$ only outside the unit disk.

c) We begin with a characteristic polynomial with roots outside the unit disk:

$$(x-2)^3 = 0$$
$$\Rightarrow x^3 - 6x^2 + 12x - 8 = 0$$
$$\Rightarrow \tfrac{1}{8}x^3 - \tfrac{3}{4}x^2 + \tfrac{3}{2}x - 1 = 0$$
$$\Rightarrow 1 - \tfrac{3}{2}x + \tfrac{3}{4}x^2 - \tfrac{1}{8}x^3 = 0$$

From this, we know that the time-series:

$$Y_t = \tfrac{3}{2}Y_{t-1} - \tfrac{3}{4}Y_{t-2} + \tfrac{1}{8}Y_{t-3} + e_t$$

is a AR(3) series. To find an invertible MA(4) series, we proceed similarly:

$$(x-2)^4 = 0$$
$$\Rightarrow x^4 - 8x^3 + 24x^2 - 32x + 16 = 0$$
$$\Rightarrow \tfrac{1}{16}x^4 - \tfrac{1}{2}x^3 + \tfrac{3}{2}x^2 - 2x + 1 = 0$$
$$\Rightarrow 1 - 2x + \tfrac{3}{2}x^2 - \tfrac{1}{2}x^3 + \tfrac{1}{16}x^4 = 0$$

From this, we know that the time-series:

$$Y_t = -2e_{t-1} + \tfrac{3}{2}e_{t-2} - \tfrac{1}{2}e_{t-3} + \tfrac{1}{16}e_{t-4} - e_t$$

is an invertible MA(4) time series. Thus, an invertible and stationary ARMA(3,4) process is given by:

$$Y_t = \tfrac{3}{2}Y_{t-1} - \tfrac{3}{4}Y_{t-2} + \tfrac{1}{8}Y_{t-3} + e_t - 2e_{t-1} + \tfrac{3}{2}e_{t-2} - \tfrac{1}{2}e_{t-3} + \tfrac{1}{16}e_{t-4}$$

A process which appears like ARMA(3, 4) but fails stationarity is

$$Y_t = \frac{1}{2}Y_{t-1} + \frac{1}{2}Y_{t-2} + \frac{1}{2}Y_{t-3} + e_t - 2e_{t-1} + \frac{3}{2}e_{t-2} - \frac{1}{2}e_{t-3} + \frac{1}{16}e_{t-4}$$

Since $\sum_{i=1}^{3} \frac{1}{2} = \frac{3}{2} > 1$.

d) Let $Y_t = e_t + t + Y_{t-1}$

$$\begin{aligned}\mu_t &= E[e_t + t + Y_{t-1}] \\ &= t + E[Y_{t-1}] \\ &= t + (t-1) + E[Y_{t-2}] \\ &= \sum_{n=0}^{t} (t-n)\end{aligned}$$

And thus μ_t clearly depends on time.

Also:

$$\begin{aligned}\operatorname{cov}(Y_t, Y_t) &= \operatorname{var}(Y_t) \\ &= \operatorname{var}(e_t + t + Y_{t-1}) \\ &= \operatorname{var}(e_t) + \operatorname{var}(t) + \operatorname{var}(Y_{t-1}) \\ &= \sigma_e^2 + \operatorname{var}(Y_{t-1}) \\ &\vdots \\ &= t\sigma_e^2\end{aligned}$$

Thus, the autocovariance clearly changes with time.

e.) Any ARIMA series can be written as:

$$\phi(B)(1-B)^d Y_t = \theta(B) e_t$$

Since the function $t!$ has no polynomial expansion, the series

$$Y_t = t!\, Y_{t-1} + e_t$$

cannot be made stationary by differencing, since no polynomial $\phi(B)$ can be constructed which will match the behavior of $t!$

Problem 2)

a.) $Y_t = 4Y_{t-1} - 6Y_{t-2} + 4Y_{t-3} - Y_{t-4} + e_t + 0.9e_{t-1}$

$\Rightarrow Y_t - 4Y_{t-1} + 6Y_{t-2} - 4Y_{t-3} + Y_{t-4} = e_t + 0.9e_{t-1}$

$\Rightarrow (1 - 4B + 6B^2 - 4B^3 + B^4)Y_t = e_t(1 + 0.9B)$

$\Rightarrow (1-B)^4 Y_t = e_t(1+0.9B)$

$\Rightarrow Y_t \sim ARIMA(0,4,1)$

$\theta = -0.9$

b.) $Y_t = Y_{t-1} - 0.8Y_{t-2} + e_t - 0.7e_{t-1} + 0.1e_{t-3}$

Note that

1.) $\phi_1 + \phi_2 = 1 - 0.8 = 0.2 < 1$

2.) $\phi_2 - \phi_1 = -0.8 - 1 = -1.8 < 1$

3.) $|\phi_2| = 0.8 < 1$

Thus, the model is already a stationary ARMA(2,a) model.

To fully specify it, we must study the autocovariance of the MA portion:

$$\gamma_0 = var(Y_t) = var(e_t - 0.7e_{t-1} + 0.1e_{t-3})$$
$$= \sigma_e^2 + 0.049\sigma_e^2 + 0.01\sigma_e^2$$
$$= (1 + 0.049 + 0.001)\sigma_e^2$$
$$= (1.05)\sigma_e^2$$

Let $\delta(x,y) = \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases}$, then:

$$\gamma_k = cov(Y_t, Y_{t-k}) = cov(e_t - \theta_1 e_{t-1} - \theta_3 e_{t-3}, e_{t-k} - \theta_1 e_{t-k-1} - \theta_3 e_{t-k-3})$$

$$= \sigma_e^2 \delta(0,k) - \theta_1 \sigma_e^2 \delta(1,k) + \theta_1^2 \sigma_e^2 \delta(0,k) - \theta_3 \sigma_e^2 \delta(3,k) + \theta_3 \theta_1 \sigma_e^2 \delta(2,k) + \theta_3^2 \sigma_e^2 \delta(0,k)$$

$$= \sigma_e^2 \left[\delta(0,k)(1 + \theta_1^2 + \theta_3^2) + \delta(1,k)(-\theta_1) + \delta(2,k)(\theta_3\theta_1) + \delta(3,k)(-\theta_3) \right]$$

$$\Rightarrow \gamma_k = \begin{cases} (1 + \theta^2 + \theta_3^3)\sigma_e^2 & k = 0 \\ -\theta_1 \sigma_e^2 & k = 1 \\ \theta_3 \theta_1 \sigma_e^2 & k = 2 \\ -\theta_3 \sigma_e^2 & k = 3 \end{cases}$$

Thus, the model is ARIMA(2, 0, 3),

$\phi_1 = 1$

$\phi_2 = -0.8$

$\theta_1 = 0.7$

$\theta_2 = 0$

$\theta_3 = -0.1$

c.) $Y_t = 0.9Y_{t-2} + e_t - 0.5e_{t-1} - 0.3e_{t-2}$

To characterise $0.9Y_{t-2} + e_t$ we note that whether AR(1) or AR(2), it is stationary since.

$$|\phi| < 1$$
$$\phi_2 - \phi_1 < 1$$

Thus:

$$\begin{aligned} Cov(Y'_t, Y'_{t-k}) &= E(Y'_t Y'_{t-k}) \\ &= E[(\phi Y_{t-2} + e_t) Y'_{t-k}] \\ &= E[\phi Y'_{t-2} Y'_{t-k}] \\ &= \phi \gamma_{k-2} \end{aligned}$$

Taking $\phi_1 = 0$, $\phi_2 = 0.9$,

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$$

And we obtain the Yule-walker equation of AR(2).
Thus:

$$Y_t \sim ARIMA(2, 0, 2)$$
$$\phi_1 = 0, \quad \phi_2 = 0.9$$
$$\theta_1 = 0.7, \quad \theta_2 = 0.3$$

d.) $Y_t = e_t - e_{t-1} - 2e_{t-4}$

This is already a stationary MA(q) process. To identify q we find the autocovariance;

$$\mathrm{cov}(Y_t, Y_{t-k}) = \mathrm{cov}(e_t - e_{t-1} - 2e_{t-4},\ e_{t-k} - e_{t-k-1} - 2e_{t-k-4})$$

$$= \sigma_e^2 \delta(0,k) - \sigma_e^2 \delta(1,k) + \sigma_e^2 \delta(0,k) - 2\sigma_e^2 \delta(4,k)$$
$$+ 2\sigma_e^2 (3,k) + 4\sigma_e^2 \delta(0,k)$$

$$= \sigma_e^2 [\delta(0,k)(1+1+4) + \delta(1,k)(-1) + \delta(3,k)(2)$$
$$+ \delta(4,k)(-2)]$$

$$\Rightarrow \gamma_k = \begin{cases} 6 & k=0 \\ -1 & k=1 \\ 0 & k=2 \\ 2 & k=3 \\ -2 & k=4 \\ 0 & k>4 \end{cases}$$

Thus, This model is ARIMA(0,0,4) with
$\theta_1 = 1$, $\theta_2 = 0$, $\theta_3 = 0$, $\theta_4 = 2$.

e.) $Y_t = e_{t-4} - 0.6e_{t-5}$

Again, we study the covariance

$$\mathrm{cov}(Y_t, Y_{t-k}) = \mathrm{cov}(e_{t-4} - 0.6e_{t-5},\ e_{t-k-4} - 0.6e_{t-k-5})$$

$$= \sigma_e^2 \delta(0,k) - 0.6\sigma_e^2 \delta(1,k) + 0.036\sigma_e^2 \delta(0,k)$$

$$\Rightarrow \gamma_k = \begin{cases} 1.036\sigma_e^2 & k=0 \\ -0.6\sigma_e^2 & k=1 \end{cases}$$

Thus, this series is ARIMA(0,0,1), with
$\theta = 0.6$.

Problem 3)

a) ARMA(1,1)

$$Y_t = \phi_1 = Y_{t-1} + e_t - \theta e_{t-1}$$

We note that:

$$E(Y_t) = 0$$

$$E(Y_t e_t) = \phi E(Y_{t-1} e_t) + E(e_t^2) - \theta E(e_{t-1} e_t)$$
$$= \sigma_e^2$$
$$E(Y_t e_{t-1}) = \phi E(Y_{t-1} e_t) + E(e_t, e_{t-1}) - \theta E(e_{t-1}^2)$$
$$= \phi\sigma_e^2 - \theta\sigma_e^2$$
$$= (\phi - \theta)\sigma_e^2$$

Thus,

$$\mathrm{Cov}(Y_t, Y_{t-k}) = E(Y_t Y_{t-k})$$
$$= E(\phi Y_{t-1} Y_{t-k} + e_t Y_{t-k} - \theta e_{t-1} Y_{t-k})$$
$$= \phi \gamma_{k-1}$$

$$\mathrm{Var}(Y_t) = E(Y_t Y_t)$$
$$= E(\phi Y_t Y_{t-1} + e_t Y_t - \theta e_{t-1} Y_{t-1})$$
$$= \phi\gamma_1 + \sigma_e^2 - \theta(\phi - \theta)\sigma_e^2$$
$$= \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2$$

$$\gamma_1 = E(Y_t Y_{t-1})$$
$$= E(\phi_1 Y_{t-1} Y_{t-1} + e_t Y_{t-1} - \theta e_{t-1} Y_{t-1})$$
$$= \phi\gamma_0 - \theta\sigma_e^2$$

$$\Rightarrow \begin{cases} \gamma_0 = \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2 \\ \gamma_1 = \phi\gamma_0 - \theta\sigma_e^2 \\ \gamma_k = \phi\gamma_{k-1} \end{cases}$$

$$\Rightarrow \gamma_0 = \phi[\phi\gamma_0 - \theta\sigma_e^2] + [1-\theta(\phi-\theta)]\sigma_e^2$$
$$= \phi^2\gamma_0 - \phi\theta\sigma_e^2 + [1-\theta(\phi-\theta)]\sigma_e^2$$
$$\Rightarrow \gamma_0(1-\phi^2) = [\phi\theta + 1 - \theta(\phi-\theta)]\sigma_e^2$$
$$= [1-\theta(2\phi-\theta)]\sigma_e^2$$

$$\Rightarrow \gamma_0 = \frac{1-\theta(2\phi-\theta)}{1-\phi^2}\sigma_e^2$$

$$= \frac{\theta^2-2\phi\theta+1}{1-\phi^2}\sigma_e^2$$

$$\Rightarrow \gamma_k = \phi\gamma_{k-1}$$

$$= \phi^{k-1}\gamma_1$$

$$= \phi^{k-1}(\phi\gamma_0 - \theta\sigma_e^2)$$
$$= \phi^k\gamma_0 - \phi^{k-1}\theta\sigma_e^2$$
$$= \frac{\theta^2-2\phi\theta+1}{1-\phi^2}\phi^k\sigma_e^2 - \phi^{k-1}\theta\sigma_e^2$$
$$= \left(\frac{\theta^2-2\phi\theta+1}{1-\phi^2}\phi^k - \phi^{k-1}\theta\right)\sigma_e^2$$
$$= \left(\frac{\theta^2-2\phi\theta+1}{1-\phi^2}\phi - \theta\right)\phi^{k-1}\sigma_e^2$$

$$= \left(\frac{\theta^2\phi - 2\phi^2\theta + \phi - \theta + \phi^2\theta}{1-\phi^2}\right)\phi^{k-1}\sigma_e^2$$

Thus,

$$\boxed{\gamma_k = \frac{\theta^2\phi - \phi^2\theta + \phi - \theta}{1-\phi^2}\,\phi^{k-1}\,\sigma_e^2}$$

$$\boxed{\gamma_1 = \frac{\theta^2\phi - \phi^2\theta + \phi - \theta}{1-\phi^2}\,\sigma_e^2}$$

$$\boxed{\gamma_0 = \frac{\theta^2 - 2\phi\theta + 1}{1-\phi^2}\,\sigma_e^2}$$

And, given $\rho_k = \gamma_k / \gamma_0$:

$$\boxed{\rho_k = \frac{\theta^2\phi - \phi^2\theta + \phi - \theta}{\theta^2 - 2\phi\theta + 1}\,\phi^{k-1}}$$

b.) MA(q)

$$\gamma_k = \text{cov}(Y_t, Y_{t-k})$$

$$= \text{cov}(e_t - \theta_1 e_{t-1} \dots - \theta_q e_{t-q},\ e_{t-k} - \theta_1 e_{t-k-1} - \dots - \theta_q e_{t-k-q})$$

$$= \sigma^2 \delta(0,k) - \theta_1 \sigma_e^2 \delta(1,k) + \theta_1^2 \sigma_e^2 \delta(0,k) + \dots - \theta_q \sigma_e^2 \delta(q,k) + \dots$$
$$+ \theta_q^2 \sigma_e^2 \delta(k,0)$$

$$= \sigma_e^2 \delta(0,k)[1 + \theta_1^2 + \theta_2^2 + \dots] + \sigma_e^2 \delta(1,k)[-\theta_1 + \theta_2\theta_1 + \theta_3\theta_2 + \dots \theta_q\theta_{q-1}]$$
$$+ \sigma_e^2 \delta(2,k)[-\theta_2 - \theta_3\theta_1 - \theta_4\theta_2 - \dots - \theta_q\theta_{q-2}] + \dots$$

$$\Rightarrow \gamma_k = \begin{cases} (1 + \theta_1^2 + \dots + \theta_q^2)\sigma_e^2 & k = 0 \\ \sigma_e^2 \left(\sum_{n=k}^{q} \theta_n \theta_{n-k}\right) + \theta_k \sigma_e^2 & 0 < k \leq q \end{cases}$$

$$\Rightarrow \rho_k = \frac{\left(\sum_{n=k}^{q} \theta_n \theta_{n-k}\right) + \theta_k}{1 + \sum_{n=1}^{q} \theta_n^2}$$

c.) AR(p)

$$\gamma_k = \text{cov}(Y_t, Y_{t-k}) = E(Y_t Y_{t-k})$$

$$= E(\phi_1 Y_{t-1} Y_{t-k}) + E(\phi_2 Y_{t-2} Y_{t-k}) + \dots + E(\phi_p Y_{t-p} Y_{t-k})$$
$$+ E(e_t Y_{t-k})$$
$$= \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \dots + \phi_p \gamma_{k-p}$$

Dividing by γ_0 we obtain

$$\rho_k = \phi_1 \gamma_{k-1}/\gamma_0 + \phi_2 \gamma_{k-2}/\gamma_0 + \dots + \phi_p \gamma_{k-p}/\gamma_0$$
$$= \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \dots + \phi_p \rho_{k-p}$$

Finally, we find γ_0 by

$$\gamma_0 = E(Y_t Y_t) = E(\phi_1 Y_{t-1} Y_t + \phi_2 Y_{t-2} Y_t + \dots + e_t Y_t)$$
$$= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \dots + \phi_p \gamma_p + E(e_t Y_t)$$

And note that

$$E(e_t Y_t) = E(\phi_1 Y_{t-1} e_t + \dots + \phi_p Y_{t-p} e_t + e_t e_t)$$
$$= \sigma_e^2$$

Thus, $\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \dots + \phi_p \gamma_p + \sigma_e^2$

And we obtain:

$$\begin{cases} \gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \dots + \phi_p \gamma_p + \sigma_e^2 \\ \gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \dots + \phi_p \gamma_{k-p} \\ \rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \dots + \phi_p \rho_{k-p} \end{cases}$$