Problem 2)

a.) Write the general form of an ARIMA(p,d,q) process.

Let $\nabla^d Y_t = W_t$. Then

$$W_t = \theta_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \dots + \phi_p W_{t-p} + e_t - \theta_1 e_{t-1} + \theta_2 e_{t-2} - \dots - \theta_q e_{t-q} \quad \sim ARIMA(p,d,q)$$

b.) Write the general form of an ARIMA(p,d,q) process using the backshift operator.

$$\varphi(B)(1-B)^d Y_t = \theta(B) e_t$$

c.) State the necessary and sufficient conditions for stationarity and invertibility of ARMA$(1,1)$ and ARMA$(2,2)$ processes that do not use statements about the characteristic equation.

ARMA$(1,1)$:

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

We have stationarity iff $|\phi| < 1$
We have invertibility iff $|\theta| < 1$

ARMA$(2,2)$:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

We have the following necessary conditions for stationarity:

1.) $\phi_1 + \phi_2 < 1$
2.) $|\phi_2| < 1$

Adding the following 3rd condition makes the set of 3 conditions sufficient and necessary:

3.) $\phi_2 - \phi_1 < 1$

In a similar manner, the process is invertible iff the roots of the MA characteristic equation:

$$1 - \theta_1 x - \theta_2 x^2 = 0$$

all lie outside the unit disk. Thus, we can come to the conclusion that the following three conditions are necessary and sufficient for invertibility:

1.) $\theta_1 + \theta_2 < 1$
2.) $|\theta_2| < 1$
3.) $\theta_2 - \theta_1 < 1$

d.) Give an example of a stationary ARMA (3,1) process and a non-stationary process that resembles ARMA (3,1):

We can construct a stationary ARMA(3,1) process by starting with a characteristic equation with roots outside the unit disk:

$$(x-2)(x-3)(x-4) = 0$$
$$\Rightarrow x^3 - 9x^2 + 26x - 24 = 0$$
$$\Rightarrow 1 - \frac{26}{24}x + \frac{9}{24}x^2 - \frac{1}{24}x^3 = 0$$
$$\Rightarrow 1 - \frac{13}{12}x + \frac{3}{8}x^2 - \frac{1}{24}x^3 = 0$$
$$\Rightarrow \phi_1 = -\frac{13}{12},\ \phi_2 = \frac{3}{8},\ \phi_3 = -\frac{1}{24}$$

Thus,

$$Y_t = -\frac{13}{12}Y_{t-1} + \frac{3}{8}Y_{t-2} - \frac{1}{24}Y_{t-3} + e_t - \frac{1}{2}e_{t-1}$$

Is a stationary ARMA(3,1) process.

A process which appears like an ARMA(3,1) process but is not stationary is

$$Y_t = \frac{1}{2}Y_{t-1} + \frac{1}{4}Y_{t-2} + \frac{1}{4}Y_{t-3} + e_t$$

Since it violates $\sum \phi_i < 1$.

e) Give an example of a nonstationary time-series with $corr(Y_t, Y_{t-k})$ and $Var(Y_t)$ free of t for all $k \geq 1$.

The simplest such series is of the form

$$Y_t = e_t + t$$

Since:

$$E(Y_t) = t$$

$$Cov(Y_t, Y_{t-k}) = Cov(e_t + t, e_{t-k} + t - k)$$
$$= Cov(e_t, e_{t-k}) + Cov(e_t, (t-k)) + Cov(t, e_{t-k})$$
$$+ Cov(t, t-k)$$
$$= 0$$

4.) Give an example of a non-stationary time-series which can be made stationary after differencing 4 times.

Any such time-series must follow:

$$\nabla^4 Y_t \sim ARIMA(p, 4, q)$$

Thus, we pick a simple ARMA model:

$$\nabla^4 Y_t = e_t - 0.1 e_{t-1}$$

$$\Rightarrow (1-B)^4 Y_t = e_t - 0.1 e_{t-1}$$

$$\Rightarrow (1 - 4B + 6B^2 - 4B^3 + B^4) Y_t = e_t - 0.1 e_{t-1}$$

$$\Rightarrow Y_t - 4Y_{t-1} + 6Y_{t-2} - 4Y_{t-3} + Y_{t-4} = e_t - 0.1 e_{t-1}$$

$$\Rightarrow Y_t = 4Y_{t-1} - 6Y_{t-2} + 4Y_{t-3} - Y_{t-4} + e_t - 0.1 e_{t-1}$$

And Y_t is visibly nonstationary since:

$$\begin{aligned} \phi_1 + \phi_2 + \phi_3 + \phi_4 &= 4 - 6 + 4 - 1 \\ &= 8 - 7 \\ &= 1 \not< 1 \end{aligned}$$

Problem 2| Identify the specific ARMA models

a.) $Y_t = 3Y_{t-1} - 3Y_{t-2} + Y_{t-3} + e_t - 0.1e_{t-1}$

$\Rightarrow Y_t - 3Y_{t-1} + 3Y_{t-2} - Y_{t-3} = e_t - 0.1e_{t-1}$
$\Rightarrow (1 - 3B + 3B^2 - B^3)Y_t = e_t - 0.1e_{t-1}$
$\Rightarrow (1-B)^3 Y_t = e_t - 0.1e_{t-1}$
$\Rightarrow \nabla^3 Y_t = e_t - 0.1e_{t-1}$

Thus, $Y_t \sim ARIMA(0,3,1)$

b.) $Y_t = 1.2Y_{t-1} - 0.25Y_{t-2} + e_t - 0.6e_{t-1} - 0.2e_{t-2}$

Note that:
$\phi_1 + \phi_2 = 1.2 - 0.25$
$= 0.95 < 1$
$\phi_2 - \phi_1 = -0.25 - 1.2$
$= -1.45 < 1$
$|\phi_2| = |-0.25|$
$= 0.25 < 1$

Thus, $Y_t \sim ARIMA(2,0,2)$

c.) $Y_t = 0.6Y_{t-2} + e_t - 0.7e_{t-1} - 0.2e_{t-2}$

Note that $Y_t \sim ARMA(1,2)$, thus
$Y_t \sim ARIMA(1,0,2)$

d.) $Y_t = e_t - 0.1e_{t-1} - 0.3e_{t-2} + 0.4e_{t-4}$

This is simply an $MA(4)$ model, thus
$Y_t \sim ARIMA(0,0,4)$

e.) $Y_t = e_{t-2} - 0.4\, e_{t-4}$

Clearly, this is a MA(q) model. To find q, we must study its behavior:

$$\gamma_k = \mathrm{Cov}(Y_t, Y_{t-k})$$

$$= \mathrm{Cov}(e_{t-2} - 0.4e_{t-4},\ e_{t-2-k} - 0.4e_{t-4-k})$$

$$= \mathrm{Cov}(e_{t-2}, e_{t-2-k}) + \mathrm{Cov}(e_{t-2}, -0.4e_{t-4-k}) + \mathrm{Cov}(-0.4e_{t-4}, e_{t-2-k})$$
$$+ \mathrm{Cov}(-0.4e_{t-4}, -0.4e_{t-4-k})$$

$$= \sigma_e\, \delta(t-2, t-2-k) - 0.4\sigma_e\, \delta(t-2, t-4-k) - 0.4\sigma_e\, \delta(t-4, t-2-k)$$
$$+ 0.016\, \sigma_e\, \delta(t-4, t-4-k)$$

$$= \sigma_e\, \delta(0,k) + \sigma_e\, \delta(-2,k) + \sigma_e\, \delta(2,k) + \sigma_e\, \delta(0,k)$$
$$= \sigma_e\, \delta(0,k) + \sigma_e\, \delta(2,k) + \sigma_e\, \delta(0,k)$$

$$\Rightarrow \gamma_k = \begin{cases} 2\sigma_e & k=0 \\ 0 & k=1 \\ \sigma_e & k=2 \end{cases}$$

Thus, this time-series behaves as MA(2)

Problem 3) Derive the autocovariance and autocorrelation for the following processes:

a.) MA(1)

$$Y_t = -\theta_1 e_{t-1} + e_t$$

$$\gamma_k = \text{corr}(Y_t, Y_{t-k})$$
$$= \text{corr}(-\theta_1 e_{t-1} + e_t, -\theta_1 e_{t-k-1} + e_{t-k})$$
$$= \theta_1^2 \text{corr}(e_{t-1}, e_{t-k-1}) - \theta_1 \text{corr}(e_{t-1}, e_{t-k}) - \theta_1 (e_t, e_{t-k-1})$$
$$+ \text{corr}(e_t, e_{t-k})$$
$$= \theta_1^2 \sigma_e \delta(-1, -k-1) - \theta_1 \sigma_e \delta(-1, -k) - \theta_1 \sigma_e \delta(0, -k-1) + \sigma_e \delta(0, k)$$
$$= \theta_1^2 \sigma_e \delta(0, k) - \theta_1 \sigma_e \delta(1, k) - \theta_1 \sigma_e \delta(-1, k) + \sigma_e \delta(0, k)$$

$$\Rightarrow \gamma_k = \begin{cases} (1+\theta_1^2)\sigma_e & k=0 \\ -\theta_1 \sigma_e & k=1 \\ 0 & k>1 \end{cases}$$

$$\Rightarrow \rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} -\dfrac{\theta_1}{1+\theta_1^2} & k=1 \\ 0 & k>1 \end{cases}$$

b.) MA(2)

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

$$\gamma_k = cov(Y_t, Y_{t-k}) = cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-k} - \theta_1 e_{t-k-1} - \theta_2 e_{t-k-2})$$

$$= cov(e_t, e_{t-k}) - \theta_1 corr(e_{t-1}, e_{t-k}) + \theta_1^2 corr(e_{t-1}, e_{t-k-1})$$

$$+ \theta_1 \theta_2 \underbrace{corr(e_{t-1}, e_{t-k-2})}_{\to 0} - \theta_2 corr(e_{t-2}, e_{t-k}) + \theta_2 \theta_1 corr(e_{t-2}, e_{t-k-1})$$

$$+ \theta_2^2 corr(e_{t-2}, e_{t-k-2})$$

$$= \sigma_e \delta(0,k) - \theta_1 \sigma_e \delta(1,k) + \theta_1^2 \sigma_e \delta(0,k) - \theta_2 \sigma_e \delta(2,k)$$

$$+ \theta_2 \theta_1 \sigma_e \delta(1,k) + \theta_2^2 \sigma_e \delta(0,k)$$

$$= \delta(0,k)[\sigma_e + \theta_1^2 \sigma_e + \theta_2^2 \sigma_e] + \delta(1,k)[-\theta_1 \sigma_e + \theta_2 \theta_1 \sigma_e]$$

$$+ \delta(2,k)[-\theta_2 \sigma_e]$$

$$\Rightarrow \gamma_k = \begin{cases} \sigma_e(1 + \theta_1^2 + \theta_2^2) & k = 0 \\ \sigma_e \theta_1(\theta_2 - 1) & k = 1 \\ -\theta_2 \sigma_e & k = 2 \\ 0 & k > 2 \end{cases}$$

$$\Rightarrow \rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{\theta_1(\theta_2 - 1)}{1 + \theta_1^2 + \theta_2^2} & k = 1 \\ \dfrac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} & k = 2 \\ 0 & k > 2 \end{cases}$$

c.) AR(1)

$$Y_t = \phi_1 Y_{t-1} + e_t$$

$$E(Y_t) = E(\phi_1 Y_{t-1} + e_t)$$

$$\Rightarrow E(Y_t) = \phi_1 E(Y_{t-1})$$

$$\Rightarrow E(Y_t) = \phi_1 E(Y_t)$$

$$\Rightarrow E(Y_t) = 0$$

Thus:

$$\begin{aligned}
\gamma_k &= cov(Y_t, Y_{t-k}) \quad , k > 0 \\
&= E(Y_t Y_{t-k}) - E(Y_t)E(Y_{t-k}) \\
&= E(Y_t Y_{t-k}) \\
&= E[(\phi_1 Y_{t-1} + e_t) Y_{t-k}] \\
&= E[\phi_1 Y_{t-1} Y_{t-k} + e_t Y_{t-k}] \\
&= \phi_1 E[Y_{t-1} Y_{t-k}] + E[e_t Y_{t-k}] \\
&= \phi_1 \gamma_{k-1}
\end{aligned}$$

$$\Rightarrow \gamma_k = \phi_1 \gamma_{k-1}$$

$$\Rightarrow \gamma_k = \phi_1^2 \gamma_{k-2}$$

$$\Rightarrow \gamma_k = \phi_1^k \gamma_0$$

And

$$\begin{aligned}
\gamma_0 &= Var(Y_t) \\
&= E(Y_t^2) - E(Y_t)^2 \\
&= E(Y_t^2) \\
&= E[\phi_1^2 Y_{t-1}^2 + 2\phi_1 Y_{t-1} e_t + e_t^2] \\
&= \phi_1^2 E[Y_{t-1}^2] + E(e_t^2)
\end{aligned}$$

$$\Rightarrow \gamma_0 = \phi_1^2 \gamma_0 + \sigma_e^2$$

$$\Rightarrow \gamma_0 - \phi_1^2 \gamma_0 = \sigma_e^2$$

$$\boxed{\begin{aligned} &\Rightarrow \gamma_0 = \frac{\sigma_e^2}{1-\phi_1^2} \\ &\Rightarrow \gamma_k = \frac{\phi_1^k}{1-\phi_1^2} \sigma_e \quad , k > 0 \end{aligned}}$$

And thus,

$$\boxed{\rho_k = \phi^k}$$

d) AR(2)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

$$E(Y_t) = E(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t)$$
$$= \phi_1 E(Y_{t-1}) + \phi_2 E(Y_{t-2})$$
$$\Rightarrow \mu_t = \phi_1 \mu_t + \phi_2 \mu_t$$
$$\Rightarrow \mu_t = \mu_t(\phi_1 + \phi_2)$$
$$\Rightarrow \mu_t = 0$$

Thus, $cov(Y_t, Y_{t-k}) = E(Y_t Y_{t-k})$

$$= \phi_1 E(Y_{t-1} Y_{t-k}) + \phi_2 E(Y_{t-2} Y_{t-k}) + E(e_t Y_{t-k})$$
$$\Rightarrow \gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$$
$$\Rightarrow \rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$$

We can compute γ_0 by:

$$Var(Y_t) = Var(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t)$$
$$= Var(\phi_1 Y_{t-1} + \phi_2 Y_{t-2}) + cov(\phi_1 Y_{t-1} + \phi_2 Y_{t-2}, e_t) + Var(e_t)$$
$$= \phi_1^2 Var(Y_{t-1}) + \phi_2^2 Var(Y_{t-2}) + \phi_1\phi_2 cov(Y_{t-1}, Y_{t-2}) + \sigma_e^2$$
$$\Rightarrow \gamma_0 = \phi_1^2 \gamma_0 + \phi_2^2 \gamma_0 + \phi_1\phi_2 \gamma_1 + \sigma_e^2$$

Thus we have:

$$\begin{cases} \gamma_0 = \phi_1^2 \gamma_0 + \phi_2^2 \gamma_0 + \phi_1 \phi_2 \gamma_1 + \sigma_e^2 \\ \gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1 \end{cases}$$

$$\Rightarrow \gamma_1 (1 - \phi_2) = \phi_1 \gamma_0$$
$$\Rightarrow \gamma_1 = \frac{\phi_1 \gamma_0}{1 - \phi_2}$$

$$\Rightarrow \gamma_0 = \phi_1^2 \gamma_0 + \phi_2^2 \gamma_0 + \phi_1 \phi_2 \left(\frac{\phi_1 \gamma_0}{1-\phi_2} \right) + \sigma_e^2$$

$$\Rightarrow \gamma_0 - \phi_1^2 \gamma_0 - \phi_2^2 \gamma_0 - \phi_1 \phi_2 \left(\frac{\phi_1 \gamma_0}{1-\phi_2} \right) = \sigma_e^2$$

$$\Rightarrow \gamma_0 \left(1 - \phi_1^2 - \phi_2^2 - \phi_1 \phi_2 \cdot \frac{\phi_1}{1-\phi_2} \right) = \sigma_e^2$$

$$\Rightarrow \gamma_0 = \frac{\sigma_e^2}{1 - \phi_1^2 - \phi_2^2 - \frac{\phi_1^2 \phi_2}{1-\phi_2}}$$

$$= \frac{\sigma_e^2 (1-\phi_2)}{(1-\phi_2)(1-\phi_1^2-\phi_2^2) - \phi_1^2 \phi_2}$$

$$= \frac{\sigma_e^2 (1-\phi_2)}{1 - \phi_1^2 - \phi_2^2 - \phi_2 + \phi_1^2 \phi_2 - \phi_2^3 - \phi_1^2 \phi_2}$$

$$= \frac{\sigma_e^2 (1-\phi_2)}{(1-\phi_2)^2 - \phi_1^2} \cdot \left(\frac{1-\phi_2}{1+\phi_2} \right)$$

e.) ARMA(1,1)

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

Before proceeding, note that:

$E(Y_t) = 0$

$$E(Y_t e_t) = \phi E(Y_{t-1} e_t) + E(e_t^2) - \theta E(e_{t-1} e_t)$$
$$= \sigma_e^2$$

$$E(Y_t e_{t-1}) = \phi E(Y_{t-1} e_t) + E(e_t e_{t-1}) - \theta E(e_{t-1}^2)$$
$$= \phi \sigma_e^2 - \theta \sigma_e^2$$
$$= (\phi - \theta)\sigma_e^2$$

Thus,

$$cov(Y_t, Y_{t-k}) = E(Y_t Y_{t-k})$$
$$= E(\phi_1 Y_{t-1} Y_{t-k} + e_t Y_{t-k} - \theta e_{t-1} Y_{t-k})$$
$$= \phi_1 \gamma_{k-1}$$

$$Var(Y_t) = E(Y_t Y_t)$$
$$\Rightarrow \gamma_0 = E(\phi_1 Y_{t-1} Y_t + e_t Y_t - \theta e_{t-1} Y_t)$$
$$= \phi_1 \gamma_1 + \sigma_e^2 - \theta(\phi - \theta)\sigma_e^2$$
$$= \phi_1 \gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2$$

$$\gamma_1 = E(Y_t Y_{t-1})$$
$$= E(\phi_1 Y_{t-1} Y_{t-1} + e_t Y_{t-1} - \theta e_{t-1} Y_{t-1})$$
$$= \phi_1 \gamma_0 - \theta \sigma_e^2$$

$$\Rightarrow \begin{cases} \gamma_0 = \phi_1 \gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2 \\ \gamma_1 = \phi_1 \gamma_0 - \theta \sigma_e^2 \\ \gamma_k = \phi \gamma_{k-1} \end{cases}$$

$$\Rightarrow \gamma_0 = \phi[\phi_1\gamma_0 - \theta\sigma_e^2] + [1-\theta(\phi-\theta)]\sigma_e^2$$
$$\Rightarrow \gamma_0 = \phi^2\gamma_0 - \phi\theta\sigma_e^2 + \quad 1-\theta(\phi-\theta)]\sigma_e^2$$
$$\Rightarrow \gamma_0(1-\phi^2) = [-\phi\theta + \quad 1-\theta(\phi-\theta)]\sigma_e^2$$
$$= [1-\theta(2\phi-\theta)]\sigma_e^2$$
$$\Rightarrow \gamma_0 = \frac{1-\theta(2\phi-\theta)}{1-\phi^2}\sigma_e^2$$
$$\Rightarrow \gamma_0 = \frac{\theta^2-2\phi\theta+1}{1-\phi^2}\sigma_e^2$$

Thus,

$$\gamma_k = \phi\gamma_{k-1}$$
$$= \phi^{k-1}\gamma_1$$
$$= \phi^{k-1}(\phi_1\gamma_0 - \theta\sigma_e^2)$$
$$= \phi^k\gamma_0 - \phi^{k-1}\theta\sigma_e^2$$
$$= \frac{\theta^2-2\phi\theta+1}{1-\phi^2}\phi^k\sigma_e^2 - \phi^{k-1}\theta\sigma_e^2$$
$$= \left(\frac{\theta^2-2\phi\theta+1}{1-\phi^2}\phi^k - \phi^{k-1}\theta\right)\sigma_e^2$$
$$= \left(\frac{\theta^2-2\phi\theta+1}{1-\phi^2}\phi - \theta\right)\phi^{k-1}\sigma_e^2$$
$$= \left(\frac{\theta^2\phi - 2\phi^2\theta + \phi - \theta + \phi^2\theta}{1-\phi^2}\right)\phi^{k-1}\sigma_e^2$$

$$\boxed{\Rightarrow \begin{cases} \gamma_k = \dfrac{\theta^2\phi - \phi^2\theta + \phi - \theta}{1-\phi^2}\phi^{k-1}\sigma_e^2 \\ \gamma_1 = \dfrac{\theta^2\phi - \phi^2\theta + \phi - \theta}{1-\phi^2}\sigma_e^2 \\ \gamma_0 = \dfrac{\theta^2-2\phi\theta+1}{1-\phi^2}\sigma_e^2 \end{cases}}$$

and,

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

$$= \phi^{k-1}(\phi_1 \gamma_0 - \theta\sigma_e^2)\, {}^1\!/\gamma_0$$

$$= \phi^k - \theta\sigma_e^2/\gamma_0$$

$$\boxed{\rho_k = \phi^k - \frac{1-\phi^2}{\theta^2 - 2\phi\theta + 1}\,\theta \qquad k \geq 1}$$