Multiscale Analysis of Complex Time Series: Integration of Chaos and Random Fractal Theory, and Beyond
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CHAPTER 5

BASICS OF FRACTAL GEOMETRY

Fractal phenomena are situations in which a part of a system or object is exactly or statistically similar to another part or to the whole. Such phenomena may be associated with concrete geometrical objects such as clouds, mountains, trees, lightning, and so on, or associated with dynamical, spatial, or temporal variations such as turbulent wave motions in the ocean or variations of a stock market with time. Thus, it is no wonder that fractal geometry provides a solid and elegant framework for the study of many natural and man-made phenomena in various scientific and engineering domains.

Mathematically speaking, fractals are characterized by power-law relations over a wide range of scales. Such power-law relations are often called scaling laws. If a certain phenomenon can be fully characterized by a single power-law scaling relation, then it is called monofractal; otherwise, it is called multifractal. In the latter case, the number of scaling laws can be infinite. In this chapter, we describe several key concepts of fractal geometry.

5.1 THE NOTION OF DIMENSION

One of the key concepts in fractal geometry is the notion of dimension. While this concept was briefly touched on in Sec. 2.1, here we treat it in some depth.

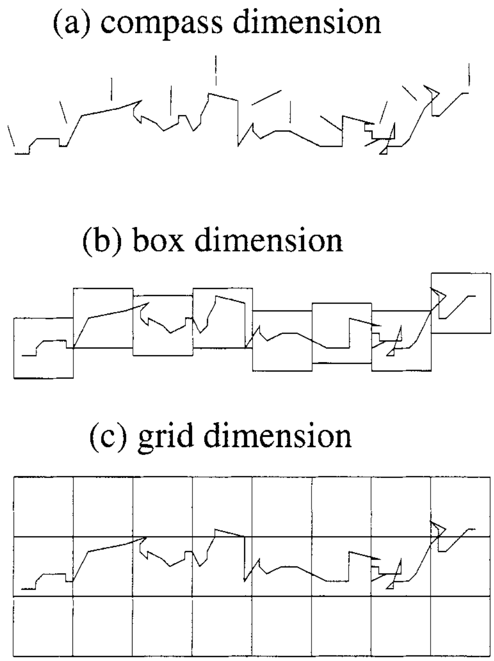

Figure 5.1. Three ways to estimate the fractal dimension.

The major idea discussed in Sec. 2.1 is that the dimension of an object is obtained by covering it with small boxes. That leads us to Eq. (2.2) and an understanding of topological dimension for isolated points, a line, an area, and a volume (which is $0 - D$, $1 - D$, $2 - D$, and $3 - D$, respectively). We have also discussed the consequence of a jagged mountain trail with $1 < D < 2$. Let us now resume the discussion of that topic more systematically.

There are a number of ways of measuring the length of an irregular curve. Three methods are shown in Fig. 5.1. While in many situations they give very similar results, occasionally, they may yield different values. One example is fractional Brownian motion, which will be discussed in the next chapter.

To deepen our understanding, in the remainder of this section we shall discuss the concept of the Hausdorff-Besicovitch dimension. To facilitate this discussion, we remind readers that when covering an object by boxes, the size of the boxes does not have to be the same. The only requirement is that the size of the largest box is bounded (which serves as ϵ).

The Hausdorff-Besicovitch dimension is related to the α-covering measure. Since our purpose is to find the dimension of an object E, let us cover it with boxes of linear length ϵ. What is the measuring unit for such a procedure? If E is a line, the unit is simply ϵ; if E is an area, the unit is ϵ^2. In general, the unit is $\epsilon^{d(E)}$. Since the dimension of E is unknown, let us denote the measuring unit by $\mu = \epsilon^\alpha$ and try out a number of different α. Let us again consider a square $L \times L$

and partition it into grids of linear length ϵ. There are a total of $N = (L/\epsilon)^2$ boxes. Then we have the total measure

$$M = N\mu = L^2\epsilon^{\alpha-2}.$$

Now if we choose $\alpha = 1$, then $M \to \infty$ as $\epsilon \to 0$. This means that the length of a square is infinite, which makes sense. If we try $\alpha = 3$, then $M \to 0$ as $\epsilon \to 0$. This means that the volume of a square is 0, which is also correct. By this argument, it is clear that a finite value for M can be obtained only when $\alpha = 2$, the true dimension of the square.

With the above argument, we can define the Hausdorff-Besicovitch dimension. First, we need the α-covering measure, which is simply the summation of the total measure in all the "boxes" denoted by V_i, $i = 1, 2, \cdots$, subject to the condition that the union of the boxes covers the object E, while the size of the largest V_i is not greater than ϵ:

$$m^\alpha(E) = \lim_{\epsilon\to 0} \inf\{\sum (diamV_i)^\alpha : \cup V_i \supset E,\ diamV_i \le \epsilon\}. \tag{5.1}$$

The notation inf indicates that this covering is minimal: without this requirement, there would be infinitely many coverings. With Eq. (5.1), $dimE$ can then be defined by

$$dimE = \inf\{\alpha : m^\alpha(E) = 0\} = \sup\{\alpha : m^\alpha(E) = \infty\}. \tag{5.2}$$

The Hausdorff-Besicovitch dimension is the value of α such that the measure jumps from zero to infinity. For $\alpha = dimE$, this measure can be any finite value.

5.2 GEOMETRICAL FRACTALS

In this section, we discuss two examples of geometrical fractals, the Cantor set and its variants and the Koch curves.

5.2.1 Cantor sets

The standard Cantor set is one of the prototypical fractal objects. It is obtained by first splitting a line segment into thirds and deleting the middle third. This step is then repeated, deleting the middle third of each remaining segment iteratively. See Fig. 5.2(a). As we shall see in Chapter 13 when discussing chaotic maps, such a process can be related to the iteration of a nonlinear discrete map. The removed middle thirds can be related to the intervals that make the map diverge to infinity. Therefore, the remaining structures are invariant points of the map. At the limiting stage, $n \to \infty$, the Cantor set consists of infinitely many isolated points, with topological dimension 0. At any stage n, one needs $N(\epsilon) = 2^n$ boxes of length $\epsilon = (\frac{1}{3})^n$ to cover the set. Hence the fractal dimension for the Cantor set is

$$D = -\ln N(\epsilon)/\ln\epsilon = \ln 2/\ln 3.$$

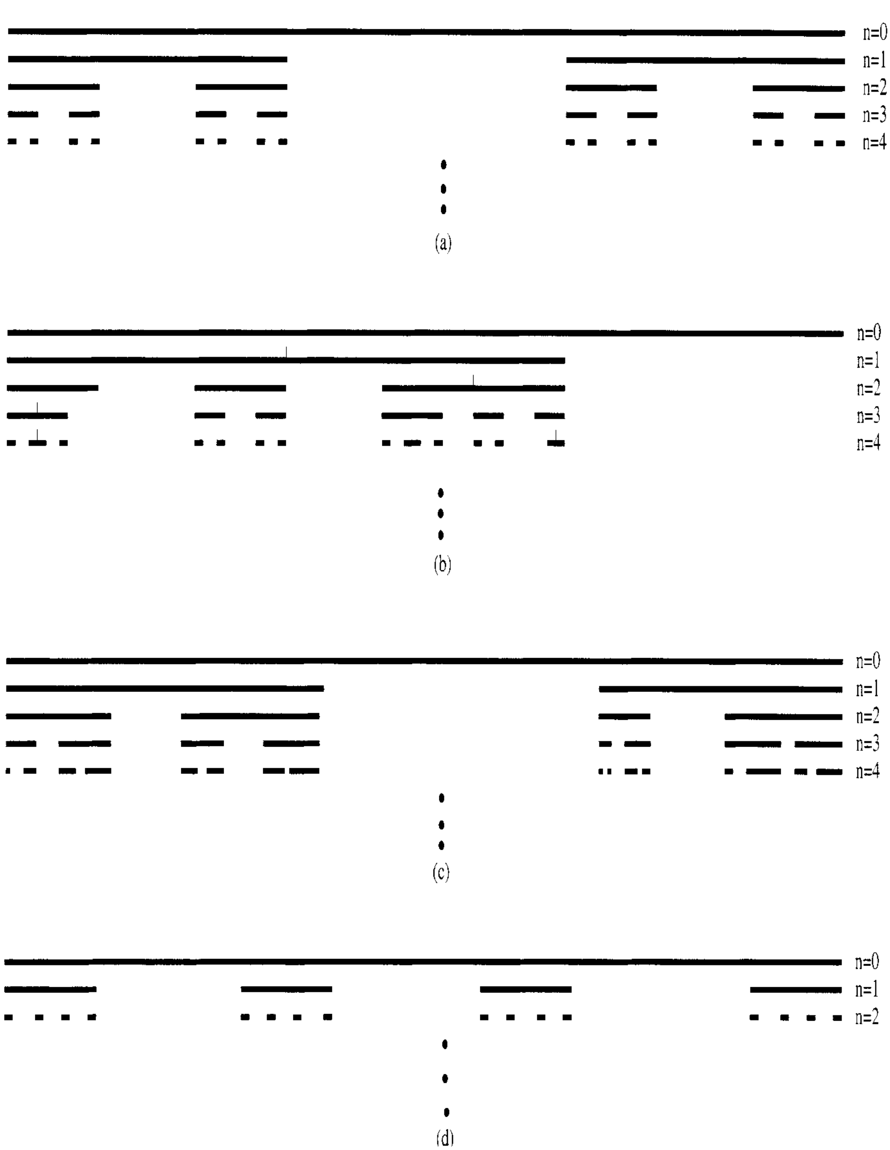

Figure 5.2. Examples of Cantor sets: (a) standard, (b) and (c) random, and (d) regular, with the same dimension as (a). See the text for details.

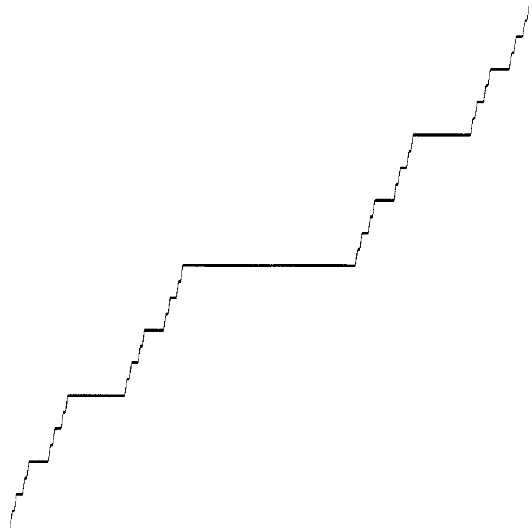

Figure 5.3. Devil's staircase constructed based on the sixth stage of the Cantor set.

Since $0 < D < 1$, the Cantor set is often called a dust. To better appreciate the self-similar feature of the Cantor set, one can construct a function, called a devil's staircase, which is simply the summation of the Cantor set — whenever there is a gap, the function remains constant. Figure 5.3 shows a devil's staircase based on the sixth stage of the Cantor set. The staircase is richest when the limiting Cantor set is used for the construction.

The fractal dimension of the Cantor set can also be computed by exploiting the self-similar feature. Denote the number of intervals needed to cover the Cantor set at a certain stage with scale ϵ by $N(\epsilon)$. When the scale is reduced by 3, $N(\epsilon/3)$ is doubled. Since $N(\epsilon/3)/N(\epsilon) = 3^D = 2$, one immediately gets $D = \ln 2/\ln 3$.

A simple variation of the standard Cantor set is obtained by dividing each interval into three equal parts and deleting one of them at random (see Fig. 5.2(b)). Another random Cantor set is obtained by modifying the middle third of the Cantor set (Fig. 5.2(a)) so that a middle interval of random length (Fig 5.2(c)) is removed from each segment at each stage. When some regulations are imposed on the length distribution of these subintervals, the fractal dimension of these random Cantor sets can be readily computed. One way of imposing such regulations is to require that the ratio of the subinterval and its immediate parent interval follows some distribution that is stage-independent. Such a regulation is essentially a multifractal construction, as we shall see later.

The foregoing discussion suggests that two different geometrical fractals may have the same fractal dimension. To further appreciate this point, we have shown in Fig. 5.2(d) a different type of regular Cantor set. It is obtained by retaining four equally spaced segments whose length is $1/9$th of the preceding segment. Denote

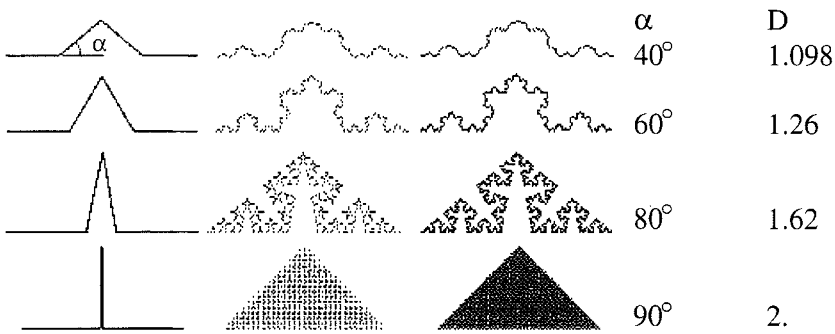

Figure 5.4. Von Koch curves

the number of segments at a certain stage with length scale ϵ by $N(\epsilon)$. When the scale is reduced by 9, $N(\epsilon/9)$ is quadrupled. Here, D is again $\ln 2/\ln 3$, since $N(\epsilon/9)/N(\epsilon) = 9^D = 4$.

5.2.2 Von Koch curves

The triadic Von Koch curve is constructed iteratively, just as the standard Cantor set is. One starts with an initiator, which is a straight line of unit length. One then superimposes an equilateral triangle on the middle third of the initiator. Finally, one removes the middle third. This completes one iteration. The resulting structure of four lines is called the generator. See the leftmost of the second row in Fig. 5.4. At the second iteration, each edge serves as an initiator and is replaced by the corresponding generator. Generators may be based on an equiangular triangle, with two edges being one third the length of the initiator and spanning an angle of α with the initiator. See the leftmost column of Fig. 5.4. The standard triadic Von Koch curve corresponds to $\alpha = 60^0$. Figure 5.4 shows the curves for four different α's of two fairly large stages. Such curves clearly are self-similar. What are their fractal dimensions?

Let us denote the length scale at iteration i by ϵ_i and the number of curves by $N(\epsilon_i)$. At iteration $i-1$, the length is $\epsilon_{i-1} = 2\epsilon_i(1+\cos\alpha)$, while $N(\epsilon_{i-1}) = N(\epsilon_i)/4$. Using Eq. (2.2), we immediately get

$$D = \frac{\ln 4}{\ln[2(1+\cos\alpha)]}.$$

In particular, $D = \ln 4/\ln 3 = 1.2618\cdots$ for the triadic Von Koch curve, and $D = 2$ when $\alpha = 90°$. The latter implies that the resulting curve is plane-filling. A moment's thought should convince one that this is indeed the case, since the resulting curve densely covers the triangle shown in the rightmost column of the last row in Fig. 5.4.

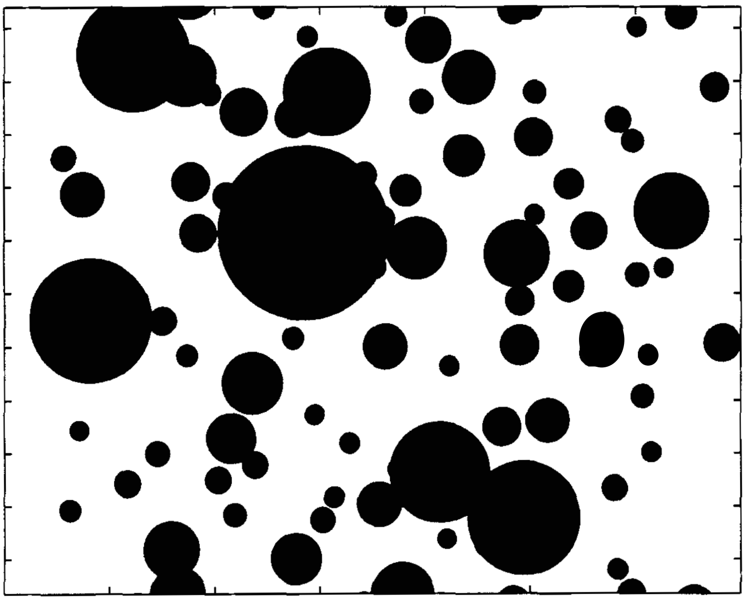

Figure 5.5. Random fractal of discs with a Pareto-distributed size: $P[X \geq x] = (1.8/x)^{1.8}$.

5.3 POWER LAW AND PERCEPTION OF SELF-SIMILARITY

The most salient property of fractal objects or phenomena is self-similarity. This is characterized by a power-law scaling relation in a wide range of temporal or spatial scales. Let us now examine how the power law is related to the perception of self-similarity.

Imagine a very large number of balls flying around in the sky. See Fig. 5.5. The size of the balls follows a power-law distribution,

$$p(r) \sim r^{-\alpha}.$$

Being human, we will instinctly focus on balls whose size is comfortable for our eyes — too small balls cannot be seen, while too large balls block our vision. Now let us assume that we are most comfortable with the scale r_0. Of course, our eyes are not sharp enough to tell the differences between scales r_0 and $r_0 + dr$, $|dr| \ll r_0$. Nevertheless, we are quite capable of identifying scales such as $2r_0$, $r_0/2$, etc. Which aspect of the flying balls may determine our perception? This is essentially given by the relevant abundance of the balls of sizes $2r_0$, r_0, and $r_0/2$:

$$p(2r_0)/p(r_0) = p(r_0)/p(r_0/2) = 2^{-\alpha}.$$

Note that the above ratio is independent of r_0. Now suppose we view the balls through a microscope, which magnifies all the balls by a scale of 100. Now our eyes will be focusing on scales such as $2r_0/100$, $r_0/100$, and $r_0/200$, and our perception will be determined by the relative abundance of the balls at those scales. Because of the power-law distribution, the relative abundance will remain the same — so does our perception.

While Fig 5.5 is intended to serve as a schematic, it could refer to lunar craters seen by projection, or holes in a piece of cheese, or a metal microfoam — foams with tiny bubbles surrounded by solid walls, such as an aluminum microfoam, seen by projection. The latter is of considerable current interest, since the presence of bubbles reduces thermal conductivity, allowing foams to act as good insulators and better resist thermal shock. Foams are also good acoustic insulators.

5.4 BIBLIOGRAPHIC NOTES

Comprehensive introductions to fractal geometry may be found in, e.g., [130, 198, 294], as well as [124].

5.5 EXERCISES

1. The dataset "crack.dat", downloadable at http://www.gao.ece.ufl.edu/GCTH_Wileybook/, describes a certain crack in a material. You could think of it as a crack on the ground or in a wall. Write a code (using whatever language you prefer) to compute $N(\epsilon)$ for different ϵ, plot $N(\epsilon)$ vs. ϵ on a log-log scale, and find the fractal dimension for this crack. How well is the scaling defined here?

2. Imagine that you are observing plants, breaking ocean waves, mountain ranges, ink stains, wrinkled or torn paper, etc. Discuss and write down qualitative observations about one or more of these objects. You might consider the following:
   How does the object fill space?
   Is its use of space dense or sparse?
   Are its edges smooth or jagged?
   What is similar throughout different parts of the object?
   What is random or different throughout different parts of the object?
   How does the object as a whole compare with its individual parts?
   What geometric shapes do you see in the object? Are the shapes similar to circles, lines, ovals, spheres?

3. Design a probability problem and associate each term on the left-hand side of the following equation with a probability:
   $$\frac{1}{3} + 2 \cdot \left(\frac{1}{3}\right)^2 + 4 \cdot \left(\frac{1}{3}\right)^3 + \cdots = 1.$$
   This is an excellent way of proving that the summation is 1.

4. Design another Cantor set with dimension $\ln 2/\ln 3$ (one possibility is to break each piece in stage i into eight pieces in stage $i+1$).

5. Reproduce Fig. 5.5. (Hint: the centers of the circles could be chosen to be the regular grid points perturbed by Gaussian random variables; the radii of the circles follow a power law. Alternatively, you could use a Poisson process to go from the center of one circle to the center of another. Although seemingly simple, a similar and popular model has been developed for network traffic: messages arrive according to a Poisson process, and the size of the messages follows a power law.)