Multiscale Analysis of Complex Time Series: Integration of Chaos and Random Fractal Theory, and Beyond
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CHAPTER 7

STABLE LAWS AND LEVY MOTIONS

The central limit theorem states that the sum of many iid random variables of finite variance is a Gaussian random variable. What happens if the condition of finite variance is dropped? In particular, one can ask: When will the distribution for the sum of the random variables and those being summed have the same functional form? This is exactly the problem asked by Paul Levy in the 1920s, and the answer is given by Levy stable laws. Consequently, the central limit theorem has been generalized. This subject has found numerous applications in fields as diverse as astrophysics, physics, engineering, finance, economics, and ecology, among many others. In fact, the applications are so broad that a book much bigger than ours will be needed to cover all of them. For this reason, no specific applications will be discussed here. Instead, we shall focus on the basic theory and provide some (casual) references about applications in the Bibliographic Notes at the end of the chapter.

To stimulate readers' curiosity about the material in this chapter, let us consider a random event, which can be described by a random variable Y and is a function of a number of unknown factors $X_1, X_2, \cdots, X_n$, $Y = f(X_1, X_2, \cdots, X_n)$. If $f(0, \cdots, 0) = 0$, then, in the first approximation using Taylor series expansion, we have

$$Y = \sum_{i=1}^{n} c_i X_i,$$

where $c_i = \frac{\partial f}{\partial X_i}\Big|_{X_1=X_2=\cdots=X_n=0}$. If the unknown factors play equivalent roles, then $c_1 = c_2 = \cdots = c_n = c$, and Y is simply the summation of the iid random variables $X_1, \cdots, X_n$. In some sense, normal distributions and the central limit theorem describe daily mundane life. Many lucky people live such a life happily. However, occasionally one has to take an unplanned journey, during which many unexpected and exciting (or terrible) things happen. Such a journey could be related to hate, love, patriotism, and so on, as illustrated by numerous classic poems, novels, and movies. Could stable laws and Levy motions describe some aspects of such unusual journeys?

7.1 STABLE DISTRIBUTIONS

Let us first introduce the equivalence relation: Two random variables X and Y are equivalent if their distributions are the same. This is denoted as

$$X \stackrel{d}{=} Y.$$

Example 1: If X is a $[0, 1]$ random variable, then

$$1 - X \stackrel{d}{=} X.$$

From example 1, it is clear that X and Y are not required to be pairwise independent.

We can also introduce the similarity relation: Two random variables X and Y are similar

$$Y \stackrel{s}{=} X$$

if there exist constants a and $b > 0$ such that

$$Y \stackrel{d}{=} a + bX.$$

Example 2: If $X \sim N(0,1)$, $Y \sim N(\mu, \sigma^2)$, then $Y \stackrel{d}{=} \mu + \sigma X$; hence, $Y \stackrel{s}{=} X$.

We can now state the definition of stable random variables. There are a few equivalent forms. While some can be derived from others, for simplicity we simply state them here.

Definition 1. A random variable Y is stable if

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} a_n + b_n Y, \tag{7.1}$$

where a_n is real, $b_n > 0$, and $Y_1, Y_2, \cdots$ are independent random variables, each having the same distribution as Y.

Definition 2. A stable random variable is called strictly stable if Eq. (7.1) holds with $a_n = 0$:

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} b_n Y, \tag{7.2}$$

where $b_n > 0$ and $Y_1, Y_2, \cdots$ are independent random variables, each having the same distribution as Y.

Observing Eq. (7.1) and rewriting

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{m} Y_i + \sum_{i=1+m}^{n} Y_i,$$

one readily sees that $\sum_{i=1}^{m} Y_i$ and $\sum_{i=1+m}^{n} Y_i$ both have to be similar to Y. We thus arrive at the following definition.

Definition 3. A random variable Y is stable if and only if for any arbitrary positive constants b' and b'' there exist constants a and $b > 0$ such that

$$b'Y_1 + b''Y_2 \stackrel{d}{=} a + bY, \tag{7.3}$$

where Y_1 and Y_2 are independent and $Y_1 \stackrel{d}{=} Y_2 \stackrel{d}{=} Y$.

Example 3: Let us examine the stability of Gaussian random variables. Assume that $Y \sim N(\mu, \sigma^2)$; then $a + bY \sim N(b\mu + a, (b\sigma)^2)$. Similarly, $b'Y_1 \sim N(b'\mu, (b'\sigma)^2))$ and $b''Y_1 \sim N(b''\mu, (b''\sigma)^2))$. By the addition rule for independent Gaussian random variables, we have $b\mu + a = (b' + b'')\mu$, $(b\sigma)^2 = (b'\sigma)^2 + (b''\sigma)^2$.

Example 4: Cauchy distribution. In Chapter 3, we derived the Cauchy distribution centered at $x = 0$:

$$f(x) = \frac{l}{\pi(l^2 + x^2)}.$$

The more general form is centered at $x = \delta$ and is given by

$$f(x) = \frac{l}{\pi[l^2 + (x - \delta)^2]}, \quad -\infty < x < \infty.$$

It is often denoted by Cauchy(l, δ). It is a stable distribution, as can be readily verified by using convolution (see exercise 2).

Example 5: Levy distribution. Let $X \sim N(0, 1)$. Consider the random variable $Y = X^{-2}$. The distribution for Y is

$$P[Y < x] = P[X^{-2} < x] = 2P[X > x^{-1/2}] = \sqrt{\frac{2}{\pi}} \int_{1/\sqrt{x}}^{\infty} e^{-u^2/2} du.$$

Differentiating the above expression with respect to x, we have

$$p_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{y^{3/2}} e^{-\frac{1}{2y}}, \quad y > 0. \tag{7.4}$$

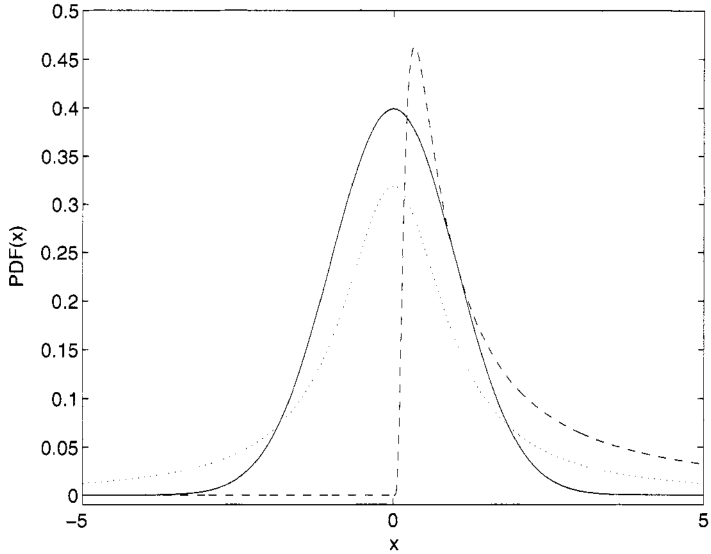

Figure 7.1. The normal (solid), Cauchy (dotted), and Levy (dashed) distributions.

This is the Levy distribution. The most general form, denoted by Levy(γ, δ), can be parameterized as

$$p_Y(y) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{(y-\delta)^{3/2}} e^{-\frac{\gamma}{2(y-\delta)}}, \quad y > \delta,$$

which corresponds to $X \sim N(\mu, \sigma^2)$. Its tail is

$$p_Y(y) \sim y^{-3/2}, \qquad y \to \infty.$$

The Levy distribution is also stable (see exercise 3).

The normal, Cauchy, and Levy distributions are the only stable distributions with closed form formulas for the densities. To better appreciate their functional dependence, Fig. 7.1 plots these three distributions. While the lack of a closed form formula for general stable distributions may be considered a drawback, fortunately characteristic functions for the stable distributions have a simple form. Although the general form for the characteristic functions can be derived based on the above definitions, for simplicity we present them as another definition. This definition enables one to readily compute stable distributions using the FFT method.

Definition 4. A random variable X is stable if and only if $X \stackrel{s}{=} Z$, where Z is a random variable with the characteristic function

$$\Phi_Z(u) = E[e^{juZ}] = \begin{cases} \exp\left(-|u|^\alpha[1 - j\beta \tan(\pi\alpha/2)\text{sign}(u)]\right), & \alpha \neq 1 \\ \exp\left(-|u|^\alpha[1 + j\beta \frac{2}{\pi} \log|u|\text{sign}(u)]\right), & \alpha = 1, \end{cases} \tag{7.5}$$

where $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, and sign is the sign function defined by

$$\text{sign}(u) = \begin{cases} -1 & u < 0 \\ 0 & u = 0 \\ 1 & u > 0. \end{cases}$$

The parameter α is called the stability index. As we shall see later, it governs how fast the tail probability decays, and $\alpha > 2$ is not allowed. The parameter β characterizes the skewness of the distribution. When $\beta = 0$, the distribution for Z is symmetric around zero. Since $X \stackrel{d}{=} aZ + b$, where $a > 0$, the distribution for X has two additional parameters a and b. They are called scale and location parameters. It is clear that the characteristic function for X is simply

$$\Phi_X(u) = e^{jbu}\Phi_Z(au).$$

More explicitly, we have

$$\Phi_X(u) = E[e^{juX}] = \begin{cases} \exp(jbu - |au|^\alpha[1 - j\beta\tan(\pi\alpha/2)\text{sign}(u)]), & \alpha \neq 1 \\ \exp(jbu - |au|^\alpha[1 + j\beta\frac{2}{\pi}\log|au|\text{sign}(u)]), & \alpha = 1. \end{cases} \tag{7.6}$$

In the above expression, we have used $\text{sign}(u) = \text{sign}(au)$ because $a > 0$.

To appreciate better the characteristic function representation, we note that the characteristic functions for the normal $N(\mu, \sigma^2)$, Cauchy(l, δ), and Levy(γ, δ) distributions are obtained by recognizing that

$$(\alpha = 2, \beta = 0, a = \sigma/\sqrt{2}, b = \mu),$$

$$(\alpha = 1, \beta = 0, a = l, b = \delta),$$

and

$$(\alpha = 1, \beta = 1, a = \gamma, b = \delta),$$

respectively.

7.2 SUMMATION OF STRICTLY STABLE RANDOM VARIABLES

Let us now examine more closely Eq. (7.1). When $Y_i \sim N(0, \sigma)$, $b_n = n^{1/2}$. We wish to find b_n for arbitrary stable distributions.

To find the answer, let us use the characteristic function representation. Denote the characteristic function for Y by $\Phi_Y(u)$. Then the characteristic function for $\sum_{i=1}^n Y_i$ is $[\Phi_Y(u)]^n$, while that for b_nY is $\Phi_Y(b_nu)$. Using Eq. (7.5), we then obtain $n = b_n^\alpha$. Therefore,

$$\sum_{i=1}^n Y_i \stackrel{d}{=} n^{1/\alpha}Y. \tag{7.7}$$

Equation (7.7) states that $(Y_1+\cdots+Y_n)/n$ has the same distribution as $Y_i n^{-1+1/\alpha}$, $i=1,\cdots,n$. Therefore, $n^{-1+1/\alpha}$ acts as the normalization factor, just as $1/\sqrt{n}$ does so for the normal distributions. If we denote the distribution for $\sum_{i=1}^{n} Y_i$ as $p_n(y)$, i.e., the n-step distribution, then Eq. (7.7) means that

$$p_n(y) = p_1\left(\frac{y}{n^{1/\alpha}}\right)\frac{1}{n^{1/\alpha}}.$$

This is the basic scaling property governing the self-similarity of the processes defined by summation of stable laws (they are called Levy flight and will be discussed soon).

While Eq. (7.7) appears to be very simple, it actually shows that $\alpha > 2$ is not a valid parameter. To see this, we note that the variance for the left-hand side is $n\mathrm{Var}Y$, while that for the right-hand side is $n^{2/\alpha}\mathrm{Var}Y$. We thus have

$$n\mathrm{Var}Y \;=\; n^{2/\alpha}\mathrm{Var}Y.$$

We identify three cases:

1. When $\mathrm{Var}Y$ is finite but not zero, α has to be 2. This is the normal case.
2. When $0 < \alpha < 2$, $\mathrm{Var}Y = \infty$.
3. When $\alpha > 2$, $\mathrm{Var}Y = 0$. This means that the density function, as a function of y, must be negative for some y and thus is not a valid probability density function. Therefore, $\alpha > 2$ is not allowed.

There is another way of deriving $\mathrm{Var}Y = 0$ for $\alpha > 2$. Starting with the characteristic function $\Phi_Y(u) = \exp(-|au|^\alpha)$ and noting that

$$\mathrm{Var}Y = -\frac{d^2\Phi_Y(u)}{du^2}\bigg|_{u=0},$$

one indeed sees that $\mathrm{Var}Y = 0$. The details are left as exercise 4.

7.3 TAIL PROBABILITIES AND EXTREME EVENTS

So far as tail probabilities are concerned, one can simply examine the symmetric Levy laws with the characteristic function $\exp(-|au|^\alpha)$. Let us denote the corresponding density function by $l_\alpha(a,x)$. It can be obtained by the inverse Fourier transform and is given by

$$\begin{aligned} l_\alpha(a,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-jux)\exp(-|au|^\alpha)du \\ &= \frac{1}{\pi}\int_0^{\infty}\cos(ux)\exp[-(au)^\alpha]du. \end{aligned} \tag{7.8}$$

We can expand $\exp[-(au)^\alpha]$ to obtain

$$\begin{aligned} l_\alpha(a,x) &= \frac{1}{\pi}\int_0^\infty \cos(ux)\sum_{m=0}^\infty \frac{[-(au)^\alpha]^m}{m!}du \\ &= \frac{1}{\pi}\sum_{m=0}^\infty \frac{(-1)^m a^{\alpha m}}{m!}\int_0^\infty \cos(ux)u^{\alpha m}du. \end{aligned}$$

Utilizing the identity

$$\int_0^\infty \cos(t)t^{v-1}dt = \cos\left(\frac{\pi v}{2}\right)\Gamma(v),$$

we obtain

$$l_\alpha(a,x) = \frac{1}{\pi x}\sum_{m=1}^\infty \frac{a^{\alpha m}\sin(\alpha m\pi/2)\Gamma(\alpha m+1)(-1)^{m-1}}{m!x^{\alpha m}}.$$

The leading order term is

$$l_\alpha(a,x) \sim \frac{a^\alpha \sin(\alpha\pi/2)\Gamma(\alpha)\alpha}{\pi x^{1+\alpha}}, \qquad \text{as} \quad x\to\infty,$$

which is a power law. If one is also concerned with the tail probability for the negative x, one can replace x by $|x|$.

We now consider extreme events. For comparison purposes, we consider both power-law tails and exponential distributions.

1. Extreme events with the power-law tail.

Suppose

$$p(x) = \frac{A}{x^{1+\alpha}} \quad \text{as} \quad x\to\infty \quad \text{(PDF)}$$

$$P(x) = 1 - \frac{A}{\alpha x^\alpha} \quad \text{as} \quad x\to\infty \quad \text{(CDF)}.$$

Consider iid steps Δx_i. Let

$$\Delta x_{(n)} = \max(\Delta x_1, \Delta x_2, \cdots, \Delta x_n).$$

Then

$$\text{Prob}\{\Delta x_{(n)}\} < x\} = P(x)^n = \left(1 - \frac{A}{\alpha x^\alpha}\right)^n \qquad \text{as} \quad x\to\infty.$$

Normalizing $\Delta x_{(n)}$ by the factor $\left(\frac{A}{\alpha}n\right)^{1/\alpha}$, we have

$$Z_n = \frac{\Delta x_{(n)}}{\left(\frac{A}{\alpha}n\right)^{1/\alpha}}.$$

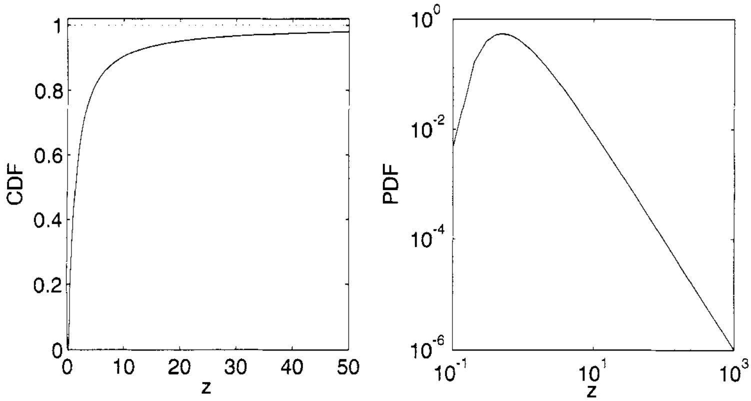

Figure 7.2. Frechet distribution with $\alpha = 1$.

Then

$$\frac{A}{\alpha x^{\alpha}} = z^{-\alpha}/n.$$

Using the identity

$$\lim_{n \to \infty} (1 - v/n)^n \to e^{-v},$$

we obtain

$$\text{Prob}(Z_n < z) = e^{-z^{-\alpha}}.$$

This is the Frechet distribution. See Fig. 7.2. For large z, the density decreases as a power law, $z^{-(\alpha+1)}$. This is evident from the right-hand plot of Fig. 7.2. Since the mean and standard deviation of $\Delta x_{(n)}$ are the mean and standard deviation of Z multiplied by the scaling factor $\left(\frac{A}{\alpha} n\right)^{1/\alpha}$, the mean and standard deviation of $\Delta x_{(n)}$ thus have the same scaling law,

$$< \Delta x_{(n)} > \sim n^{1/\alpha},$$

$$\sigma_{\Delta x_{(n)}} \sim n^{1/\alpha}.$$

Finally, let us consider the mth moment of the iid steps:

$$\frac{1}{n} \sum_{i=1}^{n} \Delta x_i^m.$$

The largest term, which is on the order of $\frac{n^{m/n}}{n}$, dominates the sum and is infinite when $m > \alpha$.

2. Extreme events for exponential distributions.

Now the CDF is

$$P(\Delta x < x) = 1 - e^{-x/x_0},$$

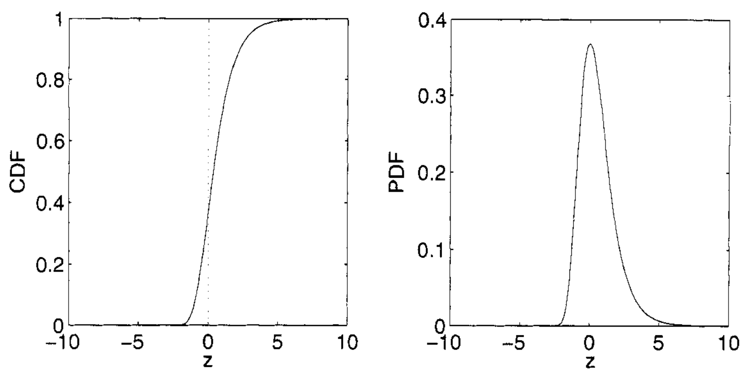

Figure 7.3. The Fischer-Tippet distribution.

where x_0 is the mean of the random variable X. Then

$$\mathrm{Prob}\{\Delta x_{(n)} < x\} = (1 - e^{-x/x_0})^n.$$

Rewriting e^{-x/x_0} as

$$\exp\left(-\frac{x - x_0 \ln n}{x_0}\right)\Big/ n,$$

letting

$$Z_n = \frac{\Delta x_{(n)} - x_0 \ln n}{x_0},$$

and taking large n limit, we get

$$P(Z_n \le z) = e^{-e^{-z}}.$$

This is the Fischer-Tippet distribution. See Fig. 7.3. Its density function is $e^{-z}e^{-e^{-z}}$, and the mean is the Euler constant $\gamma = -\int_0^\infty e^{-x} \ln x dx$. Therefore, to leading order,

$$< \Delta x_{(n)} > \sim x_0(\ln n + \gamma) + \cdots .$$

Similarly, to leading order, the variance of $\Delta x_{(n)}$ is

$$\sigma^2_{\Delta x_{(n)}} \sim x_0^2 + \pi^2/6 + \cdots .$$

7.4 GENERALIZED CENTRAL LIMIT THEOREM

The classical central limit theorem says that the normalized sum of independent identical terms with a finite variance converges to a normal distribution. To be more precise, let $X_1, X_2, \cdots$ be iid random variables with mean μ and variance σ^2. Then the classical central limit theorem states that the sample mean $\overline{X}$ =

$(X_1 + \cdots + X_n)/n$ has the same distribution as $N(\mu, \sigma^2/n)$ for large enough n. In other words,

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} Z \sim N(0,1) \quad \text{as} \quad n \longrightarrow \infty.$$

The above equation can be rewritten as

$$a_n(X_1 + \cdots + X_n) - b_n \xrightarrow{d} Z \sim N(0,1) \quad \text{as} \quad n \longrightarrow \infty,$$

where $a_n = 1/(\sigma\sqrt{n})$ and $b_n = \sqrt{n}\mu/\sigma$.

The generalized central limit theorem is concerned with the summation of random variables with infinite variance. Note that we can always group the summation $X_1 + X_2 + \cdots + X_n$ into two terms, where the first is the summation of the first k terms, while the second is the summation of the remaining $l = n - k$ terms. If the limit for the summation exists, then each term has to converge to a stable random variable and has a similar distribution. More concretely, the generalized central limit theorem can be stated as follows.

Generalized Central Limit Theorem. Let $X_1, \cdots, X_n$ be iid random variables. There exist constants $a_n > 0$, $b_n \in R$ and a nondegenerate random variable Z with

$$a_n(X_1 + \cdots + X_n) - b_n \xrightarrow{d} Z$$

if and only if Z is α-stable for some $0 < \alpha \leq 2$.

The generalized central limit theorem makes it clear that an α-stable distribution can be considered an attractor: even though the iid random variables $X_1, \cdots, X_n$ may not be α-stable, when the sum is suitably normalized and shifted, it converges to an α-stable distribution. Because of this, the random variable is said to be in the domain of attraction of the α-stable distribution. All the random variables with finite variance are in the domain of attraction of the normal distribution.

7.5 LEVY MOTIONS

A stochastic process $\{L_\alpha(t), t \geq 0\}$ is called (standard) symmetric α-stable Levy motion if

1. $L_\alpha(t)$ is almost surely 0 at the origin $t = 0$;
2. $L_\alpha(t)$ has independent increments; and
3. $L_\alpha(t) - L_\alpha(s)$ follows an α-stable distribution with characteristic function $e^{-(t-s)|u|^\alpha}$, where $0 \leq s < t < \infty$.

Observe that α-stable Levy motion is simply Bm when $\alpha = 2$. More precisely, $L_2(t) = \sqrt{2}B(t)$. The symmetric α-stable Levy motion is $1/\alpha$ self-similar. That is, for $c > 0$, the processes $\{L_\alpha(ct), t \geq 0\}$ and $\{c^{1/\alpha} L_\alpha(t), t \geq 0\}$ have the same finite-dimensional distributions. By this argument as well as Eq. (7.7), it is clear that the length of the motion in a time span of Δt, $\Delta L(\Delta t)$ is given by the following scaling:

$$\Delta L(\Delta t) \propto \Delta t^{1/\alpha}.$$

This contrasts with the scaling for fractional Brownian motion:

$$\Delta B_H(\Delta t) \propto \Delta t^H.$$

We thus note that $1/\alpha$ plays the role of H. Following the arguments of Sec. 6.4, we see that there are also two dimensions for Levy motions. One is the graph dimension, $2 - 1/\alpha$; the other is the self-similarity dimension, α.

One can also define asymmetric Levy motions by utilizing arbitrary α-stable distributions. When each step takes the same time regardless of length, the process is usually called a Levy flight. Figure 7.4 shows two examples of Levy flights with $\alpha = 1.5$ and 1. The difference between Bm's and Levy motions is that Bm's appear everywhere similarly, while Levy motions are comprised of dense clusters connected by occasional long jumps: the smaller α is, the more frequently the long jumps appear.

Let us now consider where Levy flight-like esoteric processes could occur. One situation could be this: a mosquito headed toward a giant spider web and got stuck; it struggled for a while, and luckily, with the help of a gust of wind, escaped. When the mosquito was struggling, the steps it could take were tiny; but the step leading to its fortunate escape had to be huge. As another example, let us consider (American) football games. For most of the time during a game, the offense and defense are fairly balanced, and the offense team may only be able to advance a few yards. But during an attack that leads to a touchdown, the hero getting the touchdown often "flies" tens of yards — he has somewhat escaped the defense. While these two simple examples are only analogies, remembering them could be helpful when reading research papers searching for Levy statistics in various types of problems, such as animal foraging. At this point, we should also mention that Levy flight-like patterns have been used as a type of screen saver for Linux operating systems.

A Levy motion is called a Levy walk when the time taken for each step is proportional to its length. Levy flights and walks have very different scaling behaviors: the former are characterized by memoryless jumps governed by a heavytail, while the latter, now having a fixed step size, have attained serial correlations within each huge jump.

7.6 SIMULATION OF STABLE RANDOM VARIABLES

In Chapter 3 (see Eq. (3.28)), we discussed simulation of normal random variables. For the Cauchy distribution, the inverse function method discussed in Chapter 3 can

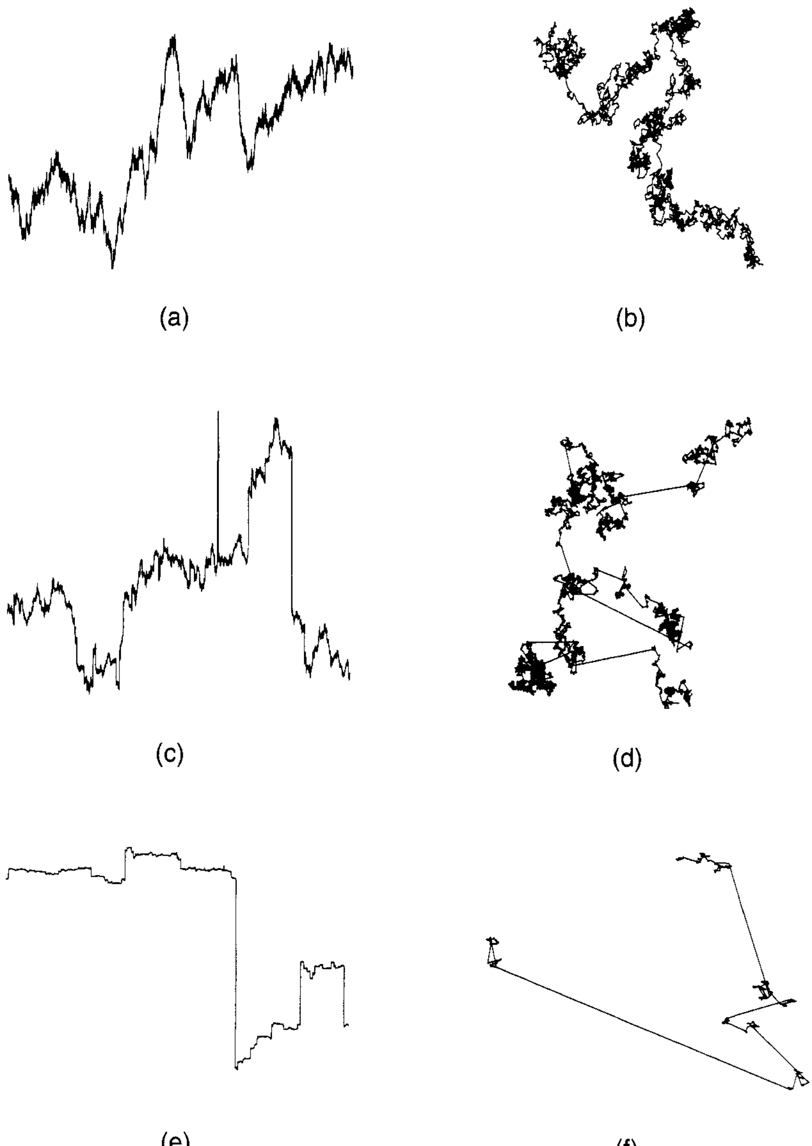

Figure 7.4. One- and two-dimensional Bm's (a,b) vs. Levy flights with $\alpha = 1.5$ and 1 for (c,d) and (e,f), respectively.

be readily applied. More precisely,

$$X = l \tan[\pi(U - 1/2)] + \delta \tag{7.9}$$

is Cauchy(l, δ).

Simulation of the Levy distribution using the definition is also straightforward. That is,

$$X = \gamma \frac{1}{Z^2} + \delta \tag{7.10}$$

is Levy(γ, δ) if $Z \sim N(0,1)$.

The first breakthrough in the simulation of stable distributions was made by Kanter [247], who provided a direct method for simulating a stable distribution with $\alpha < 1$ and $\beta = 1$. The approach was later generalized to the general case by Chambers et al. [68]. We first describe the algorithm for constructing a standard stable random variable $X \sim S(\alpha, \beta; 1)$.

Theorem: Let Θ and W be independent, with Θ uniformly distributed on $(-\pi/2, \pi/2)$ and W exponentially distributed with mean 1. For $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, and $\theta_0 = \arctan[\beta \tan(\pi\alpha/2)]/\alpha$,

$$Z = \begin{cases} \frac{\sin \alpha(\theta_0+\Theta)}{(\cos \alpha\theta_0 \cos\Theta)^{1/\alpha}} \left\{ \frac{\cos[\alpha\theta_0 + (\alpha-1)\Theta]}{W} \right\}^{(1-\alpha)/\alpha} & \alpha \neq 1 \\ \frac{2}{\pi}\left[(\frac{\pi}{2} + \beta\Theta)\tan\Theta - \beta \ln\left(\frac{\frac{\pi}{2} W \cos\Theta}{\frac{\pi}{2} + \beta\Theta} \right) \right] & \alpha = 1 \end{cases} \tag{7.11}$$

has a $S(\alpha, \beta; 1)$ distribution.

For a symmetric stable distribution with $\beta = 0$, the above expression can be greatly simplified. To simulate an arbitrary stable distribution $S(\alpha, \beta, \gamma, \delta)$, we can simply take the following linear transform,

$$Y = \begin{cases} \gamma Z + \delta & \alpha \neq 1 \\ \gamma Z + \frac{\pi}{2}\beta\gamma \ln\gamma + \delta & \alpha = 1, \end{cases} \tag{7.12}$$

where Z is given by Eq. (7.11).

7.7 BIBLIOGRAPHIC NOTES

There is a huge literature on the subject discussed here. It includes a few books, such as [381, 455], among others. An interesting general paper is [239]. An interesting website with a good discussion of stable laws is
http://www.quantlet.com/mdstat/scripts/csa/html/node235.html.
In fact, this is part of a handbook about statistics in finance. See also John Nolan's website http://academic2.american.edu/~jpnolan/stable/stable.html, *where one can* find his draft first chapter for a book being written, and a file with many references

emphasizing applications in finance. The 2001 class notes of Professor Martin Z. Bazant of MIT also have some materials relevant to this chapter. Readers keen on a physics perspective are referred to two very entertaining general review articles by a few pioneers [260, 403], as well as two topical reviews focusing on fractional dynamics [305, 306]. See also [472, 473] on the application of Levy statistics to random water waves. For simulation of stable random variables, refer to [68, 247].

Since the publication of Mandelbrot's pioneering work on applying Levy statistics to economics [296], Levy statistics have found numerous applications in many fields. In fact, the book by Uchaikin and Zolotarev [455] has nine chapters on applications. Here are some of the applications:

- Fluid mechanics: In 1993, Henry Swinney considered a rotating container of fluid shaped like a washer. When the container rotated faster, vortices appeared in the fluid. It was shown that the tracer particles followed Levy flights between the vortices with $\alpha = 1.3$ [407].

- Device physics: Levy statistics have been found in Josephson junctions [184], at liquid-solid interfaces [411], and, more recently, in electrical conduction properties due to chaotic electron trajectories [308].

- Foraging movements of animals: This is an active area of research. The animals studied include albatrosses, microzooplankton, seabirds, spider monkeys, and others. See [35, 144, 362, 458, 459]. One of the most important topics here is how to optimize the success of random searches. An interesting finding is that when the food or target density is low, a Levy walk is an optimal solution.

- Art: In the so-called automatic painting developed by the Surrealist art movement, artists paint with such speed that any conscious involvement is thought to be eliminated. One of the masterpieces created by this technique is Jackson Pollock's *Autumn Rhythm*, produced by dripping paint onto large horizontal canvases. Taylor et al. [429] found that Pollock's paintings are fractal and that his motions can be described as Levy flights. This might offer a clue to the mysterious behavior of unconsciousness.

7.8 EXERCISES

1. Using the convolution formula, show that the normal distribution is stable.

2. Using the convolution formula, show that the Cauchy distribution is stable.

3. Using the convolution formula, show that the Levy distribution is stable.

4. Taking the second derivative of the structure function $\Phi_Y(u) = \exp(-|au|^\alpha)$, show that $\mathrm{Var}Y = 0$ when $\alpha > 2$.

5. In Sec.7.3, we considered tail distribution. Starting from Eq. (7.8) and expanding $\cos(ux)$, show that for small x,

$$l_\alpha(a,x) = \frac{1}{\pi\alpha a}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\Gamma\Big(\frac{2M+1}{\alpha}\Big)\Big(\frac{x}{a}\Big)^{2m}.$$

Further, show that the leading order term is

$$l_\alpha(a,x) \sim \frac{\Gamma(1/\alpha)}{\pi\alpha a}.$$

Thus, the center gets sharper and higher as $\alpha \to 0$.

6. Reproduce Fig. 7.4.