Loop Invariants and Insertion-sort COMS10017 - Algorithms 1

Dr Christian Konrad

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Case $A[i] \leq m_i$:

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• **Termination:** When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.

At the end of the loop, i.e., after iteration n-1 (or before a virtual iteration n) $m=m_n=\max\{A[j]:j< n\}$ \checkmark

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 integer $s \leftarrow 1$ for $j = 2, \dots, n$ do $s \leftarrow s \cdot j$ return s

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4 / 8

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Algorithm computes the factorial function

Example: Insertion Sort

Sorting Problem

- **Input:** An array A of n numbers
- Output: A reordering of A s.t. $A[0] \le A[1] \le \cdots \le A[n-1]$

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 of n numbers for $j=1,\ldots,n-1$ do $v \leftarrow A[j]$ $i \leftarrow j-1$ while $i \geq 0$ and $A[i] > v$ do $A[i+1] \leftarrow A[i]$ $i \leftarrow i-1$ $A[i+1] \leftarrow v$

Insertion-Sort

5 / 8

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15	7	3	9	8	1

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$$i = 0$$
 1
 $j = 2$
 3
 4
 5

 7
 15
 15
 9
 8
 1

$$v \leftarrow 3$$

Require: Array
$$A$$
 of n numbers for $j=1,\ldots,n-1$ do $v \leftarrow A[j]$ $i \leftarrow j-1$ while $i \geq 0$ and $A[i] > v$ do $A[i+1] \leftarrow A[i]$ $i \leftarrow i-1$ $A[i+1] \leftarrow v$

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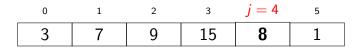
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Loop Invariant: At beginning of iteration j of the outer **for** loop, the subarray A[0,j-1] consists of the elements originally in A[0,j-1], but in sorted order

7/8

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7/8

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7/8

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- **Termination:** After iteration j = n 1 (i.e., before iteration j = n) the loop invariant states that A is sorted. \checkmark

Worst-case Runtime:

We have two nested loops

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E.g., if input is already sorted