Video 24: Elements of Dynamic Programming

COMS10017 - Algorithms 1

Dr Christian Konrad

Solving a Problem with Dynamic Programming:

Elements of Dynamic Programming

Solving a Problem with Dynamic Programming:

1. Identify optimal substructure

Elements of Dynamic Programming

Solving a Problem with Dynamic Programming:

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Discussion:

- Steps 1 and 2 requires studying the problem at hand
- Steps 3 and 4 are usually straightforward

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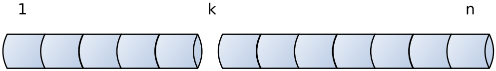

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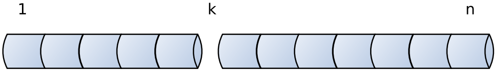

- Matrix-Chain-Parenthesization:

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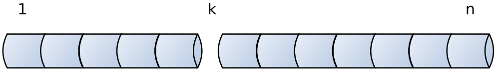

- MATRIX-CHAIN-PARENTHESIZATION: If in OPT final multiplication is $A_{1k} \times A_{(k+1)n}$ then OPT contains optimal parenthesizations of $A_1 \times \cdots \times A_k$ and $A_{k+1} \times \cdots \times A_n$

$$(A_1 \times (A_2 \times A_3)) \times ((A_4 \times A_5) \times A_6)$$

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$$\boxed{\begin{aligned} m[i,j] &= \min_{i \leq k < j} m[i,k] + m[k+1,j] \\ &+ \text{“cost for computing } A_{ik} \times A_{(k+1)j}\text{”} \end{aligned}}$$

Compute Optimal Costs

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length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \le k \le i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
-	-	-	-	-	-	-	-	-	-	-

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$m[2] = \max\{p_1 + m_1, p_2 + m_0\} = \max\{1 + 1, 5 + 0\} = 5$

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$$m[3] = \max\{p_1+m_2, p_2+m_1, p_3+m_0\} = \max\{1+5, 5+1, 8+0\} = 8$$

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$m[4] = \max\{p_1 + m_3, p_2 + m_2, p_3 + m_1, p_4 + m_0\} = \max\{1 + 8, 5 + 5, 8 + 1, 9\} = 10$

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0	1	5	8	10	-	-	-	-	-	-

$m[5] = \max\{p_1 + m_4, p_2 + m_3, p_3 + m_2, p_4 + m_1, p_5 + m_0\} = \max\{1 + 10, 5 + 8, 8 + 2, 9 + 1, 10\} = 13$

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The maximum revenue obtainable for a pole of length 10 is 30

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But how can we find out how to cut the pole?

Step 4: Construct Optimal Solution

Keep Track of Optimal Choices: store optimal choices in array s

Require: Integer n, array p of length n with prices
  Let $r[0 \dots n]$ be a new array
  $r[0] \leftarrow 0$
  for $j = 1 \dots n$ **do**
    $r[j] \leftarrow -\infty$
    for $i = 1 \dots j$ **do**
      $r[j] \leftarrow \max\{r[j], p[i] + r[j-i]\}$
  return $r[n]$

Algorithm $\textsc{Bottom-Up-Cut-Pole}(p,\ n)$

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      if $p[i] + r[j - i] > q$ **then**
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Algorithm BOTTOM-UP-CUT-POLE(p, n)

- $s[i]$ contains position of first cut in optimal solution
- Easy to reconstruct all cuts

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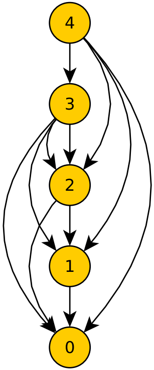

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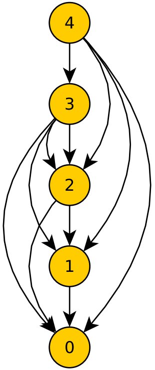

Runtime of Dynamic Programming Algorithm:

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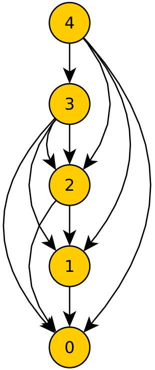

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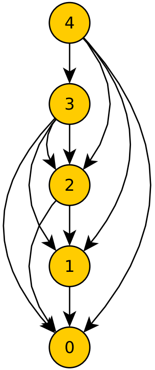

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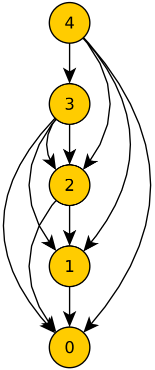

Runtime of Dynamic Programming Algorithm:

- Total number of subproblems t
- Maximum number of subproblems a subproblem depends on s
- Runtime: $O(s \cdot t)$ (assuming that computing solution takes time $O(s)$)

Fibonacci Numbers:

$$\begin{aligned} F_0 &= 0, F_1 = 1 \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2 \ . \end{aligned}$$

Require: Integer $n \geq 0$
  if $n \leq 1$ **then**
    return n
  else
    $A \leftarrow$ array of size n
    $A[0] \leftarrow 1$, $A[1] \leftarrow 1$
    for $i \leftarrow 2 \dots n$ **do**
      $A[i] \leftarrow A[i-2] + A[i-1]$
    return $A[n]$

$\textsc{DynPrgFib}(n)$

Why is this a dynamic programming algorithm?

Fibonacci Numbers - Dynamic Programming

Identify Optimal Substructure:

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- Recall: $F_n = F_{n-1} + F_{n-2}$

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Give Recursive Solution:

- Recursive solution is already given in the problem description

Fibonacci Numbers - Dynamic Programming

Identify Optimal Substructure:

- Recall: $F_n = F_{n-1} + F_{n-2}$
- (Optimal) solution to size n problem equals sum of (optimal) solutions to subproblems of sizes $n-1$ and $n-2$ ✓

Give Recursive Solution:

- Recursive solution is already given in the problem description
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Compute Optimal Costs & Compute Optimal Solution

- Cost and solution is identical for Fibonacci numbers
- There is no need to keep track of optimal choices, since there is only a single choice

Maximum Subarray Problem

Problem: MAXIMUM-SUBARRAY

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- **Input:** Array A of n numbers
- **Output:** Indices $0 \leq i \leq j \leq n-1$ such that $\sum_{l=i}^{j} A[l]$ is maximum.

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- In lecture 7 we gave a divide-and-conquer algorithm with runtime $O(n \log n)$

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Divide-and-Conquer Algorithm

- In lecture 7 we gave a divide-and-conquer algorithm with runtime $O(n \log n)$
- We will give now a faster dynamic programming algorithm

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Let A be an array of length n. Let i be the optimal solution for MAXIMUM-SUFFIX-ARRAY *on A. If $i < n-1$ then the optimal solution to* MAXIMUM-SUFFIX-ARRAY *on $A[0 \ldots n-2]$ is also i.*

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But then $\sum_{j=i'}^{n-1} A[j] > \sum_{j=i}^{n-1} A[j]$, a contradiction to the fact that i is optimal for A. □

Recursive Solution to Maximum Suffix Array

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m											

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Maximum constitutes optimal solution to MAXIMUM-SUBARRAY!

Dynamic Programming Algorithm for Maximum Subarray

Algorithm: Input is an array A of integers of length n

1. Compute dyn. prog. table for $\textsc{Maximum-Suffix-Array}$
2. Return the maximum value in the table

Require: Array A of n integers
  Let $m[0 \dots n-1]$ be a new array
  $m[0] \leftarrow A[0]$
  $q \leftarrow A[0]$
  for $i = 1 \dots n-1$ **do**
    if $m[i-1] < 0$ **then**
      $m[i] \leftarrow A[i]$
    else
      $m[i] \leftarrow A[i] + m[i-1]$
    $q \leftarrow \max\{q, m[i]\}$
  return q

Kadane's Algorithm for $\textsc{Maximum-Subarray}$

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- Runtime: $O(n)$ (n subproblems, only one subproblem needed to compute current value)
- Recall that Divide-and-Conquer solution has a runtime of $O(n \log n)$
- Observe that for $\textsc{Maximum-Subarray}$ Dynamic Programming and Divide-and-Conquer is applicable

Challenges:

- Compute max. subarray of size at most k, for some k
- Compute subarray $A[i,j]$ such that

$$\frac{\sum_{k=i}^{j} A[k]}{\sqrt{j-i+1}}$$

is maximized.