Recurrences I

COMS10017 - (Object-Oriented Programming and) Algorithms

Dr Christian Konrad

Divide-and-conquer Algorithms

Algorithmic Design Principle: Divide-and-conquer

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Examples

Quicksort, Mergesort, maximum subarray algorithm, binary search, FAST-PEAK-FINDING, ...

Example: Mergesort

Recall: Mergesort

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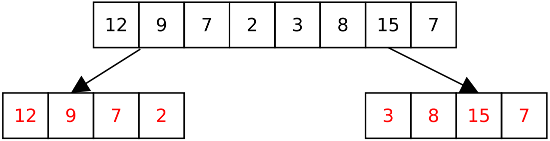
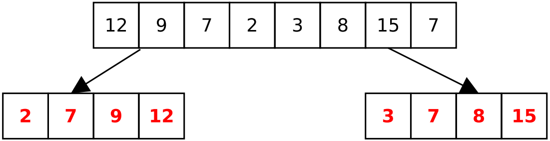
1. **Divide**
   Split input array A of length n into subarrays $A_1 = A[0, \lfloor n/2 \rfloor]$ and $A_2 = A[\lfloor n/2 \rfloor + 1, n-1]$



Example: Mergesort

Recall: Mergesort

1. **Divide** $A \rightarrow A_1$ and A_2
2. **Conquer**
   Sort A_1 and A_2 recursively using the same algorithm



Example: Mergesort

Recall: Mergesort

1. **Divide** $A \rightarrow A_1$ and A_2
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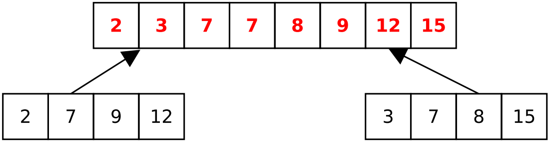
3. **Combine**
   Combine sorted subarrays A_1 and A_2 and obtain sorted array A



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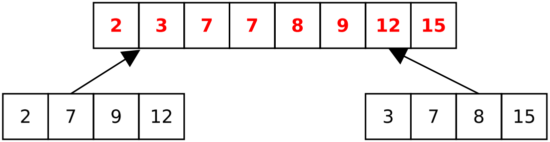

Runtime: (assuming that n is a power of 2)

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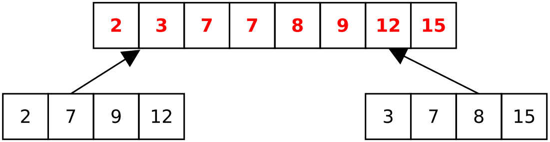

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$$T(1) = O(1)$$

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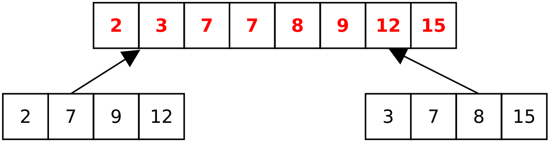

Runtime: (assuming that n is a power of 2)

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\begin{aligned}
T(1) &= O(1) \\
T(n) &= 2T(n/2) + O(n)
\end{aligned}
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How to solve Recurrences?

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  very powerful, cannot always be applied

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Step 1. Guess good upper bound

$$T(n) \leq Cn \log n$$

The Substitution Method (2)

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The base case is a problem...

The Substitution Method (3)

Recall: $T(1) = c_1$ and $T(n) = 2T(n/2) + c_2 n$
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- **Observe:** This implies $T(n) \in O(n \log n)$ (important)

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Our guess: $T(n) \le Cn \log n$ (induction step holds for $C \ge c_2$)

Solution: Choose a different base case! $n = 2$

$$\begin{aligned} T(2) &= 2T(1) + 2c_2 = 2c_1 + 2c_2 = 2(c_2 + c_1) \\ C2 \log 2 &= 2C \end{aligned}$$

Hence, for every $C \ge c_2 + c_1$, our guess holds for $n = 2$:

$$T(2) \le C2 \log 2 \ .$$

Result

- We proved $T(n) \le Cn \log n$, for every $n \ge 2$, when choosing $C \ge c_1 + c_2$
- **Observe:** This implies $T(n) \in O(n \log n)$ (important)

Asymptotic notation allows us to chose arbitrary base case

A Strange Problem

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Example: Give an upper bound on the recurrence

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$$T(n)$$

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$$T(n) \leq C\lceil n/2 \rceil + C\lfloor n/2 \rfloor + 1$$

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$f_1 : T(n)$

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$$f_1 : T(n) \le C\lceil n/2 \rceil + 1 + C\lfloor n/2 \rfloor + 1 + 1$$

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$$f_1 : T(n) \quad \le \quad C\lceil n/2 \rceil + 1 + C\lfloor n/2 \rfloor + 1 + 1 = Cn + 3$$

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(holds for every positive C)

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Verify Base Case for f_2

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Comments

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- Guessing right can be difficult and requires experience

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Comments

- Guessing right can be difficult and requires experience
- However, recursion tree method can provide a good guess!