Loop Invariants and Insertion-sort

COMS10017 - (Object-Oriented Programming and) Algorithms

Dr Christian Konrad

Loop Invariants

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  At the end of the loop, i.e., after iteration $n - 1$ (or before a virtual iteration n) $m = m_n = \max\{A[j] \ : \ j < n\}$ ✓

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Algorithm computes the factorial function

Example: Insertion Sort

Sorting Problem

- **Input:** An array A of n numbers
- **Output:** A reordering of A s.t. $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

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0	1	2	3	4	5
15	7	3	9	8	1

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0	$j = 1$	2	3	4	5
15	7	3	9	8	1

$v \leftarrow 7$

Example:

Require: Array A of n numbers
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    $i \leftarrow i - 1$

  $A[i+1] \leftarrow v$

0	1	$j = 2$	3	4	5
7	15	3	9	8	1

$v \leftarrow 3$

Example:

Require: Array A of n numbers

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0	$i = 1$	$j = 2$	3	4	5
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0	1	2	3	4	$j = 5$
3	7	8	9	15	1

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0	1	2	3	4	$j = 5$
1	3	7	8	9	15

Loop Invariant of Insertion-sort

for $j = 1, \dots, n-1$ **do**
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Loop Invariant of Insertion-sort

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for j = 1, ..., n − 1 do
    v ← A[j]
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    while i ≥ 0 and A[i] > v do
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Loop Invariant: At beginning of iteration j of the outer **for** loop, the subarray $A[0, j-1]$ consists of the elements originally in $A[0, j-1]$, but in sorted order

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- **Initialization:** $j = 1$:

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- **Initialization:** $j = 1$: subarray $A[0]$ is sorted ✓

Loop Invariant of Insertion-sort

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\end{array}
$$

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- **Maintenance:** *Informally*, element $A[j]$ is inserted at the right place within $A[0, j]$. A formal argument would require another loop invariant for the inner loop. ✓
- **Termination:** After iteration $j = n - 1$ (i.e., before iteration $j = n$) the loop invariant states that A is sorted. ✓

Worst-case Runtime of Insertion-sort

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$$\sum_{j=1}^{n-1} j \cdot O(1) = O(1) \sum_{j=1}^{n-1} j = O(1) \frac{n(n-1)}{2} = O(1)(n^2 - n) = O(n^2) \, .$$

Best-case Runtime:

Worst-case Runtime of Insertion-sort

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- The outer loop goes from $j = 1$ to $j = n - 1$
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Best-case Runtime: $O(n)$
E.g., if input is already sorted