Proofs by Induction (Recap)

COMS10018 - Algorithms

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Proofs by Induction

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Proofs by Induction and Loop Invariants

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- This is often done by induction
- We will use proofs by induction for proving loop invariants (soon) and for solving recurrences (later)

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Proofs by Induction

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1. **Statement to Prove:**
   $P(n)$ holds for all $n \in \mathbb{N}$
   (or $n \in \mathbb{N} \cup \{0\}$)
   (or n integer and $n \geq k$)
   (or similar)

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4. **Base case:** Prove that $P(1)$ holds
   Domino 1 falls

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□

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Example: $a^n = 1$, for every $a \neq 0$ and n nonnegative integer

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2. Induction hypothesis: $a^m = 1$, for every $0 \leq m \leq n$ (strong induction)
3. Induction step:

$$a^{n+1} = a^{2n-(n-1)} = \frac{a^{2n}}{a^{n-1}} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1 \ldots .$$

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Problem: a^1 is computed as $\frac{a^0 a^0}{a^{-1}}$ and induction hypothesis does not holds for $n = -1$!