Quicksort

COMS10018 - Algorithms

Dr Christian Konrad

Quicksort

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Heapsort	$O(n \log n)$	$O(n \log n)$	no	yes

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- Very efficient in practice!
- *In place version of Mergesort:*

$A[0, \lfloor \frac{n}{2} \rfloor] \leftarrow \textsc{MergeSort}(A[0, \lfloor \frac{n}{2} \rfloor])$
$A[\lfloor \frac{n}{2} \rfloor + 1, n-1] \leftarrow \textsc{MergeSort}(A[\lfloor \frac{n}{2} \rfloor, n-1])$
$A \leftarrow \textsc{Merge}(A)$
return A

recursive calls in Mergesort

Mergesort versus Quicksort

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- *Mergesort:* First solve subproblems recursively, then merge their solutions

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- *Mergesort:* First solve subproblems recursively, then merge their solutions
- *Quicksort:* First partition problem into two subproblems in a clever way so that no extra work is needed when combining the solutions to the subproblems, then solve subproblems recursively

Quicksort

Divide and Conquer Algorithm:

Quicksort

Divide and Conquer Algorithm:

- **Divide:** Chose a good *pivot* $A[q]$. Rearrange A such that every element $\leq A[q]$ is left of $A[q]$ in the resulting ordering and every element $> A[q]$ is right of $A[q]$ in the resulting ordering. Let p be the new position of $A[q]$.

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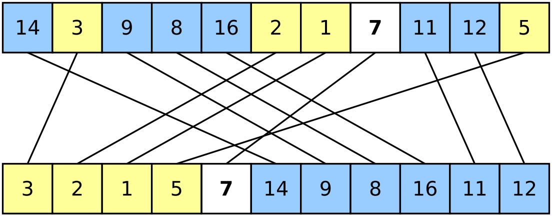

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- **Combine:** No work needed

Quicksort (2)

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1. We need to be able to rearrange the elements around the pivot in $O(n)$ time
2. What is a good pivot? Ideally we would like to obtain subproblems of equal/similar sizes

The Partition Step

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Require: Array A of length n
  x ← A[n − 1]
  i ← −1
  for j ← 0 . . . n − 1 do
    if A[j] ≤ x then
      i ← i + 1
      exchange A[i] with A[j]
  return i
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PARTITION

The Partition Step

Partition Step:

- **Input:** Array A of length n
- **Output:** Partitioning around pivot $A[n-1]$

Require: Array A of length n
   $x \leftarrow A[n-1]$
   $i \leftarrow -1$
   for $j \leftarrow 0 \dots n-1$ **do**
      if $A[j] \leq x$ **then**
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         exchange $A[i]$ with $A[j]$
   return i

PARTITION

Pivot: Algorithm assumes pivot is $A[n-1]$ (if different pivot $A[q]$ is used: swap $A[q]$ with $A[n-1]$).

Example

$x \leftarrow A[n-1]$
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1. Elements left of i (including i) are smaller or equal to x:

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2. Elements right of i (excluding i) and left of j (excluding j) are larger than x:

$$\text{For } i+1 \leq k \leq j-1 : A[k] > x$$

Proof of Loop Invariant

1. Left of i (including i): smaller equal to x
2. Right of i and left of j (excl. j): larger than x

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x \leftarrow A[n-1] \\
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\quad\quad i \leftarrow i+1 \\
\quad\quad \text{exchange } A[i] \text{ with } A[j]
\end{array}
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Initialization: $i = -1, j = 0$

i	j												
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Initialization: $i = -1, j = 0$ ✓

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Proof of Loop Invariant (2)

1. Left of i (including i): smaller equal to x
2. Right of i and left of j (excl. j): larger than x

$x \leftarrow A[n-1]$
$i \leftarrow -1$
for $j \leftarrow 0 \dots n-1$ **do**
  if $A[j] \leq x$ **then**
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Maintenance: Two cases:

1. $A[j] > x$: Then j is incremented

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Proof of Loop Invariant (3)

1. Left of i (including i): smaller equal to x
2. Right of i and left of j (excl. j): larger than x

```
x ← A[n − 1]
i ← −1
for j ← 0 . . . n − 1 do
    if A[j] ≤ x then
        i ← i + 1
        exchange A[i] with A[j]
```

Termination: (useful property showing that algo. is correct)

- $A[i]$ contains pivot element x that was located initially at position $n-1$
- All elements left of $A[i]$ are smaller equal to x
- All elements right of $A[i]$ are larger than x

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- $A[i]$ contains pivot element x that was located initially at position $n-1$
- All elements left of $A[i]$ are smaller equal to x
- All elements right of $A[i]$ are larger than x

Require: array A of length n
  if $n \leq 1$ **then**
    return A
  else
    $i \leftarrow \text{Partition}(A)$
    $\textsc{Quicksort}(A[0, i-1])$
    $\textsc{Quicksort}(A[i+1, n-1])$

Algorithm $\textsc{Quicksort}$

Quicksort

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Algorithm $\textsc{Quicksort}$

What is the runtime of Quicksort?