

Mergesort

COMS10018 - Algorithms

Dr Christian Konrad

Definition of the Sorting Problem

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- **Output:** A reordering of A s.t. $A[0] \leq A[1] \leq \dots \leq A[n-1]$

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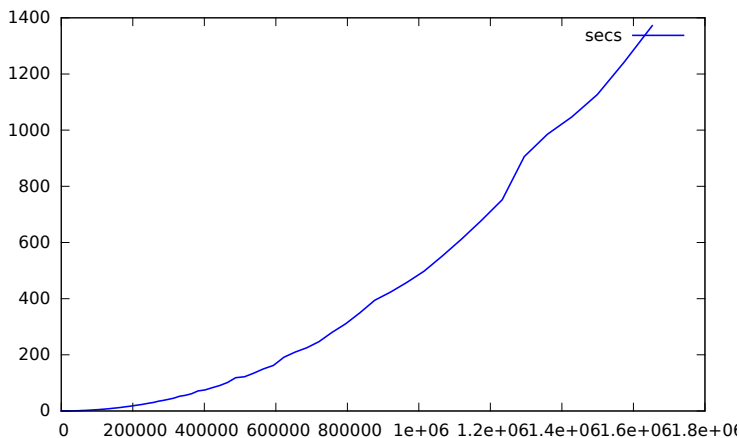
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Insertion Sort

- Worst-case runtime $O(n^2)$
- Surely we can do better?!

Insertion sort in Practice on Worst-case Instances



n	46929	102428	364178	1014570
secs	1.03084	4.81622	61.2737	497.879

Properties of a Sorting Algorithm

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Definition (in place)

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-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	----------

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family name	first name	data of birth	role
Smith	Peter	02.10.1982	lecturer
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Observe: Stability makes more sense when sorting complex data as opposed to numbers

Merge Sort

Key Idea:

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Merge Operation

- Copy left half of A to new array B
- Copy right half of A to new array C
- Traverse B and C simultaneously from left to right and write the smallest element at the current positions to A

Example: Merge Operation

A

1	4	9	10	3	5	7	11
---	---	---	----	---	---	---	----

Example: Merge Operation

A

1	4	9	10	3	5	7	11
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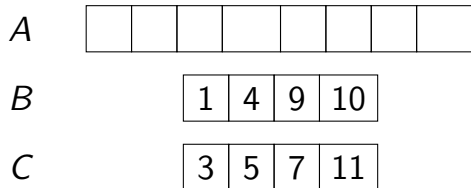
B

1	4	9	10
---	---	---	----

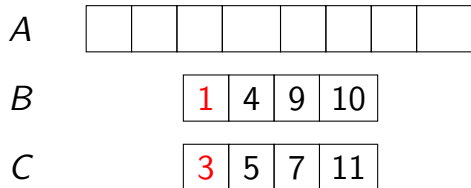
C

3	5	7	11
---	---	---	----

Example: Merge Operation



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<i>A</i>	1						
<i>B</i>		1	4	9	10		
<i>C</i>		3	5	7	11		

Example: Merge Operation

<i>A</i>	1	3						
<i>B</i>			1	4	9	10		
<i>C</i>			3	5	7	11		

Example: Merge Operation

<i>A</i>	1	3	4				
<i>B</i>		1	4	9	10		
<i>C</i>		3	5	7	11		

Example: Merge Operation

A

1	3	4	5				
---	---	---	---	--	--	--	--

B

1	4	9	10
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3	5	7	11
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Example: Merge Operation

A

1	3	4	5	7			
---	---	---	---	---	--	--	--

B

1	4	9	10
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C

3	5	7	11
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Example: Merge Operation

<i>A</i>	1	3	4	5	7	9	10	11
<i>B</i>	1	4	9	10				
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Merge Operation

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- **Input:** An array A of integers of length n (n even) such that $A[0, \frac{n}{2} - 1]$ and $A[\frac{n}{2}, n - 1]$ are sorted
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Analysis: Merge Operation

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Divide and Conquer!

Merge Sort: A Divide and Conquer Algorithm

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Require: Array A of n numbers

if $n = 1$ **then**

return A

$A[0, \lfloor \frac{n}{2} \rfloor] \leftarrow \text{MERGESORT}(A[0, \lfloor \frac{n}{2} \rfloor])$

$A[\lfloor \frac{n}{2} \rfloor + 1, n-1] \leftarrow \text{MERGESORT}(A[\lfloor \frac{n}{2} \rfloor + 1, n-1])$

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Merge Sort: A Divide and Conquer Algorithm

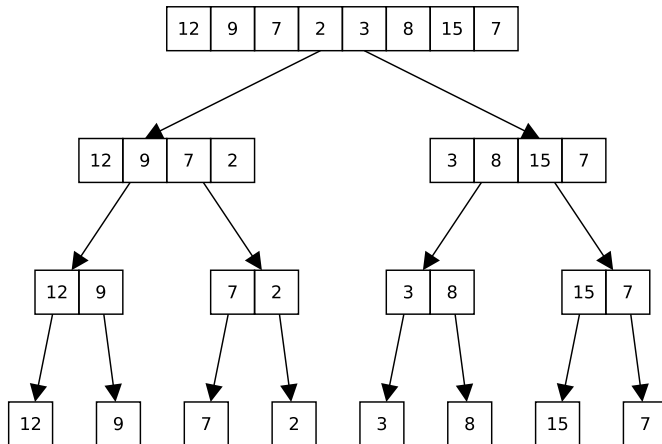
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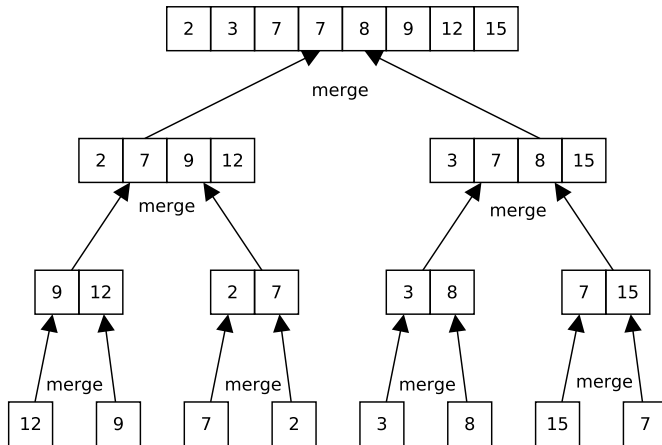
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- **Conquer** the subproblems by solving them recursively. If the subproblems are small enough, just solve them in a straightforward manner.
- **Combine** the solutions to the subproblems into the solution for the original problem.

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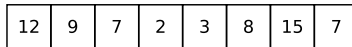
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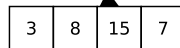
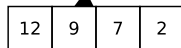
- How many levels?
- How many nodes per level?
- Time spent per node?

Number of Levels

Level 1



Level 2



Level 3



Level 4



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Number of Levels:

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$$\frac{n}{2^{l-1}} \leq 1$$

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- Array length in last but one level $l - 1$ is 2: $\lceil \frac{n}{2^{l-2}} \rceil = 2$

$$\frac{n}{2^{l-2}} > 1$$

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- Array length in last but one level $l - 1$ is 2: $\lceil \frac{n}{2^{l-2}} \rceil = 2$

$$\frac{n}{2^{l-2}} > 1 \Rightarrow n > 2^{l-2}$$

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- Array length in last level l is 1: $\lceil \frac{n}{2^{l-1}} \rceil = 1$

$$\frac{n}{2^{l-1}} \leq 1 \Rightarrow n \leq 2^{l-1} \Rightarrow \log(n) + 1 \leq l$$

- Array length in last but one level $l - 1$ is 2: $\lceil \frac{n}{2^{l-2}} \rceil = 2$

$$\frac{n}{2^{l-2}} > 1 \Rightarrow n > 2^{l-2} \Rightarrow \log(n) + 2 > l$$

Number of Levels (2)

Level i :

- 2^{i-1} nodes (at most)
- Array length in level i is $\lceil \frac{n}{2^{i-1}} \rceil$ (at most)
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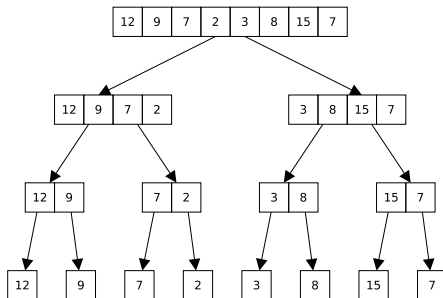
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Hence, $l = \lceil \log n \rceil + 1$.

Runtime of Merge Sort

Sum up Work:

- Levels:
 $l = \lceil \log n \rceil + 1$
- Nodes on level i :
at most 2^{i-1}
- Array length in level i :
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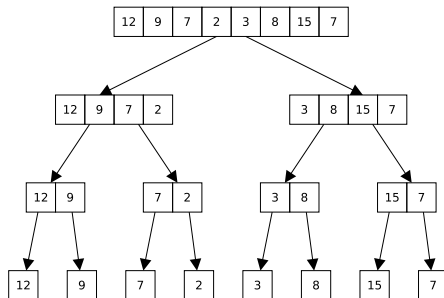


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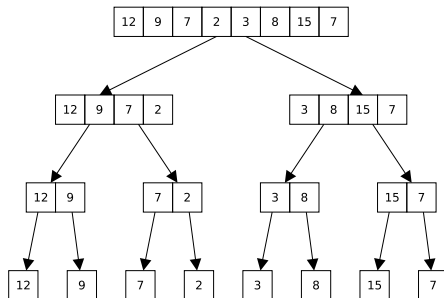
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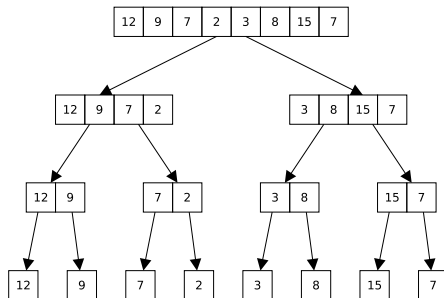
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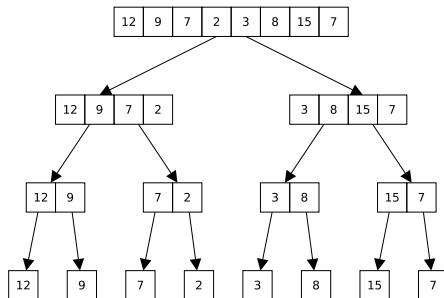
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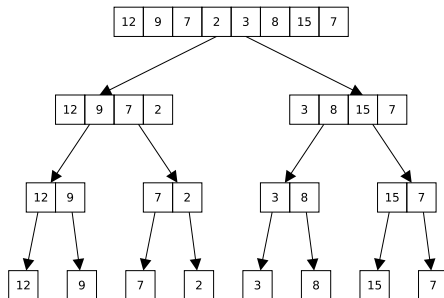
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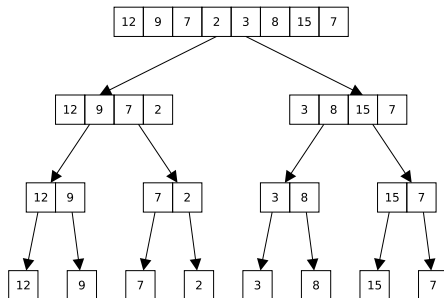
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Runtime of Merge Sort

Sum up Work:

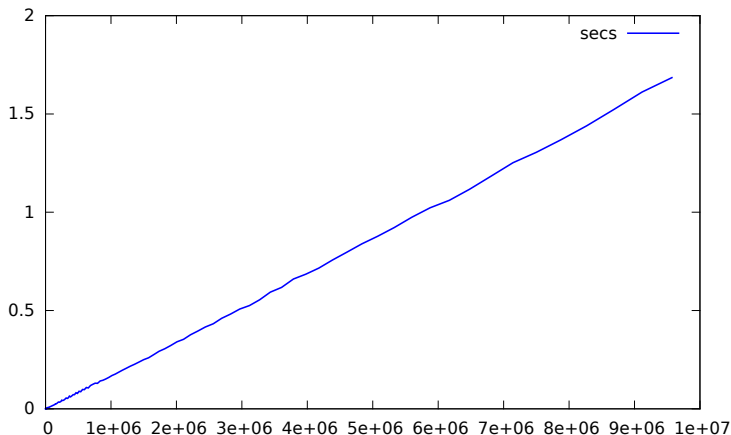
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Merge sort in Practice on Worst-case Instances



n	46929	102428	364178	1014570
secs	1.03084	4.81622	61.2737	497.879 (Insertion-sort)
secs	0.007157	0.015802	0.0645791	0.169165 (Merge-sort)

Stability and In Place Property?

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- Merge sort is stable

Stability and In Place Property?

Stability and In Place Property?

- Merge sort is stable
- Merge sort does not sort in place

Divide and Conquer Algorithm:

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Then:

A has a runtime of $O(n \log n)$.