Countingsort and Radixsort

COMS10018 - Algorithms

Dr Christian Konrad

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Idea

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- For each element $x \in \{0, 1, \ldots, k\}$, count $\#$ elements $\leq x$

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- For each element $x \in \{0, 1, \ldots, k\}$, count $\#$ elements $\leq x$
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Idea

- For each element $x \in \{0, 1, \dots, k\}$, count $\#$ elements $\leq x$
- Put elements $A[i]$ directly into correct position
- **Difficulty:** Multiple elements have the same value

Algorithm

Require: Array A of n integers from $\{0, 1, 2, \dots, k\}$, for some integer k
Let $C[0 \dots k]$ be a new array with all entries equal to 0
Store output in array $B[0 \dots n-1]$

for $i = 0, \dots, n-1$ **do** {Count how often each element appears}
    $C[A[i]] \leftarrow C[A[i]] + 1$
for $i = 1, \dots, k$ **do** {Count how many smaller (or equal) elements appear}
    $C[i] \leftarrow C[i] + C[i-1]$
for $i = n-1, \dots, 0$ **do**
    $B[C[A[i]] - 1] \leftarrow A[i]$
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- $C[A[i]]$: Number of elements *smaller or equal* to $A[i]$
- Decrementing $C[A[i]]$: Next element of value $A[i]$ should be left of the current one

Counting Sort: Example

Example: $n = 8$, $k = 5$

	0	1	2	3	4	5	6	7
A	2	5	3	0	2	3	0	3

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Analysis: Countingsort

Runtime:

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for $i = 1, \ldots, k$ **do**
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Runtime:

$$O(n) + O(k) + O(n) = O(n + k)$$

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- This beats the lower bound for comparison-based sorting

```
for i = 0, ..., n − 1 do
    C[A[i]] ← C[A[i]] + 1
for i = 1, ..., k do
    C[i] ← C[i] + C[i − 1]
for i = n − 1, ..., 0 do
    B[C[A[i]] − 1] ← A[i]
    C[A[i]] ← C[A[i]] − 1
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Stable? In-place? Yes, it is stable,

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Stable? In-place? Yes, it is stable, No, not in-place

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Input: Array A of d digits integers, each digit is from the set $\{0, 1, \ldots, b-1\}$

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- $b = 2$, $d = 5$. E.g. 01101 (binary numbers)

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Examples

- $b = 2$, $d = 5$. E.g. 01101 (binary numbers)
- $b = 10$, $d = 4$. E.g. 9714

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Idea

- Iterate through the d digits

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Examples

- $b = 2$, $d = 5$. E.g. 01101 (binary numbers)
- $b = 10$, $d = 4$. E.g. 9714

Idea

- Iterate through the d digits
- Sort according to the current digit

Radixsort (2)

Radixsort Algorithm

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for $i = 1, \ldots, d$ **do**
  Use a stable sort algorithm to sort array A on digit i

(least significant digit is digit 1)

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Example

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657
839
436
720
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> **for** $i = 1, \dots, d$ **do**
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(least significant digit is digit 1)

Example

$$
\begin{array}{ccc}
329 & & 72\mathbf{0} \\
457 & & 35\mathbf{5} \\
657 & & 43\mathbf{6} \\
839 & \rightarrow & 45\mathbf{7} \\
436 & & 65\mathbf{7} \\
720 & & 32\mathbf{9} \\
355 & & 83\mathbf{9}
\end{array}
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Example

$$
\begin{array}{ccccc}
329 & & 72\mathbf{0} & & 7\mathbf{2}0 \\
457 & & 35\mathbf{5} & & 3\mathbf{2}9 \\
657 & & 43\mathbf{6} & & 4\mathbf{3}6 \\
839 & \rightarrow & 45\mathbf{7} & \rightarrow & 8\mathbf{3}9 \\
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Lemma

We are given n d-digit numbers in which each digit can take on up to b possible values. Radixsort correctly sorts these numbers in $O(d(n+b))$ time if the stable sort (e.g. Countingsort) it uses takes $O(n+b)$ time.

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Proof

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Proof Runtime is obvious. Correctness follows by induction on the columns being sorted. □

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Proof Runtime is obvious. Correctness follows by induction on the columns being sorted. □

Observe: If $d = O(1)$ and $b = O(n)$ then the runtime is $O(n)$!