The Fibonacci Numbers
COMS10018 - Algorithms

Dr Christian Konrad

Fibonacci Numbers

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F_1 &= 1 \\
F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2 \; .
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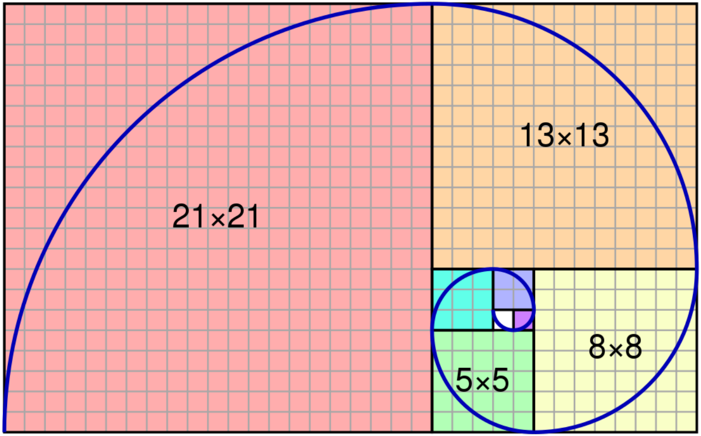
- Fibonacci heaps (data structure)
- Appear in analysis of algorithms (e.g. Euclid's algorithm)
- Appear everywhere in nature
- Interesting and instructive computational problem



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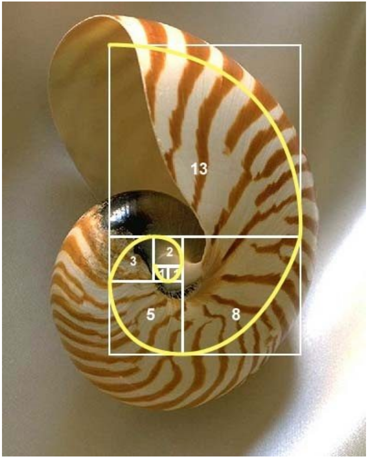

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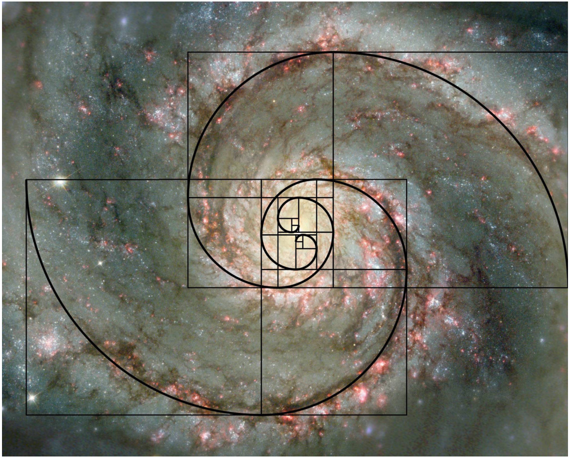

source: brian koberlein

Computing the Fibonacci Numbers

Naïve Algorithm

Require: Integer $n \geq 0$
  if $n \leq 1$ **then**
    return n
  else
    return $\textsc{Fib}(n-1) + \textsc{Fib}(n-2)$

$\textsc{Fib}(n)$

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Runtime:

- Without recursive calls, runtime is $O(1)$
- Hence, runtime is $O($"number of recursive calls"$)$

Runtime Analysis

Define Recurrence:

$T(n)$: number of recursive calls to $\textsc{Fib}$ when called with parameter n

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- We will use the recursion tree technique to obtain a guess for an upper bound

Runtime Analysis

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```
if n ≤ 1 then
   return n
else
   return Fib(n−1) + Fib(n−2)
```

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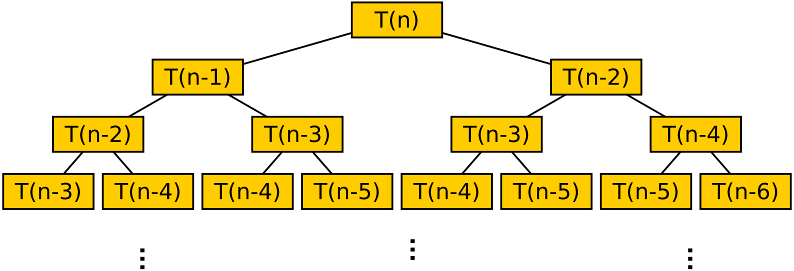
How to Solve this Recurrence?

- We will use the recursion tree technique to obtain a guess for an upper bound
- We will verify the guess with the substitution method

Recursion Tree for T



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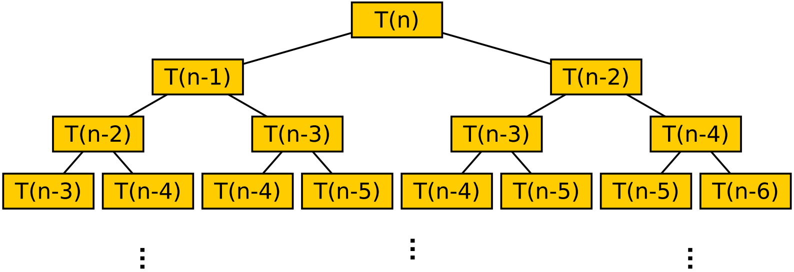

Observe:

- Each node contributes 1
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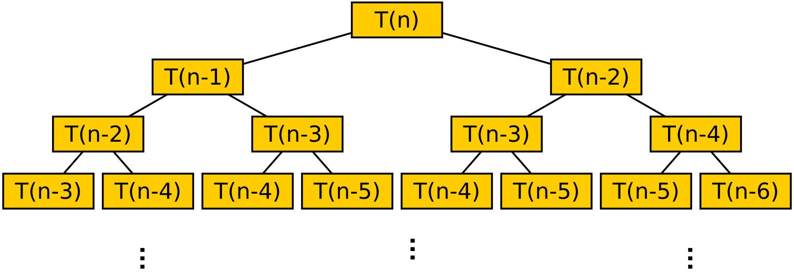

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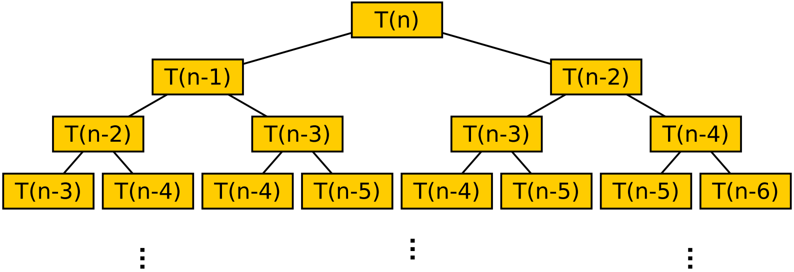

Observe:

- Each node contributes 1
- Hence, $T(n)$ equals number of nodes
- Number of levels of recursion tree: n
- Our guess: $T(n) \leq c^n$ (we believe $c \leq 2$)

Verification with the Substitution Method

Recall:

$$\begin{aligned} T(0) &= T(1) = 1 \\ T(n) &= 1 + T(n-1) + T(n-2) \ , \ \text{for } n \geq 2 \quad . \\ \text{Our guess: } T(n) &\leq c^n \end{aligned}$$

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$$T(n) = 1 + T(n-1) + T(n-2) \leq 1 + c^{n-1} + c^{n-2}$$

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- The additive 1 prevents us from getting a similar form as c^n
- Try different guess: $T(n) \leq c^n - 1$

Verification with the Substitution Method (2)

New Guess: $T(n) \leq c^n - 1$

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- $c^0 - 1 = 0$ and $c^1 - 1 \approx 0.61$ ✗

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We proved $T(n) \leq 2 \cdot (\frac{1+\sqrt{5}}{2})^n - 1$. Hence $T(n) \in O\left((\frac{1+\sqrt{5}}{2})^n\right)$.

Fibonacci Numbers: Closed-form Expression

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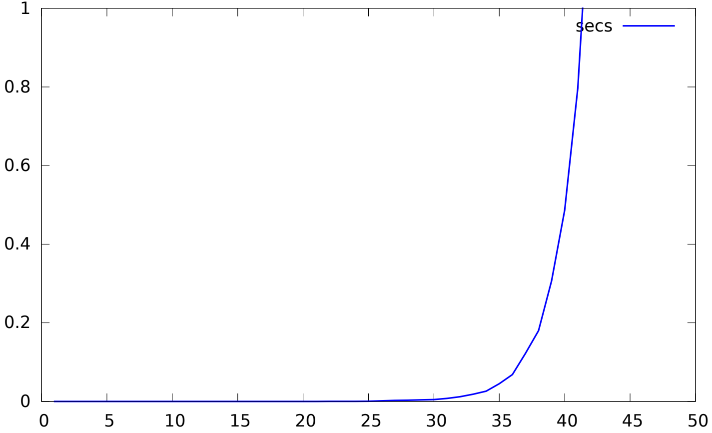
- Large numbers involved
- Impractical

Experiments



Exponential growth

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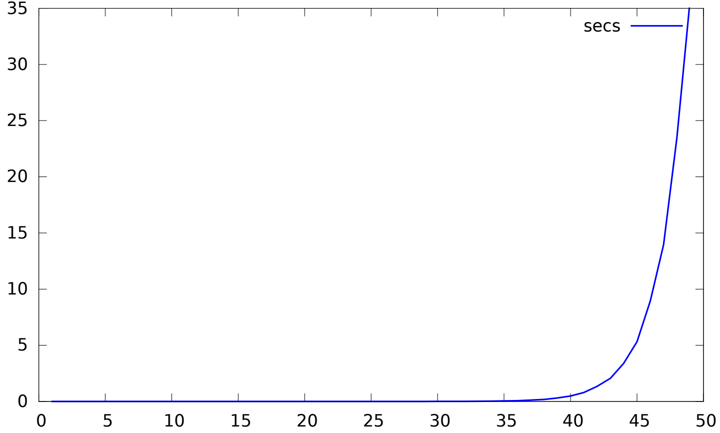

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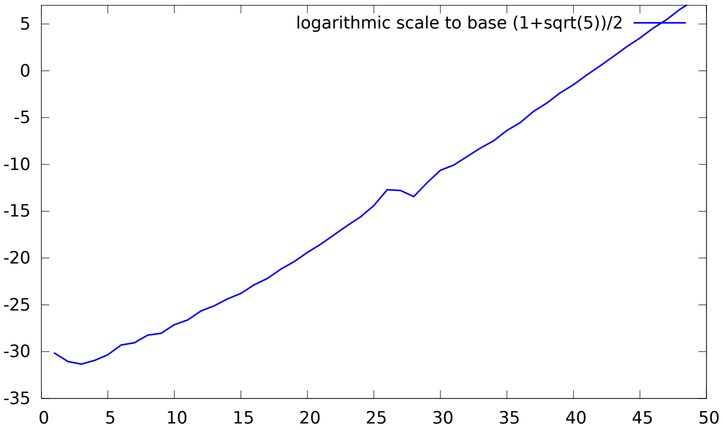
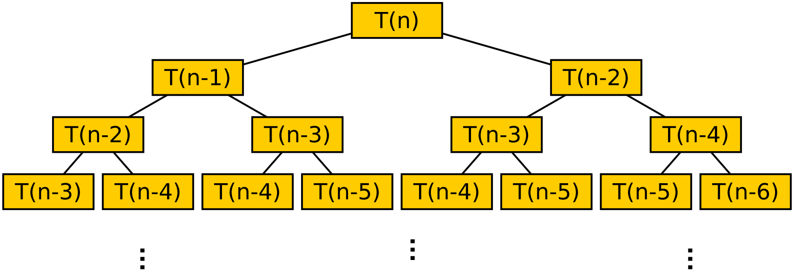


Logarithmic scale

Why is this Algorithm so slow?



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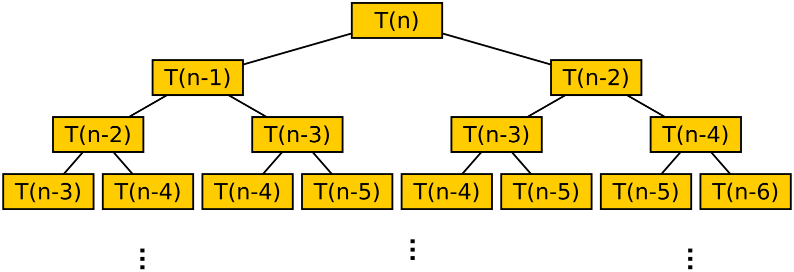

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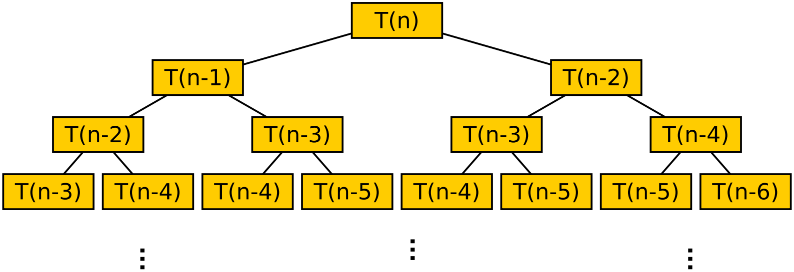

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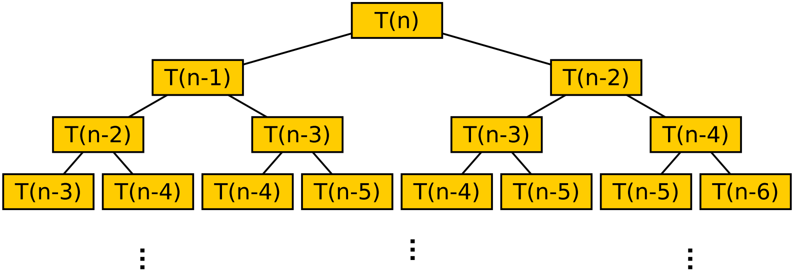
- We compute solutions to subproblems many times ($T(i)$ is computed often, for most values of i)
- How can we avoid this?

Why is this Algorithm so slow?



Discussion:

- We compute solutions to subproblems many times ($T(i)$ is computed often, for most values of i)
- How can we avoid this?

Dynamic Programming!

Dynamic Programming Solution

Dynamic Programming (will be discussed in more detail later)

- Store solutions to subproblems in a table
- Compute table bottom up

```
Require: Integer n ≥ 0
  if n ≤ 1 then
    return n
  else
    A ← array of size n
    A[0] ← 1, A[1] ← 1
    for i ← 2 ... n do
      A[i] ← A[i − 2] + A[i − 1]
    return A[n]
```

DynPrgFib(n)

Dicussion

Analysis:

- DynPrgFib() runs in time $O(n)$
- It uses space $\Theta(n)$ since it uses an array of size n

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Can we reduce the space to $O(1)$?

Improvement:

- Observe that when $T(i)$ is computed, the values $T(1), T(2), \ldots, T(i-3)$ are no longer needed
- Only store the last two values of T

Improved Algorithm

Require: Integer $n \geq 0$
  if $n \leq 1$ **then**
    return n
  else
    $a \leftarrow 0$
    $b \leftarrow 1$
    for $i \leftarrow 2 \ldots n$ **do**
      $c \leftarrow a + b$
      $a \leftarrow b$
      $b \leftarrow c$
    return c

ImprovedDynPrgFib(n)

Correctness: via loop invariant!