Trees

COMS10018 - Algorithms

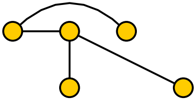
Dr Christian Konrad

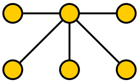
Trees

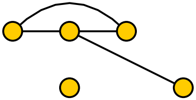
Definition: A *tree* $T = (V, E)$ of size n is a tuple consisting of

$$V = \{v_1, v_2, \dots, v_n\} \text{ and } E = \{e_1, e_2, \dots, e_{n-1}\}$$

with $|V| = n$ and $|E| = n - 1$ with $e_i = \{v_j, v_k\}$ for some $j \neq k$ s.t. for every pair of vertices v_i, v_j $(i \neq j)$, there is a path from v_i to v_j. V are the nodes/vertices and E are the edges of T.







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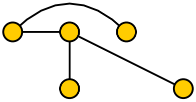

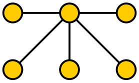

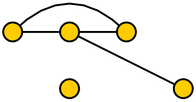

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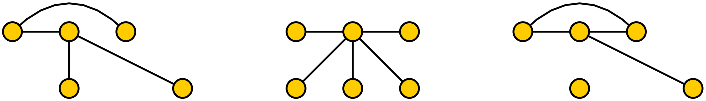

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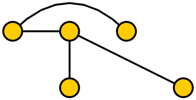

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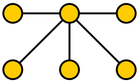

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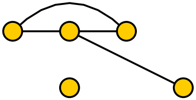

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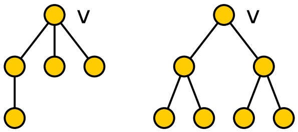
Rooted Trees

Definition: (rooted tree) A *rooted* tree is a triple $T = (v, V, E)$ such that $T = (V, E)$ is a tree and $v \in V$ is a designated node that we call the *root* of T.



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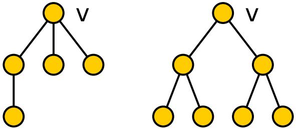

Definition: (leaf, internal node) A *leaf* in a tree is a node with exactly one incident edge. A node that is not a leaf is called an *internal node*.

Children, Parent, and Degree

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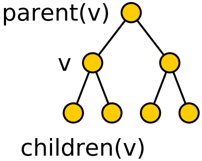

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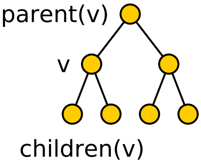

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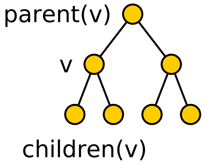

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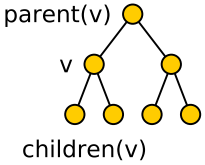

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$$\sum_{v \in V} \deg(v) = 2 \cdot |E| = 2(n-1) \ .$$

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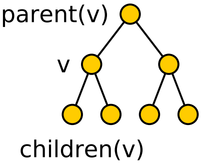

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- The *level* of a vertex v is the length of the unique path from the root to v plus 1.

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a contradiction to the fact that $\sum_{v \in V} \deg(v) = 2(n-1)$ in every tree. $\square$

Binary Trees

Definition: (k-ary tree) A (rooted) tree is *k-ary* if every node has at most k children. If $k = 2$ then the tree is called binary. A k ary tree is

- *full* if every internal node has exactly k children,
- *complete* if all levels except possibly the last is entirely filled (and last level is filled from left to right),
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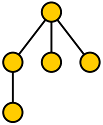

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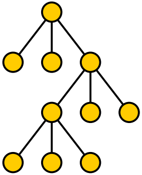

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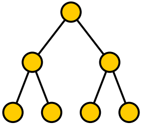

perfect binary tree

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Remark: The runtime of many algorithms that use tree data structures depends on the height of these trees. We are therefore interested in using complete/perfect trees.