The Maximum Subarray Problem

COMS10018 - Algorithms

Dr Christian Konrad

Generalizing the Analysis

Divide and Conquer Algorithm:

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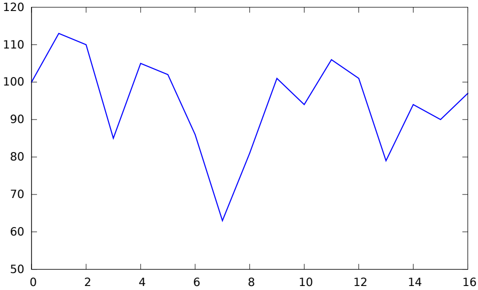
Then:

$$\boxed{\textbf{A} \text{ has a runtime of } O(n \log n)\ .}$$

Maximum Subarray Problem

Buy Low, Sell High Problem

- **Input:** An array of n integers
- **Output:** Indices $0 \leq i < j \leq n-1$ such that $A[j] - A[i]$ is maximized



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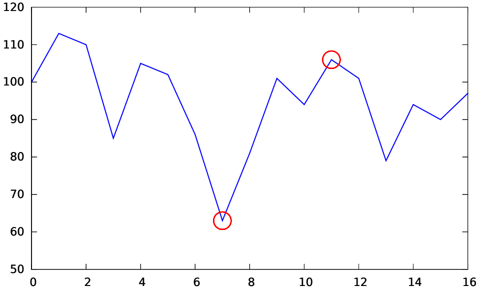

Maximum Subarray Problem

Focus on Array of Changes:

Day	0	1	2	3	4	5	6	7	8	9	10	11
\$	100	113	110	85	105	102	86	63	81	101	94	106
Δ		13	-3	-25	20	-3	-16	-23	18	20	-7	12

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- Compute subarrays for every pair i, j
- There are $O(n^2)$ pairs, computing the sum takes time $O(n)$.

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Compute maximum subarrays in left and right halves of initial array

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Three cases:

Divide and Conquer Algorithm for Maximum Subarray

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Combine:
Given maximum subarrays in L and R, we need to compute maximum subarray in A

Three cases:

1. Maximum subarray is entirely included in L ✓
2. Maximum subarray is entirely included in R ✓
3. Maximum subarray crosses midpoint, i.e., i is included in L and j is included in R

Divide and Conquer Algorithm for Maximum Subarray

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Maximum Subarray Crosses Midpoint:

Divide and Conquer Algorithm for Maximum Subarray

Maximum Subarray Crosses Midpoint:

- Find maximum subarray $A[i,j]$ such that $i \leq \frac{n}{2}$ and $j > \frac{n}{2}$ (assume that n is even)

Divide and Conquer Algorithm for Maximum Subarray

Maximum Subarray Crosses Midpoint:

- Find maximum subarray $A[i,j]$ such that $i \leq \frac{n}{2}$ and $j > \frac{n}{2}$ (assume that n is even)
- Observe that: $\sum_{l=i}^{j} A[l] = \sum_{l=i}^{\frac{n}{2}} A[i] + \sum_{l=\frac{n}{2}+1}^{j} A[l]$.

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We can solve these subproblems in time $O(n)$. (how?)

Maximum Subarray Problem - Summary

Require: Array A of n numbers
  if $n = 1$ **then**
    return A
  Recursively compute max. subarray S_1 in $A[0, \lfloor \frac{n}{2} \rfloor]$
  Recursively compute max. subarray S_2 in $A[\lfloor \frac{n}{2} \rfloor + 1, n - 1]$
  Compute maximum subarray S_3 that crosses midpoint
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Recursive Algorithm for the Maximum Subarray Problem

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- Conquer step requires $O(n)$ time
- Identical to Merge Sort, runtime $O(n \log n)$!