Dynamic Programming - Matrix Chain Parenthesization

COMS10018 - Algorithms

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Matrix Multiplication

Problem: MATRIX-MULTIPLICATION

1. **Input:** Matrices A, B with $A.columns = B.rows$
2. **Output:** Matrix product $A \times B$

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Example:

$$\begin{array}{cc} & q \\ p & \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 2 & 6 \\ 0 & 9 \end{pmatrix} \end{array} \times \begin{array}{cc} & r \\ q & \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix} \end{array} = \begin{array}{cc} r & \\ \begin{pmatrix} 6 & 2 & 4 \\ 0 & 1 & 2 \\ 12 & 2 & 4 \\ 18 & 0 & 0 \end{pmatrix} & p \end{array}$$

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- $p \times q$ matrix times $q \times r$ matrix gives a $p \times r$ matrix
- $(A \times B)_{i,j}$ = row i of A times column j of B

Algorithm for Matrix-Multiplication

Algorithm: $(A \times B)_{i,j} =$ row i of A times column j of B

Require: Matrices A, B with $A.columns = B.rows$

Let C be a new $A.rows \times B.columns$ matrix
for $i \leftarrow 1 \ldots A.rows$ **do**
  for $j \leftarrow 1 \ldots B.columns$ **do**
    $C_{ij} \leftarrow 0$
    for $k \leftarrow 1 \ldots A.columns$ **do**
      $C_{ij} \leftarrow C_{ij} + A_{ik} \cdot B_{kj}$
return C

Algorithm $\textsc{Matrix-Multiply}(A, B)$

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- Three nested loops: $O(A.rows \cdot B.columns \cdot A.columns)$

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```
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  Let C be a new A.rows × B.columns matrix
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  return C
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- Multiplying two $n \times n$ matrices: runtime $O(n^3)$

Background: Faster Matrix Multiplication

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Important Problem:

- Many algorithms rely on fast matrix multiplication

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Important Problem:

- Many algorithms rely on fast matrix multiplication
- Better bound for matrix multiplication improves many algorithms

The Matrix-chain Multiplication Problem

Problem: MATRIX-CHAIN-MULTIPLICATION

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1. **Input:** A sequence (chain) of n matrices $A_1, A_2, A_3, \dots, A_n$

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1. **Input:** A sequence (chain) of n matrices $A_1, A_2, A_3, \ldots, A_n$
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$$(A_1 \times A_2) \times A_3 = A_1 \times (A_2 \times A_3)$$

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Exploit Associativity: Parenthesize $A_1 \times A_2 \times A_3 \times \dots A_n$ so as to minimize the number of scalar multiplications (and thus the runtime)

Order matters

Example:

Order matters

Example: Three matrices A_1, A_2, A_3 with dimensions

$$A_1 : 10 \times 100 \quad A_2 : 100 \times 5 \quad A_3 : 5 \times 50$$

$(p_0 = 10, p_1 = 100, p_2 = 5, p_3 = 50)$

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- $A_1 \times A_2 = A_{12}$ requires $10 \cdot 100 \cdot 5 = 5000$ multiplications

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- $A_1 \times A_2 = A_{12}$ requires $10 \cdot 100 \cdot 5 = 5000$ multiplications
- $A_{12} \times A_3$ requires $10 \cdot 5 \cdot 50 = 2500$ multiplications

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- Total: 7500 multiplications

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Computation of $A_1 \times (A_2 \times A_3)$:

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Computation of $A_1 \times (A_2 \times A_3)$:

- $A_2 \times A_3 = A_{23}$ requires $100 \cdot 5 \cdot 50 = 25000$ multiplications

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Example: Three matrices A_1, A_2, A_3 with dimensions

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- $A_1 \times A_2 = A_{12}$ requires $10 \cdot 100 \cdot 5 = 5000$ multiplications
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- Total: 7500 multiplications

Computation of $A_1 \times (A_2 \times A_3)$:

- $A_2 \times A_3 = A_{23}$ requires $100 \cdot 5 \cdot 50 = 25000$ multiplications
- $A_1 \times A_{23}$ requires $10 \cdot 100 \cdot 50 = 50000$ multiplications
- Total: 75000 multiplications

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How many Parenthesizations $P(n)$ are there?

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- We write: A_{ij} for the product $A_i \times A_{i+1} \times \cdots \times A_j$
- There is a final matrix multiplication: $A_{1k} \times A_{(k+1)n}$, for some $1 \leq k \leq n-1$.

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$$P(n) = \begin{cases} 1 & \text{if } n = 1 \ , \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \ . \end{cases}$$

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$$A_1 \times A_{24} \quad A_{12} \times A_{34} \quad A_{13} \times A_4$$

Number of Parenthesizations

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$P(4)$

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Number of Parenthesizations

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&= P(3) + 1 + P(3)
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\end{aligned}$$

1. $A_1 \times ((A_2 \times A_3) \times A_4)$
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3. $(A_1 \times A_2) \times (A_3 \times A_4)$
4. $((A_1 \times A_2) \times A_3) \times A_4$
5. $(A_1 \times (A_2 \times A_3)) \times A_4$

Number of Parenthesizations (2)

A Bound on the Number of Parenthesizations:

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \ , \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \ . \end{cases}$$

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Dynamic Programming!

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An optimal solution to **P** contains within it optimal solutions to subproblems of **P**.

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  Proof.

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  Proof. Suppose it did not contain optimal parenthesizations of $A_1 \times A_2 \times \cdots \times A_k$ and of $A_{k+1} \times A_{k+2} \times \ldots A_n$. Then picking optimal parenthesizations of the two subproblems would give better solution to initial instance. $\square$

Recursive Solution

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- Then:

$$m[i,j] = m[i,k] + m[k+1,j] + \underbrace{p_{i-1}p_kp_j}_{\text{cost of multiplying } A_{ik} \times A_{(k+1)j}}$$

(A_{ik}: $p_{i-1} \times p_k$ matrix, $A_{(k+1)j}$: $p_k \times p_j$ matrix)

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  (A_{ik}: $p_{i-1} \times p_k$ matrix, $A_{(k+1)j}$: $p_k \times p_j$ matrix)
- Since we do not know k, we try out all possibilities and choose the best solution:

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \ , \\ \min_{i \le k < j}\{m[i,k] + m[k+1,j] + p_{i-1}p_kp_j\} & \text{if } i < j \ . \end{cases}$$

Computing the Optimal Costs

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \ , \\ \min_{i \le k < j}\{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & \text{if } i < j \ . \end{cases}$$

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- This yields an $O(n^3)$ time algorithm

Dynamic Programming Algorithm

Require: Integer n, vector of dimensions of matrices p so that matrix A_i has dimensions $p_{i-1} \times p_i$

Let $m[1 \ldots n, 1 \ldots n]$ be a new array
for $i \leftarrow 1 \ldots n$ **do**
  $m[i,i] \leftarrow 0$
for $\ell \leftarrow 2 \ldots n$ **do** {chain length}
  for $i \leftarrow 1 \ldots n - \ell + 1$ **do** {left position}
    $j \leftarrow i + \ell - 1$ {right position}
    $m[i,j] \leftarrow \infty$
    for $k \leftarrow i \ldots j - 1$ **do**
      $m[i,j] \leftarrow \min\{m[i,j], m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\}$
return m

Algorithm $\textsc{Matrix-Chain-Value}(n, p)$

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Algorithm MATRIX-CHAIN-VALUE(n, p)

Runtime: $O(n^3)$

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Algorithm $\textsc{Matrix-Chain-Value}(n, p)$

Runtime: $O(n^3)$ (by evaluating $\sum_{\ell=2}^{n}\sum_{i=1}^{n-\ell+1}\sum_{k=1}^{i+\ell-2} O(1)$)

Runtime Evaluation

Useful Formula:

$$\sum_{i=a}^{b} 1 = b - a + 1$$

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$$\sum_{l=2}^{n} \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} O(1) = O(1) \cdot \sum_{l=2}^{n} \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} 1$$

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$$\leq \quad O(1) \cdot \sum_{l=1}^{n} \sum_{i=1}^{n} \sum_{k=1}^{n} 1$$

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$$\begin{aligned} & \sum_{l=2}^{n} \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} O(1) = O(1) \cdot \sum_{l=2}^{n} \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} 1 \\ \leq \quad & O(1) \cdot \sum_{l=1}^{n} \sum_{i=1}^{n} \sum_{k=1}^{n} 1 = O(1) \cdot \sum_{l=1}^{n} \sum_{i=1}^{n} n \end{aligned}$$

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$$\begin{aligned}
\sum_{l=2}^{n} \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} O(1) &= O(1) \cdot \sum_{l=2}^{n} \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} 1 \\
&\leq O(1) \cdot \sum_{l=1}^{n} \sum_{i=1}^{n} \sum_{k=1}^{n} 1 = O(1) \cdot \sum_{l=1}^{n} \sum_{i=1}^{n} n = O(1) \cdot n \sum_{l=1}^{n} \sum_{i=1}^{n} 1
\end{aligned}$$

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$$\begin{aligned}
& \sum_{l=2}^{n} \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} O(1) = O(1) \cdot \sum_{l=2}^{n} \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} 1 \\
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= \quad & O(1) \cdot n \sum_{l=1}^{n} n
\end{aligned}$$

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&= O(1) \cdot n \sum_{l=1}^{n} n = O(1) \cdot n^2 \sum_{l=1}^{n} 1
\end{aligned}$$

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&= O(1) \cdot n \sum_{l=1}^{n} n = O(1) \cdot n^2 \sum_{l=1}^{n} 1 = O(1) \cdot n^2 \cdot n = O(1) n^3
\end{aligned}$$

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= \quad & O(1) \cdot n \sum_{l=1}^{n} n = O(1) \cdot n^2 \sum_{l=1}^{n} 1 = O(1) \cdot n^2 \cdot n = O(1) n^3 \\
= \quad & O(n^3) \ .
\end{aligned}$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1				
2				
3				
4				

for $i \leftarrow 1 \ldots n$ **do**
    $m[i, i] \leftarrow 0$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2		0		
3			0	
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Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3			0	
4				0

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$l = 2, i = 1, j = 2$

$$m[1, 2] = m[1, 1] + m[2, 2] + p_0 p_1 p_2 = 0 + 0 + 3 \cdot 7 \cdot 6 = 126$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

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  for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i, j] \leftarrow \infty$
    for $k \leftarrow i \dots j - 1$ **do**
      $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j\}$

$l = 2, i = 2, j = 3$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3		84	0	
4				0

for $l \leftarrow 2 \ldots n$ **do**
  for $i \leftarrow 1 \ldots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i, j] \leftarrow \infty$
    for $k \leftarrow i \ldots j - 1$ **do**
      $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j\}$

$l = 2, i = 2, j = 3$

$$m[2, 3] = m[2, 2] + m[3, 3] + p_1 p_2 p_3 = 0 + 0 + 7 \cdot 6 \cdot 2 = 84$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3		84	0	
4				0

for $l \leftarrow 2 \dots n$ **do**
  for $i \leftarrow 1 \dots n - l + 1$ **do** $\{$left position$\}$
    $j \leftarrow i + l - 1$ $\{$right position$\}$
    $m[i, j] \leftarrow \infty$
    for $k \leftarrow i \dots j - 1$ **do**
      $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j\}$

$l = 2, i = 3, j = 4$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3		84	0	
4			108	0

for $l \leftarrow 2 \ldots n$ **do**
  for $i \leftarrow 1 \ldots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i, j] \leftarrow \infty$
    for $k \leftarrow i \ldots j - 1$ **do**
      $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j\}$

$l = 2, i = 3, j = 4$

$$m[3, 4] = m[3, 3] + m[4, 4] + p_2 p_3 p_4 = 0 + 0 + 6 \cdot 2 \cdot 9 = 108$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3		84	0	
4			108	0

for $l \leftarrow 2 \dots n$ **do**
  for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i,j] \leftarrow \infty$
    for $k \leftarrow i \dots j - 1$ **do**
      $m[i,j] \leftarrow \min\{m[i,j], m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\}$

$l = 3, i = 1, j = 3$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4			108	0

for $l \leftarrow 2 \ldots n$ **do**
  for $i \leftarrow 1 \ldots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i,j] \leftarrow \infty$
    for $k \leftarrow i \ldots j - 1$ **do**
      $m[i,j] \leftarrow \min\{m[i,j], m[i,k] + m[k+1,j] + p_{i-1}p_kp_j\}$

$l = 3, i = 1, j = 3$

$$\begin{aligned} m[1,1] + m[2,3] + p_0p_1p_3 &= 0 + 84 + 3 \cdot 7 \cdot 2 = 84 + 42 = \color{red}{106} \\ m[1,2] + m[3,3] + p_0p_2p_3 &= 126 + 0 + 3 \cdot 6 \cdot 2 = 126 + 36 = 162 \end{aligned}$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4			108	0

for $l \leftarrow 2 \ldots n$ **do**
  for $i \leftarrow 1 \ldots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i,j] \leftarrow \infty$
    for $k \leftarrow i \ldots j - 1$ **do**
      $m[i,j] \leftarrow \min\{m[i,j], m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\}$

$l = 3, i = 2, j = 4$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4		210	108	0

for $l \leftarrow 2 \ldots n$ **do**
  for $i \leftarrow 1 \ldots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i,j] \leftarrow \infty$
    for $k \leftarrow i \ldots j - 1$ **do**
      $m[i,j] \leftarrow \min\{m[i,j], m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\}$

$l = 3, i = 2, j = 4$

$$
\begin{aligned}
m[2,2] + m[3,4] + p_1 p_2 p_4 &= 0 + 108 + 7 \cdot 6 \cdot 9 = 108 + 378 = 486 \\
m[2,3] + m[4,4] + p_1 p_3 p_4 &= 84 + 0 + 7 \cdot 2 \cdot 9 = 84 + 36 = 210
\end{aligned}
$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4		210	108	0

for $l \leftarrow 2 \dots n$ **do**
  for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i, j] \leftarrow \infty$
    for $k \leftarrow i \dots j - 1$ **do**
      $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j\}$

$l = 4, i = 1, j = 4$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4	160	210	108	0

for $l \leftarrow 2 \ldots n$ **do**
  for $i \leftarrow 1 \ldots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i,j] \leftarrow \infty$
    for $k \leftarrow i \ldots j - 1$ **do**
      $m[i,j] \leftarrow \min\{m[i,j], m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\}$

$$
\begin{aligned}
m[1,1] + m[2,4] + p_0 p_1 p_4 &= 0 + 210 + 3 \cdot 7 \cdot 9 = 399 \\
m[1,2] + m[3,4] + p_0 p_2 p_4 &= 126 + 108 + 3 \cdot 6 \cdot 9 = 396 \\
m[1,3] + m[4,4] + p_0 p_3 p_4 &= 106 + 0 + 3 \cdot 2 \cdot 9 = 160
\end{aligned}
$$

Optimal Solution of Example

Example: $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

Optimal Solution of Example

Example: $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

- Algorithm outputs value of optimal solution: $m[1, 4] = 160$

Optimal Solution of Example

Example: $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

- Algorithm outputs value of optimal solution: $m[1,4] = 160$
- We would like to know the optimal parenthesization as well

$$((A_1 \times A_2) \times A_3) \times A_4$$

Optimal Solution of Example

Example: $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

- Algorithm outputs value of optimal solution: $m[1, 4] = 160$
- We would like to know the optimal parenthesization as well

$$((A_1 \times A_2) \times A_3) \times A_4$$

$\rightarrow$ Modify algorithm to keep track of parameters that give minimum in array s

Keep Track of Optimal Choices

Require: Integer n, vector of dimensions of matrices p so that matrix A_i has dimensions $p_{i-1} \times p_i$

Let $m[1 \dots n, 1 \dots n]$ be a new array
for $i \leftarrow 1 \dots n$ **do**
  $m[i,i] \leftarrow 0$
for $l \leftarrow 2 \dots n$ **do** {chain length}
  for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i,j] \leftarrow \infty$
    for $k \leftarrow i \dots j - 1$ **do**
      $m[i,j] \leftarrow \min\{m[i,j], m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\}$
return m, s

Algorithm $\textsc{Matrix-Chain-Value}(n, p)$

Keep Track of Optimal Choices

Require: Integer n, vector of dimensions of matrices p so that matrix A_i has dimensions $p_{i-1} \times p_i$

Let $m[1 \dots n, 1 \dots n]$ and $s[1 \dots n, 2 \dots n]$ be new arrays
for $i \leftarrow 1 \dots n$ **do**
  $m[i, i] \leftarrow 0$
for $l \leftarrow 2 \dots n$ **do** {chain length}
  for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}
    $j \leftarrow i + l - 1$ {right position}
    $m[i, j] \leftarrow \infty$
    for $k \leftarrow i \dots j - 1$ **do**
      $q \leftarrow m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$
      if $q < m[i, j]$ **then**
        $m[i, j] \leftarrow q$
        $s[i, j] \leftarrow k$
return m

Algorithm $\textsc{Matrix-Chain-Order}(A, B)$

Print Optimal Parenthesization

Using s to find Optimal Parenthesization

Require: Array s, positions i, j
  if $i = j$ **then**
    print "A_i"
  else
    print "("
    $\textsc{Print-Optimal-Parens}(s, i, s[i, j])$
    $\textsc{Print-Optimal-Parens}(s, s[i, j] + 1, j)$
    print ")"

Algorithm $\textsc{Print-Optimal-Parens}(s, i, j)$

Call $\textsc{Print-Optimal-Parens}(s, 1, n)$ to obtain parenthesization