Loop Invariants and Insertion-sort

COMS10018 - Algorithms

Dr Christian Konrad

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Algorithm computes the factorial function

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Sorting Problem

- **Input:** An array A of n numbers
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0	1	2	3	4	5
15	7	3	9	8	1

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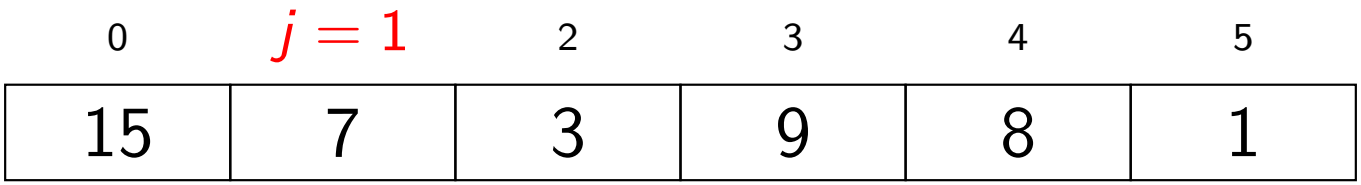

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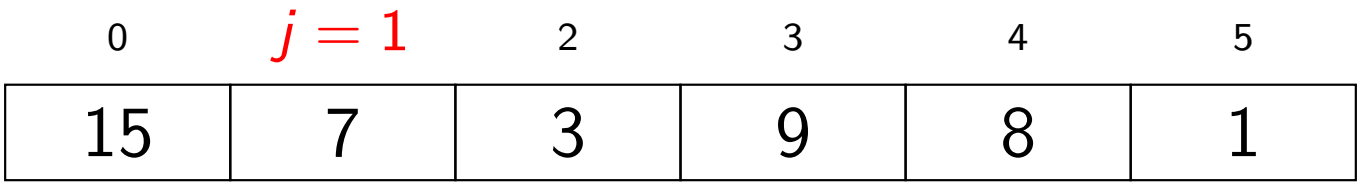

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$i = 0$	$j = 1$	2	3	4	5
15	7	3	9	8	1

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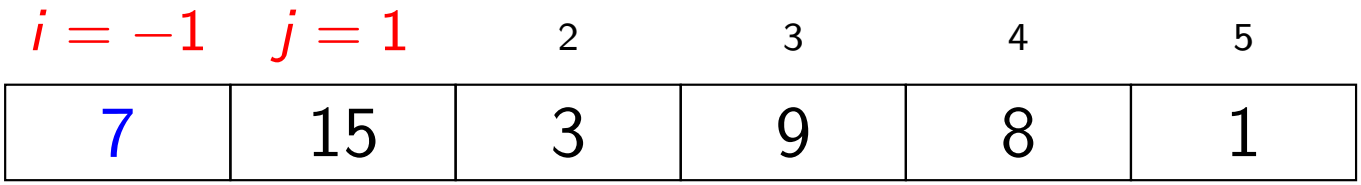

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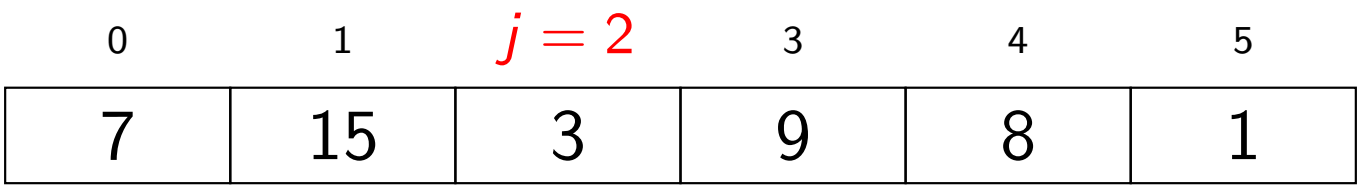

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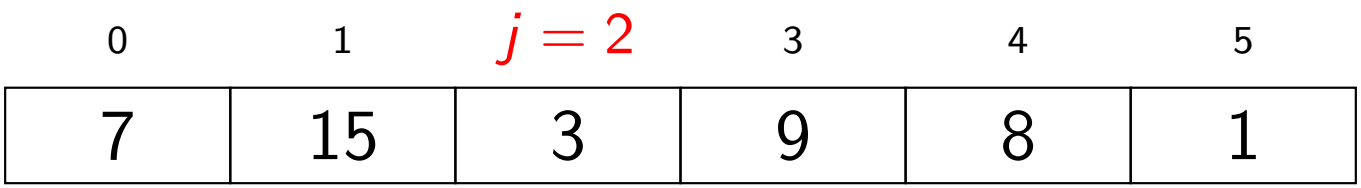

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Example:

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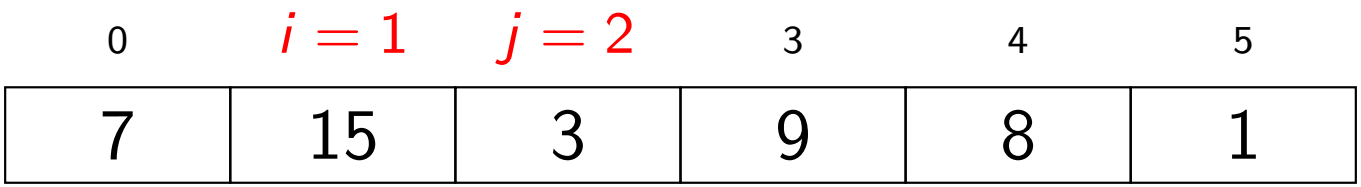

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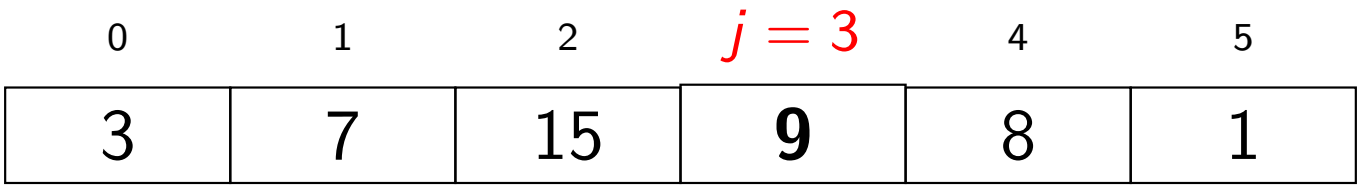

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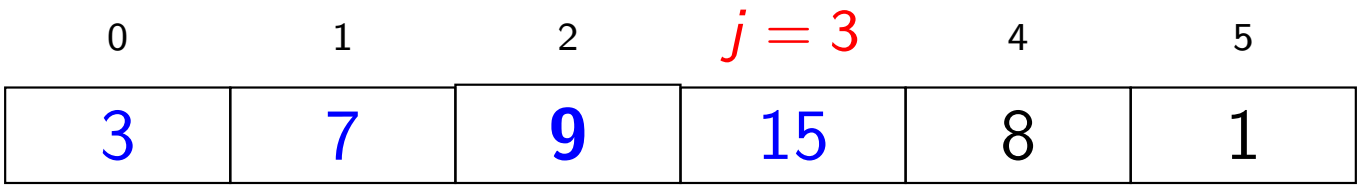

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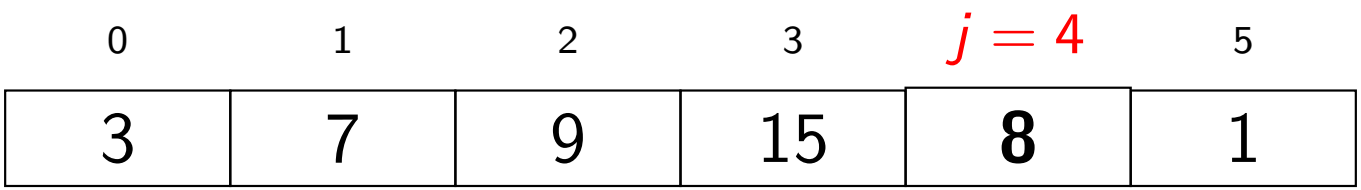

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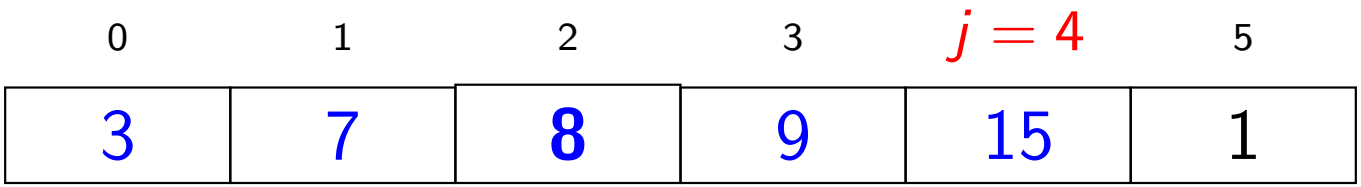

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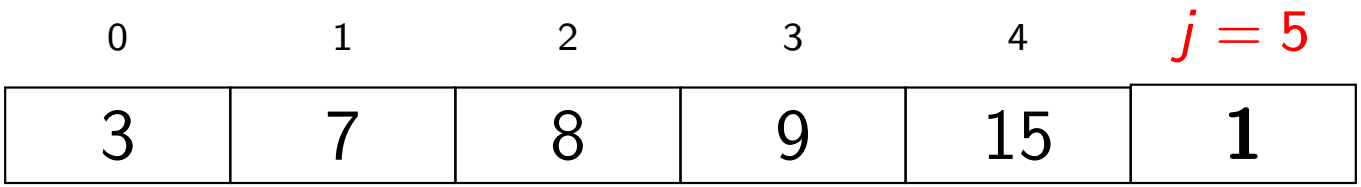

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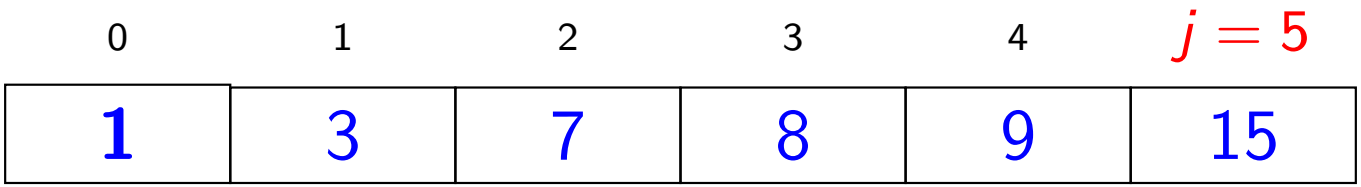

Loop Invariant of Insertion-sort

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for j = 1, ..., n − 1 do
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- **Termination:** After iteration $j = n - 1$ (i.e., before iteration $j = n$) the loop invariant states that A is sorted. ✓

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Best-case Runtime: $O(n)$
E.g., if input is already sorted