The Fibonacci Numbers
COMS10018 - Algorithms

Dr Christian Konrad

The Fibonacci Numbers

Fibonacci Numbers

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F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2 \ .
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Why are they important?

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- Appear everywhere in nature
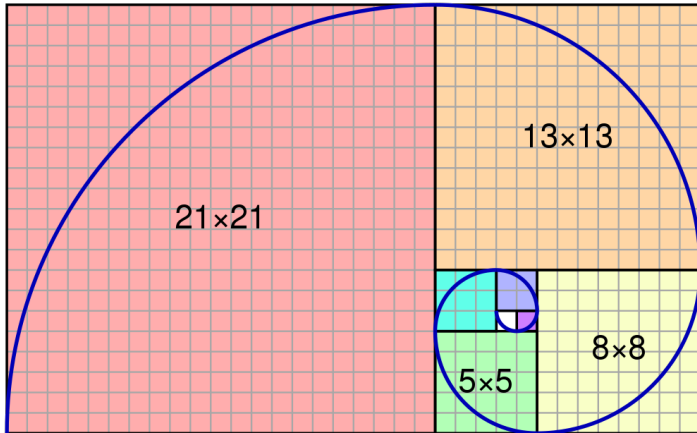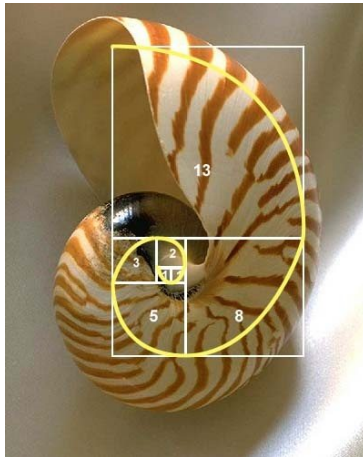
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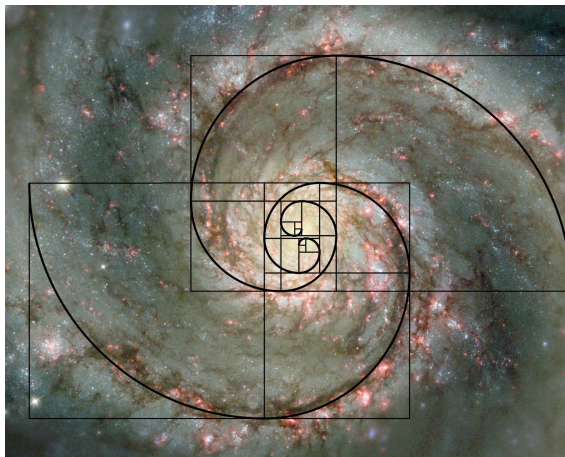
Why are they important?

- Fibonacci heaps (data structure)
- Appear in analysis of algorithms (e.g. Euclid's algorithm)
- Appear everywhere in nature
- Interesting and instructive computational problem

source: wikipedia

source: realworldmathematics at wordpress

source: brian koberlein

Computing the Fibonacci Numbers

Naïve Algorithm

> **Require:** Integer $n \geq 0$
>   **if** $n \leq 1$ **then**
>   **return** n
>   **else**
>   **return** $\text{FIB}(n-1) + \text{FIB}(n-2)$
>
> $\text{FIB}(n)$

Computing the Fibonacci Numbers

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Require: Integer n ≥ 0
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$$\text{F\textsc{ib}}(n)$$

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Computing the Fibonacci Numbers

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- Without recursive calls, runtime is $O(1)$

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What is the runtime of this algorithm?

Runtime:

- Without recursive calls, runtime is $O(1)$
- Hence, runtime is $O(\text{"number of recursive calls"})$

Runtime Analysis

Define Recurrence:

$T(n)$: number of recursive calls to FIB when called with parameter n

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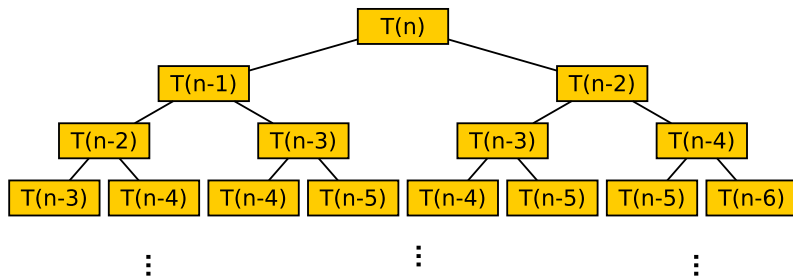- We will verify the guess with the substitution method
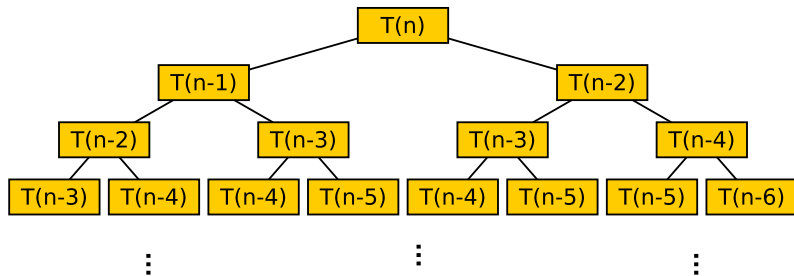
Observe:

- Each node contributes 1
- Hence, $T(n)$ equals number of nodes
- Number of levels of recursion tree: n
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Recursion Tree for T

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- Hence, $T(n)$ equals number of nodes
- Number of levels of recursion tree: n
- Our guess: $T(n) \leq c^n$ (we believe $c \leq 2$)

Recall:

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- It is required that $1 + c^{n-1} + c^{n-2} \leq c^n$

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- It is required that $1 + c^{n-1} + c^{n-2} \leq c^n$
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- Try different guess: $T(n) \leq c^n - 1$

New Guess: $T(n) \leq c^n - 1$

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$$T(n) \;=\; 1 + T(n-1) + T(n-2)$$

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Verification with the Substitution Method (2)

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Select smallest possible c:

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- $c^0 - 1 = 0$ and $c^1 - 1 \approx 0.61$ ✗

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Base Case:

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Verification with the Substitution Method (3)

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We proved $T(n) \leq 2 \cdot (\frac{1+\sqrt{5}}{2})^n - 1$. Hence $T(n) \in O\left((\frac{1+\sqrt{5}}{2})^n \right)$.

Fibonacci Numbers: Closed-form Expression

Golden Ratio:

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.61803$$

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- Large numbers involved

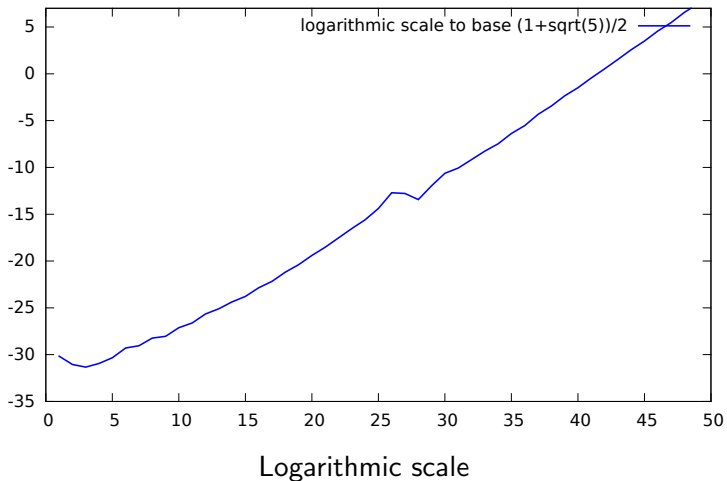# Fibonacci Numbers: Closed-form Expression

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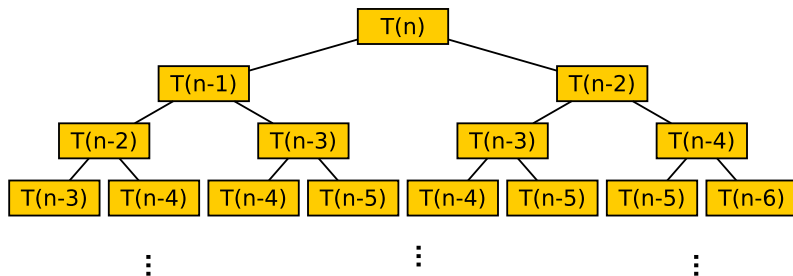- Large numbers involved
- Impractical

Experiments

Exponential growth

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Experiments
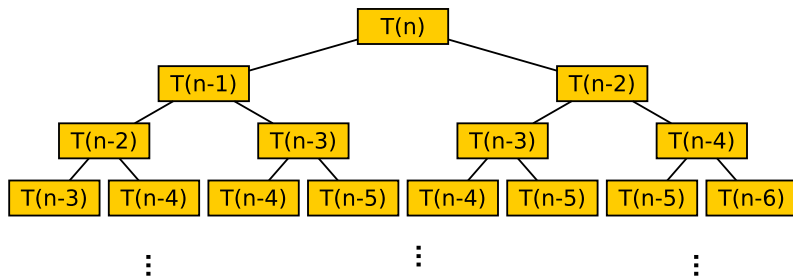

Logarithmic scale
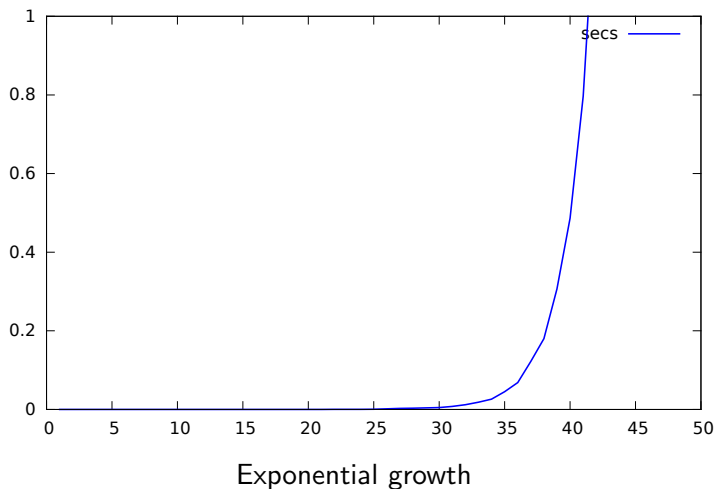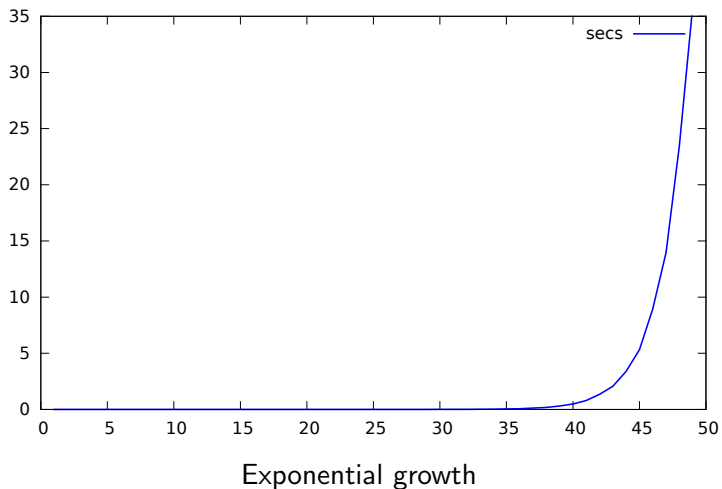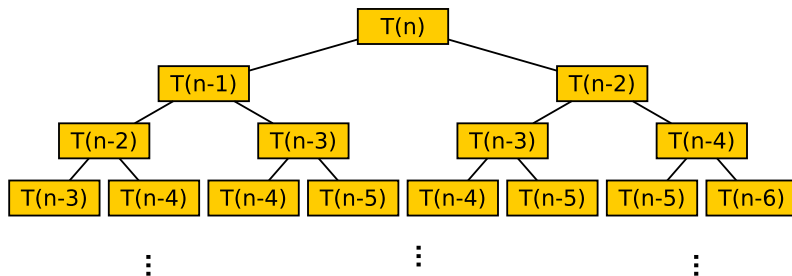
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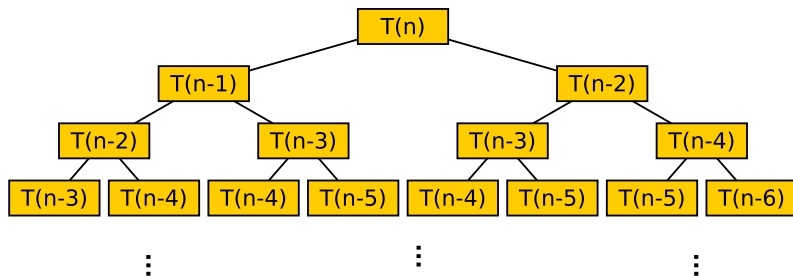- We compute solutions to subproblems many times ($T(i)$ is computed often, for most values of i)

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- How can we avoid this?

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- We compute solutions to subproblems many times ($T(i)$ is computed often, for most values of i)
- How can we avoid this?

Dynamic Programming!

Dynamic Programming Solution

Dynamic Programming (will be discussed in more detail later)

- Store solutions to subproblems in a table
- Compute table bottom up

```
Require: Integer n ≥ 0
  if n ≤ 1 then
    return n
  else
    A ← array of size n
    A[0] ← 1, A[1] ← 1
    for i ← 2 . . . n do
      A[i] ← A[i − 2] + A[i − 1]
    return A[n]
```

$\mathrm{DYNPRGFIB}(n)$

Dicussion

Analysis:

- DynPrgFib() runs in time $O(n)$
- It uses space $\Theta(n)$ since it uses an array of size n

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Dicussion

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- It uses space $\Theta(n)$ since it uses an array of size n

    Can we reduce the space to $O(1)$?

Improvement:

- Observe that when $T(i)$ is computed, the values $T(1), T(2), \ldots, T(i-3)$ are no longer needed
- Only store the last two values of T

Improved Algorithm

```
Require: Integer n ≥ 0
  if n ≤ 1 then
    return  n
  else
    a ← 0
    b ← 1
    for i ← 2 . . . n do
      c ← a + b
      a ← b
      b ← c
    return  c
```

$\mathrm{ImprovedDynPrgFib}(n)$

Correctness: via loop invariant!