

# INDUCTION

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Reading: PFPL, §2.4, 2.5, 2.6, 3.1

Recall the rules that generate the natural numbers:

$$\frac{\text{ZERO}}{\text{zero nat}} \qquad \frac{\text{Succ} \quad n \text{ nat}}{\text{succ}(n) \text{ nat}}$$

These rules define all the ways of constructing a natural number. Thus, if we *prove* that the truth of a property is preserved by these two rules, it must be that we have proved it for all natural numbers.

This is called the *principle of induction*.

Stated for the natural numbers, the principle of induction is as follows:

Let  $\mathcal{P}$  be a property of the natural numbers. If

- $\mathcal{P}(\text{zero})$ , and
- whenever  $\mathcal{P}(n)$  we know that  $\mathcal{P}(\text{succ}(n))$

then  $\mathcal{P}(n)$  for all  $n \text{ nat}$ .

Every set of rules generates an associated induction principle. Thus, the usual principle of natural number induction is implicitly generated by the rules **ZERO** and **Succ**.

We can use induction to prove results.

**Claim 1.** If  $\text{succ}(n) \text{ nat}$  then  $n \text{ nat}$ .

To prove this it suffices to prove the following property by induction.

$\mathcal{P}(n)$ : “If  $n \text{ nat}$  and  $n = \text{succ}(x)$  for some  $x$ , then  $x \text{ nat}$ .”

## 1 Derivable and admissible rules

Statements like **Claim 1** are often written as rules themselves:  $\frac{\text{succ}(n) \text{ nat}}{n \text{ nat}}$

This rule is not one of the defining rules of natural numbers. Rather, it is an admissible rule.

A rule is **admissible** if whenever we have a derivation of the premises, then we know we can construct a derivation of the conclusion.

In this particular instance, given a derivation of  $\text{succ}(n) \text{ nat}$ , we can construct a derivation of  $n \text{ nat}$  by trimming the last line of the derivation. For  $n \stackrel{\text{def}}{=} \text{succ}(\text{zero})$ :

$$\frac{\frac{\text{zero nat}}{\text{succ}(\text{zero}) \text{ nat}}}{\text{succ}(\text{succ}(\text{zero})) \text{ nat}} \rightsquigarrow \frac{\text{zero nat}}{\text{succ}(\text{zero}) \text{ nat}}$$

This is what the proof of **Claim 1** implicitly does.

Admissible rules imply some non-trivial reasoning. When there is no such requirement, a rule is called derivable.

A rule is **derivable** if we can use a derivation of its premise as a building block in deriving its conclusion.

For example, the following rule is derivable:  $\frac{n \text{ nat}}{\text{succ}(\text{succ}(n)) \text{ nat}}$

Indeed, if we have a derivation of the premise, all it takes is two uses of the rule **Succ**:

$$\frac{\frac{\vdots}{n \text{ nat}}}{\text{succ}(\text{succ}(n)) \text{ nat}} \rightsquigarrow \frac{\frac{\frac{\vdots}{n \text{ nat}}}{\text{succ}(n) \text{ nat}} \text{ Succ}}{\text{succ}(\text{succ}(n)) \text{ nat}} \text{ Succ}$$

There is no need to perform induction to show that a rule is derivable.

## 2 Simultaneous induction

The principle of induction also applies to the simultaneous generation of judgements in a natural way. Recall the mutually inductive generation of the judgments  $n$  even and  $n$  odd by the rules

$$\begin{array}{ccc} \text{EVENZ} & \text{ODD} & \text{EVEN} \\ \frac{}{\text{zero even}} & \frac{n \text{ even}}{\text{succ}(n) \text{ odd}} & \frac{n \text{ odd}}{\text{succ}(n) \text{ even}} \end{array}$$

The associated induction principle is as follows:

Let  $\mathcal{P}$  be a property of even numbers, and let  $\mathcal{Q}$  be a property of odd numbers. If

- $\mathcal{P}(\text{zero})$ , and
- whenever  $n$  even and  $\mathcal{P}(n)$  we have  $\mathcal{Q}(\text{succ}(n))$ , and
- whenever  $n$  odd and  $\mathcal{Q}(n)$  we have  $\mathcal{P}(\text{succ}(n))$ ,

then  $\mathcal{P}(n)$  for all  $n$  even, and  $\mathcal{Q}(n)$  for all  $n$  odd.

For example, we can prove that

**Claim 2.** If  $n$  even then either  $n = \text{zero}$  or  $n = \text{succ}(x)$  where  $x$  odd.

We cannot prove this by a simple induction; we need to **strengthen** the inductive hypothesis. The proof amounts to performing simultaneous induction over the following predicates  $\mathcal{P}$  and  $\mathcal{Q}$ .

$\mathcal{P}(n)$ : “If  $n$  even then either  $n = \text{zero}$  or  $n = \text{succ}(x)$  where  $x$  odd.”

$\mathcal{Q}(n)$ : “If  $n$  odd then  $n = \text{succ}(x)$  where  $x$  even.”

The claim itself amounts to  $\mathcal{P}(n)$  for all  $n$  even.