Problem Sheet 1

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1. Write down a derivation of the judgement

$$\mathsf{succ}(\mathsf{succ}(\mathsf{succ}(\mathsf{zero})))\ \mathsf{odd}$$

2. (i) Write down the rules that generate lists of natural numbers.

   (ii) Write down the associated induction principle.

   (iii) In your notation, write a derivation of the judgement that `[0, 1]` is a list.

3. Prove that the following rule is derivable.

$$\frac{n\ \mathsf{even}}{\mathsf{succ}(\mathsf{succ}(n))\ \mathsf{even}}$$

4. Prove that the following rule is admissible.

$$\frac{n\ \mathsf{even}}{n\ \mathsf{nat}}$$

   (You might need to strengthen this statement a bit.)

5. ($*$) All the judgements we have seen up to this point have been *unary*, in the sense that they referred to only one entity. For example, the judgement n nat only refers to the object n.

   However, judgements can have arbitrary *arity*, and can thus define arbitrary relations between an arbitrary number of objects. For example, the following *ternary* judgment $\mathsf{sum}(a, b, c)$ defines a relation between three objects: a, b and c.

$$\textsc{Base}\ \frac{b\ \mathsf{nat}}{\mathsf{sum}(\mathsf{zero}, b, b)} \qquad\qquad \textsc{Ind}\ \frac{\mathsf{sum}(a, b, c)}{\mathsf{sum}(\mathsf{succ}(a), b, \mathsf{succ}(c))}$$

   The judgement $\mathsf{sum}(a, b, c)$ can be written in more familiar notation as $a + b = c$.

   Such judgements can be used—amongst countless other things—to define functions. This exercise is about showing that the above rules define the addition function.

   (i) Write down a derivation of $\mathsf{sum}(\mathsf{succ}(\mathsf{zero}), \mathsf{succ}(\mathsf{zero}), \mathsf{succ}(\mathsf{succ}(\mathsf{zero})))$.

   (ii) Restate the above rules as a Haskell function on the data type

```
data Nat = Zero | Succ Nat
```

Does your code use pattern matching? Discuss its relation to the rules given above.

(iii) Prove that if $\mathsf{sum}(a, b, c)$ then a nat, b nat, and c nat.

(iv) (Existence) Prove that if a nat and b nat then there exists a c nat such that $\mathsf{sum}(a, b, c)$.

(v) (Uniqueness) Prove that if $\mathsf{sum}(a, b, c)$ and $\mathsf{sum}(a, b, c')$ it must be that $c = c'$.

(vi) Conclude that $\mathsf{sum}(a, b, c)$ indeed defines a function on natural numbers.