Problem Sheet 5

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The following questions are about PCF.

1. Draw derivations that evidence the following typing judgements.
   (i) $\vdash \mathsf{fix}(n : \mathsf{Nat}.\, \mathsf{succ}(n)) : \mathsf{Nat}$
   (ii) $\vdash \mathsf{plus} : \mathsf{Nat} \rightharpoonup \mathsf{Nat} \rightharpoonup \mathsf{Nat}$ (as in Lecture 10)
2. Will the following terms eventually reduce to values? If so, write down transition sequences.

   You may find it useful to introduce some shorthands so you do not end up writing the same thing over and over. E.g. you can define $u \stackrel{\text{def}}{=} \mathsf{fix}(x : \tau.\, e)$, and then abbreviate the conclusion of D-Fix to the much more economical reduction $u \longmapsto e[u/x]$.
   (i) $\mathsf{plus}(\mathsf{succ}(\mathsf{zero}))(\mathsf{succ}(\mathsf{zero}))$ (as in Lecture 10)
   (ii) $\mathsf{fix}(x : \mathsf{Nat}.\, x)$
3. (i) Does the term $\mathsf{fix}(x : \mathsf{Nat}.\, \mathsf{succ}(x))$ reduce to a value?
   (ii) Suppose we change the rule Val-Succ to read

   $$\textsc{Val-Succ}\quad \frac{}{\mathsf{succ}(e)\ \mathsf{val}}$$

   Does it reduce to a value in that case? How is the evaluation of the term in 2(i) affected?
   (iii) Which of the two behaviours more closely approximates that of Haskell and the data type `Nat`?
4. Complete the definition of `times :: Nat -> Nat -> Nat` from Lecture 10. Then translate it to PCF.

   [Hint: use `plus` and plus respectively.]
5. This question is about proving progress and preservation for PCF.
   (i) State the substitution lemma for PCF. Prove the case of the fixed point rule.
   (ii) State the inversion lemma for the pre-term $\mathsf{fix}(x : \tau.\, e)$.
   (iii) State the progress and preservation theorems for PCF. Prove the cases for the fixed point rules.
6. ($*$) This question is about the uniqueness of typing in PCF.
   (i) State the uniqueness of typing for PCF (cf. problem sheet 2).
   (ii) Prove the inductive case for the fixed point rule.
   (iii) Would uniqueness of typing hold if we replaced the fixed point rule with the following one?

   $$\textsc{Fix}\quad \frac{\Gamma, x : \tau \vdash e : \tau}{\Gamma \vdash \mathsf{fix}(x.\, e) : \tau}$$

   If yes, give a proof. If not, give a counterexample.