

INVERSION & STRUCTURAL RULES

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Reading: PFPL, §4.3

1 Inversion

The nature of the typing rules for the language is such that the shape of a term places strong restrictions on its type. For example, we cannot imagine that the type of a term of the form $\text{plus}(e_1; e_2)$ is Str . After all, the only rule that allows us to derive a typing judgement of the form $\Gamma \vdash \text{plus}(e_1; e_2) : \tau$ forces $\tau \stackrel{\text{def}}{=} \text{Num}$.

Such facts about type systems are often called **inversion lemmata**. They are formally stated as follows.

Lemma 1 (Inversion). Suppose $\Gamma \vdash e : \tau$.

1. If $e = \text{plus}(e_1; e_2)$ then it must be that

- $\tau = \text{Num}$
- $\Gamma \vdash e_1 : \text{Num}$
- $\Gamma \vdash e_2 : \text{Num}$

2. ...

The lemma has a case for each construct of the language; you fill in the rest.

In practice, the proof of the inversion lemma is *by inspection* ('look at the rules, there can't be another way!'). More formally, the lemma can be shown by induction on the typing derivation.

2 Weakening

Suppose that $x : \sigma \vdash e : \tau$, i.e. that e computes a value of type τ if x is of type σ . It is reasonable to expect that for any **fresh variable** y (i.e. a variable that doesn't already occur in the term e), the typing judgement $x : \sigma, y : \rho \vdash e : \tau$ should hold as well, no matter what the type ρ is. In other words: assuming random free variables that we do not use at all should not influence the type of a program. This property is called **weakening**, and it holds in the majority of programming languages.

The reason it holds is that we may systematically thread a fresh variable across a derivation. For example,

$$\frac{\frac{x : \text{Num} \vdash x : \text{Num} \quad x : \text{Num} \vdash \text{num}[1] : \text{Num}}{x : \text{Num} \vdash \text{plus}(x; \text{num}[1]) : \text{Num}} \quad \frac{}{x : \text{Num}, y : \text{Num} \vdash y : \text{Num}}}{x : \text{Num} \vdash \text{let}(\text{plus}(x; \text{num}[1]); y. y) : \text{Num}}$$

can be systematically transformed by adding the binding $z : \text{Str}$ everywhere to obtain the derivation

$$\frac{\frac{x : \text{Num}, z : \text{Str} \vdash x : \text{Num} \quad x : \text{Num}, z : \text{Str} \vdash \text{num}[1] : \text{Num}}{x : \text{Num}, z : \text{Str} \vdash \text{plus}(x; \text{num}[1]) : \text{Num}} \quad \frac{}{x : \text{Num}, z : \text{Str}, y : \text{Num} \vdash y : \text{Num}}}{x : \text{Num}, z : \text{Str} \vdash \text{let}(\text{plus}(x; \text{num}[1]); y. y) : \text{Num}}$$

Formally, we state and prove by induction on the typing derivation that

Lemma 2 (Weakening). If $\Gamma \vdash e : \tau$ and x is fresh then $\Gamma, x : \sigma \vdash e : \tau$.

3 Substitution

We have read a judgement $x : \sigma \vdash e : \tau$ as saying that $e : \tau$ if we assume that x stands for a program of type σ . A term $\vdash e : \sigma$ whose context is empty is called a **closed term**; that means the program e has no free variables.

If we are given a closed term $\vdash e : \sigma$, and a term $x : \sigma \vdash u : \tau$, then we should somehow be able to ‘plug-in’, or **substitute**, the term $e : \sigma$ for the free variable $x : \sigma$ in u . This is the same process that we know from mathematics as e.g. plugging in $x \stackrel{\text{def}}{=} 5$ in the expression $x^2 + 3x + 1$ to obtain $5^2 + 3 * 5 + 1$.

We write the resulting term as $u[e/x]$. This is **not a construct of the programming language**. Rather, it is some notation we use to signify the substitution of one expression in another (as above). We sometimes call such things **metatheoretic operations**.¹ Formally, substitution is defined by induction on pre-terms.

$$\begin{aligned} z[e/x] &\stackrel{\text{def}}{=} \begin{cases} e & \text{if } z \equiv x \\ z & \text{if } z \not\equiv x \end{cases} & \text{let}(e_1; y. e_2)[e/x] &\stackrel{\text{def}}{=} \text{let}(e_1[e/x]; y. e_2[e/x]) \\ (\text{num}[n])[e/x] &\stackrel{\text{def}}{=} \text{num}[n] & \text{plus}(e_1; e_2)[e/x] &\stackrel{\text{def}}{=} \text{plus}(e_1[e/x]; e_2[e/x]) \\ (\text{str}[s])[e/x] &\stackrel{\text{def}}{=} \text{str}[s] & \text{cat}(e_1; e_2)[e/x] &\stackrel{\text{def}}{=} \text{cat}(e_1[e/x]; e_2[e/x]) \end{aligned}$$

The missing cases for $\text{times}(e_1; e_2)$ and $\text{len}(e_1)$ are analogous.

The fact the term $\text{let}(e_1; y. e_2)$ binds y in e_2 is treacherous! Consider what should happen in the following cases.

$$\text{let}(x + \text{num}[1]; y. x + y)[y/x] \qquad \text{let}(y; y. \text{len}(y))[\text{str}[\text{‘hi’}]/y]$$

In the first case, we risk **variable capture**; we must first α -rename the term to $\text{let}(x + \text{num}[1]; z. x + z)$. In the second case, we risk **referential clash**; we must first α -rename the term to $\text{let}(y; z. \text{len}(z))$.

Bound variables are always a source of trouble. It is common to adopt the **Barendregt convention**:² when substituting we assume that everything has been silently α -renamed in a way that is advantageous for us, and that will not cause the two problems exemplified above. Thus, when writing $u[e/x]$ we will assume that the variable x does not occur bound anywhere in the term u , because we can α -rename it if it does.

Substitution interacts very well with typing. The following is perhaps the most important result in this unit.

Lemma 3 (Substitution). If $\Gamma \vdash e : \tau$ and $\Gamma, x : \tau \vdash u : \sigma$, then $\Gamma \vdash u[e/x] : \sigma$.

Proof. By induction on the derivation of $\Gamma, x : \tau \vdash u : \sigma$. We show only the most involved case, viz. that of LET.

If the derivation of $\Gamma, x : \tau \vdash u : \sigma$ ends with LET, then we know that, for some type σ_1 it has the form

$$\frac{\frac{\vdots}{\Gamma, x : \tau \vdash e_1 : \sigma_1} \quad \frac{\vdots}{\Gamma, x : \tau, y : \sigma_1 \vdash e_2 : \sigma_2}}{\underbrace{\Gamma, x : \tau \vdash e_1 : \sigma_1}_{\text{“}\Gamma\text{”}} \quad \underbrace{\text{let}(e_1; y. e_2) : \sigma_2}_{\text{“}u\text{”}} \quad \underbrace{\sigma_2}_{\text{“}\sigma\text{”}}} \text{LET}$$

By the **induction hypothesis (IH)** applied to the assumption $\Gamma \vdash e : \tau$ and the derivation of $\Gamma, x : \tau \vdash e_1 : \sigma_1$ we obtain a derivation of $\Gamma \vdash e_1[e/x] : \sigma_1$. By the assumption $\Gamma \vdash e : \tau$ and **weakening** we get $\Gamma, y : \sigma_1 \vdash e : \tau$. We can then apply the IH to that and the second subtree to obtain a derivation of $\Gamma, y : \sigma_1 \vdash e_2[e/x] : \sigma_2$.

Using a single instance of the LET rule we can put these two together to obtain a derivation

$$\frac{\frac{\vdots}{\Gamma \vdash e_1[e/x] : \sigma_1} \quad \frac{\vdots}{\Gamma, y : \sigma_1 \vdash e_2[e/x] : \sigma_2}}{\Gamma \vdash \text{let}(e_1[e/x]; y. e_2[e/x]) : \sigma_2} \text{LET}$$

But $(\text{let}(e_1; y. e_2))[e/x]$ is by definition of substitution exactly the term $\text{let}(e_1[e/x]; y. e_2[e/x])$ in this derivation, so we have shown the result! \square

¹This term has its origins in logic. We are studying a little programming language which we call our **theory**. Anything we prove about it this programming language using maths and our minds is a **metatheoretic** statement, i.e. a statement *about* the theory itself.

²Originally coined by Dutch logician Henk Barendregt (b. 1947). Called the **identification convention** in PFPL.