Problem Sheet 2

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The following questions are about the language of numbers and strings.

1. Write down the abstract syntax tree for the pre-term $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\,\mathsf{plus}(i; n)); \mathsf{num}[2])$.

2. Assume $\Sigma \stackrel{\text{def}}{=} \{0, 1\}$. Write a program that
   - has a free variable x of type Str,
   - appends the string 0110 to x,
   - computes the length of the compound string, and
   - adds that number to itself.

   Your program should not mention the string literal $\mathsf{str}[0110]$ more than once.

3. Produce a typing derivation for the following terms, assuming that $\Sigma \stackrel{\text{def}}{=} \{0, 1\}$.
   (i) $x : \mathsf{Str} \vdash x : \mathsf{Str}$
   (ii) $\vdash \mathsf{plus}(\mathsf{num}[1]; \mathsf{num}[1]) : \mathsf{Num}$
   (iii) $x : \mathsf{Str} \vdash \mathsf{cat}(x; \mathsf{str}[01]) : \mathsf{Str}$
   (iv) $x : \mathsf{Str}, n : \mathsf{Num} \vdash \mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\,\mathsf{plus}(i; n)); \mathsf{num}[2]) : \mathsf{Num}$

4. Perform the following substitutions, step-by-step.
   (i) $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\,\mathsf{plus}(i; n)); \mathsf{num}[2])[i/x]$
   (ii) $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\,\mathsf{plus}(i; n)); \mathsf{num}[2])[\mathsf{num}[0]/n]$
   (iii) $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\,\mathsf{plus}(i; n)); \mathsf{num}[2])[i/n]$

5. State the cases of the inversion lemma for the following constructs:
   (i) $\mathsf{len}(e)$
   (ii) $\mathsf{let}(e_1; x.\,e_2)$

6. Prove the weakening lemma for the programming language of numbers and strings.

7. $(*)$ Complete the proof of substitution from Lecture 4.

   [Hint: In the case of variables, consider various cases: is it the variable I'm substituting for, or is it not? Also, you will have to use weakening, so assume that you have proven that already.]

8. Prove that types are unique, i.e. that for every context Γ and pre-term e there exists at most one τ such that $\Gamma \vdash e : \tau$.

   [Hint: assume that there exist two, and prove that they must be the same.]

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