Induction

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Reading: PFPL, §2.4, 2.5, 2.6, 3.1

1 Induction

Recall the rules that generate the natural numbers:

$$\textsc{Zero}\ \frac{}{\mathsf{zero}\ \mathsf{nat}} \qquad \textsc{Succ}\ \frac{n\ \mathsf{nat}}{\mathsf{succ}(n)\ \mathsf{nat}}$$

These rules define all the ways of constructing a natural number. Thus, if we *prove* that the truth of a property is preserved by these two rules, it must be that we have proved it for all natural numbers.

This is called the *principle of induction.*

Stated for the natural numbers, the principle of induction is as follows:

> Let $\mathcal{P}$ be a property of the natural numbers. If
>
> - $\mathcal{P}(\mathsf{zero})$, and
> - whenever $\mathcal{P}(n)$ we know that $\mathcal{P}(\mathsf{succ}(n))$
>
> then $\mathcal{P}(n)$ for all n nat.

Every set of rules generates an associated induction principle. Thus, the usual principle of natural number induction is implicitly generated by the rules Zero and Succ.

We can use induction to prove results.

Claim 1. If $\mathsf{succ}(n)$ nat then n nat.

To prove this it suffices to prove the following property by induction.

> $\mathcal{P}(n)$: "If n nat and $n = \mathsf{succ}(x)$ for some x, then x nat."

2 Simultaneous induction

The principle of induction also applies to the simultaneous generation of judgements in a natural way. Recall the mutually inductive generation of the judgments n even and n odd by the rules

$$\textsc{EvenZ}\ \frac{}{\mathsf{zero}\ \mathsf{even}} \qquad \textsc{Odd}\ \frac{n\ \mathsf{even}}{\mathsf{succ}(n)\ \mathsf{odd}} \qquad \textsc{Even}\ \frac{n\ \mathsf{odd}}{\mathsf{succ}(n)\ \mathsf{even}}$$

The associated induction principle is as follows:

> Let $\mathcal{P}$ be a property of even numbers, and let $\mathcal{Q}$ be a property of odd numbers. If
>
> - $\mathcal{P}(\mathsf{zero})$, and
> - whenever n even and $\mathcal{P}(n)$ we have $\mathcal{Q}(\mathsf{succ}(n))$, and

- whenever n odd and $\mathcal{Q}(n)$ we have $\mathcal{P}(\text{succ}(n))$,

then $\mathcal{P}(n)$ for all n even, and $\mathcal{Q}(n)$ for all n odd.

For example, we can prove that

Claim 2. If n even then either $n = \text{zero}$ or $n = \text{succ}(x)$ where x odd.

We cannot prove this by a simple induction; we need to **strengthen** the inductive hypothesis. The proof amounts to performing simultaneous induction over the following predicates $\mathcal{P}$ and $\mathcal{Q}$.

$\mathcal{P}(n)$: "If n even then either $n = \text{zero}$ or $n = \text{succ}(x)$ where x odd."

$\mathcal{Q}(n)$: "If n odd then $n = \text{succ}(x)$ where x even."

The claim itself amounts to $\mathcal{P}(n)$ for all n even.