

# STATICS

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Reading: PFPL, §4.1, 4.2

## 1 The phase distinction

The lifetime of a computer program is divided into two phases:

- the **static** phase — which comprises everything that occurs *before* running a program; and
- the **dynamic** phase — which comprises everything that happens when a program is actually run.

Thus, the **statics** of a program include things such as lexing, parsing, type-checking, static analysis, etc. In contrast, the **dynamics** of a program include its runtime behaviour: final value, side-effects, exceptions, etc.

In this unit both the statics and the dynamics of a PL will be specified in a fairly idealised, mathematical manner. We will use **abstract syntax** as the syntax of our program; this will absolve us from having to deal with lexing, parsing, grammars, and so on. Our only statics will be a **type system** for this abstract syntax.

Correspondingly, our dynamics will be given by specifying the **operational semantics** of our programs. These will also be presented in a mathematical style, by specifying a little **abstract machine** that evaluates a program.

## 2 Typing judgements

In this unit we will concern ourselves with **typing judgments** of the following form:

$$\underbrace{\Gamma}_{\text{context}} \vdash \underbrace{e}_{\text{term}} : \underbrace{\sigma}_{\text{type}}$$

A typing judgement is a ternary relation between three elements:

- the **context**—an unordered list  $\Gamma$  of variable-type bindings
- the **term**—the program  $e$  that is being typed
- the **type** of the term—which classifies what the program computes

We read  $\Gamma \vdash e : \sigma$  as “the program  $e$  has type  $\sigma$  in context  $\Gamma$ .”

The **context**  $\Gamma$  consists of (variable, type) pairs. E.g. the context  $\Gamma = x : \sigma, y : \tau$  declares two **free variables**:

- $x$ , which stands for a term of type  $\sigma$
- $y$ , which stands for a term of type  $\tau$

These are in no particular order: the context  $x : \sigma, y : \tau$  is the same as the context  $y : \tau, x : \sigma$ .

Thus, we can read the judgement  $x : \tau \vdash e : \sigma$  as follows: “assuming that the free variable  $x$  stands for a program that computes a value of type  $\tau$ , the program  $e$  computes a value of type  $\sigma$ .”

We will only say that “ $e$  is a **term**” if there exist  $\Gamma$  and  $\sigma$  such that the judgement  $\Gamma \vdash e : \sigma$  is evident. However, we will identify a larger class of programs, which we will call **pre-terms**. These will have the same ‘shape’ as terms, but they will not necessarily be well-typed. In short, the well-typed pre-terms will be called terms.

Finally, in this unit we will only consider so-called **simple types**, which will come from an inductively generated syntax (see next section).

### 3 A little language of numbers and strings

To illustrate the aforementioned concepts we will present the statics of a language of numbers and strings.

The abstract syntax, types, and pre-terms of the language are presented by the following **syntax chart**.

types	$\tau ::=$	Num	numbers
		Str	strings
pre-terms	$e ::=$	$x$	variables
		$\text{num}[n]$	numeral
		$\text{str}[s]$	string literals
		$\text{plus}(e_1; e_2)$	$e_1 + e_2$ addition
		$\text{times}(e_1; e_2)$	$e_1 * e_2$ multiplication
		$\text{cat}(e_1; e_2)$	$e_1 \mathbin{++} e_2$ concatenation
		$\text{len}(e)$	$ e $ length
		$\text{let}(e_1; x. e_2)$	$\text{let } x \Leftarrow e_1 \text{ in } e_2$ let-definition

This notation is sometimes called an extended **Backus-Naur form**. It generates **syntax trees**.

The first symbol represents the **syntactic category** (e.g. type  $\tau$ , expression  $e$ , etc.).

The second column (immediately to the right of  $::=$ ) is the **abstract syntax**: it corresponds closely to the way you would represent the expression in a high-level functional programming language as an abstract syntax tree. Subscripted occurrences (e.g.  $e_1, e_2$ ) are recursive occurrences of the same syntactic element. For example,  $\text{cat}(e_1; e_2)$  is an expression, provided  $e_1$  and  $e_2$  are also expressions. We tacitly assume  $n \in \mathbb{N}$  and  $s \in \Sigma^*$  for some alphabet  $\Sigma$ . We also tacitly assume that variables  $x$  come from some predetermined, infinite supply.

The third column is the **concrete syntax**: it is a user-friendly abbreviation for the abstract syntax.

In this language a type  $\tau$  is either a Num or a Str. A pre-term  $e$  is given by one of the many forms listed above.

The following rules generate the typing judgements, and hence the well-typed **terms** of the language.

$\frac{\text{VAR}}{\Gamma, x : \sigma \vdash x : \sigma}$	$\frac{\text{NUM} \quad n \in \mathbb{N}}{\Gamma \vdash \text{num}[n] : \text{Num}}$	$\frac{\text{STR} \quad s \in \Sigma^*}{\Gamma \vdash \text{str}[s] : \text{Str}}$
$\frac{\text{PLUS} \quad \Gamma \vdash e_1 : \text{Num} \quad \Gamma \vdash e_2 : \text{Num}}{\Gamma \vdash \text{plus}(e_1; e_2) : \text{Num}}$	$\frac{\text{TIMES} \quad \Gamma \vdash e_1 : \text{Num} \quad \Gamma \vdash e_2 : \text{Num}}{\Gamma \vdash \text{times}(e_1; e_2) : \text{Num}}$	
$\frac{\text{CAT} \quad \Gamma \vdash e_1 : \text{Str} \quad \Gamma \vdash e_2 : \text{Str}}{\Gamma \vdash \text{cat}(e_1; e_2) : \text{Str}}$	$\frac{\text{LEN} \quad \Gamma \vdash e : \text{Str}}{\Gamma \vdash \text{len}(e) : \text{Num}}$	$\frac{\text{LET} \quad \Gamma \vdash e_1 : \sigma_1 \quad \Gamma, x : \sigma_1 \vdash e_2 : \sigma_2}{\Gamma \vdash \text{let}(e_1; x. e_2) : \sigma_2}$

Some points about variables and binding:

- Writing  $\Gamma, x : \sigma$  insinuates that  $x$  does not occur elsewhere in  $\Gamma$ .
- $x$  is **bound** within  $e_2$  in  $\text{let}(e_1; x. e_2)$ . Thus, it is subject to  $\alpha$ -conversion.

An example derivation; for any  $s \in \Sigma^*$ :

$x : \text{Str} \vdash x : \text{Str}$	$\frac{s \in \Sigma^*}{x : \text{Str} \vdash \text{str}[s] : \text{Str}}$	$\frac{x : \text{Str}, y : \text{Str} \vdash y : \text{Str}}{x : \text{Str}, y : \text{Str} \vdash \text{len}(y) : \text{Num}}$	$\frac{1 \in \mathbb{N}}{x : \text{Str}, y : \text{Str} \vdash \text{num}[1] : \text{Num}}$
$x : \text{Str} \vdash \text{cat}(x; \text{str}[s]) : \text{Str}$		$x : \text{Str}, y : \text{Str} \vdash \text{plus}(\text{len}(y); \text{num}[1]) : \text{Num}$	
$x : \text{Str} \vdash \text{let}(\text{cat}(x; \text{str}[s]); y. \text{plus}(\text{len}(y); \text{num}[1])) : \text{Num}$			

In words: if we plug in a program that computes a string for  $x : \text{Str}$ , this program will append the string  $s \in \Sigma^*$  to it; it will then compute its length, and add 1 to it.