# Inversion & Structural Rules

Alex Kavvos

*Reading: PFPL, §4.3*

## 1 Inversion

The nature of the typing rules for the language is such that the shape of a term places strong restrictions on its type. For example, we cannot imagine that the type of a term of the form $\mathsf{plus}(e_1; e_2)$ is Str. After all, the only rule that allows us to derive a typing judgement of the form $\Gamma \vdash \mathsf{plus}(e_1; e_2) : \tau$ forces $\tau \stackrel{\text{def}}{=} \mathsf{Num}$.

Such facts about type systems are often called **inversion lemmata**. They are formally stated as follows.

**Lemma 1** (Inversion)**.** Suppose $\Gamma \vdash e : \tau$.

1. If $e = \mathsf{plus}(e_1; e_2)$ then it must be that
   - $\tau = \mathsf{Num}$
   - $\Gamma \vdash e_1 : \mathsf{Num}$
   - $\Gamma \vdash e_2 : \mathsf{Num}$
2. …

The lemma has a case for each construct of the language; you fill in the rest.

In practice, the proof of the inversion lemma is *by inspection* ('look at the rules, there can't be another way!'). More formally, the lemma can be shown by induction on the typing derivation.

## 2 Weakening

Suppose that $x : \sigma \vdash e : \tau$, i.e. that $e$ computes a value of type $\tau$ if $x$ is of type $\sigma$. It is reasonable to expect that for any **fresh variable** $y$ (i.e. a variable that doesn't already occur in the term $e$), the typing judgement $x : \sigma, y : \rho \vdash e : \tau$ should hold as well, no matter what the type $\rho$ is. In other words: assuming random free variables that we do not use at all should not influence the type of a program. This property is called **weakening**, and it holds in the majority of programming languages.

The reason it holds is that we may systematically thread a fresh variable across a derivation. For example,

$$\dfrac{\dfrac{x : \mathsf{Num} \vdash x : \mathsf{Num} \qquad x : \mathsf{Num} \vdash \mathsf{num}[1] : \mathsf{Num}}{x : \mathsf{Num} \vdash \mathsf{plus}(x; \mathsf{num}[1]) : \mathsf{Num}} \qquad \dfrac{}{x : \mathsf{Num}, y : \mathsf{Num} \vdash y : \mathsf{Num}}}{x : \mathsf{Num} \vdash \mathsf{let}(\mathsf{plus}(x; \mathsf{num}[1]); y.\, y) : \mathsf{Num}}$$

can be systematically transformed by adding the binding $z : \mathsf{Str}$ everywhere to obtain the derivation

$$\dfrac{\dfrac{x : \mathsf{Num}, z : \mathsf{Str} \vdash x : \mathsf{Num} \qquad x : \mathsf{Num}, z : \mathsf{Str} \vdash \mathsf{num}[1] : \mathsf{Num}}{x : \mathsf{Num}, z : \mathsf{Str} \vdash \mathsf{plus}(x; \mathsf{num}[1]) : \mathsf{Num}} \qquad \dfrac{}{x : \mathsf{Num}, z : \mathsf{Str}, y : \mathsf{Num} \vdash y : \mathsf{Num}}}{x : \mathsf{Num}, z : \mathsf{Str} \vdash \mathsf{let}(\mathsf{plus}(x; \mathsf{num}[1]); y.\, y) : \mathsf{Num}}$$

Formally, we state and prove by induction on the typing derivation that

**Lemma 2** (Weakening)**.** If $\Gamma \vdash e : \tau$ and $x$ is fresh then $\Gamma, x : \sigma \vdash e : \tau$.

## 3 Substitution

We have read a judgement $x : \sigma \vdash e : \tau$ as saying that $e : \tau$ if we assume that $x$ stands for a program of type $\sigma$.

A term $\vdash e : \sigma$ whose context is empty is called a **closed term**; that means the program $e$ has no free variables.

If we are given a closed term $\vdash e : \sigma$, and a term $x : \sigma \vdash u : \tau$, then we should somehow be able to 'plug-in', or **substitute**, the term $e : \sigma$ for the free variable $x : \sigma$ in $u$. This is the same process that we know from mathematics as e.g. plugging in $x \stackrel{\text{def}}{=} 5$ in the expression $x^2 + 3x + 1$ to obtain $5^2 + 3 * 5 + 1$.

We write the resulting term as $u[e/x]$. This is **not a construct of the programming language**. Rather, it is some notation we use to signify the substitution of one expression in another (as above). We sometimes call such things **metatheoretic operations**.[1] Formally, substitution is defined by induction on pre-terms.

$$z[e/x] \stackrel{\text{def}}{=} \begin{cases} e & \text{if } z \equiv x \\ z & \text{if } z \not\equiv x \end{cases} \qquad \mathsf{let}(e_1; y.\, e_2)[e/x] \stackrel{\text{def}}{=} \mathsf{let}(e_1[e/x]; y.\, e_2[e/x])$$

$$(\mathsf{num}[n])[e/x] \stackrel{\text{def}}{=} \mathsf{num}[n] \qquad \mathsf{plus}(e_1; e_2)[e/x] \stackrel{\text{def}}{=} \mathsf{plus}(e_1[e/x]; e_2[e/x])$$

$$(\mathsf{str}[s])[e/x] \stackrel{\text{def}}{=} \mathsf{str}[s] \qquad \mathsf{cat}(e_1; e_2)[e/x] \stackrel{\text{def}}{=} \mathsf{cat}(e_1[e/x]; e_2[e/x])$$

The missing cases for $\mathsf{times}(e_1; e_2)$ and $\mathsf{len}(e_1)$ are analogous.

The fact the term $\mathsf{let}(e_1; y.\, e_2)$ binds $y$ in $e_2$ is treacherous! Consider what should happen in the following cases.

$$\mathsf{let}(x + \mathsf{num}[1]; y.\, x + y)[y/x] \qquad \mathsf{let}(y; y.\, \mathsf{len}(y))[\mathsf{str}[\text{'hi'}]/y]$$

In the first case, we risk **variable capture**; we must first $\alpha$**-rename** the term to $\mathsf{let}(x + \mathsf{num}[1]; z.\, x + z)$. In the second case, we risk **referential clash**; we must first $\alpha$-rename the term to $\mathsf{let}(y; z.\, \mathsf{len}(z))$.

Bound variables are always a source of trouble. It is common to adopt the **Barendregt convention**:[2] when substituting we assume that everything has been silently $\alpha$-renamed in a way that is advantageous for us, and that will not cause the two problems exemplified above. Thus, when writing $u[e/x]$ we will assume that the variable $x$ does not occur bound anywhere in the term $u$, because we can $\alpha$-rename it if it does.

Substitution interacts very well with typing. The following is perhaps the most important result in this unit.

**Lemma 3** (Substitution)**.** If $\Gamma \vdash e : \tau$ and $\Gamma, x : \tau \vdash u : \sigma$, then $\Gamma \vdash u[e/x] : \sigma$.

*Proof.* By induction on the derivation of $\Gamma, x : \tau \vdash u : \sigma$. We show only the most involved case, viz. that of Let.

If the derivation of $\Gamma, x : \tau \vdash u : \sigma$ ends with Let, then we know that, for some type $\sigma_1$ it has the form

$$\dfrac{\dfrac{\vdots}{\Gamma, x : \tau \vdash e_1 : \sigma_1} \qquad \dfrac{\vdots}{\Gamma, x : \tau, y : \sigma_1 \vdash e_2 : \sigma_2}}{\underbrace{\Gamma, x : \tau}_{\text{“}\Gamma\text{”}} \vdash \underbrace{\mathsf{let}(e_1; y.\, e_2)}_{\text{“}u\text{”}} : \underbrace{\sigma_2}_{\text{“}\sigma\text{”}}}\ \textsc{Let}$$

By the **induction hypothesis (IH)** applied to the assumption $\Gamma \vdash e : \tau$ and the derivation of $\Gamma, x : \tau \vdash e_1 : \sigma_1$ we obtain a derivation of $\Gamma \vdash e_1[e/x] : \sigma_1$. By the assumption $\Gamma \vdash e : \tau$ and **weakening** we get $\Gamma, y : \sigma_1 \vdash e : \tau$. We can then apply the **IH** to that and the second subtree to obtain a derivation of $\Gamma, y : \sigma_1 \vdash e_2[e/x] : \sigma_2$.

Using a single instance of the Let rule we can put these two together to obtain a derivation

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash e_1[e/x] : \sigma_1} \qquad \dfrac{\vdots}{\Gamma, y : \sigma_1 \vdash e_2[e/x] : \sigma_2}}{\Gamma \vdash \mathsf{let}(e_1[e/x]; y.\, e_2[e/x]) : \sigma_2}\ \textsc{Let}$$

But $(\mathsf{let}(e_1; y.\, e_2))[e/x]$ is by definition of substitution exactly the term $\mathsf{let}(e_1[e/x]; y.\, e_2[e/x])$ in this derivation, so we have shown the result! □

[1] This term has its origins in logic. We are studying a little programming language which we call our **theory**. Anything we prove about it this programming language using maths and our minds is a **metatheoretic** statement, i.e. a statement *about* the theory itself.

[2] Originally coined by Dutch logician Henk Barendregt (b. 1947). Called the **identification convention** in PFPL.