Judgements

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Reading: PFPL, §2.1–2.3

1 Judgements and evidence

A **judgement** is a statement (proposition, utterance, enunciation).

An **evident judgement** is a statement for which I have evidence (proof).

For example, I can assert that 2 is a natural number: $2\ \mathsf{nat}$.

This is a judgement. It is not *evident* yet, as we do not know how to prove it.

2 Rules

To prove a judgement we have to first elaborate what we accept as evidence for it.

For example, we can write down the following **rules** for proving that something is a natural number:

$$\frac{}{\mathsf{zero}\ \mathsf{nat}}\ \textsc{Zero} \qquad \frac{n\ \mathsf{nat}}{\mathsf{succ}(n)\ \mathsf{nat}}\ \textsc{Succ}$$

Judgements above the line are called **premises**; below the line, they are called **conclusions**.

Rules with no premises are called **axioms**.

Such rules can be assembled into **derivations** that prove judgements. For example, we can define

$$2 \stackrel{\text{def}}{=} \mathsf{succ}(\mathsf{succ}(\mathsf{zero}))$$

and then prove that it is a natural number:

$$\cfrac{\cfrac{\cfrac{}{\mathsf{zero}\ \mathsf{nat}}}{\mathsf{succ}(\mathsf{zero})\ \mathsf{nat}}}{\mathsf{succ}(\mathsf{succ}(\mathsf{zero}))\ \mathsf{nat}}$$

Listing such rules tactily implies that anything they do not prove is not an evident judgement.

For example, the rules for natural numbers state that

- 0 is a natural number
- if n is a natural number then so is $n + 1$
- *nothing else is a natural number*

This circumscribes the notion of natural number and what constitutes evidence for its construction.

3 Simultaneous generation of judgements

When stating proof rules, judgements may be mixed and matched to generate more complicated evidence.

For example, the following rules generate judgements on the parity of natural numbers.

$$\textsc{EvenZ}\ \frac{}{\mathsf{zero\ even}} \qquad \textsc{Odd}\ \frac{n\ \mathsf{even}}{\mathsf{succ}(n)\ \mathsf{odd}} \qquad \textsc{Even}\ \frac{n\ \mathsf{odd}}{\mathsf{succ}(n)\ \mathsf{even}}$$

4 Induction

The rules given above describe all the ways of constructing a natural number. Thus, if we *prove* that the truth of a property is preserved by these two rules, it must be that we have proved it for all natural numbers.

This is called the *principle of induction.*

Stated for the natural numbers, the principle of induction is as follows:

> Let $\mathcal{P}$ be a property of the natural numbers. If
>
> - $\mathcal{P}(\mathsf{zero})$, and
> - whenever $\mathcal{P}(n)$ we know that $\mathcal{P}(\mathsf{succ}(n))$
>
> then $\mathcal{P}(n)$ for all n nat.

Every set of rules generates an associated induction principle. Thus, the usual principle of natural number induction is implicitly generated by the rules Zero and Succ.

We can use induction to prove results.

Claim 1. If $\mathsf{succ}(n)$ nat then n nat.

To prove this it suffices to prove the following property by induction.

> $\mathcal{P}(n)$: "If n nat and $n = \mathsf{succ}(x)$ for some x, then x nat."

5 Derivable and admissible rules

Statements like Claim 1 are often written as rules themselves: $\dfrac{\mathsf{succ}(n)\ \mathsf{nat}}{n\ \mathsf{nat}}$

This rule is not one of the defining rules of natural numbers. Rather, it is an admissible rule.

A rule is **admissible** if whenever we have a derivation of the premises, then we know we can construct a derivation of the conclusion.

In this particular instance, given a derivation of $\mathsf{succ}(n)$ nat, we can construct a derivation of n nat by trimming the last line of the derivation. For $n \stackrel{\text{def}}{=} \mathsf{succ}(\mathsf{zero})$:

$$\frac{\dfrac{\mathsf{zero\ nat}}{\mathsf{succ}(\mathsf{zero})\ \mathsf{nat}}}{\mathsf{succ}(\mathsf{succ}(\mathsf{zero}))\ \mathsf{nat}} \quad \rightsquigarrow \quad \frac{\mathsf{zero\ nat}}{\mathsf{succ}(\mathsf{zero})\ \mathsf{nat}}$$

This is what the proof of Claim 1 implicitly does.

Admissible rules imply some non-trivial reasoning. When there is no such requirement, a rule is called derivable.

A rule is **derivable** if we can use a derivation of its premise as a building block in deriving its conclusion.

For example, the following rule is derivable: $\dfrac{n\ \mathsf{nat}}{\mathsf{succ}(\mathsf{succ}(n))\ \mathsf{nat}}$

Indeed, if we have a derivation of the premise, all it takes is two uses of the rule Succ:

$$\dfrac{\vdots}{n\ \mathsf{nat}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdots}{n\ \mathsf{nat}}}{\mathsf{succ}(n)\ \mathsf{nat}}\ \textsc{Succ}}{\mathsf{succ}(\mathsf{succ}(n))\ \mathsf{nat}}\ \textsc{Succ}$$

There is no need to perform induction to show that a rule is derivable.