The Discount for Lack of Marketability Term Structure

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Abstract

I empirically estimate a discount for lack of marketability (DLOM) term structure for restriction periods up to 10 years. I model the multi-year DLOM by annually compounding the one-year DLOM plus a term premium over the restriction period. I fit the model to a sample of 5,333 private equity placement implied DLOMs between 1985 and 2017. I estimate different DLOM term structures for the last and earlier pre-IPO private equity transactions. DLOM term structures exhibit level, slope, and curvature shifts similar to an interest rate term structure. At least a quarter of the transactions in my sample were underpriced.

JEL Classification: G12, G13

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The almost six-fold growth in private equity (PE) since 1990 has stimulated market activity in unregistered common stocks.[1] Transactions include PE funds selling shares of portfolio companies, PE fund investors selling limited partnership units, and employees and funders of privately owned firms selling their shares to investors.[2] Privately owned firms with greater access to private capital are also waiting longer to go public, so more shareholders seek liquidity by selling shares in private transactions.[3] The number of so-called unicorns now exceeds 1,200 while the number of exchange-listed U.S. public operating firms has more than halved to less than 4,000 since 1997.[4] However, secondary markets for unregistered common stock are relatively illiquid due to the resale restrictions under the Securities Act of

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[1] Private equity funds invested $2.42 trillion in 59,900 private deals globally in 2022, up from $190 billion in 2,200 private deals in 1990 (McKinsey, 2023).

[2] PE fund sponsors have formed funds specifically to purchase the investments that other PE funds wish to cast off and the PE fund LP units that institutional investors wish to dispose of. Likewise, a secondary market has developed for shares of privately held firms that are expected to go public.

[3] Firms are staying private longer due to the increased availability of private equity, especially for technology firms; the fourfold increase in the maximum number of shareholders before a firm must publish its financial statements publicly under the U.S. Jumpstart Our Business Startups (JOBS) Act of 2012; and the generally higher IPO multiples the public equity market has at times accorded larger tech firms.

[4] Privately owned firms whose total equity value exceeds $1 billion are known as 'unicorns.' CB Insights reported there were 1,225 unicorns globally as of July 2023 with an aggregate value of approximately $3.845 trillion. See https://www.cbinsights.com/research-unicorn-companies. Last accessed August 2, 2023. Between the dot.com peak in 1997 and year-end 2020, the number of U.S. operating firms listed on a U.S stock exchange fell by more than 50% from 7,665 to 3,731, according to data on Professor Jay Ritter's website. Available at https://site.warrington.ufl.edu/ritter/files/number-of-listed-firms-on-US-exchanges.pdf. Last accessed August 22, 2023.

1933 (1933 Act).[5] Restricted shares are not freely tradable in the public market and consequently are valued at a discount to their value if they were freely tradable. The desire to transact in unregistered common stock coupled with the lack of regular market prices to facilitate price discovery have increased interest in discount for lack of marketability (DLOM) models that can be used reliably in pricing restricted shares when there are multi-year restriction periods.

I empirically estimate DLOM term structures for restriction periods up to 10 years for valuing unregistered shares. The fitted DLOM term structure should be of interest because financial economists and business appraisers typically value restricted stock by calculating the freely traded value of the stock assuming it trades in a liquid market free of restrictions and subtracting an estimated DLOM to reflect the loss of timing flexibility due to the resale or transfer restrictions (Cornell, 1993; Pratt, 2001; Damodaran, 2012; Pratt and Grabowski, 2014; and Ben-Rephael, Kadan, and Wohl, 2015). The DLOM generally increases with the length of the restriction period, which is captured within the estimated DLOM term structures. I find that the probability distribution for the length of the restriction period for unregistered shares is approximately log-normal but has different parameters for the last private equity transaction preceding a firm's IPO and earlier transactions. I estimate a separate DLOM term structure for each of these samples of pre-IPO private equity transactions.

Recent regulatory and private market changes will likely increase investors' interest in a fitted DLOM term structure. The Jumpstart Our Business Startups (JOBS) Act, enacted in 2012, expanded the availability of private equity capital and reduced the urgency of going public to raise funds, which has

[5] For example, Nasdaq Private Market reported transaction volume of $4.5 billion in 2020 (up from $1.1 billion in 2016) involving liquidity programs for 90 private companies. See https://www.nasdaq.com/solutions/nasdaq-private-market/End-of-Year-Report-2020. Last accessed November 20, 2021.

potentially increased the need for DLOM models to price unregistered shares. The Regulation A small issue exemption under the Securities Act of 1933 had become ineffectual by the early 2000s because of its low $5 million annual offering limit (Knyazeva, 2016). The JOBS Act directed the SEC to adopt rules exempting securities offerings of up to $50 million within a 12-month period from the 1933 Act registration requirements. Final SEC rules became effective June 19, 2015, which expanded Regulation A (referred to as Regulation A+) into two tiers: Tier 1, for securities offerings of up to $20 million in a 12-month period, and Tier 2, for securities offerings of up to $50 million in a 12-month period (SEC, 2015).[6] The SEC raised the limit on Regulation A+ offerings from $50 million to $75 million and loosened the restrictions on "general solicitation" and "testing the waters" in rules that became effective March 15, 2021.[7] An unregistered Regulation A+ issue could become a viable alternative to a small registered IPO.

Nasdaq, Goldman Sachs, Morgan Stanley, and other banks formed a centralized market platform to trade the common stock of private firms in 2021. Nasdaq Private Market enables private equity market participants to conduct tender offers and block trades and engage in continuous trading while

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[6] Regulation A is the so-called 'small issue' exemption from the 1933 Act's securities registration requirements. The Regulation A offering limitation had not been raised (above $5 million) since 1992. Shareholders are limited in the aggregate to selling (through Regulation A secondary offerings) no more than 30% of the dollar amount of the offering in the following 12-month period.

[7] General solicitation rules permit a firm that is planning a private equity capital raise to market the unregistered issue publicly under tight restrictions that are designed to prevent fraud. Testing the waters rules allow a private firm to contact prospective investors to gauge interest in a private equity raise, again under tight restrictions. See Zanki, Tom, "4 Ways Private Capital-Raising Rules Are Loosening Up," *Law360* (April 23, 2021). Available at https://www.law360.com/securities/articles/1378287/4-ways-private-capital-raising-rules-are-loosening-up. Last accessed August 22, 2023.

complying with the restrictions on transferring unregistered shares. They can use a fitted DLOM term structure to assess the reasonableness of the DLOM implicit in the market price of an unregistered stock.

This paper fits into the equity pricing literature and improves DLOM estimation. DLOMs were initially estimated from the discounted prices of letter stock (SEC, 1971; Wruck, 1989; Silber, 1991; Hertzel and Smith, 1993; and Hertzel et al., 2002).[8] Put option DLOM valuation models were developed to capitalize on the seminal BSM put option pricing model (Black and Scholes, 1973; and Merton, 1973). The earliest put option DLOM model (Alli and Thompson, 1991) measures the DLOM with respect to the loss of the flexibility to sell shares at today's freely traded market price when the shareholder cannot sell the stock until the end of some restriction period. It does not measure the DLOM accurately because it assumes a fixed strike price for the duration of the restriction period, which gives a restricted stockholder absolute downside protection (barring default by the option writer). The price of the otherwise identical unrestricted stock will change during the restriction period due to normal market forces. A stockholder's loss of timing flexibility should be measured relative to the opportunity she would have to sell at any of these market prices were there no transfer restrictions.

Two more recent put option DLOM models, the lookback put and average-strike put option models, avoid this shortcoming by adjusting the strike price to reflect the market price of the unrestricted stock during the restriction period. Longstaff (1995) obtains an upper bound on the DLOM by modeling the value of marketability as the price of a lookback put option. The model assumes investors have perfect market-timing ability, which is generally consistent with empirical evidence that private information

[8] Letter stock consists of unregistered shares of common stock public firms sell privately, which purchasers can later resell subject to the resale restrictions imposed by Rule 144 under the 1933 Act. It is not registered for resale under the 1933 Act and therefore cannot be freely traded in the public market. It must be placed privately with accredited (sophisticated) investors.

enables insiders to time the market and realize excess returns (Gompers and Lerner, 1998). However, it is inconsistent with evidence that outside investors, at least on average, do not have any special ability to outperform the market (Graham and Harvey, 1996; and Barber and Odean, 2000). Consequently, the lookback put option model is likely to overstate the DLOM when investors do not have valuable private information about the stock.

Finnerty (2012, 2013) models the DLOM as the value of an average-strike put option. An investor is not assumed to have any special market-timing ability; instead, the model assumes an investor would, in the absence of any transfer restrictions, be equally likely to sell the shares anytime during the restriction period. Finnerty (2012) compares the model-predicted discounts to the discounts observed in a sample of 208 discounted common stock private placements and finds that the average-strike put option model discounts are generally consistent with the observed private placement discounts after adjusting for the information, ownership concentration, and overvaluation effects that accompany a stock private placement (Wruck, 1989; Hertzel and Smith, 1993; and Hertzel et al., 2002). However, the model becomes increasingly less accurate as the restriction period lengthens beyond two years, especially for high-volatility stocks.

I annually compound the one-year average-strike DLOM plus a term premium over the restriction period to model the DLOM for longer restriction periods. I fit the model to a sample of 5,333 private equity placement implied DLOMs between 1985 and 2017 and derive *ex post* DLOM term structures for restriction periods up to 10 years for the last pre-IPO and earlier pre-IPO transactions. The estimated DLOM term structures exhibit level, slope, and curvature shifts similar to an interest rate term structure.

This paper is also related to the literature on pricing illiquid assets. The DLOM plays an equivalent economic role to Amihud and Mendelson's (1986) liquidity risk premium in illiquid asset pricing. Stock transfer restrictions, such as those imposed by Rule 144, are one of the more often-cited factors responsible for a restricted share's relative lack of liquidity. Liquidity refers to the relative ease with which an asset holder can convert the asset into cash without losing some of the asset's intrinsic value. A liquid market gives an asset holder the flexibility to sell the asset anytime she chooses without sacrificing any intrinsic value. An asset that lacks marketability also lacks liquidity because the marketability restrictions inhibit the holder from selling the asset quickly for its full intrinsic value.

Lack of marketability and lack of liquidity entail similar losses of resale or transfer flexibility but with different root causes. Lack of marketability arises when legal or contractual restrictions on transfer prevent, or at least severely impair, an asset holder's ability to sell the asset or transfer it until the restriction period lapses.[9] Lack of liquidity arises from thin trading in the asset. Lack of liquidity, like lack of marketability, imposes a loss of timing flexibility because an asset holder cannot dispose of the asset quickly in an illiquid market unless she is willing to accept a reduction in value. Longstaff (2017) models illiquidity as a restriction on the set of stopping rules an investor can apply when selling a thinly traded asset and develops an option-theoretic framework to determine an upper bound on the illiquidity discount. This loss of flexibility could also be modeled as the loss of value of a foregone put option by applying the model developed in this paper provided one could reasonably estimate the time horizon T at which the illiquidity constraints on the thinly traded asset are removed, as in the Longstaff (2017) model.

I. Rationale for the Put Option Characterization of a DLOM

Resale and transfer restrictions entail a loss of timing flexibility because a shareholder must postpone sale or transfer until the restrictions lapse unless the shareholder can find an accredited investor who is willing to buy the shares. This loss of flexibility imposes a cost, which has been modeled as the

[9] The 1933 Act's legal restrictions on offering unregistered securities to non-accredited investors and the contractual prohibition on transferring employee stock options exemplify legal and contractual marketability restrictions, respectively.

value of a foregone put option that will expire when the marketability restriction lapses (Longstaff, 1995, 2001; Kahl, Liu, and Longstaff, 2003; and Finnerty, 2012, 2013). I utilize the average-strike put option DLOM model (Finnerty, 2012) to develop a multi-period DLOM model that could apply to restriction periods of any specified length L. I model the L-year DLOM as the value of the one-year DLOM plus a term premium compounded annually over L years and then fit the multi-period DLOM model to a sample of estimated DLOMs for restriction periods up to 10 years.

The average-strike put option DLOM model assumes the firm's unrestricted shares trade continuously in a frictionless market. The firm also has restricted shares outstanding, and all the firm's shares are identical except for the transfer restrictions, which prevent an investor from selling the restricted shares for a period of length L. Any cash dividends are paid continuously during [0, L] at a continuously compounded rate $q \geq 0$, which is proportional to $V(t)$, the value of a share of common stock without transfer restrictions. V follows a geometric diffusion process of the form

$$dV = (\mu - q)Vdt + \sigma VdZ \tag{1}$$

where μ and σ are constants that measure the continuously compounded mean return and volatility, respectively, and Z is a standard Wiener process. The continuously compounded riskless interest rate r is constant and the same for all maturities during [0, L]. No shareholder has any special market-timing ability. The model assumes, in the absence of resale restrictions, investors would be equally likely to sell the shares anytime during the restriction period. A shareholder can sell the registered shares at t, $0 < t < L$, and reinvest the proceeds in the riskless asset until L.

Since a shareholder can sell the registered shares at t, $0 < t < L$, and reinvest the proceeds in the riskless asset until L, in a risk-neutral world, she would be indifferent between selling the share immediately for $V(t)$ and selling it forward for delivery at L with forward price $e^{(r-q)(L-t)}V(t)$. Suppose further that the investor would want to sell the unregistered shares prior to L were it not for the resale restrictions. Since the investor lacks any special timing ability, assume that the investor would be equally likely to sell unrestricted shares at $N + 1$ discrete points in time and that these points are equally spaced, so that the investor considers selling at $t = 0$, $t = L/N$, $t = 2L/N, \ldots, t = NL/N = L$. In a risk-neutral world, such an investor would be indifferent between holding a registered share and holding an unregistered share plus a series of forward contracts all expiring at L. If the investor's transfer restriction risk exposure could be perfectly hedged, or if this risk is idiosyncratic with respect to the investor's securities portfolio, then the unregistered shares would be priced on a risk-neutral basis.

While price risk is hedgeable, liquidity risk is not. Because of the restrictions on hedging, the investor bears an opportunity cost due to the transfer restrictions if

$$\frac{1}{N+1}\sum_{j=0}^{N}\left[e^{(r-q)L(N-j)/N}V(jL/N)\right] > V(L) \tag{2}$$

but realizes an opportunity gain if the inequality is reversed. Inequality (2) suggests that the DLOM should be zero if the value of the investor's potential for economic gain exactly offsets the cost of the investor's potential for economic loss during [0, L]. In particular, if (a) the stock's return distribution is symmetrical, (b) the investor is risk neutral or can costlessly fully hedge her risk exposure from holding the stock, and (c) she has adequate liquidity from other sources such that the transfer restrictions do not cause her to miss any positive-NPV investment opportunities or to bear any other illiquidity costs, then the value of the potential upside and the cost of the potential downside offset, and the investor would not require a DLOM to purchase the restricted share. In effect, the restricted share would be equivalent to a long forward contract for the delivery at L of an otherwise identical unrestricted share when the transfer restrictions expire.

Empirical evidence indicating that common stock return distributions exhibit long-term negative skewness and fat tails provides a rationale for the put option characterization of the DLOM, which has not previously been addressed in the literature. Investors are generally risk averse, and liquidity risk cannot be hedged. So inequality (2) should generally hold. Engle (2009, 2011) shows that long-term negative skewness is a consequence of asymmetric volatility; negative returns lead to higher volatility than comparable positive returns with the result that potential market declines are more extreme than possible market increases.[10] Skewness is negative for every horizon and increasingly negative for longer horizons. Harvey and Siddique (2000) furnish empirical evidence that U.S. equity returns for stocks in CRSP (NYSE, AMEX, and NASDAQ) are systematically negatively skewed, this skewness is economically important and commands an average risk premium of 3.60%, and the returns of the smallest stocks by market capitalization exhibit the most pronounced negative skewness based on stock return data for the period 1963 to 1993. Ze-To (2008) furnishes empirical evidence that the S&P 500 Index returns were negatively skewed and heavily tailed during the 1991 to 2005 period. Bali (2007) and Bali and Theodossiou (2007) document the negatively skewed fat-tailed distributions of the returns on U.S. stocks as proxied by the Dow Jones Industrial Average for the period 1896-2000 and by the S&P 500 Index for the period 1950 to 2000, respectively. Cotter (2004) documents the negatively skewed fat-tailed distributions of European equity index returns.

Given the economically significant negative skewness and fat tails that characterize common stock return distributions, it is reasonable to expect a risk averse investor to require a DLOM to compensate for the incremental cost of having to bear the excess of the cost of potential extreme downside loss over the value of potential extreme upside gain during a marketability restriction period, except in the special case where he is committed to holding the stock for the entire restriction period and doing so does

[10] Guidolin and Timmermann (2006) define a term structure of risk, which has an important implication for stock return distributions. The risk of unexpected extreme stock price movements increases over longer time horizons, and negative skewness consequently increases with longer time horizons (Engle, 2009, 2011).

not impose any potential liquidity cost.

II. DLOM Model for Longer Restriction Periods

There are numerous forms of selling or liquidity restrictions that can prevent a shareholder from freely selling her shares for greater than two years. Since the accuracy of the average-strike put option DLOM model deteriorates as the restriction period lengthens beyond about two years, it is useful to develop a model for longer restriction periods.

A. Need to Accommodate Longer Restriction Periods

The transfer of unregistered securities is restricted but not strictly prohibited. A stockholder can transfer unregistered shares of common stock by relying on one of the exemptions from registration under the 1933 Act. Specialized secondary markets have developed to enable holders of unregistered common stock to find potential buyers.[11] Nevertheless, a buyer of unregistered shares faces the same restrictions as the seller. Thus, it is reasonable to assume that the original holder cannot transfer the security for the duration of the initial restriction period even though that is not strictly correct, unless a secondary market platform offers a viable means of selling the security. Secondary markets for restricted shares transform

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[11] Nasdaq Private Market (previously SecondMarket) and Forge Global (formerly SharesPost) are two of the more prominent secondary markets for unregistered shares of private firms. Nasdaq Private Market operates a web-based auction platform that supports liquidity programs through which private firms can tender for or buy back unregistered stock from shareholders and employees. It has conducted more than 450 liquidity programs involving over $29 billion of transactions since 2013, including 90 programs with $4.5 billion of transactions in 2020. Forge Global, which developed the first online secondary market for private firm shares, provides company research and data, facilitates secondary trading in unregistered shares among accredited investors, and operates a private stock loan program all in the shares of leading private technology firms. It has completed over $12 billion of transactions in more than 500 private firms since 2009. Information concerning these markets is available at https://www.nasdaq.com/ and https://www.forgeglobal.com/, respectively. Last accessed August 22, 2023.

a security that lacks marketability into one that is marketable but lacks liquidity. They can reduce the effective length of the restriction period but are unlikely to reduce it to zero because the legal restrictions remain.

Selling and liquidity restrictions can inhibit share sales by corporate officers, directors, entrepreneurs, private equity investors, venture capitalists, and certain other classes of shareholders for periods that sometimes extend over several years. They are often designed to resolve moral hazard and adverse selection problems (Kahl, Liu, and Longstaff, 2003). Such resale restrictions have caused stockholders to suffer large losses during severe market downturns. Executive restricted stock plans typically prohibit share sales for between three- and five-year vesting periods (Simon, 2016). SEC Rule 144 places resale restrictions on shareholders of public firms who hold unregistered shares and on affiliates who hold control shares (SEC, 2016). The SEC has progressively relaxed the restrictions and shortened the restriction periods on Rule 144 sales over the past 20 years making them less problematic but they are still economically significant because of the size of the DLOM involved. Private equity investors and venture capitalists hold unregistered shares, which they cannot sell until they either register the stock for an IPO or sell them to another private investor or in a change-of-control transaction. Private equity investors and venture capitalists typically hold an investment for between five and seven years, and investors in a private equity fund typically hold their fund investment for 10 years (Metrick and Yasuda, 2010). Entrepreneurs may hold their equity investments for even longer periods.

Even some contractual restrictions that are usually designed for relatively short periods can sometimes restrict share sales for much longer periods. For example, IPO lock-up periods typically extend for 180 days from the date of the IPO but the issuer and its underwriters sometimes agree to a much longer lock-up period (Kahl, Liu, and Longstaff, 2003). Merger agreements typically impose multi-year lock-up periods on the selling firm’s corporate officers and key employees to discourage them from leaving the target firm and taking with them their valuable human capital, which may have been the main motivation behind the merger (Kahl, Liu, and Longstaff, 2003). Lastly, Rule 144 restrictions are generally much more onerous for 'affiliates,' who include corporate executives, directors, and 5% or greater shareholders who exercise some degree of control by having the power to direct the management and policies of the firm, than other shareholders (SEC, 2016). An affiliate cannot sell her shares for at least six months after acquiring her stock.[12] Thereafter, the number of shares she can sell during any three-month period is limited to one percent of the firm's outstanding common stock or, if the shares are listed on a stock exchange, the greater of one percent of the firm's outstanding common stock and the stock's reported average weekly trading volume during the four weeks preceding the sale. Consequently, for smaller firms with less actively traded shares, it may take several years before an affiliate can sell her entire shareholding. In general, for privately held firms, resale limitations restrict share sales until an IPO, a change-of-control transaction, or some other liquidity event occurs, and the timing of such an event, and even whether it will ever occur, are uncertain.

B. Average-Strike Put Option DLOM Model

The average-strike put option formulation models the DLOM as the value of a put option for which the expected strike price is the arithmetic average of the risk-neutral forward prices. This option represents the option to exchange a package of forward contracts on a share for the underlying share. It can be evaluated as the value of an option to exchange one asset for another. The option payoff function contains the sum of a set of correlated log-normal random variables. Finnerty (2012) approximates its probability distribution as the probability distribution for a log-normal random variable using Wilkinson's method,

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[12] The firm must be current with its filings under the Securities Exchange Act of 1934 (the 1934 Act). If it is not subject to the reporting requirements under the 1934 Act or is not current with its filings, the minimum holding period is one year.

derives the moment-generating function for the bivariate normal distribution for the average of the risk-neutral forward prices and the price of the underlying unrestricted share, and then applies Hull's (2009) generalization of Margrabe's (1978) expression for the value of the option to exchange one asset for another when the stock is dividend-paying to obtain the average-strike put option model for the value, *D(L)*, of the DLOM:

$$D(L) = V_0 e^{-qL}\left\{N\left(\frac{v\sqrt{L}}{2}\right) - N\left(-\frac{v\sqrt{L}}{2}\right)\right\} \tag{3}$$

$$v\sqrt{L} = \left[\sigma^2 L + ln\left[2\left\{e^{\sigma^2 L} - \sigma^2 L - 1\right\}\right] - 2ln\left[e^{\sigma^2 L} - 1\right]\right]^{1/2} \tag{4}$$

where $N(\cdot)$ is the cumulative standard normal distribution function.

The volatility v in equation (4) depends directly on the volatility σ of the underlying unrestricted share of stock. Margrabe's (1978) expression for the value of the option to exchange one asset for another corresponds in the case of equations (3) and (4) to the exchange of a set of forward contracts to deliver a restricted share for the unrestricted share where the forward contracts have the same underlying unrestricted share. Accordingly, $v\sqrt{L}$ in equation (4) is the volatility of the ratio of the price of the unrestricted share at L to the average forward contract price calculated over the interval $[0, L]$. The volatility of this ratio is less than σ because the average forward contract price is directly related to the unrestricted share price. The two components of the ratio pull in opposite directions, making the value of the ratio less volatile than the price of the underlying share. Table 1 illustrates the relationship between v and σ for a range of assumed parameter values.

The average-strike put option DLOM model (3) – (4) has an important limitation, which results in its worsening performance as the restriction period lengthens beyond two years. The logarithmic approximation effectively caps the DLOM causing the model's accuracy to worsen as L increases. The volatility parameter v^2 behaves like $ln2/L$ for large L, which approaches zero as L becomes very large, causing the average-strike put option DLOM model to have a 32.28% effective upper bound (Brooks, 2016).

C. Multi-Period DLOM Model

Calculating a DLOM is more challenging when there is no legal or contractual restriction on the holder's ability to sell or transfer the asset because the length of the restriction period is uncertain.[13] For example, the market for an asset may be poorly developed, making it difficult, time-consuming, and therefore expensive to find a buyer for it. The asset is marketable but there is no specified expiration date for the effective restrictions on freely selling it. Applying a put-option-based DLOM model would require estimating how long the effective restriction period should be expected to last. Likewise, the expected length of the restriction period when a privately held firm's unregistered common shares are being valued must be estimated because the lack of registration imposes a marketability restriction of uncertain length. But if the private firm's business is sufficiently well-developed, then it may be reasonable to estimate the amount of time that is likely to elapse before an IPO or a change-of-control transaction might occur. I assume the length of the restriction period can be estimated.

This section develops a multi-year DLOM model by modelling the DLOM for a restriction period of specified length L as the value of the 1-year average-strike put option DLOM plus a term premium τ compounded forward L years. I use the 1-year DLOM because the marketability discounts calculated from the average-strike put option model are more consistent with empirical private placement discounts for 1-year than for 2-year restriction periods (Finnerty, 2012). Set $L = 1$ in equations (3) - (4) to obtain a 1-year marketability discount formula:

[13] Even with a marketability restriction, the length of the restriction period should be adjusted upward to reflect how long it is expected to take to sell all the shares after the resale restriction lapses.

$$D(1) = V_0 e^{-q}\left[N\left(\frac{v}{2}\right) - N\left(-\frac{v}{2}\right)\right] \tag{5}$$

$$v = \left[\sigma^2 + ln\left[2\left\{e^{\sigma^2} - \sigma^2 - 1\right\}\right] - 2ln\left[e^{\sigma^2} - 1\right]\right]^{1/2} \tag{6}$$

I use equations (5) - (6) as the basis for the multi-year DLOM model.

To simplify the development of the model, I begin with the special case of no marketability term premium. I add a term premium later in this section. The one-year percentage marketability discount is defined as $P = D(1)/V_0$ where $D(1)$ is given by equation (5). It is easier to work in continuous time. The discrete marketability discount P can be expressed in terms of an equivalent continuously compounded DLOM for a one-year restriction period, denoted Δ, which I will call the base one-year restriction period DLOM, according to the equation

$$P = 1 - e^{-\Delta}. \tag{7}$$

Use equations (5) and (7) to express Δ as

$$\Delta = -ln[1 - P] = -ln\left\{1 - e^{-q}\left[N\left(\frac{v}{2}\right) - N\left(-\frac{v}{2}\right)\right]\right\} \tag{8}$$

where v is given by equation (6).

One might think of the percentage discount per year for restriction periods of different lengths as forming a DLOM term structure in the following sense. A DLOM term structure is a term structure of marketability restriction risk, which is a natural application of Guidolin and Timmermann's (2006) term structure of risk. Engle (2011) notes that risk measures are computed for horizons of varying length extending from one day to many years. Risk measures take into account that the losses that might result from extreme moves in the price of an underlying asset take time to unfold, which generally leads to long-term risks exceeding short-term risks. This phenomenon suggests that it is reasonable to expect marketability risk to increase with the length of the marketability restriction period; by implication, the DLOM should also increase as the length of the restriction period increases.

Engle (2011) also notes that the term structure of risk is analogous to the term structure of interest rates and the term structure of volatility (Cox, Ingersoll, and Ross, 1981; Fama, 1984, 1990; and Fabozzi, 2016). I will draw on this analogy to the term structure of interest rates in describing the DLOM term structure. In doing so, I distinguish between the DLOM for restriction period of length L, which represents a schedule of price discounts as a function of L of the type appraisers use to value restricted shares of common stock, and the DLOM per year (DLOMPY), which represents a schedule of average annual discounts per year.

If the DLOMPY is the same for every value of L, then the DLOMPY term structure implied by equation (8) is flat and equal to Δ. Just as the interest rate term structure and the illiquidity risk premium term structure can be upward-sloping, downward-sloping, or hump-shaped (Chen, Lesmond, and Wei, 2007), it seems reasonable to expect that the DLOMPY term structure should also exhibit the same variety of shapes because it is related to the term structure of risk. I find that the DLOMPY term structure can exhibit any of these shapes. The shape of the DLOM term structure is different, however, and is generally monotonically increasing and concave.

Assume the DLOMPY compounds continuously at the constant rate Δ over a restriction period of L years. The DLOM for an L-year restriction period, denoted $D^*(\Delta, L)$, can be expressed as

$$D^*(\Delta, L) = 1 - e^{-\Delta L} = 1 - \left[1 - e^{-q}\left\{N\left(\frac{v}{2}\right) - N(-\frac{v}{2})\right\}\right]^L \tag{9}$$

where v is given by equation (6). Δ can be interpreted as the rate at which the value of the asset decays due to lack of marketability as L increases.[14] Note the basic properties: $D^*(\Delta, 0) = 0$ and $\lim_{L\to\infty} D^*(\Delta, L) = 1$ for all positive values for Δ. As one would expect, the DLOM is zero when there are no marketabilty

[14] The value of the asset net of the discount is $V - D^*V = (1 - D^*)V = Ve^{-\Delta L}$, which decreases at the continuously compounded proportional rate $-\Delta$ as L increases.

restrictions, and it approaches 100% in the limit as L increases without bound. For the special case of a non-dividend-paying stock, $q = 0$, and equation (9) simplifies to

$$D^*(\Delta, L) = 1 - \left[2N(-\tfrac{v}{2})\right]^L \tag{10}$$

Equation (9) provides the percentage price discount, the loss of fair market value per dollar of freely traded asset value, attributable to the L-period marketability restrictions. It offers another way of characterizing the term structure of marketability risk premia by expressing these premia as percentage price discounts that vary as a function of L. Figure 1 illustrates the behavior of the price discount $D^*(\Delta, L)$ as a function of the restriction period L for different values of the stock price volatility σ. The DLOM term structure $D^*(\Delta, L)$ is a concave monotonically increasing function of L. For a restriction period of length L, $D^*(\Delta, L)$ is larger for greater v, the volatility of the ratio of the unrestricted share price to the average forward contract price over [0, L]. One can interpret each $D^*(\Delta, L)$ curve in Figure 1 as a DLOM term structure for an issuer's restricted stock given the volatility σ of the issuer's unrestricted stock. When $v = 0$, $D^*(\Delta, L) = 0$; the DLOM equals zero regardless of the length of the restriction period when the volatility of the price ratio is zero. As v increases, the DLOM increases and reaches the following maximum for given L, which is illustrated in Figure 1:

$$\lim_{v \to \infty} D^*(\Delta, L) = 1 - [1 - e^{-q}]^L \tag{11}$$

$D^*(\Delta, L)$ would approach, but not exceed, 100% in the limit as v becomes infinite.[15]

The average-strike put option model DLOMs obtained from equation (3) and the DLOMs obtained from equation (9) are similar when the restriction period is two years or less and the stock volatility is 30% or less. The two models agree at $L = 1$ by design. For restriction periods less (greater) than one year, the basic model DLOM given by equation (3) is greater (less) than the DLOM given by equation (9). The

[15] When $q = 0$, $D^*(L) = 1$ in the limit for all values of L as $v \to \infty$.

basic model DLOMs exhibit greater concavity, which causes the DLOMs from the two models to diverge more widely for any given restriction period L as σ increases.

D. DLOM Term Premium

Next, I introduce a term premium into the model to reflect a risk averse investor's reaction to a more prolonged exposure to the risk of an increasingly negatively skewed fat-tailed return distribution as L increases. Building on the analogy between the interest rate term structure and the DLOMPY term structure, equation (9) implies that market participants expect the one-year DLOM given by equation (8) to remain the same each year for L years under a pure expectations theory. However, the DLOMPY term structure might be upward-sloping if risk averse restricted stock investors require a marketability term premium to compensate for the greater risk of an increasingly negatively skewed fat-tailed stock return distribution as L increases and the greater uncertainty associated with being resale restricted for a longer period (Engle, 2009, 2011). This term premium might exhibit irregular shapes if restricted stock investors self-select into restriction-period preferred habitats or if legal investment restrictions or investment policy constraints effectively segment the market for restricted stock into specific maturity sectors.

The DLOMPY can change as L increases if there is a term premium (discount, if negative) $\tau(\Delta, L)$, which may be a function of the base 1-year restriction period DLOM Δ given by equation (8) and the length of the restriction period L. The DLOMPY is the sum of Δ and $\tau(\Delta, L)$. Marketability restrictions can affect the price of a restricted share much like the effect that diminished liquidity has on the price of a fully marketable unrestricted share even though the source of the impairment in value is different. The behavior of illiquidity yield spreads is thus suggestive of how DLOMPY term premia might be expected to behave.

The finance literature quantifies illiquidity yield spreads for bonds that vary with the length of the restriction period (Amihud and Mendelson, 1986; Chen, Lesmond, and Wei, 2007; Koziol and Sauerbier, 2007; Goyenko, Subrahmanyam, and Ukhov, 2011; and Kempf, Korn, and Uhrig-Homburg, 2012). The term structure of illiquidity yield spreads can be upward-sloping, downward-sloping, or a hump-shaped function of maturity much like the interest rate term structure (Chen, Lesmond, and Wei, 2007). The illiquidity yield spread tends to increase with risk as measured by the volatility of security returns (Koziol and Sauerbier, 2007), and the illiquidity spread, like the credit spread, is wider for lower-rated bonds due to the greater risk (Chen, Lesmond, and Wei, 2007). I find that the empirical DLOMPY term structure similarly can be upward-sloping, downward-sloping, or hump-shaped.

Goyenko, Subrahmanyam, and Ukhov (2011) investigate the Treasury bond market and the illiquidity premia reflected in the yields of the off-the-run Treasury issues. They find that the term structure of illiquidity premia steepens during recessions and flattens during expansions, which suggests that these term structures systematically shift. Likewise, Kempf, Korn, and Uhrig-Homburg (2012) find substantial variation in the level and shape of the illiquidity premium term structure over the economic cycle. I document actual DLOMPY and DLOM term structure shifts that are like an interest rate term structure's level, slope, and curvature shifts later in the paper.

I interpret $\tau > 0$ as indicative of private equity investor aversion to the incremental riskiness of longer-duration restriction periods, including more prolonged exposure to the risk of an increasingly negatively skewed fat-tailed return distribution as L increases. The longer the restriction period, the longer the forced holding period, during which a greater number of value-decreasing events are possible; $\tau > 0$ implies that investors are increasingly adverse to this negative event risk the longer the holding period, which increases their required DLOMPY. A longer restriction period L for a stock investment means that it will have a longer duration, which increases the restricted stock's sensitivity to (i) changes in stock prices generally, (ii) the expected increase in downside investment risk (due to the longer restriction period's greater exposure to the risk of increasingly negative skewness), (iii) shifts in the equity risk premium, or (iv) changes in the riskless rate (for example, as reflected in CAPM). The length of the restriction period L could interact with marketability risk to magnify the impact on the DLOM and increase τ as L increases.

The marketability discount $D^*(\Delta, L)$ in equation (9) can be modified by adding a duration-specific marketability term premium $\tau(\Delta, L)$ to the base 1-year restriction period DLOM Δ in equation (9). The modified marketability discount, $\widehat{D}(\Delta, L)$, is:

$$\widehat{D}(\Delta, L) = 1 - e^{-(\Delta + \tau(\Delta, L))L} = 1 - e^{-\tau(\Delta, L)L}\left[1 - e^{-q}\left\{N\left(\frac{v}{2}\right) - N\left(-\frac{v}{2}\right)\right\}\right]^{L} \tag{12}$$

$\Delta + \tau(\Delta, L)$ approximates the rate at which the value of a restricted asset decays due to lack of marketability as L increases.[16] The expressions for $D^*(\Delta, L)$ for the special cases in equations (10) and (11) are similarly modifed and the resulting formulas are interpreted similarly to equation (12).

Equation (12) suggests that investor risk aversion could lead to greater marketability term premia for longer restriction periods ($\tau > 0$), in which case $\widehat{D}(\Delta, L)$ exceeds $D^*(\Delta, L)$. $\widehat{D}$ is a concave increasing function of each parameter. I find that τ is positive and decreasing for the DLOMs in my sample.

III. Empirical Estimation of DLOM Term Structures

This section uses a 27-year sample of DLOMs estimated from 5,333 pre-IPO private sales of unregistered common stock and private stock options grants to estimate DLOM term structures and quantify the marketability term premium $\tau(\Delta, L)$. The DLOM term structures for the last pre-IPO private equity transactions are different from those for the earlier pre-IPO private equity transactions, and I estimate separate DLOM term structures. I do not have information concerning the length of the restriction period the issuers and investors assumed at the time they negotiated the private equity

[16] The exact proportional rate of decay is $\Delta + \tau(\Delta, L) + L \cdot \delta\tau/\delta L$. The proportional rate of decay would be $\Delta + \tau(\Delta, L)$ exactly if the term premium were constant with respect to L.

transactions. Thus, the DLOM term structures provide the *ex post* relationship between the DLOM and the realized *L*. I illustrate how the fitted models can be used to estimate a DLOM for a pending sale of unregistered common stock later in the paper.

A. DLOM Sample

My sample consists of U.S. private equity transactions in which privately held firms sold unregistered common stock or granted stock options prior to going public. I measure the DLOM implied by the difference between the IPO price stated in the firm's IPO prospectus and the prices disclosed in the IPO prospectus at which the firms previously had sold unregistered common shares or granted options exercisable for the common shares. I adjust the IPO price to take into account factors that would cause the price of the firm's shares to change between the private transaction date and the IPO date to control for the prices' non-contemporaneity. I measure *L* as the amount of time elapsed between the private equity transaction date and the IPO date. I investigate the statistical properties of *L*. I use the estimated DLOMs to empirically fit DLOM term structures. I control for industry fixed effects and an IPO market fixed effect, and I test for any date of issue fixed effects. I also provide fitted DLOMPY term structures in the appendix.

The estimated DLOM term structures may tend to understate the typical DLOMs for private common stock sales because every firm in my DLOM sample succeeded in going public. Firms that go public are typically financially stronger and more profitable and tend to have superior growth prospects -- and lower financial risk -- than private firms that do not go public because the latter group includes firms that were not deemed sufficiently attractive by IPO underwriters and investors to enable them to complete an IPO.

I obtained my sample of private equity transactions from the Valuation Advisors Lack of

Marketability Discount Study (Valuation Advisors, 2017).[17] The Valuation Advisors database reports company name, SIC code, the private transaction date and share price, the length of the time interval between the private transaction date and the IPO date, the type of private equity transaction, the IPO date and IPO share price, the marketability discount expressed as the difference between the IPO share price and the private transaction share price, and a very limited amount of firm financial data. It provides details concerning pre-IPO primary sales of restricted common stock, options, and convertible preferred stock, which Valuation Advisors compiles from the historical private equity transaction data that a firm furnishes in its IPO registration statement filed with the SEC.[18] When an issuer makes multiple transactions within any 3-month period or multiple transactions more than one year prior to its IPO date, Valuation Advisors includes only the highest-priced transaction (and by implication, the one with the smallest discount) for the relevant interval. All stock and option transactions are adjusted for stock splits between the private equity transaction date and the IPO date.

The Valuation Advisors database listed 13,090 pre-IPO private equity transactions for the period between June 1985 and April 2017. I excluded the 1,400 non-U.S transactions because they are less likely to be affected by marketability restrictions imposed by U.S. securities laws than U.S. transactions. I also excluded 3,266 transactions involving convertible preferred stock. Even when the conversion feature is based on the estimated fair market value of the underlying common stock, the transaction price reflects the optionality of the convertible preferred stock, and the fair market value of the underlying common stock is not provided in the database. I also excluded 2,339 private equity transactions for which I could

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[17] Information regarding this database is available at https://www.bvresources.com/products/valuation-advisors-lack-of-marketability-study. Last accessed August 22, 2023.

[18] The database excludes IPOs by real estate investment trusts, master limited partnerships, limited partnerships, closed-end funds, or mutual bank or insurance company conversions.

not identify the isuer's ticker symbol, which I needed to be able to calculate σ during the one-year period post-IPO for equations (5) - (6) in the DLOM calculation.

Some private equity placements take place at a premium to the stock's freely traded value. Allen and Phillips (2000) find that 59 percent of what they term 'strategic' private placements take place at a premium stock price, and strategic private buyers on average pay a 6 percent premium. I excluded 694 private placements for which the calculated DLOM is negative because these premiums in price are unrelated to the marketability of the stock. To improve the quality of the data, I deleted 43 last pre-IPO private equity transaction outliers with $L > 3$ years and 15 earlier pre-IPO transaction outliers with $L > 10$ years.[19] After these exclusions, the final sample contains 5,333 private sales of restricted common stock and stock options of private firms for which the implied DLOM is nonnegative.

Table 2 provides a summary description of my sample. The private equity transactions took place between June 1985 and April 2017 and involved 1,609 firms, which went public between March 1986 and June 2017. Panel A furnishes the annual revenue, annual net income, and year-end total assets, book value of equity, and market capitalization of equity for the fiscal year in which the firm went public. I have distinguished between sample firms that only sold common stock or issued options at a calculated DLOM and those that had separately sold at least one other private placement at a premium (which was not included in the sample transactions). The mix of firms exhibits a wide size range, for example, with revenue varying between zero and $136 billion during the IPO year. Nearly two-thirds were unprofitable in their IPO year, although this percentage is slightly smaller for the roughly eight percent of the sample

[19] These filters are consistent with the view that IPOs are anticipated up to 3 years ahead and liquidity events for earlier pre-IPO transactions are anticipated up to 10 years ahead. In both cases, the DLOMs for the outliers that were dropped would suggest that the DLOM is inversely related to L when L exceeds the indicated limit, which is counterintuitive.

firms that had at least one private sale at a calculated premium. Roughly a tenth of the sample firms had negative book equity at the end of their IPO year.

Panel B provides a breakdown of the sample by industry and by IPO year. Panel C provides a breakdown of the private equity placements by year of issue and by industry. The industry breakdown is based on the French (2023) 12-industry classification scheme. Two industries predominate: business equipment, which includes computers and related devices (33% of the firms and 36% of the private equity transactions), and healthcare (22% of the firms and 27% of the private equity transactions). No other industry accounts for more than 9.4% of the firms or 8.2% of the transactions. Panel D shows how the length of the restriction period varies by industry, and Panel E reports the frequency of multiple private equity transactions pre-IPO by sample firms.

Common stock price volatility is the primary determinant of the value of the base one-year restriction period DLOM Δ, which is critical in the DLOM (and DLOMPY) models. Stock price volatility cannot be calculated until a firm goes public. I calculated the common stock price volatility during the 12 months following the firm's IPO. Just under half the firms had 12-month common stock price volatility between 50% and 100%, and more than a quarter had 12-month volatility greater than 100%. Given this volatility profile, after fitting each term structure, I plot illustrative DLOM (and DLOMPY in the appendix) term structures for volatilities of 25%, 50%, 75%, and 100%.

B. DLOM Estimation

The Valuation Advisors database reports the "true fair market value" of the common stock for each private equity transaction in accordance with SEC guidelines (Valuation Advisors, 2017). Valuation Advisors accounts for the sale of 'cheap stock' or grants of 'cheap options' by adding back the transaction's deferred compensation component, which a firm must disclose in its final IPO registration statement. The deferred compensation component equals the difference between the fair market value of the stock and either the stock issue price (for stock sales) or the option strike price (for option grants) when the stock is underpriced.

The Valuation Advisors database does not make any adjustment for the change in the value of the shares that would be expected to occur between the private equity transaction date and the IPO date, for example, as a private firm grows its business and improves its profitability as it approaches a planned IPO. I calculated the implied DLOMs by deflating the IPO price back to the private equity transaction date based on the average monthly total returns for the common stocks of firms in the smallest quartile of the same industry to account for the small-firm effect, IPO underpricing, and the fact that pre-IPO firms are more closely comparable to firms in the bottom quartile than to the entire industry (Emery, Finnerty, and Stowe, 2018). Cornell (1993) describes an essentially equivalent approach. I fit the DLOM models to the implied DLOMs based on the adjusted IPO prices.

I employed the French (2023) 48-industry classification scheme to assign each firm in my sample to an industry. Using CRSP data, for each month between a sample firm's private equity transaction date and its IPO date, I ranked all the firms classified in the same industry that traded every day during the month by market capitalization based on the average daily closing stock price and the average number of shares outstanding for the month. I calculated the monthly total return including dividends for each stock in the bottom quartile and averaged them to get the average monthly return r_i. I compounded the average monthly returns and obtained the following IPO-price-adjusted estimate for $\widehat{D}$ in equation (12):

$$\widehat{D}[est] = \frac{\left[{}^{IPO}/_{(1+r_1)(1+r_2)\ldots(1+r_M)}\right] - S}{\left[{}^{IPO}/_{(1+r_1)(1+r_2)\ldots(1+r_M)}\right]} \tag{13}$$

The adjusted IPO price in brackets proxies for the effect of market, industry, and company-specific factors that would affect the price of the stock between the private equity transaction and IPO dates. The monthly returns r_1 and r_M are calculated for the relevant fraction of the month. S is the private transaction price, which incorporates the implied DLOM the private equity purchasers required. $\widehat{D}[est]$ for a range of values for L defines an empirical DLOM term structure.

A very buoyant IPO market at the time of a firm's IPO could bias $\widehat{D}[est]$ upwards if it tends to boost IPO prices and could accelerate IPOs. Both factors would affect the fitted DLOM term structure by increasing $\widehat{D}[est]$. I control for this IPO effect by including a control variable IPO_t in the DLOM term structure estimation models, which is equal to the number of IPOs in the year the issuer went public divided by the average annual number of IPOs during the sample period. I have flagged the years in Panels B and C of Table 2 that had an above-average number of IPOs.[20] If there is an IPO effect, I expect the coefficient of this control variable to be positive and statistically significant.

C. Evidence of the Sale of Cheap Stock

$\widehat{D}[est]$ could be unusually large if the issuer sold the stock or granted options at a heavily discounted price but failed to adjust fully for this heavy discount when it reported its pre-IPO private equity transactions in its IPO prospectus. I control for unreasonably large calculated DLOMs by winsorizing the DLOMs for each L by imposing the lookback put option model (Longstaff, 1995) DLOM as an upper bound.

The lookback put option model provides an upper bound on the DLOM because it assumes perfect market-timing ability. Table 3 reports the number of private equity transactions for which the calculated DLOM was initially above the upper bound. $\widehat{D}$ can exceed the upper bound for either of at least two reasons. A firm could sell 'cheap' stock or grant 'cheap' stock options, which would cause the actual discount to be greater than the appropriate discount. Second, the parties to the private equity transaction could negotiate a share price assuming that it will take much longer for a liquidity event to occur than

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[20] The above-average IPO amounts occur early in the sample period. The number of IPOs per year generally declined during the sample period as many private firms delayed or eschewed altogether IPOs in favor of private equity financing.

what actually transpired. Overestimating L would result in an *ex post* discount that exceeds the DLOM that would have been agreed if the parties had estimated L accurately *ex ante*.

Firms sometimes sell cheap stock or grant cheap options to officers, directors, or affiliates prior to their IPO. However, SEC Regulation S-K requires firms to disclose in the IPO prospectus the terms of the sale or grant and the sale/grant date aggregate fair market value for each transaction within the past five years for which there is a 'substantial disparity' between the IPO public offering price and the effective sale price per share (for a stock sale) or strike price (for an option grant). The SEC staff often issues comment letters on the IPO registration statement asking firms to justify past transaction prices that seem unusually low compared to the IPO price (Melbinger, 2014). They typically focus on the most recent pre-IPO transaction for which any intentional underpricing is likely to be most evident The SEC normally requires firms to report any discount the SEC finds excessive as share-based compensation. Valuation Advisors (2017) accounts for cheap stock by adding back to the reported share price or option strike price the amount of the deferred compensation per share a firm reports in its final IPO registration statement for any underpriced pre-IPO stock or option transactions (Valuation Advisors, 2017). The threat of such disclosure should at least partly restrain the practice of unreported underpricing, but it seems unlikely that this disclosure requirement will completely eliminate undisclosed cheap stock sales or option grants due to the lack of transparency in pricing illiquid shares. I find evidence of such unreported underpricing, especially in the DLOM for the last private equity transaction preceding an IPO.

Any investor tendency to overestimate L should be less problematic for the last pre-IPO transactions because presumably the firm's IPO should be within sight. As illustrated in Figure 4, L for those transactions is bunched within zero to 1 year, and very few exceed 2 years. If firms do sell cheap stock and grant cheap options, I would expect the incidence of underpricing to be greater in the last pre-IPO transaction because it is the last opportunity pre-IPO to grant cheap stock. But I also expect the degree of underpricing to be smaller because it is closer in time to the IPO, and thus, any underpricing should be easier for the SEC to detect.

Table 3 reports the number of underpriced private equity transactions, those with $\widehat{D}$ above the upper bound. Of the last pre-IPO transactions, 37.70% had $\widehat{D}$ above the upper bound but only 22.12% of the earlier pre-IPO transactions did. The difference of 15.58% is statistically significant at the .01 level. The median excess is 15.05% for the last pre-IPO transactions but 16.93% for the earlier pre-IPO transactions. The difference of 1.88% is statistically significant at the .05 level. The significantly greater underpricing in the earlier pre-IPO transactions is consistent with firms being concerned that the SEC will focus its monitoring on the most recent transactions.

D. Probability Distributions for *L*

The restriction period for unregistered common stock will last until the stock is registered for public resale or until it is exchanged for cash or registered securities in a change-of-control transaction. L is uncertain. Suppose an investor wants to purchase the common stock of a privately held firm that intends to go public after the firm has developed its product line sufficiently that the stock market would be receptive to its IPO. The investor might estimate an IPO date after identifying similar past IPOs and assume that L will end on the date following the firm's IPO when the lockup on selling insider shares could be expected to expire.[21] Specifying a probability distribution for L and using it to calculate a weighted average DLOM would be another way to address this uncertainty.

The distribution of elapsed times for L might be quite different for firms that are doing one last round of financing before going public. The last pre-IPO private equity transactions in my sample have restriction periods that are concentrated in the interval $L \leq 1$. I performed a Chow test of the null

[21] A six-month post-IPO lockup period is typical.

hypothesis that there is no structural break at $L = 1$ against the alternative hypothesis that there is such a structural break and obtained $F = 18.58$, which is statistically significant at the .01 level. I conclude that the distribution of the sample DLOMs has a break point at $L = 1$. The more interesting economic issue is whether the last pre-IPO transactions and the earlier pre-IPO transactions have different DLOM distributions. The cheap stock results reported in section C indicate that if a firm is going to sell cheap stock or grant cheap stock options to key employees or other insiders, it is more likely to do so in the last private placement that precedes its IPO because that transaction affords the last opportunity to do so privately. Thus, the last pre-IPO transactions in my sample are economically different from the earlier pre-IPO transactions.

Metrick (2007) models the same liquidity events in calculating the value of a venture capitalist's (VC) random-expiration call option. Metrick assumes an exponential distribution for the exercise date. The insights from Metrick (2007), confirmed by a data plot, suggest the exponential distribution as a candidate for the distribution of L. I also considered the gamma and Weibull distributions, which are generalizations of the exponential distribution (Balasooriya and Abeysinghe, 1994). Further insights from Siswadi and Quesenberry (1982), again confirmed by a data plot, suggest that the log-normal distribution is also a candidate.

Panel D of Table 2 shows how the distribution of L differs between the 1,589 last pre-IPO private equity transactions and the 3,744 earlier pre-IPO private equity transactions. It also shows how it varies with industry within each sub-sample. Figure 2 provides the parameter estimates for each candidate distribution and the results of performing a standard Kolmogorov-Smirnov goodness-of-fit test and stricter goodness-of-fit tests due to Lilliefors (1967, 1969) for the last pre-IPO private equity transactions in Panel

A and for the earlier pre-IPO private equity transactions in Panel B.[22] The log-normal distribution provides a better fit than the other three distributions of L for the last pre-IPO private equity transactions. The hypothesis that the actual distribution of L for the last pre-IPO transactions is log-normal cannot be rejected at conventional levels. None of the four distributions provides a close fit to the distribution of L for the earlier pre-IPO transactions, although the log-normal distribution seems to provide a slightly better fit than the other three distributions. In any case, the two log-normal distributions, which have very different shape parameters (-0.97 versus 0.48), are statistically significantly different at the .01 level based on a Mann-Whitney U test. Due to this difference, I estimate DLOM term structures separately for the last pre-IPO and earlier pre-IPO private equity transaction sub-samples.

E. Marketability Term Premium

The marketability term premium τ plays a critical role in determining the term structure $\widehat{D}(\varDelta, L)$ in equation (12). Rewrite equation (12) to express the continuously compounded DLOM as:

$$-\ln\left[1-\widehat{D}\right] = \varDelta L + \tau(\varDelta, L) \cdot L \tag{14}$$

Equation (14) defines the DLOM term structure with $\tau(\varDelta, L) \cdot L$ measuring the impact of the marketability term premium. Allowing τ to be a function of $\varDelta$ permits τ to increase with the stock's volatility because $\varDelta$ is directly related to σ. This behavior would be consistent with the tendency of the illiquidity yield spread to increase with the volatility of security returns (Koziol and Sauerbier, 2007).

First, I investigate the behavior of the term premium in the DLOM sample. Re-express equation (14) to get the following expression for τ:

[22] The Lilliefors (1967) test for normality and the Lilliefors (1969) test for exponentiality correct for the bias inherent in performing the Kolmogorov-Smirnov test on a hypothetical distribution that assumes the true distribution has the parameter values estimated from the sample data.

$$\tau(\Delta, L) = -\Delta - ln\,[1 - \widehat{D}]/L \ . \tag{15}$$

Table A1 in the appendix provides the median values for τ within successive restriction period ranges and separately for each industry. The median values for τ are positive, as expected, for each industry within almost every restriction period range. The term premia are very similar for the two largest industries, business equipment and healthcare. The term premium is generally downward-sloping, which implies that the DLOMPY is similarly downward-sloping, for every industry. The differences in industry term premia suggest that there are systematic industry effects in the DLOM.

F. Estimation of the *Ex Post* DLOM Term Structures

Next, I fit the DLOM term structures. I focus on the DLOM term structure because it is the basic measure of the lack of marketability. I provide the DLOMPY term structures in Figure A1 in the appendix. The functional form chosen for $\tau(\Delta, L)$ allows enough degrees of freedom to permit τ to be upward-sloping, downward-sloping, concave, or convex with respect to L. I control for industry fixed effects in the term structure regression models by including industry dummies because of the systematic industry effects evident in Table A1. I interact each industry dummy with L. I also allow for the possibility that a buoyant IPO market in the year a firm in my sample goes public might bias upward the calculated value for $\widehat{D}[est]$ in equation (13). I define the control variable IPO_t as the ratio of the number of U.S. IPOs in the year t in which the firm went public to the average annual number of IPOs for the years in the sample period, 1995 to 2017, when any of the issuing firms in the sample went public. I fit the following equation to the adjusted DLOM data for the full sample and each sub-sample:

$$\begin{aligned} -ln\big[1 - \widehat{D}\big] = c_0 + c_1 L + c_2 L^2 + c_3 L^3 + c_4 \Delta L + b_1 I_1 L + b_2 I_2 L \\ + b_3 I_3 L + \cdots + b_{11} I_{11} L + \ e_1 IPO_t \end{aligned} \tag{16}$$

I_j are the industry dummies. The omitted industry is the miscellaneous category, Industry 12 (Other). Interacting I_j with L adjusts the DLOM per period, $-ln\big[1 - \widehat{D}\big]/L$, for industry fixed effects. When c_4

differs from 1, the empirical term structure conforms to equation (14). Comparing equations (14) and (16), I expect c_0 to be statistically insignificant.

The sample exhibits clustering of error terms because most of the sample firms have more than one private equity transaction, as reported in Panel E of Table 2. All private equity transactions conducted by a particular firm have the same IPO date in the calculation of $\widehat{D}[est]$ in equation (13). I adjust for the clustering of error terms by calculating asymptotically consistent estimators and robust standard errors.

Table 4 provides the regression results of fitting equation (16) to the full sample of 5,333 transactions, the sub-sample of 1,589 last pre-IPO private equity transactions, and the sub-sample of 3,744 earlier pre-IPO private equity transactions. The adjusted R^2 values are within the range from 0.40 to 0.50, and the F-statistic for each regression is statistically significant at the .01 level. The coefficient of the IPO_t control variable is highly statistically significant at the .01 level in all three regressions, which indicates the important effect the quality of the IPO market has on the pricing of IPOs. The regression results for the last pre-IPO transactions differ from those for the earlier pre-IPO transactions. As expected, the intercept is not statistically significant at the .10 level for the full sample or the last pre-IPO transactions regressions, but it is statistically significant at the .01 level for the earlier pre-IPO transactions regression. All three restriction period coefficients are statistically significant at the .01 level in the full sample regression. The coefficients of L^2 and L^3 are statistically significant at the .05 level in the last pre-IPO transactions regression, but the coefficient of L is not statistically significant at the .10 level. The coefficients of L and L^2 are statistically significant at the .01 level in the earlier pre-IPO transactions regression, but the coefficient of L^3 is not statistically significant at the .10 level.

The coefficient of the ΔL interaction term is highly statistically significant at the .01 level in all three regressions and is also statistically significantly greater than 1 at the .01 level in all three regressions,[23] which indicates that the term premium is positively related to the volatility of the firm's stock. In other words, the term premium increases with the stock's risk as proxied by its volatility, which is consistent with Koziol and Sauerbier (2007). The coefficients of the three restriction period terms are generally consistent with the values and the more or less steadily declining pattern of τ that is evident in Table A1.

Figure 3 plots the DLOM term structure for the full sample, the last pre-IPO transactions sample, and the earlier pre-IPO transactions sample. Each term structure is plotted for $\sigma = 0.25$, 0.50, 0.75, and 1.00. These volatilities reflect the range of volatilities in the private equity transaction sample in the year following their IPOs: 2% have σ less than 0.25, 20% have σ between 0.25 and 0.50, 33% have σ between 0.50 and 0.75, 19% have σ between 0.75 and 1.00, and 26% have σ greater than 1.00. Appendix Figure A1 plots the DLOMPY term structures for the same four volatilities.

The DLOM term structures for $\sigma = 0.50$, 0.75, and 1.00 are monotonically increasing concave functions of L except the DLOM term structure for the last pre-IPO transactions for $\sigma = 0.50$ in Panel B, which has an inflection point at $L = 1$ and is concave to the left and convex to the right of this point. The generally upward-sloping concave behavior, at least for higher-volatility stocks, conforms to expectations. The DLOM term structures for $\sigma = 0.25$, which characterizes only a very small portion of the sample, are initially monotonically increasing concave functions of L (for $L \leq 5$ for the full sample and the earlier pre-IPO transactions and for $L \leq 0.5$ for the last pre-IPO transactions) but exhibit divergent behavior for larger L (concave decreasing for the full sample and the earlier pre-IPO transactions and decreasing to $L = 1.5$

[23] The t-statistics for the test of the null hypothesis that the coefficient of ΔL equals 1.0 against the alternative hypothesis that it is greater than 1.0 are 4.31 for the full sample, 7.90 for the last pre-IPO transactions sample, and 3.63 for the earlier pre-IPO transactions sample, all of which are significant at the .01 level.

and thereafter turning convex and increasing for the last pre-IPO transactions). All four term structures shift upward as the risk proxy Δ increases.

Systematic industry effects are evident in the regression models for the full sample and for the earlier pre-IPO transactions in Table 4. The industry effects are significant for non-durables (industry 1), durables (industry 2), manufacturing (industry 3), energy (industry 4), utilities (industry 8), retail and wholesale enterprises (industry 9), and financial firms (industry 11), all of which have a negative coefficient that is statistically significant at the .05 level in both models. They are strongest for non-durables and utilities, for which the coefficient in both models is statistically significant at the .01 level. The two largest components of the sample, business equipment (industry 6) and healthcare (industry 10), do not exhibit statistically significant industry effects. The DLOMs for firms in industries 1, 2, 3, 4, 8, 9, and 11 are generally smaller than the DLOMs for firms in industries 6 and 10, which is not surprising in view of the comparatively higher technology and other business risks in business equipment (computers and other electronic devices) and healthcare. None of the industry coefficients is significant at the .10 level in the last pre-IPO transactions regression, which may be due to the relatively short restriction periods limiting the industry effects that are evident for longer restriction periods.

G. Estimating DLOMs

The DLOM models estimated in Table 4 can be used to calculate a DLOM consistent with past experience when L is fixed (by contract or regulation) or can be estimated. They can also be used to estimate a DLOM when L is uncertain but can be assumed to follow a particular probability distribution, such as the log-normal distribution fitted to DLOM data in Figure 2. In that case, use the selected regression model to calculate DLOMs for a range of discrete values for L and then calculate the DLOM as a probability-weighted average of these estimates. Longstaff (2017) uses the same weighted-average approach to model a random liquidity horizon.

To calculate a DLOM, select the appropriate model in Table 4, estimate L, use equation (8) to calculate Δ, substitute Δ and L into the selected regression model to estimate $(\Delta + \tau(\Delta, L))L$, and then substitute $(\Delta + \tau(\Delta, L))L$ into equation (12). The historical data in Panel D of Table 2 can assist in estimating L. The choice of DLOM model depends on whether the equity private placement might be the firm's last prior to going public. All three models in Table 4 adjust for industry fixed effects. I suppress the *IPO* control variable because the future state of the IPO market is unpredictable at the time of a private equity transaction.

If the private equity transaction is expected to be the firm's last before it goes public, apply the last pre-IPO private equity transactions model in Table 4. Suppressing the IPO control variable, the model simplifies to

$$\widehat{D} = 1 - exp\left[-\left(0.0383 + 0.0282L - 0.3814L^2 + 0.1220L^3 + 4.2100\Delta L + b_j I_j L\right)\right] \tag{17}$$

$b_j I_j L$ is the fixed effect adjustment for the stock issuer's industry.

For example, suppose a private nondurable goods producer is undertaking one last private equity raise before it plans to go public through an IPO in approximately two years. It plans to sell 3,000,000 common shares; investors estimate a \$10.00 freely traded stock price; no dividends are expected; comparable publicly traded stocks have a volatility of 35%; IPO investment bankers typically require a six-month lockup period; and the midpoint of the estimated period it would take to sell the block is 0.111 years. Substituting $\sigma = 0.35$ into equation (6) gives $v = 0.2$. Substituting $v = 0.2$ and $q = 0$ into equation (8) gives $\Delta = 0.083$. Substituting $\Delta = 0.083$, $L = 2.611$, and $b_1 =$ - 0.0283 (Table 4, Industry 1) into equation (17) gives $\widehat{D} = 40.67\%$. The private transaction price of the stock is \$5.933.

If instead the IPO is a long way off but L can nevertheless be reasonably estimated, use the earlier pre-IPO private equity transactions model in Table 4, which simplifies to

$$\widehat{D} = 1 - exp\left[-\left(0.1582 + 0.2877L - 0.0472L^2 + 0.0016L^3 + 1.5764\Delta L + b_j I_j L\right)\right] \tag{18}$$

For example, suppose a manufacturing firm would like to sell 4,000,000 common shares privately, and purchasers estimate a \$5.00 freely traded stock price and a five-year holding period. Comparable publicly traded stocks have a volatility of 45%, and the firm is not expected to pay a cash dividend before it goes public. Substituting $\sigma = 0.45$ into equation (6) gives $v = 0.2554$. Substituting $v = 0.2554$ and $q = 0$ into equation (8) gives $\Delta = 0.1072$. Substituting $\Delta = 0.1072$, $L = 5.0$, and $b_3 = -0.1243$ (Table 4, Industry 3) into equation (18) gives $\widehat{D} = 56.82\%$. The private transaction price of the stock is \$2.159.

If L can be reasonably estimated but it is uncertain whether the equity private placement might be the firm's last before an IPO, use the model for the full sample of private equity transactions in Table 4, which simplifies to

$$\widehat{D} = 1 - exp\left[-\left(0.0001 + 0.4413L - 0.0861L^2 + 0.0041L^3 + 1.6833\Delta L + b_j I_j L\right)\right] \tag{19}$$

For example, suppose the manufacturing firm in the previous example was not sure the 4,000,000-share sale would be its last before it goes public. Substituting the same variables into equation (19) gives $\widehat{D} = 58.50\%$. The private transaction price is \$2.075, which differs from the estimate from equation (18) by less than 4%. As illustrated in Figure 3, the portions of the two DLOM term structures are very similar in the restriction interval [3,10] because there are no last pre-IPO equity transactions with $L > 3$.

A more significant source of uncertainty concerns the length of the restriction period. I illustrate the usefulness of the DLOM models when the length of the restriction period is uncertain by extending the previous example. The DLOM model for the earlier pre-IPO private transactions is valid for $0 \leq L \leq 10$. Partition this interval into subintervals of width 1.0. Use equation (18) to estimate $\widehat{D}$ for L corresponding to the midpoint of each subinterval and also for $L = 10$ to allow for the possibility that L might actually exceed 10 years. L has the log-normal probability distribution in Panel B in Figure 2 for earlier pre-IPO private equity transactions. Calculate the log-normal probability associated with each subinterval.[24] Then calculate the log-normal-probability-weighted average of the estimated DLOMs to get $\widehat{D}_{avg} = 47.35\%$.[25] Longstaff (2017) uses the same weighted-average approach to model a random liquidity horizon but assumes the horizon is exponentially distributed. This weighted average is less than the 5-year DLOM because the DLOM term structure is upward-sloping and Prob $[L \leq 5] = 0.87$. Consequently, the private transaction price of the stock is $2.633.

These DLOM estimates are conservative because of sample selection bias. Every sample firm eventually went public. Private firms that eventually go public are likely to be financially stronger on average than those that never do, which should reduce the DLOM. Gauging how this bias affects the DLOM estimates would require a sample of private equity transactions for firms that did not go public. I leave this for future research.

In each model, $\widehat{D}$ is a monotonically increasing concave function of Δ, and thus stock price volatility, over the entire restriction period. The DLOM term structure for a more volatile stock lies above the term structure for a less risky stock, consistent with financial theory. Next, I investigate whether the DLOM term structures are stationary.

IV. Factors Responsible for DLOM Term Structure Shifts

I have characterized the DLOM-L (and DLOMPY-L) relationship as forming a term structure analogous to an interest rate term structure. This section investigates whether these term structures

[24] The log-normal probabilities are 9.6%, 37.1%, 21.9%, 11.9%, 6.8%, 4.1%, 2.6%, 1.7%, 1.2%, 0.8%, and 2.5% for [0,1], [1,2], [2,3], [3,4], [4,5], [5,6], [6,7], [7,8], [8,9], [9,10], and [10,∞), respectively.

[25] I made one further adjustment in calculating $\widehat{D}_{avg}$. I required the DLOM for each subinterval to be no less than the preceding subinterval's DLOM in keeping with the shape of the DLOM term structure. Not applying this requirement would have resulted in $\widehat{D}_{avg} = 46.09\%$.

exhibited level, slope, and curvature shifts typical of an interest rate term structure during the sample period.

A. Possible DLOM Term Structure Shifting

In section III, I identified a statistically significant IPO date fixed effect, which reflects the impact of market conditions at the time of the IPO on $\widehat{D}[est]$ in equation (13). The size of the DLOM could also depend on the state of the private equity market at the issue date. The term structure regression models in Table 4 can be augmented to test for evidence of date of issue fixed effects, which would suggest that the DLOM term structure shifts if they are significant. If the term structure shifts, then each DLOM model should be re-estimated for the time period in which it is to be applied.

I added annual time dummy variables based on the year of the private equity placement to the term structure regression models in Table 4. There are 28 years when the transactions in my sample occurred, 1985 and 1989 through 2017. 2016 is the holdout year. I find that six of the date of issue fixed effects dummies are statistically significant at the .05 level or greater in the regression for the full sample and four are statistically significant at the .05 level or greater in the regression for the last pre-IPO private equity transactions sample, but 13 are statistically significant at the .05 level or greater in the regression for the earlier pre-IPO private equity transactions sample.[26] This evidence of date of issue fixed effects suggests that the DLOM term structure shifted during the sample period.

B. Latent Factors Underlying Term Structure Movements

Principal components analysis (PCA) provides an empirical description of the behavior of the entire term structure of interest rates in terms of three or four common latent factors (Knez, Litterman, and Scheinkman, 1994; Wu, 2006; Gürkaynak and Wright, 2012; and Tuckman and Serrat, 2012). The

[26] Results are available on request from the author.

three most important latent factors underlying interest rate term structure shifts are usually described as level, slope, and curvature. Some studies have identified a fourth common latent factor but there does not seem to be agreement as to how best to characterize it (Knez, Litterman, and Scheinkman, 1994).

The level factor represents parallel shifts in the term structure in which interest rates of all maturities move up or down together by approximately the same amount. These movements are often explained in terms of an increasing (decreasing) long-term inflation rate causing an upward (downward) parallel term structure shift. The slope factor represents a steepening or flattening movement in which short-term interest rates drop and long-term interest rates simultaneously rise, or *vice versa*. These movements are often explained in terms of faster (slower) economic growth causing a steepening (flattening) shift. The curvature factor represents increasing or decreasing curvature of the term structure in which short-term and long-term interest rates increase and simultaneously intermediate-term interest rates decrease, or *vice versa*. Litterman and Scheinkman (1991) and Litterman, Scheinkman, and Weiss (1991) have explained these movements in terms of increasing (decreasing) interest rate volatility causing greater (lesser) curvature.

Knez, Litterman, and Scheinkman (1994) find that the three-factor model explains about 86% of the total variation in the U.S. interest rate term structure, and the four-factor model explains about 90%. The level, slope, and curvature factors explain about 63%, 10%, and 13% of the total variation, respectively. Other studies of sovereign interest rate term structures find that level shifts explain the vast majority of term structure movements, that slope changes are usually more significant than curvature changes, and that the fourth factor is usually much less important than the first three.

C. DLOM Term Structure Shifts

The DLOM term structure is of interest because market participants typically calculate a DLOM (price discount) when estimating the fair market value of a share of restricted stock given the (estimated) restriction period (Cornell, 1993; Pratt, 2001; Damodaran, 2012; and Ben-Rephael, Kadan, and Wohl, 2015). I applied PCA for restriction periods of five different lengths, $[0,6]$, $[0,7]$, $[0,8]$, $[0,9]$, and $[0,10]$, to determine the relative importance of the fundamental common latent factors that are responsible for the annual shifts in the DLOM term structure. Each restriction range is divided into annual intervals. The implied marketability discounts are winsorized using the Longstaff (1995) model price as the upper bound, and the discounts are averaged when there is more than one private equity transaction within the same annual restriction interval during the calender year. Each calender year included in the analysis has at least 90 private equity transactions that span the entire indicated restriction period range to ensure an adequate-sized annual transaction sample.

Figure 4 furnishes the PCA factor sensitivities for the DLOM term structure for the [0,7] restriction period range, and appendix Figure A2 provides the same factor sensitivities for the DLOMPY term structure. The upper panel of Figure 4 plots the factor sensitivities, and the lower panel provides the numerical values. The pattern of the factor sensitivities is consistent with level, slope, and curvature interpretations of factors 1, 2, and 3, respectively. The factor 1 sensitivities are all positive and within a narrow range betweem 0.1795 and 0.4567, which is consistent with an approximately parallel shift upward (or downward) in the DLOMs for all restriction periods when an event occurs that triggers a change in the DLOM. The factor 2 sensitivities are negative for the relatively short restriction period ranges and positive for the relatively long restriction period ranges. The short period DLOMs and the long period DLOMs move in opposite directions and steepen (or flatten) the DLOM term structure when a DLOM-changing event occurs. The factor 3 sensitivities are positive for the relatively longest and relatively shortest DLOMs and negative for the intermediate-term DLOMs. The intermediate-term DLOMs move in the opposite direction from the longest and shortest DLOMs, which move together when a DLOM-changing event occurs.

Table 5 compares the PCA results for term structure shifts for the five restriction period ranges (and appendix Table A2 provides a similar PCA for the DLOMPY term structure). Factor 1 accounts for between 36% and 56% of the variation in the DLOM term structure; the first three factors account for between 74% and 90%, and the first four factors account for between 85% and 95%, which are consistent with the relative importance of the three or four main factors underlying interest rate term structure shifts (Knez, Litterman, and Scheinkman, 1994). Figure 4 and Table 5 are consistent with the interpretation of the DLOM-L relationship as a term structure analogous to an interest rate term structure.

D. How Have the DLOM Term Structures Actually Shifted?

This section investigates how the DLOM term structures shifted during the sample period. I fit term structures (16) for four subperiods: (1) 1992-2000, which had the highest frequency of IPOs during the overall sample period (1359 private equity transactions); (2) 2001-2006, which immediately preceded the Great Recession (1151 transactions); (3) 2007-2010, which encompassed the Great Recession (1108 transactions); and (4) 2011-2017, which followed the Great Recession (1713 transactions). I fitted term structures separately for all pre-IPO transactions, the last pre-IPO transactions, and the earlier pre-IPO transactions again adjusting for firm fixed effects. I converted the continuously compounded DLOMs estimated for $\sigma = 0.75$ to equivalent annually compounded DLOMs and plotted the DLOM term structures in Figure 5.

The DLOM term structures generally exhibit a monotonically increasing concave shape each period for the full sample and each sub-sample. For the full sample and for the earlier pre-IPO transactions, the DLOMs are generally lower in the later two periods. The DLOMs are generally greater in 1992-2000 than in later periods, and they are lowest in 2011-2017 for the last pre-IPO transactions. This suggests that underpricing of the last private equity transaction has diminished during the sample period. The downward shift during the Great Recession when IPO market activity declined, as indicated in Table 2, could reflect IPO underwriters' greater selectivity in deciding which firms to take public because of the state of the IPO market.[27] Private equity investments in higher-quality firms would normally be expected to have less holding period uncertainty, especially as of the date of the last pre-IPO private equity transaction, and should thus entail a lower DLOM.

Finally, the DLOM term structures are lowest in 2011-2017, except for the longest realized restriction periods. The economic recovery that followed the Great Recession would be expected to stimulate rising share prices causing equity investors to become more receptive to IPOs and lowering the DLOMs across the term structure. But the greater uncertainty in the economic outlook due to the severity of the crisis would suggest even greater equity risk for longer restriction periods (Engle, 2011), likely producing the more steeply upward-sloping DLOM term structure for longer restriction periods for 2011-2017 evident in Figure 5.

V. Conclusions

Previous research has shown that the average-strike put option DLOM model calculates marketability discounts that are generally consistent with the discounts observed in letter stock private placements, which have relatively short marketability restriction periods. Building on this prior research, I modeled the L-year DLOM as the value of the 1-year average-strike put option DLOM plus a term premium compounded over L years. The restriction-period-sensitive DLOM term premium reflects a risk averse investor's more prolonged exposure to the risk of an increasingly negatively skewed fat-tailed return distribution as the length of the restriction period increases.

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[27] It is evident from Table 2 that these shifts were not the result of any change in the industry composition of IPOs between the two periods.

I empirically estimated a DLOM term structure for restriction periods up to 10 years and investigated how the DLOM term structure differs depending on whether the private equity placement might be the firm's last prior to going public. I fitted three *ex post* DLOM term structure models to a sample of equity private placement implied marketability discounts with a wide range of restriction periods up to 3 years for the last pre-IPO private equity transactions and up to 10 years for earlier pre-IPO private equity transactions. The DLOM term structures exhibit level, slope, and curvature shifts similar to an interest rate term structure.

My sample does not include any *ex ante* estimates of the (expected) restriction period, but if such information were to become available, my DLOM estimation procedure could be utilized to fit an *ex ante* DLOM term structure. My models could also be extended to allow for a possible DLOM term discount to reflect the value of access to a particular class of longer-duration assets generally offering positive-α investments but imposing a long period of restricted marketability. They could be extended further to allow for the increasing effect on the DLOM of any idiosyncratic exposure to investment-specific agency costs or the decreasing effect on the DLOM of any idiosyncratic benefit from investment-specific α due to superior fund manager investment skill or to strategic partner equity investment value by adding a measure of this effect to equations (14), (15), and (16). I leave these extensions for future research.

Appendix

A. DLOMPY Term Structures

This appendix provides the DLOMPY term structures. A DLOMPY term structure is obtained by expressing $\widehat{D}$ in equation (12) on an equivalent annually compounded basis:

$$DLOMPY = 1 - \left(1 - \widehat{D}\right)^{1/L} \tag{A1}$$

The DLOMPY term structures in Figure A1 exhibit shapes different from the DLOM term structures in Figure 3. DLOMPY for the full sample in Panel A and for the earlier pre-IPO transactions in Panel C is downward-sloping and convex over the entire range for all four volatilities. The term premia τ underlying the DLOMPY term structures in Panels A and C exhibit a humped shape [not shown[28]]. DLOMPY for the last pre-IPO transactions in Panel B is initially downward-sloping but becomes upward-sloping at L = 1.5 years. It has a convex inverted humped shape over the entire range for all four volatilities. The term premium τ is monotonically increasing [not shown]. The generally decreasing pattern of DLOMPY is consistent with the decreasing term premia in Table A1 but is inconsistent with the liquidity preference theory of the term structure. However, the increase in DLOMPY in Panel B after L = 1.5 years indicates that for the last pre-IPO private equity transactions, the behavior does conform to liquidity preference theory. In all three panels, the DLOMPY is greater at higher volatilities due to the greater risk.

B. DLOMPY Term Structure Shifts

I applied PCA to the annually compounded DLOMPY in equation (A1) for the same five restriction periods as in Table 5. Figure A2 furnishes the PCA factor sensitivities for the DLOMPY term structure for the [0,7] restriction period range. The pattern of the factor sensitivities is similar to the pattern for the DLOM term structure in Figure 4 and is again consistent with level, slope, and curvature interpretations of factors 1, 2, and 3, respectively. Table A2 compares the PCA results for term structure shifts for the five restriction period ranges. Factor 1 accounts for between 44% and 52% of the variation in the DLOMPY term structure, the first three factors account for between 81% and 87%, and the first four factors account for between 87% and 94%, which are a little narrower than the respective ranges of the factors in Table 5.

[28] Plots of the term premia are available on request from the author.

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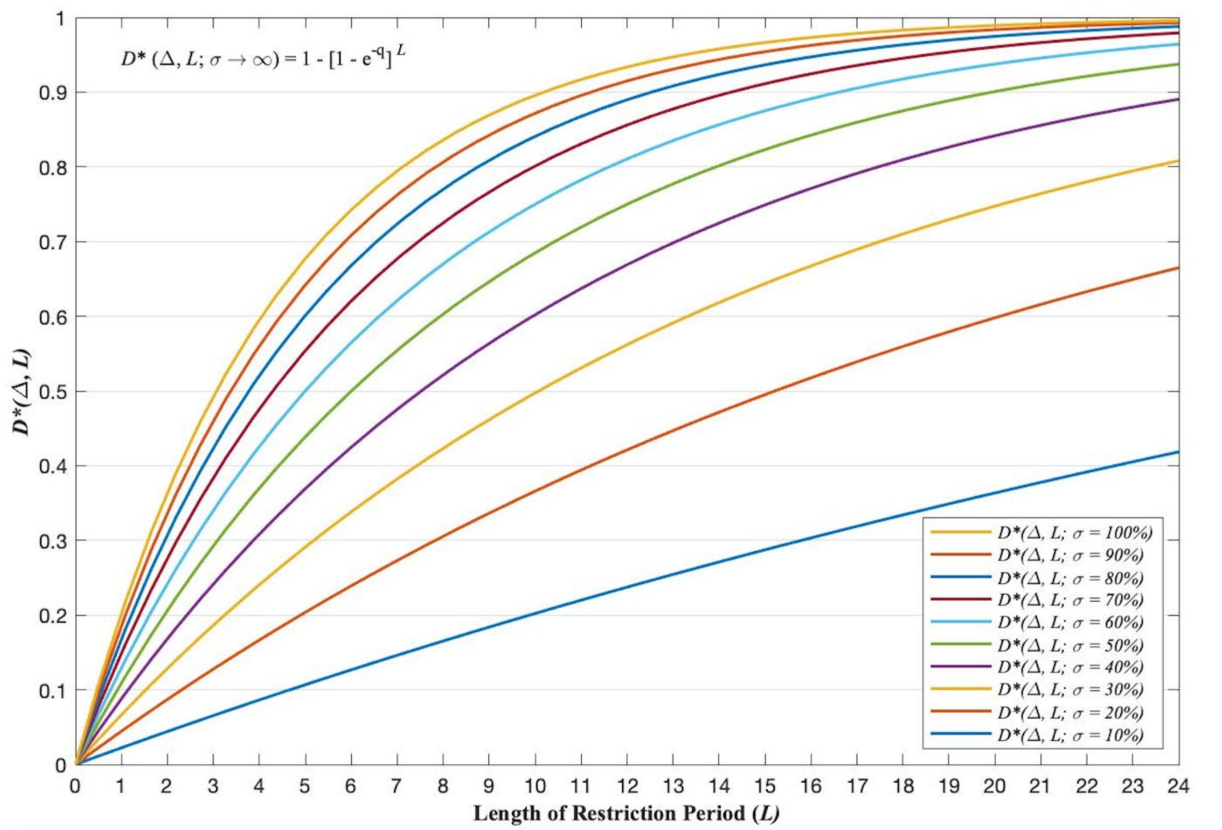
Figure 1

Term Structure of the Generalized DLOM *D*(Δ, L)*

$D^*(\Delta, L)$, the generalized DLOM expressed by equation (9), is a function of the continuously compounded percentage marketability discount for a one-year restriction period Δ and the length of the restriction period L. The value of Δ varies directly with the stock price volatility σ. The figure illustrates the concave increasing behavior of the DLOM as L increases. $D^*(\Delta, L; \sigma \to \infty)$ is the limiting case expressed by equation (11).



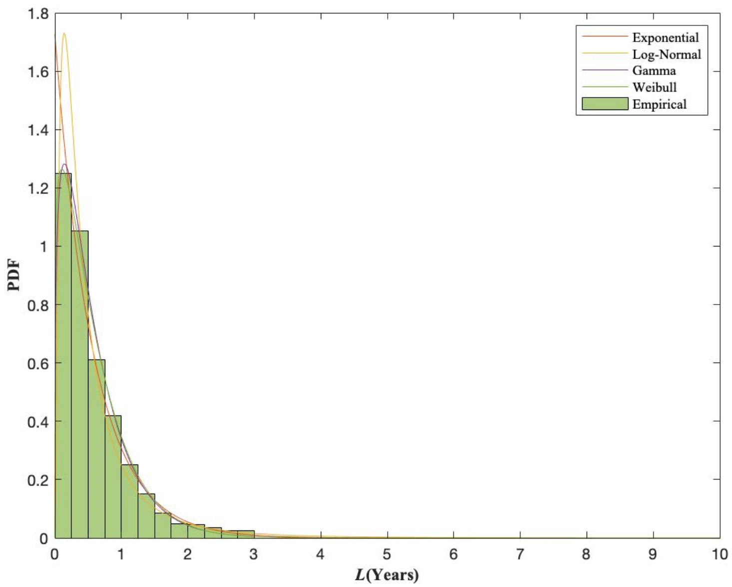
Note: Assumes q=0.03 for illustrative purposes.

Figure 2

Comparison of Distributions of L for the Last Pre-IPO Transactions and the Earlier Pre-IPO Transactions

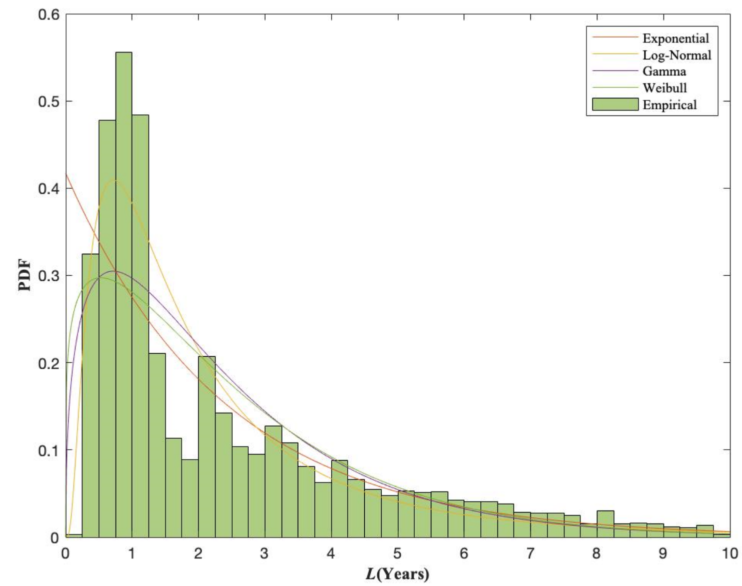
This figure fits four probability distributions, the exponential, log-normal, gamma, and Weibull distributions, to the empirical distribution of the restriction period *L* for the private equity transaction sample. Panel A fits the distributions to the 1,589 last pre-IPO private equity transactions between December 1985 and April 2017. Panel B fits the distributions to the 3,744 earlier pre-IPO private equity transactions between June 1985 and January 2017. The Kolmogorov-Smirnov test is used to test the goodness-of-fit of the fitted distributions to the empirical distribution. The results of applying stricter goodness-of-fit tests of normality (Lilliefors, 1967) and exponentiality (Lilliefors, 1969) are also provided.

Panel A. Last Pre-IPO Private Equity Transactions



	Exponential	Log-Normal	Gamma	Weibull
Shape Parameter/Mu	0.58	-0.97	1.33	1.15
Scale Parameter/Sigma	--	1	0.44	0.61
K-S statistic	0.0661	0.0378	0.0522	0.0522
Standard K-S Test p Value	0.0018	0.2032	0.0252	0.0252
Lilliefors Test p Value	0.0010	0.0010	0.0010	0.0010

Panel B. Earlier Pre-IPO Private Equity Transactions



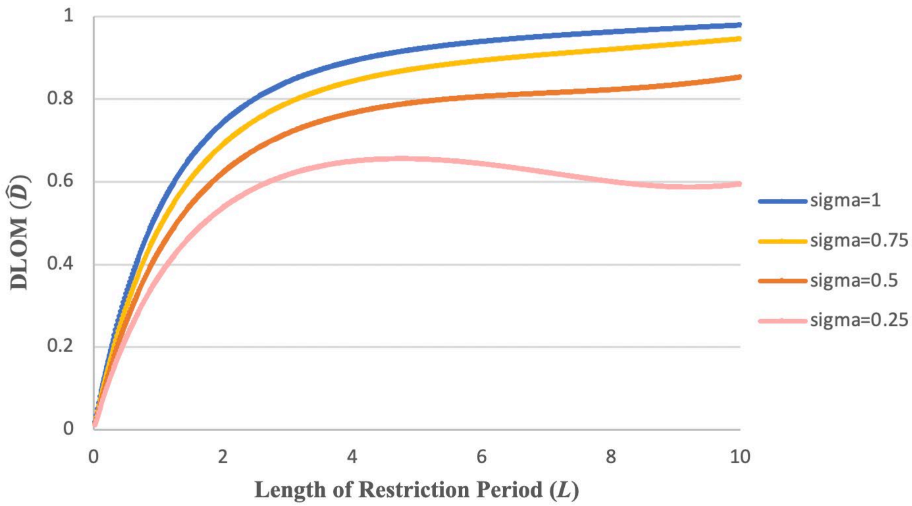
	Exponential	Log-Normal	Gamma	Weibull
Shape Parameter/Mu	2.40	0.48	1.42	1.18
Scale Parameter/Sigma	--	0.90	1.68	2.55
K-S statistic	0.1066	0.0769	0.1205	0.1140
Standard K-S Test p Value	<.0001	<.0001	<.0001	<.0001
Lilliefors Test p Value	<.0001	<.0001	<.0001	<.0001

Figure 3

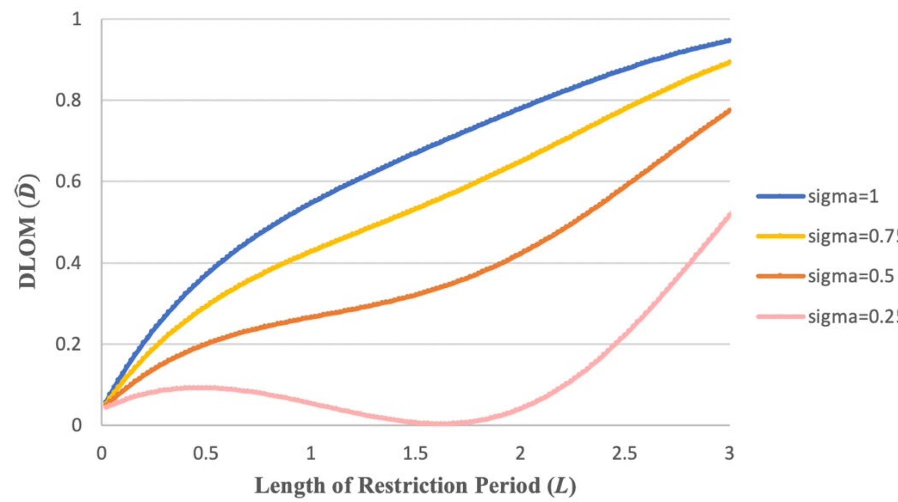
Fitted DLOM Term Structures

These figures show the DLOM term structures implied by the DLOM term structure regression models in Table 4. Adjusted Discount ≥ 0% for all private equity transactions. Adjusted Discount is winsorized at the lookback put option model (Longstaff, 1995) upper bound when Adjusted Discount would otherwise exceed this upper bound. The DLOM term structures are plotted for four representative stock price volatilities, $\sigma = 0.25$, $\sigma = 0.5$, $\sigma = 0.75$, and $\sigma = 1.0$, which reflect the volatilities of the stocks of the sample firms in the 12 months after going public. Panel A plots the DLOM term structure for all pre-IPO private equity transactions in the sample. Panel B plots the DLOM term structure for the last pre-IPO private equity transactions. Panel C plots the DLOM term structure for the earlier pre-IPO private equity transactions.

Panel A. DLOM Term Structure for All Pre-IPO Private Equity Transactions



Panel B. DLOM Term Structure for Last Pre-IPO Private Equity Transactions



Panel C. DLOM Term Structure for Earlier Pre-IPO Private Equity Transactions

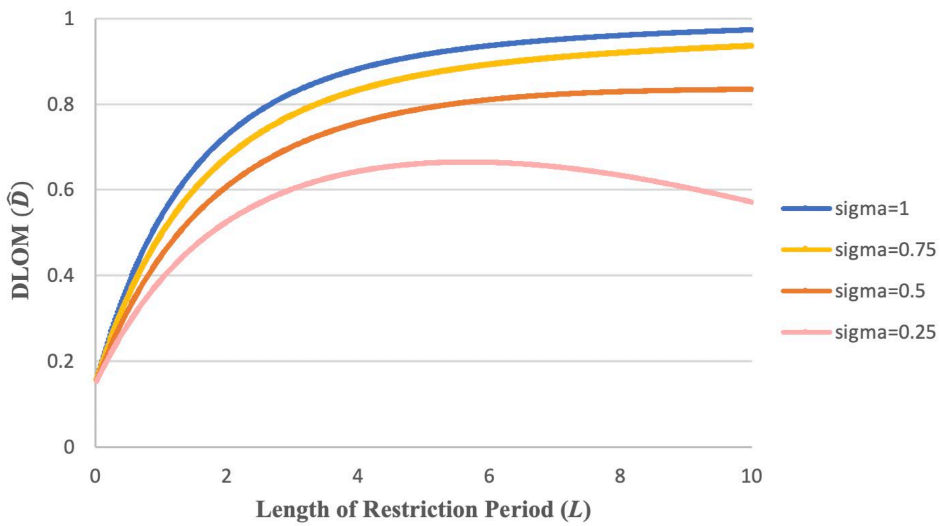
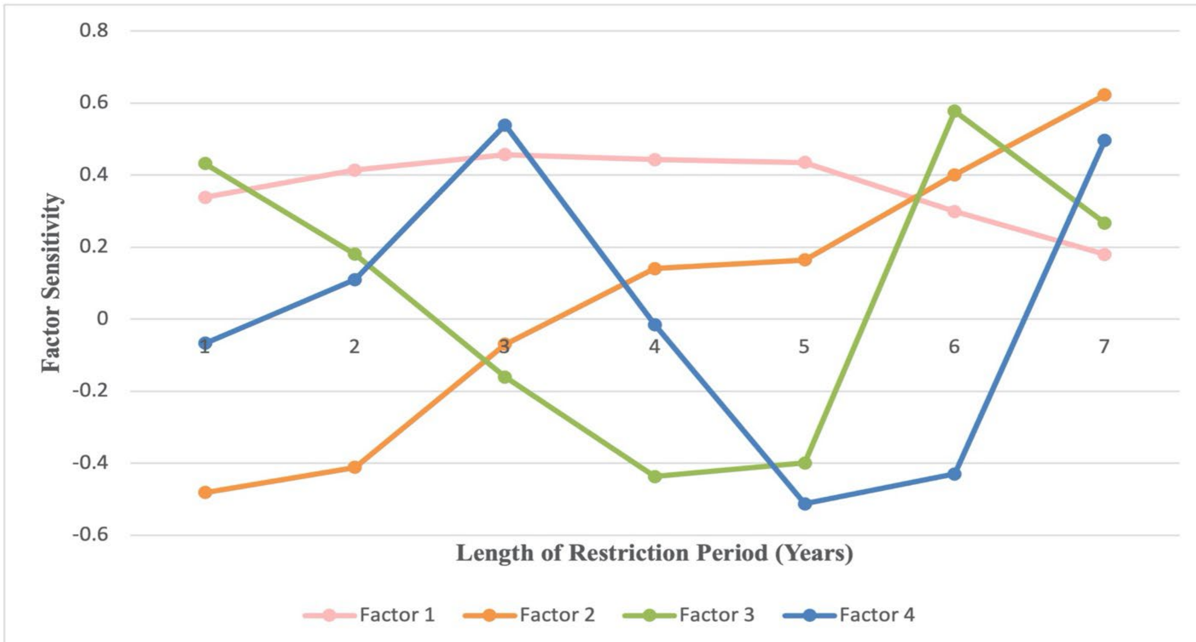


Figure 4

Factor Sensitivities for the DLOM Term Structure for the [0, 7] Restriction Period Range

Principal components analysis is used to obtain the factor sensitivities for the four most significant latent factors underlying the DLOM term structure for the [0,7] restriction period range. The sample contains 5,333 private equity transactions between June 1985 and April 2017.



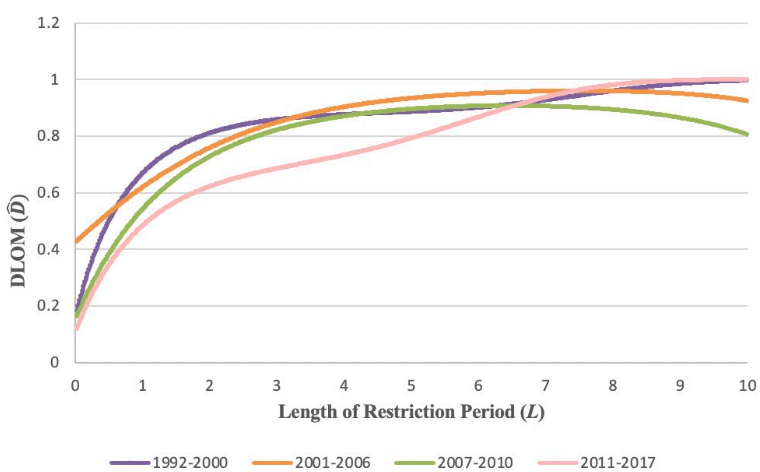
Eigenvectors							
	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
0<L<=1	0.3379	-0.4812	0.4320	-0.0671	0.1060	-0.1360	0.6583
1<L<=2	0.4134	-0.4114	0.1802	0.1094	0.3821	0.2863	-0.6224
2<L<=3	0.4567	-0.0698	-0.1608	0.5386	-0.6852	0.0323	-0.0081
3<L<=4	0.4426	0.1400	-0.4369	-0.0151	0.3394	-0.6905	-0.0370
4<L<=5	0.4349	0.1641	-0.3996	-0.5123	-0.0121	0.5581	0.2240
5<L<=6	0.2984	0.4009	0.5772	-0.4296	-0.3225	-0.2269	-0.2776
6<L<=7	0.1795	0.6221	0.2670	0.4963	0.3924	0.2422	0.2248

Figure 5

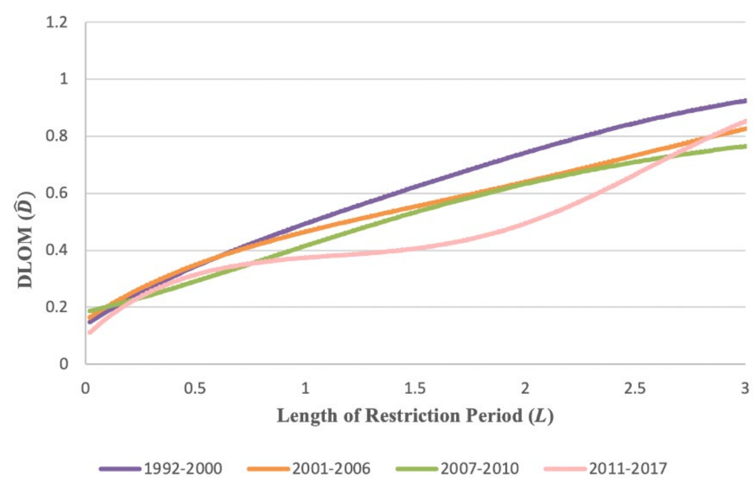
Comparison of Fitted DLOM Term Structures for Four Sub-Periods

These figures show the DLOM term structures estimated from the DLOM term structure regression models for four sub-periods: 1992 to 2000 (high frequency of IPOs), 2001 to 2006 (pre-financial crisis), 2007 to 2010 (financial crisis), and 2011 to 2017 (post-financial crisis). Adjusted Discount ≥ 0% for all private equity transactions. Adjusted Discount is winsorized at the lookback put option model (Longstaff, 1995) upper bound when Adjusted Discount would otherwise exceed this upper bound. The DLOM term structures are plotted for stock price volatility σ = 0.75, which reflects stock volatility in the 12 months after a firm goes public. Panel A plots the DLOM term structures for all pre-IPO private equity transactions in the sample. Panel B plots the DLOM term structures for the last pre-IPO private equity transactions. Panel C plots the DLOM term structures for the earlier pre-IPO private equity transactions.

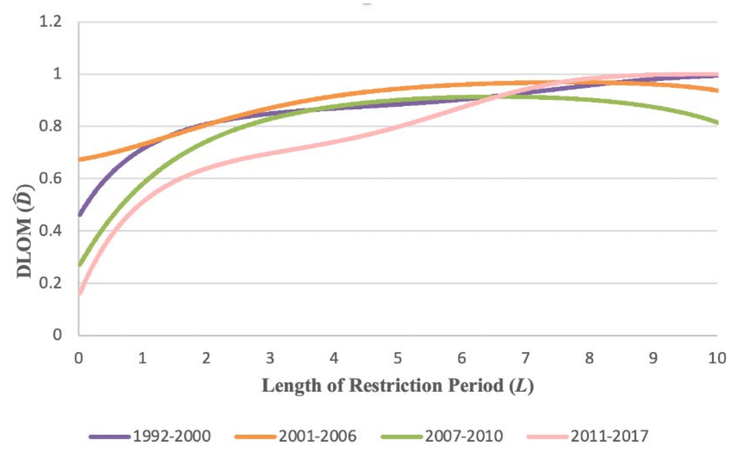
Panel A. DLOM Term Structures for All Pre-IPO Private Equity Transactions



Panel B. DLOM Term Structures for Last Pre-IPO Private Equity Transactions



Panel C. DLOM Term Structures for Earlier Pre-IPO Private Equity Transactions



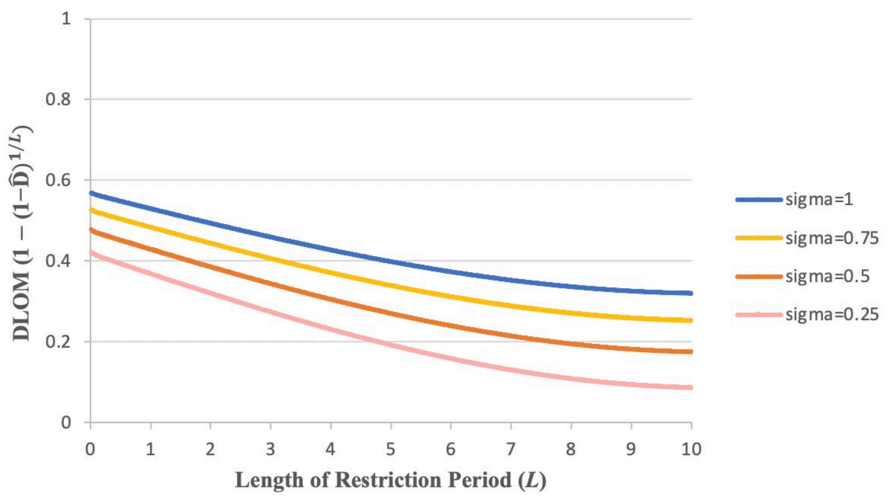
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10
DLOM ($\hat{D}$)
Length of Restriction Period (L)
1992-2000
2001-2006
2007-2010
2011-2017

Figure A1

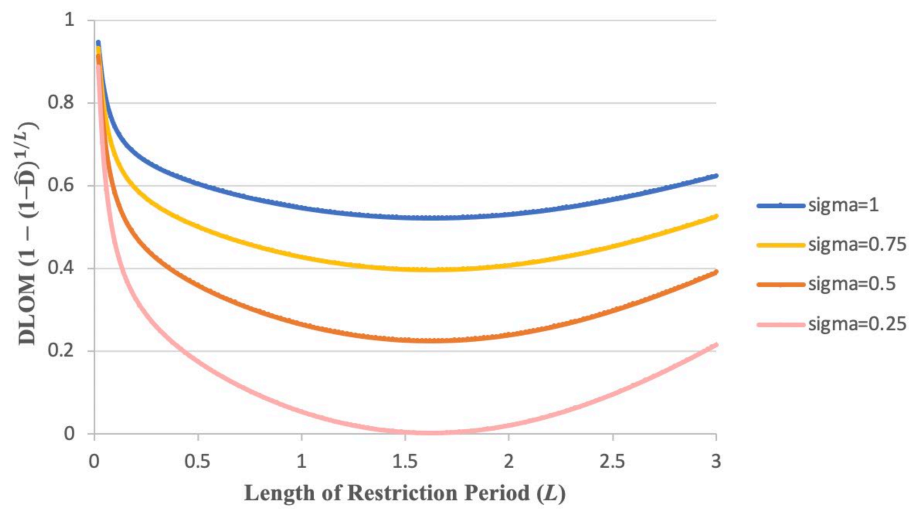
Fitted DLOMPY Term Structures

These figures show the DLOMPY term structures implied by the DLOM term structure regression models in Table 4. The DLOMPY term structure is obtained by converting $\widehat{D}$ in equation (12) to an equivalent annually compounded basis by applying equation (A1). Adjusted Discount ≥ 0% for all private equity transactions. Adjusted Discount is winsorized at the lookback put option model (Longstaff, 1995) upper bound when Adjusted Discount would otherwise exceed this upper bound. The DLOMPY term structures are plotted for four representative stock price volatilities, $\sigma = 0.25$, $\sigma = 0.5$, $\sigma = 0.75$, and $\sigma = 1.0$, which reflect the volatilities of the stocks of the sample firms in the 12 months after going public. Panel A plots the DLOMPY term structure for all pre-IPO private equity transactions in the sample. Panel B plots the DLOMPY term structure for the last pre-IPO private equity transactions. Panel C plots the DLOMPY term structure for the earlier pre-IPO private equity transactions.

Panel A. DLOMPY Term Structure for All Pre-IPO Private Equity Transactions



Panel B. DLOMPY Term Structure for Last Pre-IPO Private Equity Transactions



Panel C. DLOMPY Term Structure for Earlier Pre-IPO Private Equity Transactions

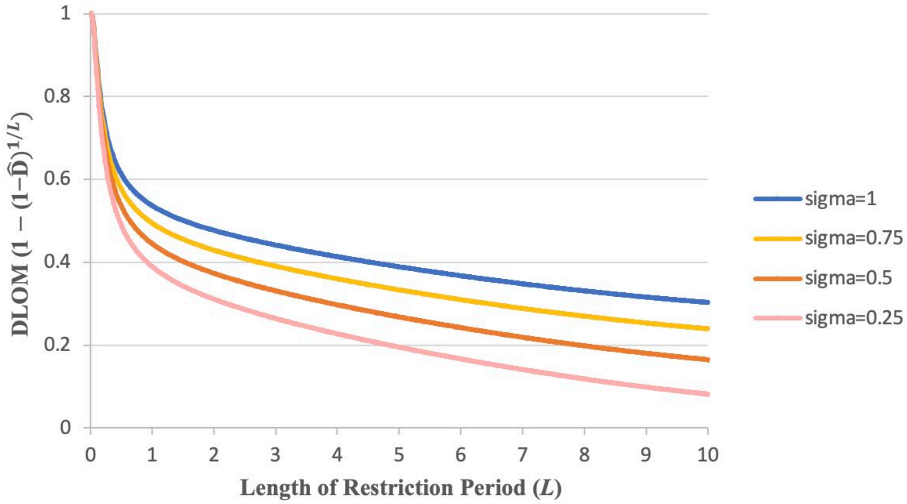
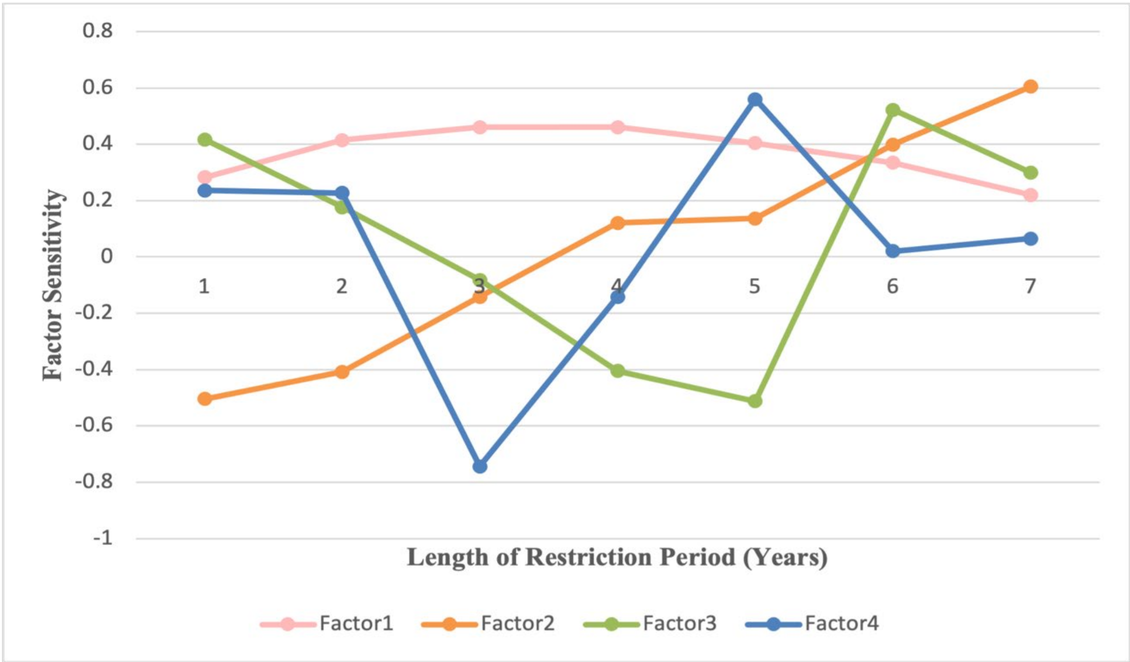

DLOM $(1-(1-\hat{D})^{1/L})$
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10
Length of Restriction Period (L)
sigma=1
sigma=0.75
sigma=0.5
sigma=0.25

Figure A2

Factor Sensitivities for the DLOMPY Term Structure for the [0, 7] Restriction Period Range

Principal components analysis is used to obtain the factor sensitivities for the four most significant latent factors underlying the annually compounded DLOMPY term structure for the [0,7] restriction period range. The sample contains 5,333 private equity transactions between June 1985 and April 2017.



Eigenvectors							
	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
0<L<=1	0.2828	-0.5048	0.4167	0.2365	0.3893	0.4113	-0.3391
1<L<=2	0.4150	-0.4083	0.1759	0.2273	-0.1428	-0.5109	0.5451
2<L<=3	0.4607	-0.1417	-0.0818	-0.7445	-0.2205	0.3629	0.1624
3<L<=4	0.4609	0.1211	-0.4050	-0.1417	0.5040	-0.4517	-0.3617
4<L<=5	0.4040	0.1369	-0.5121	0.5598	-0.3084	0.3838	0.0057
5<L<=6	0.3341	0.3989	0.5219	0.0199	-0.4711	-0.1985	-0.4417
6<L<=7	0.2200	0.6049	0.3001	0.0649	0.4565	0.2179	0.4854

Table 1

Relationship between the Volatility Parameters σ and *v* in the Average-Strike Put Option DLOM Model

This table quantifies the relationship between the volatility parameters σ and v in equations (4) and (6). σ is the price volatility of the firm's unrestricted shares. $v\sqrt{L}$ is the volatility of the ratio of the price of the unrestricted share at the end of the restriction period L to the average forward contract price calculated over the interval $[0, L]$. v is calculated from equation (4). The values for v in the column $L = 1.0$ illustrate the relationship between v and σ in equation (6).

	v			
σ	***L* = 0.5**	***L* = 1.0**	***L* = 2.0**	***L* = 5.0**
10%	5.77%	5.77%	5.76%	5.75%
20%	11.53%	11.51%	11.47%	11.35%
30%	17.26%	17.19%	17.06%	16.67%
40%	22.94%	22.79%	22.47%	21.54%
50%	28.57%	28.26%	27.66%	25.83%
60%	34.12%	33.60%	32.54%	29.43%
70%	39.59%	38.75%	37.08%	32.26%
80%	44.95%	43.70%	41.22%	34.31%
90%	50.20%	48.42%	44.91%	35.67%
100%	55.31%	52.88%	48.12%	36.48%

Table 2

Sample Description

Panel A provides a summary financial description of the 1,609 firms in the marketability discount sample. The 1,609 firms conducted 5,333 private equity transactions between June 1985 and April 2017. The financial data pertain to the fiscal year the firm went public. In the first column under each header, the sample contains 1,466 firms with IPO dates ranging from March 1986 to June 2017 for which all the pre-IPO private equity transactions were at an implied marketability discount, $\widehat{D} \geq 0$. In the second column under each header, the sample contains 122 firms with IPO dates ranging from May 1996 to April 2017 that had issued stock or granted options (not included in the sample) at a premium at least once. Panel B provides a breakdown of the sample of 5,333 private equity transactions based on the number of firms that went public each year broken down by industry. Panel C provides a breakdown of the sample based on the number of private equity transactions each year broken down by industry. Panel D shows how the length of the restriction period varies by industry for the last private equity transaction prior to an IPO and for earlier private equity transactions. Industry in Panels B, C, and D is based on the French (2023) 12-industry classification scheme. Panel E reports the clustering of private equity transactions based on the number of transactions for each of the 1,609 firms in the sample.

Panel A. Firm Descriptive Financial Data (Millions of Dollars)[a]

	Annual Revenue		Annual Net Income		Total Assets at Year-End		Book Value of Equity at Year-End		Market Capitalization at Year-End	
Adjusted Discount	Discounts Only	Discounts and Premiums	Discounts Only	Discounts and Premiums	Discounts Only	Discounts and Premiums	Discounts Only	Discounts and Premiums	Discounts Only	Discounts and Premiums
Number of Firms	1,466	122	1,466	122	1,466	122	1,466	122	1,466	122
Minimum	0	0	-3,445.07	-468.83	1.93	17.33	-8,258.01	-2,111.01	3.37	29.84
First Quartile	19.14	36.83	-27.31	-28.87	71.49	99.02	43.78	43.95	179.58	220.94
Median	69.12	103.08	-5.03	-2.50	140.15	144.72	85.02	80.25	445.57	355.61
Mean	511.38	277.35	-6.16	-5.81	831.40	585.49	190.58	113.84	1,188.94	672.77
Third Quartile	228.91	273.50	8.27	11.21	386.08	296.95	167.13	151.04	1,084.62	679.80
Maximum	135,592.00	4,738.00	6,172.00	217.00	151,828.00	18,242.88	24,277.02	1,347.76	63,141.93	5,452.04
Percentage Negative	--		62.12%	58.97%	--		10.02%	11.21%	--	

Panel B. Sample Breakdown by Number of Firms Going Public per Year by Industry

Year	1 Non-Durables	2 Durables	3 Manufact-uring	4 Energy	5 Chemicals	6 Business Equip.	7 Telcomm.	8 Utilities	9 Retail Wholesale	10 Healthcare	11 Finance	12 Other	Total
1986[b]	0	0	0	0	0	1	0	0	0	0	0	0	1
1995[b]	0	1	1	0	1	1	0	0	2	1	0	0	7
1996[b]	2	0	2	2	1	27	1	0	6	9	1	6	57
1997[b]	2	1	4	0	0	25	0	0	7	7	3	9	58
1998[b]	4	1	2	0	0	32	3	0	9	6	5	11	73
1999[b]	5	2	3	0	0	97	14	1	26	3	6	41	198
2000[b]	0	1	8	0	2	74	10	0	4	22	0	28	149
2001	1	1	1	1	0	9	0	0	1	5	2	3	24
2002	1	0	0	0	0	11	0	0	7	6	3	6	34
2003	1	0	1	0	0	9	1	0	3	3	7	8	33
2004	0	2	1	2	0	18	2	1	14	28	8	9	85
2005	3	2	4	4	2	10	4	0	7	19	15	8	78
2006	3	1	6	4	5	17	3	1	5	19	10	16	90
2007	1	1	3	2	1	31	2	1	4	24	7	16	93
2008	0	0	2	0	0	2	0	1	0	1	1	4	11
2009	0	0	2	0	1	9	1	0	3	3	0	7	26
2010	1	3	4	1	2	17	2	0	4	10	8	10	62
2011	0	1	1	1	3	18	2	0	7	9	2	6	50
2012	3	0	5	2	0	30	1	0	8	12	4	7	72
2013	0	1	1	1	2	29	1	0	9	36	10	17	107
2014	1	1	3	2	1	28	0	0	12	61	13	10	132
2015	1	1	4	0	1	19	0	0	8	40	9	8	91
2016	3	0	1	1	1	14	0	1	3	24	4	3	55
2017	0	1	1	3	0	8	0	0	2	5	1	2	23
Total	32	21	60	26	23	536	47	6	151	353	119	235	1609
Industry Percentage	1.99%	1.31%	3.73%	1.62%	1.43%	33.31%	2.92%	0.37%	9.38%	21.94%	7.40%	14.61%	100.00%

Panel C. Sample Breakdown by Number of Private Equity Transactions per Year by Industry

Year	1 Non-Durables	2 Durables	3 Manufact-uring	4 Energy	5 Chemicals	6 Business Equip.	7 Telcomm.	8 Utilities	9 Retail Wholesale	10 Healthcare	11 Finance	12 Other	Total
1985	0	0	0	0	0	2	0	0	0	0	0	0	2
1992[b]	0	2	1	0	0	2	0	0	0	0	0	1	6
1993[b]	0	1	1	0	2	10	0	0	1	1	0	0	16
1994[b]	1	0	3	1	2	16	0	0	6	4	1	3	37
1995[b]	5	1	2	3	1	47	2	0	16	10	3	6	96
1996[b]	3	2	7	0	0	64	3	0	16	23	7	15	140
1997[b]	7	1	8	0	0	88	7	0	23	19	8	26	187
1998[b]	9	6	4	1	1	164	22	0	32	18	12	54	323
1999[b]	1	6	12	0	2	180	23	1	28	37	8	93	391
2000[b]	1	2	5	1	5	72	9	0	10	27	10	21	163
1992-2000	27	21	43	6	13	643	66	1	132	139	49	219	1359
2001	3	1	0	0	0	28	1	1	12	17	12	17	92
2002	3	0	2	1	0	24	4	0	12	22	11	15	94
2003	4	2	2	2	1	35	4	1	20	37	21	18	147
2004	1	5	7	9	5	49	14	2	22	61	26	19	220
2005	4	7	15	9	7	82	13	2	11	69	26	28	273
2006	5	1	14	2	7	126	8	1	9	90	26	36	325
2001-2006	20	16	40	23	20	344	44	7	86	296	122	133	1151
2007	5	2	8	1	2	104	4	0	18	75	18	31	268
2008	3	2	11	1	5	71	3	1	19	67	15	16	214
2009	2	7	10	1	9	93	5	0	22	74	21	29	273
2010	3	5	11	6	10	121	8	0	24	99	29	37	353
2007-2010	13	16	40	9	26	389	20	1	83	315	83	113	1108
2011	3	2	5	2	3	139	5	0	30	109	20	37	355
2012	2	3	7	3	5	124	4	0	30	120	31	37	366
2013	0	4	5	1	3	119	0	1	39	194	26	39	431
2014	0	2	12	1	3	84	0	0	21	139	25	25	312
2015	5	0	6	0	2	38	0	1	9	85	8	11	165
2016	3	1	1	2	1	32	0	0	6	25	1	4	76
2017	0	0	0	0	0	5	0	0	0	3	0	0	8
2011-2017	13	12	36	9	17	541	9	2	135	675	111	153	1713
Total	73	65	159	47	76	1919	139	11	436	1425	365	618	5333
Industry Percentage	1.37%	1.22%	2.98%	0.88%	1.43%	35.98%	2.61%	0.21%	8.18%	26.72%	6.84%	11.59%	100.00%

Panel D. Distribution of Length of Restriction Period by Type of Private Equity Transaction and by Industry

	Last Pre-IPO Private Equity Transactions						Earlier Pre-IPO Private Equity Transactions					
Industry	Num	Min	25th Percentile	Median	75th Percentile	Max	Num	Min	25th Percentile	Median	75th Percentile	Max
1. Non-Durables	31	0.12	0.28	0.52	1.07	2.93	42	0.40	1.00	2.15	4.07	9.36
2. Durables	21	0.04	0.35	0.58	0.96	1.53	44	0.32	0.80	1.37	2.79	7.72
3. Manufacturing	60	0.06	0.27	0.57	1.12	2.35	99	0.28	0.78	1.40	3.13	9.70
4.Energy	25	0.08	0.42	0.78	1.33	2.81	22	0.44	0.96	1.77	2.43	6.96
5.Chemicals	23	0.02	0.20	0.41	0.84	1.96	53	0.32	0.73	1.19	2.42	9.04
6.Business Equip.	530	0.00	0.16	0.32	0.59	2.93	1389	0.18	0.79	1.21	3.10	9.81
7.Telecomm.	46	0.05	0.15	0.28	0.59	2.18	93	0.27	0.61	1.22	3.08	8.84
8.Utilities	6	0.02	0.35	0.61	0.98	1.11	5	1.65	1.66	2.05	2.46	3.37
9.Retail/Wholesale	151	0.02	0.28	0.55	0.93	2.66	285	0.27	0.88	1.61	3.13	9.36
10.Healthcare	347	0.01	0.22	0.40	0.75	2.95	1078	0.25	0.84	1.76	4.02	9.89
11.Finance	117	0.04	0.27	0.57	1.04	2.47	248	0.25	1.06	2.33	4.58	9.78
12.Other	232	0.01	0.23	0.44	0.82	2.96	386	0.25	0.81	1.29	2.99	9.60
Total	1589	0.00	0.21	0.39	0.79	2.96	3744	0.18	0.82	1.41	3.40	9.89

Panel E. Clustering of Private Equity Transactions by Issuing Firm

No. of Private Equity Transactions	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total Firms
No. of Firms	360	382	286	209	125	71	65	49	27	21	10	1	1	1	1	1,609

[a]Not all firms have financial data available on Compustat or Bloomberg for their IPO year. In the first column under each header, data were available for 1,466 of 1,486 firms. In the second column, data were available for 122 of 123 firms.
[b]Year with an above-annual-average number of IPOs for the sample period.

Table 3

Implied Marketability Discounts that Exceed the Lookback Put Option Model Upper Bound

This table reports evidence of underpricing based on the number of private equity transactions for which the implied DLOMs exceed the lookback put option model (Longstaff, 1995) upper bound. Number with MD Above Upper Bound is the number of DLOMs that exceed the upper bound. Percentage with MD Above Upper Bound is the percentage of DLOMs that exceed the upper bound. The two proportion z-test is used to test the difference between the percentage of last pre-IPO transaction DLOMs and the percentage of earlier pre-IPO transaction DLOMs that exceed the upper bound. Excess is the difference between the implied DLOM and the lookback put option model upper bound. Mean Excess (Median Excess) is the mean (median) of the Excess for those private equity transactions with a DLOM that exceeds the upper bound. A pooled t-test is used to test the difference between the Mean Excesses for the last pre-IPO private equity transactions and the earlier pre-IPO private equity transactions, and the Mann-Whitney U test is used to test the difference between the Median Excesses. A one sample t-test is used to test the Mean Excess, and a Wilcoxon signed-rank test is used to test the Median Excess. The p-value for each test statistic is reported within parentheses.

	All Pre-IPO Transactions		Last Pre-IPO Transactions		Earlier Pre-IPO Transactions		
	Number	Percentage	Number	Percentage	Number	Percentage	Difference Between Last and Earlier (p-value)
Number of Private Issues	5333		1589		3744		
Number with MD Above Upper Bound	1427		599		828		
Percentage with MD Above Upper Bound	26.76% 37.78 (<.0001)		37.70% 24.47 (<.0001)		22.12% 28.78 (<.0001)		15.58% 11.76 (<.0001)
Private Issues with MD Above Upper Bound:							
Mean Excess	19.77% 48.52 (<.0001)		18.88% 30.29 (<.0001)		20.42% 37.97 (<.0001)		-1.54% -1.86 (0.0631)
Median Excess	16.06% 509439 (<.0001)		15.05% 89850 (<.0001)		16.93% 171603 (<.0001)		-1.88% -2.11 (0.0350)
0 < Excess ≤ 5%	244	17.10%	114	19.03%	130	15.70%	
5% < Excess ≤ 10%	210	14.72%	95	15.86%	115	13.89%	
10% < Excess ≤ 15%	215	15.07%	90	15.03%	125	15.10%	
15% < Excess ≤ 20%	182	12.75%	80	13.36%	102	12.32%	
20% < Excess ≤ 25%	129	9.04%	43	7.18%	86	10.39%	
Excess > 25%	447	31.32%	177	29.55%	270	32.61%	

Table 4

Parameter Estimation for the DLOM Term Structure Models

This table reports the results of fitting the cubic regression model $-ln[1-\widehat{D}] = c_0 + c_1L + c_2L^2 + c_3L^3 + c_4\Delta L + b_1I_1L + \cdots + b_{11}I_{11}L + e_1IPO_t$. The dependent variable is the continuously compounded DLOM. L is the length of the restriction period. IPO_t is the IPO control variable. $\widehat{D} \geq 0$ for every sample transaction. I_j are industry dummy variables. Interacting I_j with L adjusts for industry fixed effects.

The DLOM is calculated as the difference between the adjusted IPO price (the IPO price discounted by the industry returns) and the transaction price divided by the adjusted IPO price by applying equation (13). Holding period is the difference between the IPO date and the private equity transaction date. The implied marketability discounts are winsorized using the Longstaff (1995) model price as the upper bound. Δ is calculated from equation (8). The industry dummy variables are based on the French (2023) 12-industry classification system. The *IPO* control variable is the ratio of the number of IPO offerings in the year the firm's IPO occurred to the average annual number of IPO offerings during the time period in which private equity transactions in the sample took place, 1985 to 2017.

The full sample contains 5,333 private equity transactions that took place between June 1985 and April 2017 after eliminating outliers consisting of 43 last pre-IPO transactions with L > 3 years and 15 earlier pre-IPO transactions with L > 10 years. The last pre-IPO transactions sample includes 1,589 private equity transactions. The earlier pre-IPO transactions sample includes 3,744 private equity transactions.

Regression errors are clustered at the issuer level. The table reports asymptotically consistent estimators. Robust standard errors are provided to the right of the estimated coefficients.

	All Private Equity Transactions			Last Pre-IPO Transactions			Earlier Pre-IPO Transactions		
Variable	Coefficient	Standard Error	t-Statistics	Coefficient	Standard Error	t-Statistics	Coefficient	Standard Error	t-Statistics
Intercept	0.0001	0.0324	0.00	0.0383	0.0289	1.32	0.1582	0.0504	3.14[a]
L	0.4413	0.0570	7.74[a]	0.0282	0.1675	0.17	0.2877	0.0717	4.01[a]
L^2	-0.0861	0.0140	-6.16[a]	-0.3814	0.1908	-2.00[b]	-0.0472	0.0177	-2.67[a]
L^3	0.0041	0.0012	3.56[a]	0.1220	0.0592	2.06[b]	0.0016	0.0014	1.17
ΔL	1.6833	0.1584	10.63[a]	4.2100	0.4063	10.36[a]	1.5764	0.1590	9.92[a]
I_1	-0.2839	0.0692	-4.10[a]	-0.0283	0.1615	-0.18	-0.2894	0.0671	-4.32[a]
I_2	-0.1637	0.0765	-2.14[b]	0.1453	0.1007	1.44	-0.1728	0.0765	-2.26[b]
I_3	-0.1178	0.0494	-2.38[b]	0.1346	0.1056	1.27	-0.1243	0.0493	-2.52[b]
I_4	-0.1304	0.0489	-2.67[a]	0.0923	0.0926	1.00	-0.1285	0.0517	-2.49[b]
I_5	0.1497	0.1041	1.44	0.0536	0.1099	0.49	0.1582	0.1088	1.45
I_6	0.0379	0.0391	0.97	-0.0060	0.0876	-0.07	0.0346	0.0401	0.86
I_7	0.1002	0.0843	1.19	-0.0608	0.1154	-0.53	0.1040	0.0861	1.21
I_8	-0.1367	0.0379	-3.60[a]	0.1298	0.1169	1.11	-0.1413	0.0399	-3.54[a]
I_9	-0.0987	0.0434	-2.27[b]	0.0484	0.0868	0.56	-0.1037	0.0448	-2.31[b]
I_{10}	0.0055	0.0381	0.15	0.0122	0.0833	0.15	0.0038	0.0390	0.10
I_{11}	-0.0818	0.0392	-2.08[b]	0.0325	0.0723	0.45	-0.0868	0.0403	-2.15[b]
IPO_t	0.2177	0.0266	8.19[a]	0.0925	0.0145	6.37[a]	0.2960	0.0401	7.38[a]
N	5333			1589			3744		
R-Square	0.4789			0.4999			0.4068		
Adj R-Square	0.4774			0.4948			0.4042		
F-statistic	305.39			98.21			159.72		
Prob (F-statistic)	<.0001			<.0001			<.0001		
Akaike info criterion	12538.62			1296.44955			9752.04565		
Schwarz criterion	12650.51			1387.75417			9857.92011		
Hannan-Quinn criterion	12577.71			1330.36572			9789.70174		
Durbin-Watson stat	1.18			2.0335			1.206		

[a] Significant at the .01 level. [b] Significant at the .05 level. [c] Significant at the .10 level.

Table 5

Principal Components Analysis of the DLOM Term Structure

Principal components analysis is applied to the implied marketability discount $\widehat{D}[est]$ in equation (13) for five restriction periods of different length, $[0, 6]$, $[0, 7]$, $[0, 8]$, $[0, 9]$, and $[0, 10]$, to determine the relative importance of the fundamental common latent factors that are responsible for the annual shifts in the DLOM term structure. The full sample contains 5,333 private equity transactions between June 1985 and April 2017. Each restriction period range is divided into annual intervals. The implied marketability discounts are winsorized using the Longstaff (1995) model price as the upper bound, and the discounts are averaged when there is more than one private equity transaction within the same annual restriction interval during the calender year. Each calender year included in the analysis has at least 90 private equity transactions that span the entire indicated restriction period range to ensure an adequate-sized annual transaction sample.

L Range	[0, 6]		[0, 7]		[0, 8]		[0, 9]		[0, 10]	
Date Range	1995-2012		1995-2011		1995-2009		1995-2008		1995-2008	
Explained	%	Cum	%	Cum	%	Cum	%	Cum	%	Cum
Factor 1	56.18%	56.18%	49.50%	49.50%	47.37%	47.37%	40.05%	40.05%	36.37%	36.37%
Factor 2	20.53%	76.72%	24.89%	74.39%	22.00%	69.37%	25.63%	65.68%	23.62%	60.00%
Factor 3	13.11%	89.83%	12.82%	87.20%	13.98%	83.35%	14.80%	80.49%	14.15%	74.14%
Factor 4	5.54%	95.36%	5.34%	92.54%	7.65%	91.00%	8.30%	88.79%	10.96%	85.10%
Additional	.	4.64%	.	7.46%	.	9.00%	.	11.21%	.	14.90%

Table A1

Distribution of Median Term Premium by Industry and Length of Restriction Period

This table provides the distribution of the median term premium expressed by equation (15), $\tau(\Delta, L) = -\Delta - ln\,(1 - \widehat{D})/L$, for the sample firms by industry and length of restriction period. Industry is based on the French (2023) 12-industry classification scheme. The sample contains 5,333 pre-IPO private equity transactions between June 1985 and April 2017.

	Median Term Premium (τ)											
Length of Restriction Period	1 Non-Durables	2 Durables	3 Manufact-uring	4 Energy	5 Chemicals	6 Business Equip.	7 Telcomm.	8 Utilities	9 Retail/ Wholesale	10 Healthcare	11 Finance	12 Other
$0 < L <= 0.1$	.	3.81	3.90	0.37	5.94	2.88	4.87	2.96	2.48	2.22	0.89	2.43
$0.1 < L <= 0.25$	1.51	0.98	0.83	0.87	1.71	1.41	1.30	.	1.30	1.41	0.78	1.20
$0.25 < L <= 0.5$	0.63	1.03	0.77	0.68	0.53	0.93	1.31	-0.03	0.69	0.83	0.35	0.89
$0.5 < L <= 0.75$	0.88	0.98	0.58	0.50	0.47	0.81	1.10	0.67	0.52	0.70	0.31	0.55
$0.75 < L <= 1$	0.26	0.60	0.36	0.34	0.38	0.68	0.89	0.12	0.35	0.61	0.22	0.46
$1 < L <= 1.5$	0.54	0.42	0.28	0.28	0.43	0.63	0.65	0.81	0.38	0.56	0.19	0.45
$1.5 < L <= 2$	0.15	0.35	0.44	0.16	0.38	0.43	0.29	0.26	0.33	0.47	0.11	0.28
$2 < L <= 2.5$	0.38	0.29	0.31	0.22	0.86	0.47	0.41	0.29	0.20	0.52	0.11	0.26
$2.5 < L <= 3$	0.25	0.23	0.22	0.42	0.10	0.39	0.31	.	0.22	0.39	0.10	0.19
$3 < L <= 4$	-0.01	0.18	0.16	0.32	0.74	0.34	0.35	0.20	0.16	0.36	0.06	0.22
$4 < L <= 5$	-0.06	-0.05	0.17	.	0.97	0.31	0.33	.	0.14	0.23	0.06	0.17
$5 < L <= 6$	-0.05	0.34	0.08	0.00	0.43	0.29	0.61	.	0.07	0.19	0.07	0.25
$6 < L <= 7$	0.00	0.04	0.07	0.03	0.11	0.25	0.20	.	0.09	0.18	0.06	0.15
$7 < L <= 8$	-0.17	-0.08	0.09	.	.	0.20	0.34	.	0.22	0.19	0.06	0.16
$8 < L <= 9$	-0.09	.	-0.01	.	0.37	0.14	0.13	.	-0.01	0.14	0.05	0.09
$9 < L <= 10$	-0.09	.	-0.01	.	0.19	0.12	.	.	-0.03	0.09	0.03	0.20

Table A2

Principal Components Analysis of the Annually Compounded DLOMPY Term Structure

Principal components analysis is applied to the annually compounded DLOMPY = $1-(1-\widehat{D})^{1/L}$ term structure for five restriction periods of different length, $[0,6]$, $[0,7]$, $[0,8]$, $[0,9]$, and $[0,10]$, to determine the relative importance of the fundamental common latent factors that are responsible for the annual shifts in the DLOMPY term structure. The full sample contains 5,333 private equity transactions between June 1985 and April 2017. Each restriction period range is divided into annual intervals. The implied marketability discounts are winsorized using the Longstaff (1995) model price as the upper bound, and the discounts are averaged when there is more than one private equity transaction within the same annual restriction interval during the calender year. Each calender year included in the analysis has at least 90 private equity transactions that span the entire indicated restriction period range to ensure an adequate-sized annual transaction sample.

L Range	[0, 6]		[0, 7]		[0, 8]		[0, 9]		[0, 10]	
Date Range	1995-2012		1995-2011		1995-2009		1995-2008		1995-2008	
Explained	%	Cum	%	Cum	%	Cum	%	Cum	%	Cum
Factor 1	51.99%	51.99%	46.03%	46.03%	45.29%	45.29%	44.07%	44.07%	47.02%	47.02%
Factor 2	21.54%	73.53%	26.04%	72.07%	25.41%	70.70%	24.54%	68.61%	22.31%	69.33%
Factor 3	13.11%	86.64%	13.88%	85.95%	13.63%	84.33%	13.37%	81.99%	12.04%	81.37%
Factor 4	6.95%	93.60%	5.94%	91.89%	5.16%	89.49%	6.40%	88.39%	5.77%	87.14%
Additional	.	6.40%	.	8.11%	.	10.51%	.	11.61%	.	12.86%