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The business valuation profession is moving toward a consensus that development and application of empirically rigorous methods for determining and testing discounts and premiums is needed. Increased availability of affordable, good quality public market data has contributed to this move toward empirical analysis. Time/volatility models used for estimating the appropriate discounts for lack of liquidity are based on options theory. These models are conceptually easy to understand and relatively simple to apply. The model parameters are directly observable from market data, making them easy to replicate and to test the results. The discount for illiquidity derived from these models reflects only the incremental return differentials between securities, which have already crossed the marketability barrier and can be legally traded. Any discount related strictly to marketability is an addition to the discount for illiquidity, not subsumed within it. This paper presents three major put option models and illustrates their application for estimating appropriate discounts for lack of liquidity.

Introduction

Traditionally, business valuation practitioners have used discount for lack of marketability (DLOM) as a generic term to indicate impairment in value due to lack of marketability and liquidity. Bajaj and Sarin[1] developed an analysis reflecting the incremental returns associated with a lack of legal marketability but concluded that other factors contributed to the observed discounted prices associated with securities, which were not freely tradable by the general public.

Since the adoption of definitions for marketability (capability and ease of transfer or salability) and liquidity (ability to readily convert an asset into cash *without significant loss of principal*), it is important to parse this discount based on the source of impairment. Marketability denotes the legal ability to sell or transfer ownership. A public, registered, and unrestricted security is fully marketable, while a public, registered, but restricted security (subject to Rule 144) is less marketable and a private, unregistered interest is least marketable (most impaired).

Lack of liquidity within a permitting public-trading framework is an intermediate level of impairment leading to a demand for incremental returns on a less liquid security over the returns demanded from a highly liquid security characterized by higher turnover of the instruments. An actively traded stock with large market capitalization is fairly liquid, with high trading volume and a low bid-ask spread. In contrast, an unrestricted, infrequently traded, small capitalization stock, while fully marketable, is relatively illiquid with a high bid-ask spread and low trading volume. Ibbotson and Chen[2] (SBBI 2009 pages 108–109) show, for example, that the returns generated by the least liquid small capitalization stocks were 13.26% greater than the returns demanded of highly liquid stocks of the same size. The liquidity effect appears to dominate size effect, as the difference between returns for small and large stocks (size increasing from S1 to S4 and liquidity increasing from L1 to L4) diminishes rapidly as liquidity increases, in fact reversing for the most liquid portfolios (Table 1).

This difference in mean returns translates to significant discounts for smaller illiquid stocks as compared to the largest, most liquid stocks, which constitute the market

[1]Mukesh Bajaj, David J. Denis, Stephen P. Ferris, and Atulya Sarin, "Firm Value and Marketability Discounts." *Journal of Corporation Law* 27 (2001):89–115.

[2]Ibbotson SBBI 2009 Classic Yearbook, Table 9-2, p. 132, ISBN 978-0-9792402-4-9.

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Table 1
Reported Mean Returns for Size and Liquidity Quartile Portfolios

Size\Liquidity	L1 (%)	L2 (%)	L3 (%)	L4 (%)	L1–L4 (%)
S1	17.35	15.99	12.42	4.09	13.26
S2	16.24	14.10	10.55	4.66	11.58
S3	14.58	13.88	11.98	7.89	6.69
S4	11.89	11.01	10.78	8.47	3.42
S1–S4	5.46	4.98	1.64	−4.38	

Table 2
Valuation Discounts for Size and Liquidity Quartile Portfolios

Size\Liquidity	L1 (%)	L2 (%)	L3 (%)	L4 (%)
S1	51.18	47.03	31.80	−107.09
S2	47.84	39.93	19.72	−81.76
S3	41.91	38.98	29.30	−7.35
S4	28.76	23.07	21.43	0.00

indexes. It is interesting to note that small but the most liquid stocks show a valuation premium as compared to larger and liquid stocks (Table 2).

Marketability does not automatically confer liquidity. Marketable securities display liquidity only in very small lots. Even the largest public stocks display average daily trading volumes less than a fraction of 1% of outstanding stock. Liquidity drops rapidly for larger blocks with rapidly climbing bid-ask spreads, large price impact, and frequent market failure. When a large block of publicly traded stock is valued, this increased bid-ask spread representing the liquidity impairment is called "blockage." A block of unregistered stock in a privately held business suffers from impairment in value from a lack of both marketability and liquidity.

Valuing the ability to liquidate readily an asset is an important and interesting issue for valuation practitioners. Market data–based models can be used to identify the impairment in value due to lack of marketability (e.g., restricted stock studies) and liquidity (option-based models). It is easy to think of cost of liquidity in terms of the bid-ask spread. The market maker sells the stock to the investor at the ask price and offers to buy back the stock at a lower bid price. The seller accepts the lower bid price in exchange for immediacy of execution, in effect exercising a put option, with the exercise price being the bid price and the put premium being the bid-ask spread. The proportional discount from the ask price is thus given by bid-ask spread/ask price. For example, if the ask price is $20 and the bid price is $15, the bid-ask spread (put option) is $5, which represents a discount of 25% from the ask price.

Table 3
Converting from Put Option Premium to Discount for Lack of Liquidity (DLOL)

Put Premium (%)	DLOL (%)
5.00	4.76
10.00	9.09
20.00	16.67
50.00	33.33
100.00	50.00
200.00	66.67
500.00	83.33

Put option–based models are being increasingly employed by practitioners to measure the price risk associated with lack of liquidity. Often, however, the value of a put option premium, estimating the cost of liquidity, is presented incorrectly as the discount for lack of liquidity. This is similar to the merger premium being treated as a discount for lack of control. Neglecting to convert the option premium to the applicable discount creates the illusion that the estimated discounts are greater than 100%, an impossible solution. Stockdale[3] makes an important observation that "Models providing results that equal or exceed 100% may have a theoretical problem because it seems logical that a discount would not reduce a value to zero or less." It appears that this conclusion may have been reached by treating the estimated option premiums as discounts for lack of liquidity. Table 3 presents the equivalent discount for lack of liquidity for calculated put option premium.

The current paper presents the underlying put option models and their correct application to estimating the discounts for lack of liquidity. In order to keep the analysis simple, we assume that the asset does not pay any dividends during the period of illiquidity. The equations presented in this paper can accommodate dividend payments, but they become slightly more complex.

Put option models estimate the price risk borne by an owner during the period of illiquidity. A put is a simple

[3]John J. Stockdale, "A Test of DLOM Computational Models," *Business Valuation Review* 27 (2008):131–137.

contract that allows the holder to liquidate the underlying asset at a predetermined price at a certain date. The price of the option, commonly called the premium, is the present value, at the risk-free rate, of the expected benefit from owning the option at maturity. A put option covers the price risk faced by a holder. There are two components of this price risk: If the price of the asset goes down during the period of illiquidity and the realized price is lower than the price at which the asset was purchased, there is a realized loss. This is the first component and is well understood (LOSS I). Potentially, a second and much larger component of the price risk is the opportunity loss that occurs when the asset increases in price during the period of illiquidity and then declines to a lower value before the asset can be liquidated (LOSS II).

An alternative approach using a collar (combination of buying a put option and selling a call option) has also been proposed to measure the cost of lack of liquidity. Under this approach, the owner sells a covered call option, giving up any and all potential gains from holding the asset, incurring LOSS II, and lowering the cost of covering LOSS I. Further, successful application of this strategy requires an ability to buy and sell large and equal amounts of puts and calls to cover the risk. Historically, the ratio of put and call contracts sold changes rapidly in line with the market expectations. The price of puts and calls would also change in response to the needed incremental buying (puts, higher) and selling (calls, lower), increasing the cost of the collar. This approach explicitly includes incurring LOSS II by taking all potential opportunity losses and, therefore, provides only a partial measure of the cost of liquidity.

The three put option models most commonly used in valuation practice are:

Black-Scholes put (BSP) (Chaffee 1993)[4]
Average price Asian put (AAP) (Finnerty 2002)[5]
Maximum price strike look back put (LBP) (Longstaff 1995)[6]

Each model is briefly described below. In each case, we use the following inputs to illustrate the use of these models for calculating the applicable discount for lack of liquidity.

- Risk-free rate 4%
- annual volatility (standard deviation) 65%
- liquidation period one year

[4]David B. H. Chaffee III, "Option Pricing as a Proxy for Discount for Lack of Marketability in Private Company Valuations—A Working Paper," *Business Valuation Review* (December 1993).
[5]John D. Finnerty, "The Impact of Transfer Restrictions on Stock Prices," *Analysis Group/Economics* (October 2002). The equations and results presented in this paper reflect a correction to the original paper as discussed by that author at the ASA Advanced Business Valuation Conference 2009.
[6]Francis A. Longstaff, "How Much Can Marketability Affect Security Values?" *Journal of Finance* (December 1995).

BSP is a simple contract. It provides protection against any realized loss in value only at maturity of the contract (LOSS I). The minimum value any asset can reach is zero. Therefore, the maximum value payable under a BSP contract is the exercise price for the put. As the time to maturity and volatility of the underlying asset increase, the likelihood of lower asset values being realized and higher option payouts being received increases. The value of the BSP put increases as the price of the underlying asset decreases, but there is no increase in value of the BSP put if the value of the underlying asset increases. Further, since there is an upper bound for the payout (the exercise price), the present value of this bounded payout decreases as the risk-free rate and the time to payout increases; therefore, the option premium for an option on an asset with a fixed level of volatility is a parabola. The price of the option increases as the volatility of the asset and the time to maturity increases, until the maximum likely payout is achieved (projecting the probability of the asset value reaching zero). At the same time as the time to maturity and the risk-free rate increase, the present value of this payout starts declining (in trading parlance, the option decays). BSP provides protection against decline in value of the asset as compared to the current price (LOSS I) but does not address the opportunity cost of not being able to liquidate the asset at the intermediate high price reached but not realized (LOSS II). Put simply, and using any of the various Black-Scholes models available at no charge on the Internet, if the market price and strike price of the put option are both assumed to be 100, with an option term of one year, annual volatility of 65%, a risk-free rate of 4%, and no dividends, the value of the put option at the contract date is 23.07 and the implied discount for lack of liquidity is 18.75%.

The Finnerty model is an application of the AAP. This contract provides a payout based on average price achieved for the asset during the life of the option. The price of the option increases as the volatility of the underlying asset and the time to maturity increases. Initially, the value of an AAP is lower than the corresponding BSP, as the payout is based on the average of gains and losses. It increases slowly with volatility, as the likelihood of achieving lower values and higher values is symmetric. Once the lower bound for negative returns (−100%) is reached, the entire increase in value comes from the potential increase in the value of the underlying asset. For larger values of the asset volatility and the time to maturity, the potential increase in value of the underlying asset dominates the growth in value. The Asian put option provides a partial coverage of the opportunity cost for not being able to liquidate at the higher prices reached during the life of the option, averaging it with the potential losses.

The equation for an Asian put premium for a non-dividend-paying asset is

$$D(T) = V\left[e^{rt}N\left(r/\upsilon\sqrt{T} + \upsilon\sqrt{T/2}\right) - N\left(r/\upsilon\sqrt{T} - \upsilon\sqrt{T/2}\right)\right]$$

and

$$\upsilon^2T = \sigma^2T + \text{Ln}\,[2(e^{\sigma^2T} - \sigma^2T - 1)] - 2\,\text{Ln}\,(e^{\sigma^2T} - 1).$$

Once again setting V to 1, $D(T)$, the option premium, becomes

$$\left[e^{rt}N\left(r/\upsilon\sqrt{T} + \upsilon\sqrt{T/2}\right) - N\left(r/\upsilon\sqrt{T} - \upsilon\sqrt{T/2}\right)\right].$$

Then the corresponding discount for lack of liquidity becomes $D(T)/1 + D(T)$. Using the same inputs as before, the value of the Asian average put becomes 16.78, and the corresponding discount is 14.37%.

The Longstaff model is an application of the look back put (LBP), a contract that pays out based on the highest value for the underlying asset achieved over the lifetime of the option. The price of the option increases as the volatility of the underlying asset and the time to maturity increase. The LBP option addresses the risk of loss in value of the asset as well as provides a full coverage of the opportunity cost for not being able to liquidate at the highest price reached during the life of the option. From the perspective of a buyer of an asset, the price risk is value given up by buying an asset that cannot be liquidated over a defined period of time. LBP is the ultimate no-regret contract as it fully compensates the buyer for the inability to sell during the period of the contract, protecting against a realized loss in value as well as the opportunity cost of not being able to sell at the intermediate high price reached (LOSS I + LOSS II). This property makes it highly desirable for the buyer to demand a discount that fully compensates for the lack of liquidity.

The simplified estimation equation for an LBP involves only two parameters: T, the time to liquidation, and σ, the standard deviation of returns. The equation defines the potential maximum value reached during the period to liquidation as

$$V(2 + \sigma^2T/2)N\left(\sqrt{\sigma^2T/2}\right) + V\sqrt{(\sigma^2T/2\Pi)}e^{(-\sigma 2T/8)}$$

where V is the current value of the asset, T is the time to liquidation, and σ^2 is the variance of returns on the asset.

N^{-} is the cumulative normal distribution

$$F(V,T) = V(2 + \sigma^2T/2)N\left(\sqrt{\sigma^2T/2}\right) + V\sqrt{(\sigma^2T/2\Pi)}e^{(-\sigma 2T/8)} - V.$$

Again, since we are interested in computing a proportional discount for lack of liquidity, we can set V to 1, and the LBP option premium becomes

$$F(T) = (2 + \sigma^2T/2)N\left(\sqrt{\sigma^2T/2}\right) + \sqrt{(\sigma^2T/2\Pi)}e^{(-\sigma 2T/8)} - 1.$$

The corresponding LBP discount for lack of liquidity becomes $F(T)/1 + F(T)$.

Using the same inputs as before, the value of the look back put becomes 63.33, and the corresponding discount is 38.77%.

These theoretical models indicate that for low levels of volatility, BSP discounts will be greater than AAP discounts because the holder of an AAP contract shares in the lower as well as higher prices achieved during the period of illiquidity. However, as volatility increases, AAP discounts will become larger than BSP discounts. LBP discounts will be greater than both the AAP and BSP discounts using the same inputs, because the exercise price of the put is set to the highest value achieved during the period of illiquidity, compensating the holder for both LOSS I and LOSS II, as defined above.

In negotiated sales of illiquid securities, BSP or AAP represents the discount offered by the seller, and LBP represents the discount demanded by the buyer. Buyers would likely demand a price incorporating the higher LBP discount, whereas sellers are likely to offer prices incorporating the lower of the BSP or AAP discount. The negotiated discount would lie in between these bounds based on the relative motivations of the buyer and seller.

The primary advantage of option-based models lies in their adaptability to varying levels of liquidity and volatility. In the valuation of privately held, nonmarketable securities, the level of volatility must be imputed from public market data. Furthermore, the impact of changing market conditions, such as the extreme volatility observed in late 2008 and early 2009, can be readily captured using these models.

Empirical Tests of the Option Models Using FMV Data

The results of empirical testing each of the three models using a subset of FMV studies data are presented in Table 4. Risk-free rates for each transaction date were obtained from the Federal Reserve Economic Data series. Volatilities were calculated using the CRSP data. The aggregate FMV data set covers a time period from July 1980 to March 2005. The range of dispersion is quite wide, and the time period covers different regimes for the required Rule 144 holding periods.

The data set was reduced to a sample of 278 transactions occurring after 1 January 1993 to meet CRSP volume-reporting requirements. Further, eleven transactions were eliminated for lack of trading around the transaction date and resulting inability to develop volatility

Table 4
FMV Data Descriptive Statistics

	Transaction Date	Transaction Discount (%)	Shares Placed	Shares Placed (%)	Volatility (%)
Low	1 July 1980	(29.5)	10,000	0.1	2.8
High	2 March 2005	81.0	25,000,000	43.7	2,024.7
Mean		22.3	1,725,200	11.3	91.6
Median		20.0	1,000,000	9.8	77.0

estimates. A total of 267 transactions remained for which complete data were available.

Observed and estimated lack of liquidity discounts using each of the three models are presented in Table 5. Theoretical relationships observed above are supported by the data. The BSP estimates are much smaller than the observed discounts, and LBP and AAP estimates are closer to the observed discounts.

These results indicate that the AAP and LBP models work for estimating the discounts for lack of liquidity and can be applied across different liquidity and volatility conditions.

In order to test the goodness of fit between the observed (FMV) and estimated discounts for lack of liquidity using the three models, a pairwise t test was performed. The results are presented in Table 6.

Look back put estimates are closest to the reported FMV discounts. This suggests that the discounts observed in the FMV sample may be driven by the buyers and that the transactions are likely to be seller initiated.

These results indicate that option-based models provide appraisers with an empirical tool to estimate applicable discounts for lack of liquidity appropriate for the block size, volatility, and interest rate environment. The results are based on directly observable data from public markets. In the case of privately held blocks of stock, it would be appropriate to use the volatility estimates from the same peer cohort as the one used to develop the required rate of return. In addition to the discount for lack of liquidity, an additional level of discount for lack of marketability will need to be estimated to completely capture the impairment in value due to lack of marketability and liquidity.

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Table 5
Estimated and Reported Discounts

Variable	N	Mean (%)	Standard Deviation (%)	Minimum (%)	Maximum (%)
Black-Scholes Put–Based Discount	267	6.75	5.09	0.01	31.74
Asian Average Put–Based Discount	267	14.95	8.54	2.52	49.06
Look Back Put–Based Discount	267	19.89	10.81	3.34	73.05
FMV Reported Discounts	267	20.29	15.83	−26.09	74.19

Table 6
Pairwise t test for Difference between Sample Means: Null Hypothesis—No Difference between Estimated and Reported Discounts for the Model

Discount Model	N	Mean	Standard Deviation	Standard Error	t Statistic	Degrees of Freedom	Probability $> t$	Null Hypothesis
Black-Scholes Put	267	6.75%	0.0513	0.0031	−14.745	266	$<.0001$	Rejected
Asian Average Put	267	14.95%	0.0855	0.0052	−5.396	266	$<.0001$	Rejected
Look Back Put	267	19.89%	0.1081	0.0066	−0.434	266	0.6647	Not rejected
FMV	267	20.29%	0.1585	0.0097				