A General Option Valuation Approach to Discount for Lack of Marketability

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A general option-based approach to estimating the discount for lack of marketability is offered. It is general enough to capture maturity, volatility, hedging availability, and investor skill as well as other important factors. The model is shown to contain the Chaffe model, the Longstaff model, and the Finnerty model as special cases. The model also contains two weighting variables that provide valuation professionals much needed flexibility in addressing the unique challenges of each non-marketable valuation assignment. Selected prior empirical results are reinterpreted with this approach.

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Numerous financial instruments cannot be easily converted into cash, such as private equity, restricted stock in publicly traded companies, and over-the-counter financial derivatives contracts. This lack of marketability normally results in a lower value when compared to an otherwise equivalent publicly traded instrument. Assigning an appropriate discount for lack of marketability (DLOM) is a challenging task requiring both quantitative analysis and qualitative judgment.

The purpose of this paper is to offer a new tool for the financial professional seeking to address this challenging task. Unlike prior models, the model presented here is general enough to capture maturity, volatility, hedging availability, and investor skill as well as other important factors and yet is easy to interpret and use. Three major contributions are provided here: First, Longstaff's (1995) model is decomposed into two separable components that can be attributed to whether the non-marketable instrument can be hedged and whether the non-marketable instrument owner possess any sort of skill related to this particular instrument (for example, market timing ability). Second, Finnerty's (2012) model is reinterpreted in a manner that can be easily justified in practice. Finally, a general option valuation-based model is introduced that provides an intellectually rigorous framework, yet remains flexible enough to be applied in practice.

According to Abrams (2010), the first tenuous evidence of DLOM can be found in the sale of Joseph for 20 pieces of silver when the going rate was 30, thus a discount of 33 percent.[1] Stockdale (2013) identifies an unnamed federal income tax case in 1934 as the first mention of DLOM. From 1934 through the 1970s, the admissible DLOM averaged between 20 to 30 percent.[2]

Based on a 1969 Securities and Exchange Commission (SEC) document, mutual funds held in excess of $3.2 billion of restricted equity securities, or about 4.4 percent of their total net assets. The

[1] See Abrams (2010) page 301 along with Genesis 37:28 and Exodus 21:32 (Bible). See also Josephus, *Antiquities*, Book XII, Chapter 11 and Leviticus 27:5.
[2] See Stockdale (2013), page 13.

SEC recognized that mutual fund managers would be tempted to report the net asset value of these restricted securities at or near the current publicly traded price, creating an instant gain upon which the manager would be compensated. For example, suppose a company with $100 publicly traded stock offered a mutual fund restricted shares at $75. Upon acquisition, the mutual fund manager would be tempted to assert the value of the restricted shares at $100, creating an instant gain of 33 percent. Thus, the SEC recognized that " ... securities which cannot be readily sold in the public market place are less valuable than securities which can be sold ... ."[3] It was not until 1971 that rigorous academic studies began to appear. The SEC conducted a detailed study of trading data on restricted stock and estimated a DLOM around 26 percent.[4] Based in part on the 1971 SEC study, the Internal Revenue Service issued Revenue Ruling 77-287 that provides some guidance on estimating an appropriate DLOM.

Discount for lack of marketability is a vital consideration in a wide array of financial valuation problems. From establishing fair market value for a business transaction to determining the estate tax liability, the DLOM is a significant factor to be addressed. Based on two different restricted stock data sources, Stockdale (2013) documents the existence of discounts from below –10 percent (premium) to above 90 percent.[5] Finnerty (2013) reports a range of discounts from –79 percent (premium) to 85 percent based on an analysis of 275 private placements of public equity.[6] Clearly, with such an enormous range of values, additional tools to address quantifying DLOM would be helpful.

In the Internal Revenue Service publication, "Discount for Lack of Marketability Job Aid for IRS Valuation Professionals" dated September 25, 2009, the authors make the following standard definitions as well as provide a few important observations: (italics and footnotes in the original)

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[3] See Securities and Exchange Commission (1969), page 3.
[4] See Securities and Exchange Commission (1971).
[5] See Stockdale (2013), Volume 1, page 53.
[6] See Finnerty (2013), Table II, page 583.

> Marketability is defined in the *International Glossary of Business Valuation Terms* as "the ability to quickly convert property to cash at minimal cost"[7]. Some texts go on to add "with a high degree of certainty of realizing the anticipated amount of proceeds".[8]
> A Discount for Lack of Marketability (DLOM) is "an amount or percentage deducted from the value of an ownership interest to reflect the relative absence of marketability."[9]
> ...
> In the alternative [non-marketable instrument], a lesser price is expected for the business interest that cannot be quickly sold and converted to cash. A primary concern driving this price reduction is that, over the uncertain time frame required to complete the sale, the final sale price becomes less certain and with it a decline in value is quite possible. Accordingly, a prudent buyer would want a discount for acquiring such an interest to protect against loss in a future sale scenario.

While there are numerous technical issues and subtle nuances, the goal of the DLOM exercise is to monetize the uncertainty surrounding the lack of marketability. Generally, it is a normative question, what ought to be the DLOM for this specific case? Obviously, there is never direct evidence for a specific case. Valuation professionals use data from numerous sources that provide indirect evidence, such as restricted stock studies. Not surprisingly, given the enormous amount of money involved in DLOM estimation, this indirect evidence often leads to vastly different valuations depending on your objective (for example, IRS or taxpayer). Also, there are many other potential discounts that are not addressed here, including minority interest discount, other transferability restrictions discounts, and nonsystematic risk discounts.

Approaches to estimating DLOM can be generally categorized as either empirical or theoretical. Empirical approaches typically focus on market evidence from restricted stock transactions, various private placements, and private investments in public equities. One then has to

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[7] *International Glossary of Business Valuation Terms*, as adopted in 2001 by American Institute of Certified Public Accountants, American Society of Appraisers, Canadian Institute of Chartered Business Valuators, National Association of Certified Valuation Analysts, and The Institute of Business Appraisers.
[8] Shannon P. Pratt, Alina V. Niculita, *Valuing a Business, The Analysis and Appraisal of Closely Held Businesses*, 5th ed (New York: McGraw Hill, 2008), p. 39.
[9] *International Glossary*.

extrapolate from the market evidence to the particular case at hand. Given the unique attributes of every case, this extrapolation can be quite tenuous.[10]

Theoretical approaches are attractive as they typically provide a parsimonious set of inputs that can be estimated for each DLOM assignment. Theoretical models are generally either based on discounted cash flow models or option valuation models. For examples of the discounted cash flow models, see Meulbroek (2005), Tabak (2002), and Stockdale (2013, Mercer's quantitative marketability discount model, p. 232-238).

In this paper, we build on existing DLOM literature related to option theory to provide a general framework for estimating DLOM that can be used in a wide array of applications. Presently, option-based DLOM approaches are rudimentary and often provide only an upper bound. Our unique approach decomposes existing models into a skill component and a hedge component. The skill component measures the DLOM attributable to the economic value lost for talented investors who suffer solely because the underlying instrument is not marketable. That is, if the underlying instrument were marketable, then this investor would be better off at the end of the non-marketable period. The hedge component measures the DLOM attributable to the inability to hedge adverse market price movements solely because the underlying instrument is not marketable.

For example, the DLOM for restricted stock of a highly skilled CEO would be significantly different from the DLOM of the same restricted stock when estimating the estate taxes if this same CEO passed away.[11] Our decomposition provides valuation professionals needed flexibility to address a wide array of DLOM valuation problems. We demonstrate that our model has existing option-based models as special cases.

The focus here is option valuation-based monetization of the lack of marketability. The remainder of the paper is organized as follows. Section I discusses the relevant option valuation-based

[10] For a concise summary of an extensive set of empirical studies, see Stockdale (2013), pages 47-49.
[11] Assuming the terms of the restriction were not contingent on the CEO being alive.

DLOM literature. In Section II, we introduce our option-based DLOM model and explain its decomposition. Section III illustrates our model as well as provides alternative interpretations of existing empirical evidence. Section IV presents our conclusions.

I. Option Valuation-Based DLOM Literature

There is a vast literature that addresses estimating DLOM. For a review of this literature as well as current DLOM estimation practices, see Stockdale (2013). We focus here solely on option valuation-based DLOM approaches.

A. European style put option

Chaffe (1993) summarizes, "if one holds restricted or non-marketable stock and purchases an option to sell those shares at the free market price, the holder has, in effect, purchased marketability for the shares. The price of the put is the discount for lack of marketability." (p. 182) The acquisition of a put option eliminates the uncertainty of the future downside risk. The shares have essentially been insured against any event that drives the price down. Recall, however, that the put option does not eliminate the benefits that accrue when the share price rises. Thus, this put option is an upper bound for DLOM, at best. Chaffe proposes using the cost of capital for the interest rate in the Black-Scholes-Merton option valuation model (BSMOVM). Chaffe further illustrates the DLOM using volatilities from 60 percent to 90 percent.

Recall the BSMOVM is based on the assumption that the underlying instrument following geometric Brownian motion implying the terminal distribution is lognormal. Remember that distributions are not observable. Often DLOM is estimated for highly volatile instruments and relatively long horizons. The terminal volatility is linear in the square root of maturity time ($\sigma\sqrt{T-t}$). Assuming a one-year horizon, when terminal volatility exceeds 100 percent, the lognormal distribution appears dubious at best. Often estimated volatility exceeds 100 percent. For example, if we have \$100 stock price with \$100 strike price and terminal volatility of 100 percent (with interest rates and dividend yield assumed to be zero), then based on the lognormal distribution assumption, although the mean is \$100, the median is \$61, and the mode is \$22 (skewness is 6.2, and excess kurtosis is 111). Given that stock returns tend to be negatively skewed, these statistics are inconsistent with observed stock price behavior. Although 100 percent terminal volatility seems high, it is equivalent to a four-year (6.25-year) horizon and 50 percent (40 percent) annual volatility. Therefore, DLOM estimates based on option-valuation approaches are used with significant professional judgment.

Several authors have extended Chaffe's work by examining longer maturity options. See, for example, Trout (2003) and Seaman (2005a, 2005b, 2007, 2009).

Interestingly, Chaffe notes "There is also a component of the discount that is related to the inability to realize an intermediate gain quickly and efficiently. For purposes of this analysis, we forego quantification of the discount factor associated with this second aspect of marketability." (p. 182) Longstaff incorporates this aspect with a lookback put option.

B. Lookback put option

Longstaff (1995) captures the intermediate gain issue by assuming the investor has skill or perfect timing ability. He develops an upper bound estimate for DLOM based on a floating strike lookback put option model. Longstaff concludes, "The results of this analysis can be used to provide rough order-of-magnitude estimates of the valuation effects of different types of marketability restrictions. In fact, the empirical evidence suggests that the upper bound may actually be a close approximation to observed discounts for lack of marketability. More importantly, however, these results illustrate that option-pricing techniques can be useful in understanding liquidity in financial markets and that liquidity derivatives have potential as tools for managing and controlling the risk of illiquidity." (p. 1774)

We present a more general version of Longstaff's model. Note Longstaff assumes the lookback maximum is expressed as $M_T = \max_{0 \le \tau \le T}\left[V_\tau e^{r(T-\tau)}\right]$, where V_τ denotes the underlying instrument's value at time τ, the risk-free interest rate is r, and the time to maturity is T. We assume the more traditional case of $\hat{M}_T = \max_{0 \le \tau \le T}[V_\tau]$ and then consider the special case of Longstaff. Longstaff's approach results in the interest rate being absent from the DLOM model.

C. Average-strike put option

Finnerty (2012, 2013) introduces an average-strike put option approach to approximating DLOM. Finnerty assumes the standard dividend adjusted geometric Brownian motion as well as assumes the instrument holder has no special skill (e.g., market-timing ability). Finnerty's model is based on assuming an average forward price that relies on the standard carry arbitrage formula. Finnerty's model effectively provides DLOM estimates roughly 50 percent lower than an equivalent plain vanilla put option (see detailed discussion in the next section). Finnerty (2012) concludes, "The average-strike put option model ... calculates marketability discounts that are generally consistent with the discounts observed in letter stock private placements, although there is a tendency to understate the discount when the stock's volatility is under 45 percent or over 75 percent, especially for longer restriction periods. The marketability discounts implied by observed private placement discounts reflects differences in stock price volatility, as option theory and the average-strike put option model predict." (p. 67)

We now turn to the general option-based DLOM and the flexibility to decompose it into component pieces.

II. Option-Based DLOM and Decomposition

Four major issues are addressed here. First, based on Longstaff's model, we decompose the standard floating strike lookback put option into two components, a plain vanilla put option and a term we define as the residual lookback portion. Second, we apply this decomposition to Longstaff's model and explore its implications by reexamining some of his results. Third, we provide a simple alternative interpretation to Finnerty's model and introduce a weighting system. Finally, we present our general options-based DLOM model and explore some simple, but extreme, cases.

A. Decomposition of Floating Strike Lookback Put Option Model

Let $LP_t\left(V_t,\hat{M}_t\right)$ denote the value at time t of a floating strike lookback put option on the underlying instrument, V_t, where $\hat{M}_t = \max_{0\le\tau\le t}\left[V_\tau\right]$ (note the maximum could have occurred prior to t). Then one way to express the option value is based on two separate components

$$L\hat{P}_t\left(V_t,\hat{M}_t\right) = P_t\left(V_t,\hat{M}_t\right) + L_t\left(V_t,\hat{M}_t\right), \tag{1}$$

where $P_t\left(V_t,\hat{M}_t\right)$ denotes the plain vanilla put portion, and $L_t\left(V_t,\hat{M}_t\right)$ denotes the residual lookback portion. We now examine each portion separately. The general option-based DLOM introduced later will be based on this decomposition.

Component 1: Plain vanilla put portion

Based on the BSMOVM, we have[12]

$$P_t\left(V_t,X\right) = Xe^{-r(T-t)}N\left(-d_2\right) - V_t e^{-\delta(T-t)}N\left(-d_1\right), \tag{2}$$

where

$$d_1 = \frac{\ln\left(\frac{V_t}{X}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{3}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}\,, \tag{4}$$

and

$$N(d) = \int_{-\infty}^{d} n(x)dx = \int_{-\infty}^{d} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$$ (Standard normal cumulative distribution function).

T Time to expiration, measured in years, the option is assumed to be evaluated at time t,

r Known, annualized, continuously compounded "risk-free" interest rate,

[12] See Black and Scholes (1973) and Merton (1973).

δ Known, annualized, continuously compounded dividend yield,

σ Known, annualized standard deviation of continuously compounded percentage change in the underlying instrument's price,

V_t Observed value of the underlying instrument at time t,

X Strike or exercise price, and

P_t Model value of a plain vanilla put option.

As we will discuss later, DLOM is influenced by dividend policy, thus our model captures this influence. Note that if the put option is at the money, then

$$P_t = V_t\left\{e^{-r(T-t)}N\left(-\frac{\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)-e^{-\delta(T-t)}N\left(-\frac{\left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right\}.$$

Further, if $r=0$, then

$$P_t = V_t\left\{N\left(\frac{\left(\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)-e^{-\delta(T-t)}N\left(\frac{\left(\delta-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right\}.$$

If $\delta=0$, then

$$P_t = V_t\left\{e^{-r(T-t)}N\left(-\frac{\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)-N\left(-\frac{\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right\}.$$

Finally, if $r=\delta=0$, then[13]

$$P_t = V_t\left\{2N\left(\frac{\sigma\sqrt{T}}{2}\right)-1\right\}. \tag{5}$$

Component 2: Residual lookback portion

[13] Recall N(–d) = 1 – N(d).

Based on the BSMOVM framework, the residual lookback portion can be expressed as[14]

$$L_t\left(V_t,\hat{M}_t\right)=V_t e^{-r(T-t)}\frac{\sigma^2}{2(r-\delta)}\left[e^{(r-\delta)(T-t)}N\left(d_1\right)-\left(\frac{V_t}{\hat{M}_t}\right)^{-2(r-\delta)/\sigma^2}N\left(d_3\right)\right]\ (r\neq\delta), \tag{6a}$$

$$L_t\left(V_t,\hat{M}_t\right)=V_t e^{-r(T-t)}\left[\frac{\sigma^2(T-t)}{2}N\left(d_1\right)+\sigma\sqrt{T-t}\,n\left(d_1\right)+\ln\left\{\frac{V_t}{\hat{M}_t}\right\}N\left(d_3\right)\right]\ (r=\delta), \tag{6b}$$

where

$$d_3=d_1-\frac{2(r-\delta)\sqrt{T-t}}{\sigma}, \text{ and} \tag{7}$$

$$n(d)=\frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} \text{ (Standard normal probability density function).}$$

Note that if the option is at the money, then

$$L_t=V_t e^{-r(T-t)}\frac{\sigma^2}{2(r-\delta)}\left[e^{(r-\delta)(T-t)}N\left(\frac{\left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)-N\left(-\frac{\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right]\ (r\neq\delta), \text{ and}$$

$$L_t=V_t e^{-r(T-t)}\left[\frac{\sigma^2(T-t)}{2}N\left(\frac{\left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)+\sigma\sqrt{T-t}\,n\left(\frac{\left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right]\ (r=\delta).$$

Again, if $r=0,\delta\neq0$, then

$$L_t=V_t\frac{\sigma^2}{2\delta}\left[e^{-\delta(T-t)}N\left(\frac{\left(-\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)-N\left(\frac{\left(\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right].$$

If $r\neq0,\delta=0$, then

[14] This well-known result can be found in Haug (2007), p. 142, Wilmott (2000), Vol. 1, p. 282, as well as other places. A detailed derivation of the floating strike lookback put is also available from the author.

$$L_t = V_t e^{-r(T-t)} \frac{\sigma^2}{2r}\left[e^{r(T-t)} N\left(\frac{\left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \right) - N\left(-\frac{\left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \right) \right].$$

Finally, if $r = \delta = 0$, then

$$L_t = V_t\left[\frac{\sigma^2(T-t)}{2} N\left(\frac{\sigma\sqrt{T-t}}{2} \right) + \sigma\sqrt{T-t}n\left(\frac{\sigma\sqrt{T-t}}{2} \right) \right]. \tag{8}$$

We will discuss these results in detail when we present the general option-based DLOM model. First, we highlight several new insights from Longstaff's model in the context of this decomposition. After exploring insights related to Finnerty's model, we will introduce the general option-based DLOM model.

B. Decomposition of Longstaff's Model

Note when $V_t = \hat{M}_t$ (at-the-money) and $r = \delta = 0$ (no dividends and no underlying instrument carry cost), then $\hat{M}_T = M_T = \max_{t \le \tau \le T}\left[V_\tau\right]$, and our results are equivalent to Longstaff's model. That is,

$$LP_t(V_t = M_t, M_t) = P_t + L_t, \tag{9}$$

where

$$P_t = V_t\left[2N\left(\frac{\sigma\sqrt{T-t}}{2} \right) - 1 \right] \text{ (Longstaff's plain vanilla put portion), and} \tag{10}$$

$$L_t = V_t\left[\frac{\sigma^2(T-t)}{2} N\left(\frac{\sigma\sqrt{T-t}}{2} \right) + \sigma\sqrt{T-t}n\left(\frac{\sigma\sqrt{T-t}}{2} \right) \right] \text{ (Longstaff's residual lookback portion).} \tag{11}$$

We now explore this decomposition of the Longstaff model. Table I replicates Longstaff's Table I with the addition of the plain vanilla put and residual lookback portions. Note that for low volatility and short horizons roughly half of the lookback put option value is comprised of the plain vanilla put portion, and the other half is the residual lookback portion.

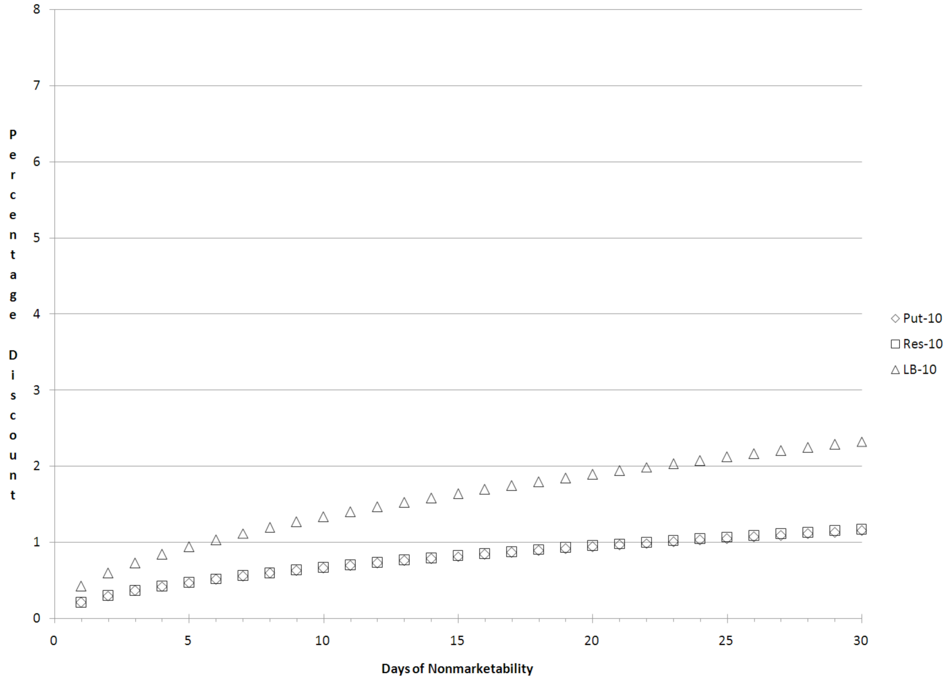
Table I
Lookback Put, Plain Vanilla Put, and Residual Lookback Portion as Percentage of Underlying Value

Marketability Restriction Period	**σ = 1.25%**			**σ = 2.50%**			**σ = 5.0%**		
	PVPut	**RLBP**	**Total**	**PVPut**	**RLBP**	**Total**	**PVPut**	**RLBP**	**Total**
1 Day	0.0263	0.0263	0.0526	0.0526	0.0526	0.1052	0.1051	0.1053	0.2104
5 Days	0.0588	0.0588	0.1176	0.1175	0.1178	0.2353	0.2351	0.2359	0.4710
10 Days	0.0831	0.0832	0.1663	0.1662	0.1667	0.3329	0.3325	0.3341	0.6666
20 Days	0.1175	0.1176	0.2353	0.2351	0.2359	0.4710	0.4702	0.4736	0.9438
30 Days	0.1440	0.1443	0.2882	0.2879	0.2892	0.5771	0.5758	0.5810	1.1569
60 Days	0.2036	0.2042	0.4078	0.4072	0.4098	0.8169	0.8143	0.8248	1.6391
90 Days	0.2493	0.2503	0.4997	0.4987	0.5026	1.0013	0.9973	1.0131	2.0104

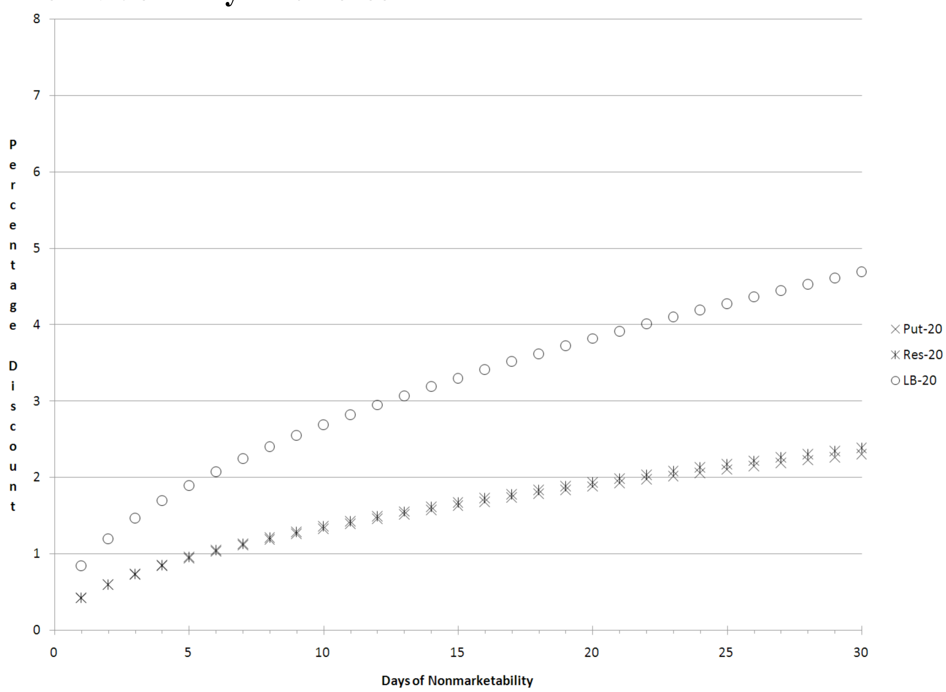
Note: According to Longstaff, the standard deviations of σ = 1.25%, 2.50%, and 5.0% correspond to the approximate historic standard deviations of returns for 6-month, 1-year, and 2-year Treasury securities in the early 1990s. Values expressed as a percent of the underlying instrument. Longstaff appeared to be using a 360-day year; hence, this table assumes the same. PVPut denotes a plain vanilla at the money put option portion, RLBP denotes the residual lookback portion of the lookback put option, and Total denotes an at the money lookback put option expressed as a percentage of the underlying instrument's value.

Figure 1 provides the same results as Longstaff's Figure 1 with the addition of the plain vanilla put and residual components. The results are presented in three panels for clarity. The highest line (– in Panel C) is the percentage DLOM based on Longstaff's model with 30 percent volatility. Note that the residual lookback portion and plain vanilla portion are roughly similar. Clearly, each component increases with days of nonmarketability. Also, the components at 20 percent volatility are roughly identical to Longstaff's DLOM at 10 percent volatility.

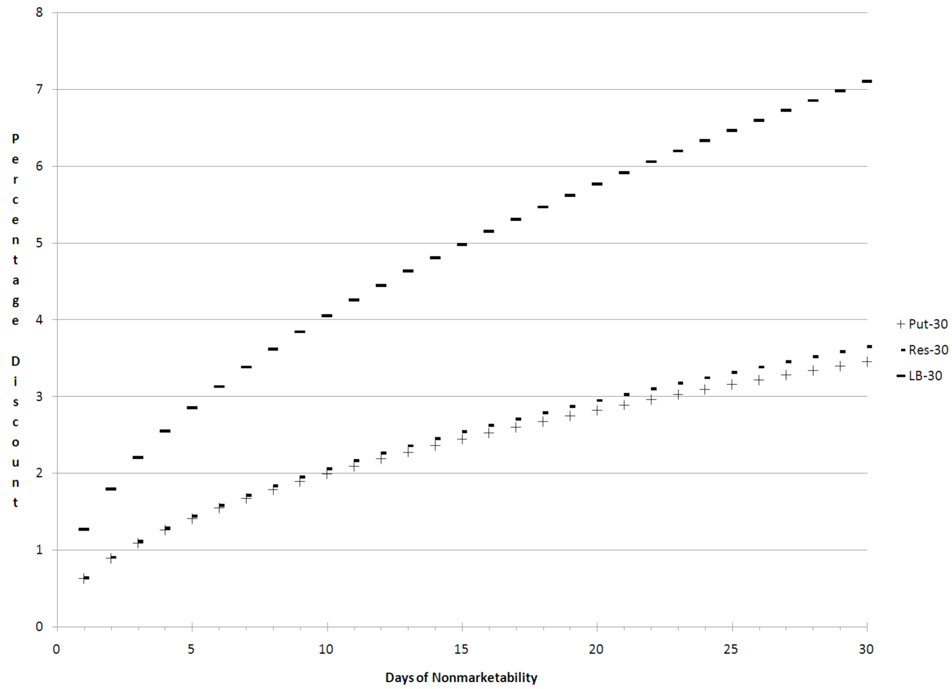
Figure 1.
Lookback Put, Plain Vanilla Put, and Residual Lookback Portion as Percentage of Underlying Value as a Function of Day Until Maturity for Different Volatilities.
Panel A. Volatility = 10 Percent



Panel B. Volatility = 20 Percent



Panel C. Volatility = 30 Percent



Note: LB denotes the lookback put value as a percentage of the underlying, Put denotes the plain vanilla put value as a percentage of the underlying, and Res denotes the residual lookback portion as a percentage of the underlying

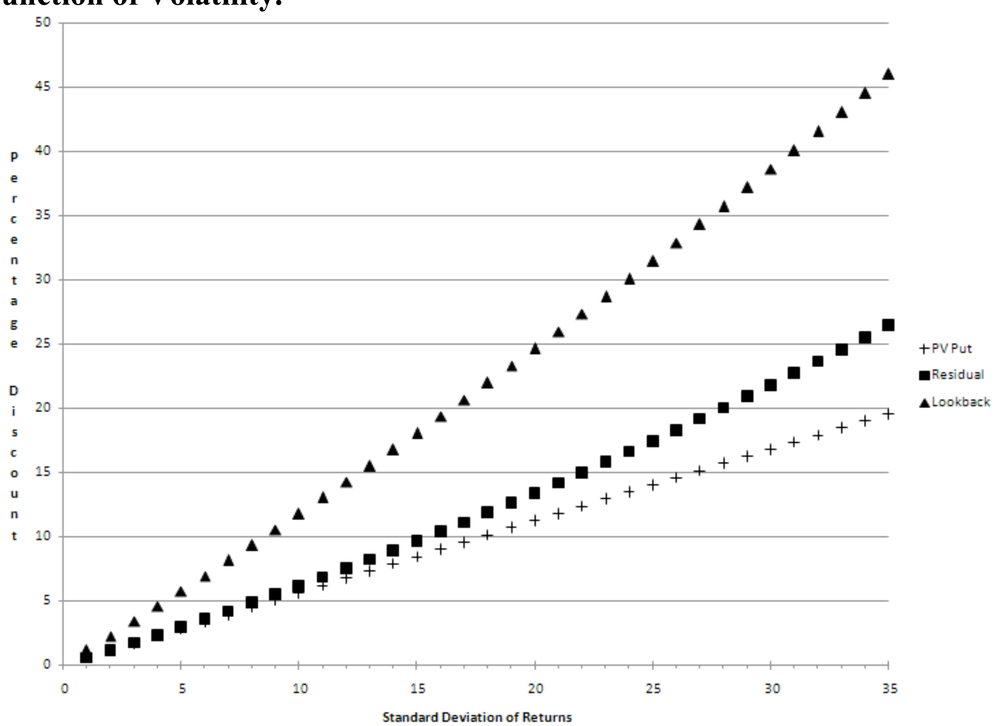
Table II replicates Longstaff's Table II with the addition of the plain vanilla put portions and residual lookback portions. Again, note that roughly half of the lookback put option value is comprised of the plain vanilla put portion, and the other half is the residual lookback portion. With longer time to maturity and higher volatilities, we see that the residual lookback portion is significantly greater than the plain vanilla put portion as illustrated in Figure 2. Figure 2 assumes a two-year restriction period consistent with Longstaff's Figure 2.

Table II
Lookback Put, Plain Vanilla Put, and Residual Lookback Portion as Percentage of Underlying Value

Marketability Restriction Period	σ = 10%			σ = 20%			σ = 30%		
	PVPut	**RLBP**	**Total**	**PVPut**	**RLBP**	**Total**	**PVPut**	**RLBP**	**Total**
1 Day	0.210	0.211	0.421	0.421	0.423	0.844	0.631	0.637	1.268
5 Days	0.470	0.474	0.944	0.940	0.954	1.895	1.410	1.442	2.852
10 Days	0.665	0.672	1.337	1.330	1.358	2.688	1.995	2.058	4.052
20 Days	0.940	0.954	1.895	1.880	1.937	3.817	2.820	2.948	5.768
30 Days	1.152	1.173	2.324	2.303	2.388	4.691	3.454	3.646	7.100
60 Days	1.629	1.671	3.299	3.256	3.427	6.683	4.883	5.270	10.153
90 Days	1.995	2.058	4.052	3.988	4.244	8.232	5.979	6.563	12.542
180 Days	2.820	2.948	5.768	5.637	6.156	11.793	8.447	9.635	18.082
1 Year	3.988	4.244	8.232	7.966	9.019	16.984	11.924	14.353	26.276
2 Years	5.637	6.156	11.793	11.246	13.396	24.643	16.800	21.805	38.605
5 Years	8.902	10.226	19.128	17.694	23.285	40.979	26.268	39.503	65.772

Note: According to Longstaff, the standard deviations of σ = 10%, 20%, and 30% correspond to the approximate historic standard deviations of equity securities." Values expressed as a percent of the underlying instrument. Longstaff appeared to be using a 360-day year; hence, this table assumes the same. PVPut denotes a plain vanilla at the money put option portion, RLBP denotes the residual lookback portion of the lookback put option, and Total denotes an at the money lookback put option expressed as a percentage of the underlying instrument's value.

Figure 2.
Lookback Put, Plain Vanilla Put, and Residual Lookback Portion as Percentage of Underlying Value as a Function of Volatility.



Note: Lookback denotes the lookback put value as a percentage of the underlying, PV Put denotes the plain vanilla put value as a percentage of the underlying, and Residual denotes the residual lookback portion as a percentage of the underlying

We exploit this decomposition to construct a more flexible DLOM tool. Next, however, we explore an alternative interpretation of Finnerty's model.

C. Alternative Interpretation of Finnerty's Model

Based on our notation and some rearranging, Finnerty's DLOM model when expressed in dollars is

$$DLOM_{\$,Finnerty} = V_t e^{-\delta(T-t)}\left[2N\left(\frac{\nu\sqrt{T-t}}{2}\right)-1\right]. \tag{12}$$

where[15]

$$\nu^2(T-t) = \sigma^2(T-t) + \ln\left\{2\left(e^{\sigma^2(T-t)} - \sigma^2(T-t) - 1\right)\right\} - 2\ln\left\{e^{\sigma^2(T-t)} - 1\right\} < \ln\{2\}. \tag{13}$$

Due to the upper limit on the volatility time term ($\nu^2(T-t) < \ln\{2\}$), the upper limit on Finnerty's DLOM model is ($\delta = 0$)

$$DLOM_{\%,Finnerty} = 2N\left(\frac{\sqrt{\ln\{2\}}}{2}\right) - 1 = 32.28\%.$$

Often DLOM is observed to exceed this upper limit, so Finnerty's model is limited.

Recall Finnerty assumes the residual lookback portion is zero. Consider a plain vanilla put option with a strike price equal to the at-the-forward price $X = F_t = V_t e^{(r-\delta)(T-t)}$; thus, we have[16]

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[15] Note that $\ln\left\{2\left(e^{\sigma^2(T-t)} - \sigma^2(T-t) - 1\right)\right\} = \ln\{2\} + \ln\left\{e^{\sigma^2(T-t)} - \sigma^2(T-t) - 1\right\}$ and the limit as $\sigma^2(T-t) \to \infty$ is $\ln\left\{e^{\sigma^2(T-t)} - 1\right\} \to \sigma^2(T-t)$ and $\ln\left\{e^{\sigma^2(T-t)} - \sigma^2(T-t) - 1\right\} \to \sigma^2(T-t)$.

[16] We assume a fully arbitraged market; hence, the equilibrium forward price is as expressed above. Clearly, in markets where arbitrage activity is limited, then this forward expression is not appropriate. Most small capitalization stocks do not have an active forward market, limiting this model's realism.

$$P_t\left(V_t, X = V_t e^{(r-\delta)(T-t)}\right) = Xe^{-r(T-t)}N(-d_2) - V_t e^{-\delta(T-t)}N(-d_1)$$
$$= V_t e^{(r-\delta)(T-t)}e^{-r(T-t)}N(-d_2) - V_t e^{-\delta(T-t)}N(-d_1) = V_t e^{-\delta(T-t)}\left[N(-d_2) - N(-d_1)\right],$$

where

$$d_1 = \frac{\ln\left(\frac{V_t}{V_t e^{(r-\delta)(T-t)}}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\sigma\sqrt{T-t}}{2}.$$

$$d_2 = -\frac{\sigma\sqrt{T-t}}{2},$$

or

$$P_t\left(V_t, X = V_t e^{(r-\delta)(T-t)}\right) = V_t e^{-\delta(T-t)}\left[N\left(\frac{\sigma\sqrt{T-t}}{2}\right) - N\left(-\frac{\sigma\sqrt{T-t}}{2}\right)\right]$$
$$= V_t e^{-\delta(T-t)}\left[2N\left(\frac{\sigma\sqrt{T-t}}{2}\right) - 1\right].$$

This simple plain vanilla at-the-forward put option result is equivalent to Finnerty's model except for the volatility term. There are two alternative interpretations for this model. First, note that $\sigma > \nu > 0$ when volatility is positive and time to maturity is positive. Thus, let $\pi_\nu \equiv \frac{\nu}{\sigma}$ denote the proportion of Finnerty's parameter when compared with the underlying instrument's volatility. Thus,

$$DLOM_{\$,Finnerty} = V_t e^{-\delta(T-t)}\left[2N\left(\frac{\pi_\nu \sigma\sqrt{T-t}}{2}\right) - 1\right]. \quad (14)$$

Table III presents this proportion for various underlying volatilities and maturities. But for very low volatility, short maturities and high volatility, long maturities the proportion is relatively stable around 57 percent. Based on Finnerty's assertion regarding letter stock, then the residual lookback put portion overstates the DLOM, and only a portion of the plain vanilla put option portion is reflected in DLOM. This interpretation makes sense because a put option provides more than just marketability; it provides protection from unanticipated declines in value.

Table III.
Finnerty's Model Volatility as Percentage of Underlying Volatility

Vol\Mat	0.1	0.2	0.3	0.4	0.5	1	2	3	4	5	10	30	100
1%	9.1%	58.1%	57.6%	57.9%	57.6%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%
5%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.6%	57.4%	56.5%
10%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.6%	57.6%	57.5%	57.5%	57.3%	56.3%	52.9%
15%	57.7%	57.7%	57.7%	57.7%	57.7%	57.6%	57.5%	57.4%	57.3%	57.2%	56.6%	54.5%	47.0%
20%	57.7%	57.7%	57.7%	57.7%	57.6%	57.5%	57.3%	57.2%	57.0%	56.8%	55.8%	51.9%	39.8%
30%	57.7%	57.6%	57.6%	57.6%	57.5%	57.3%	56.9%	56.4%	56.0%	55.6%	53.4%	45.0%	27.7%
40%	57.7%	57.6%	57.5%	57.4%	57.3%	57.0%	56.2%	55.4%	54.6%	53.9%	50.0%	37.1%	20.8%
60%	57.6%	57.4%	57.2%	57.0%	56.9%	56.0%	54.2%	52.5%	50.8%	49.0%	41.3%	25.3%	13.9%
80%	57.4%	57.1%	56.8%	56.5%	56.2%	54.6%	51.5%	48.5%	45.6%	42.9%	32.7%	19.0%	10.4%
100%	57.3%	56.8%	56.3%	55.8%	55.3%	52.9%	48.1%	43.7%	39.8%	36.5%	26.3%	15.2%	8.3%
150%	56.6%	55.6%	54.5%	53.4%	52.3%	47.0%	38.1%	31.9%	27.7%	24.8%	17.6%	10.1%	5.6%

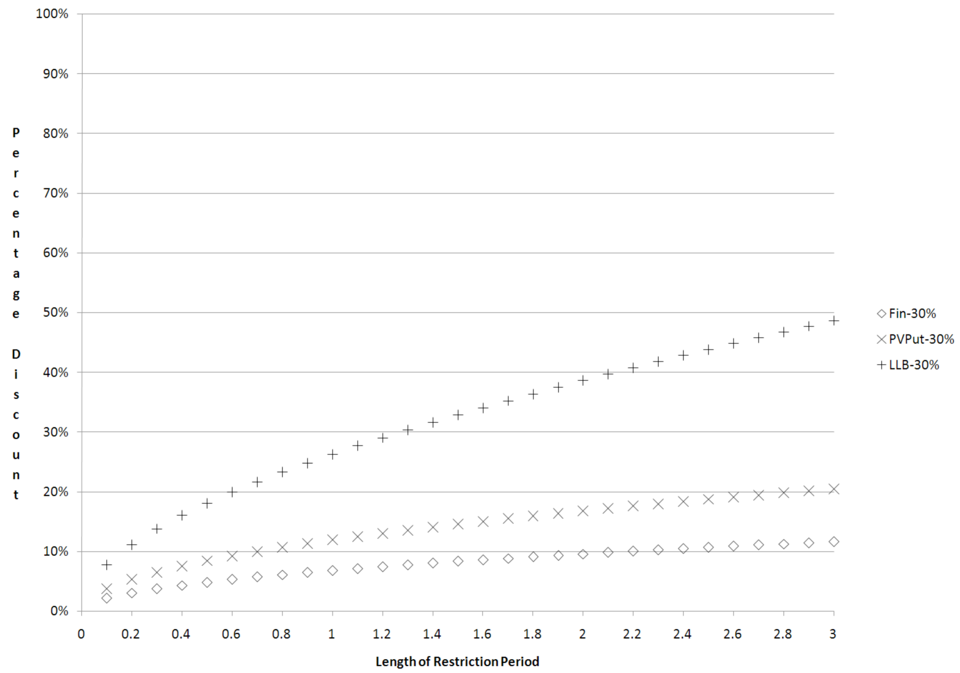
Table IV presents the proportion of Finnerty's DLOM value with respect to the equivalent plain vanilla option value for various underlying volatilities and maturities. The results are strikingly similar. For very high volatility, long maturities, the proportion is much higher than the results in Table III.

Table IV.
Finnerty's Model Values as Percentage of Plain Vanilla Put Option Values

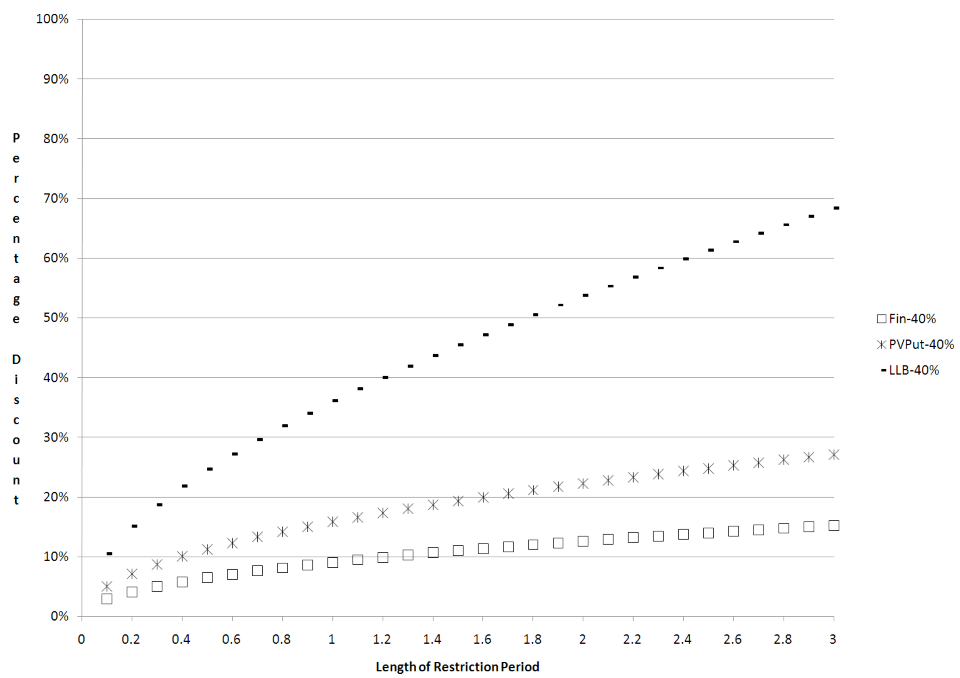
Vol\Mat	0.1	0.2	0.3	0.4	0.5	1	2	3	4	5	10	30	100
1%	9.1%	58.1%	57.6%	57.9%	57.6%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%
5%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.5%	56.9%
10%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.6%	57.6%	57.6%	57.4%	56.8%	54.5%
15%	57.7%	57.7%	57.7%	57.7%	57.7%	57.7%	57.6%	57.5%	57.4%	57.4%	57.0%	55.5%	50.4%
20%	57.7%	57.7%	57.7%	57.7%	57.7%	57.6%	57.5%	57.3%	57.2%	57.1%	56.4%	53.8%	45.3%
30%	57.7%	57.7%	57.6%	57.6%	57.6%	57.4%	57.2%	56.9%	56.6%	56.3%	54.8%	49.0%	37.2%
40%	57.7%	57.6%	57.6%	57.5%	57.5%	57.2%	56.7%	56.2%	55.6%	55.1%	52.5%	43.4%	33.8%
60%	57.6%	57.5%	57.4%	57.3%	57.2%	56.6%	55.4%	54.2%	53.0%	51.8%	46.4%	35.9%	32.4%
80%	57.5%	57.3%	57.1%	56.9%	56.7%	55.6%	53.5%	51.4%	49.4%	47.5%	40.4%	33.2%	32.3%
100%	57.4%	57.1%	56.8%	56.4%	56.1%	54.5%	51.2%	48.1%	45.3%	43.0%	36.4%	32.5%	32.3%
150%	57.0%	56.3%	55.5%	54.8%	54.0%	50.4%	44.1%	39.9%	37.2%	35.6%	32.9%	32.3%	32.3%

Figure 3 replicates Finnerty's Exhibit 2 with the inclusion of the plain vanilla put option model. The results are again presented in three panels for clarity. Clearly, Longstaff's lookback put option model results in much higher discounts when compared to the other two models. Finnerty's model is always less than the plain-vanilla put option model as his model is equivalent to using a proportionally lower volatility (π_v).

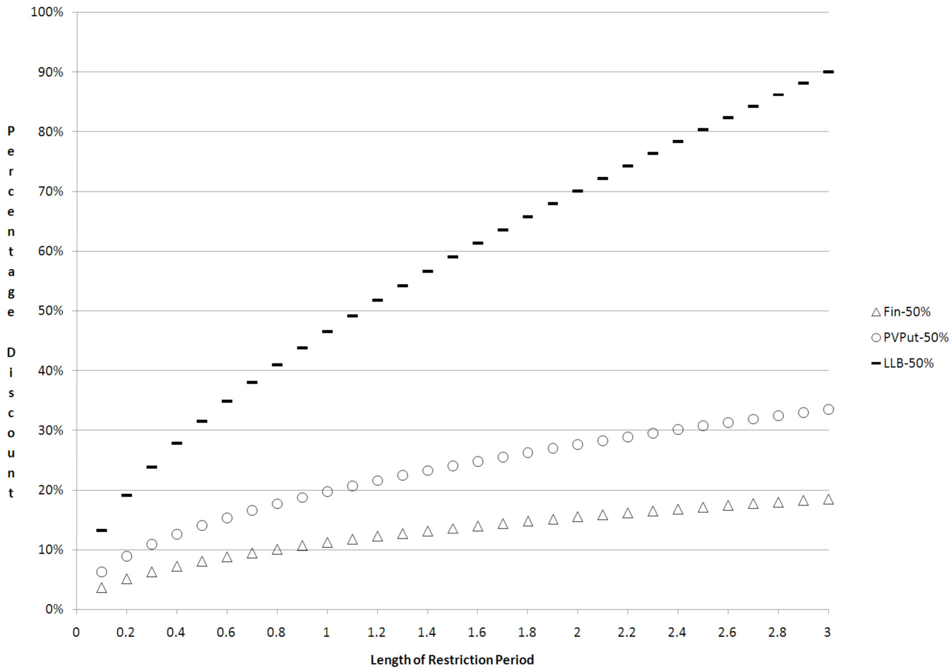
Figure 3.
Comparison of Finnerty's Model (Fin), Plain Vanilla Put Model (PVPut), and Longstaff's Lookback Put Model (LLB)
Panel A. Volatility = 30 Percent



Panel B. Volatility = 40 Percent



Panel C. Volatility = 50 Percent



Note: Fin denotes Finnerty's Model, PVPut denotes the plain vanilla put option model, and LLB denotes the Longstaff lookback put option model. 30%, 40%, and 50% denotes input volatility of the underlying instrument

Building on previous theoretical work and empirical insights, we now propose a general options-based DLOM model.

D. General Options-Based DLOM

We introduce a weighting scheme that is general enough to handle most DLOM problems. First, the degree of external hedging opportunities will directly influence DLOM. If hedging opportunities could be pursued, then the DLOM applies only to the proportion of the underlying instrument's volatility that *cannot* be hedged. Recall if 100 percent of the volatility can be hedged, then the DLOM should be zero.[17] That is, if a put option is available to purchase that eliminates the non-marketable instrument's downside risk, then a call option is likely also available to sell that will offset the cost of the put option and guarantee the future sale price.[18] We assume the external hedging

[17] Ignoring the skill argument made below.
[18] This is a simple illustration of the well-known put-call parity relationship.

opportunities are not specific to the investor. Clearly, if a particular investor is excluded from available hedging opportunities due to regulation or corporate culture, then this portion of the DLOM applies.

Second, the degree of investor skill will also directly influence DLOM. If a particular investor evidences some capacity for say market timing, then the lack of marketability imposes a significant expense. Clearly, perfect skill with active trading would result in near infinite profits in a short period of time within the standard option valuation paradigm. Therefore, we propose the following general options-based DLOM (express as a percentage of the underlying instrument's fully marketable value):

$$\mathrm{DLOM}_{i,t} = w_{\mathrm{DLOM},i,t}\frac{L\hat{P}_t}{V_t} = w_{\neg\mathrm{Hedge},t}\frac{P_t}{V_t} + w_{\mathrm{Skill},i,t}\frac{L_t}{V_t}, \tag{15}$$

where $L\hat{P}_t$, P_t, and L_t are as defined in equations (1), (2), and (6), respectively, and

$w_{\neg\mathrm{Hedge},t}$ denotes the investor *independent* proportion of the underlying instrument that cannot be hedged,

$w_{\mathrm{Skill},i,t}$ denotes the investor *dependent* proportion of the underlying instrument that reflects the investor's skill (e.g., market timing), and

$w_{\mathrm{DLOM},i,t}$ denotes the investor *dependent* proportion of the underlying instrument that reflects the aggregate of the investor's skill and underlying instrument's degree of hedging.

Note that purchasing a put option is an upper bound on DLOM when the investor is not assumed to have skill. Thus, $w_{\neg\mathrm{Hedge},t}$ also provides an adjustment mechanism to account for this over-estimate. We now consider a few extreme examples to clarify the model.

First, if the non-marketable underlying instrument can be completely hedged and the individual owner has no skill, then the DLOM is zero ($w_{\neg\mathrm{Hedge},t} = 0\%$, $w_{\mathrm{Skill},i,t} = 0\%$, thus $w_{\mathrm{DLOM},i,t} = 0\%$).

Second, now suppose that the non-marketable underlying instrument cannot be hedged at all and the individual owner has no skill, then the upper bound on DLOM is $\mathrm{DLOM}_{i,t} = \frac{P_t}{V_t}$ and

$w_{DLOM,i,t} = \frac{P_t}{L\hat{P}_t}$. This is an upper bound because purchasing a put option in this case provides more than just marketability.

Third, now suppose that the non-marketable underlying instrument can be completely hedged but the individual owner has perfect skill, though can trade only once, then the upper bound on DLOM is $DLOM_{i,t} = \frac{L_t}{V_t}$ and $w_{DLOM,i,t} = \frac{L_t}{L\hat{P}_t}$. This DLOM estimate reflects the tremendous value afforded to those with skill. This perspective explains, in part, why some private placements occur at a premium.[19] If the investor skill is unique to a particular set of underlying instruments, then the investor may be willing to pay a premium for the instrument for the opportunity to exercise her skill. The value to the investor of the position is $V_t + L_t$ and there may be times when the particular investor cannot acquire the instrument in the traded market. We focus here solely on DLOM.

Fourth, now suppose that the non-marketable underlying instrument cannot be hedged and the individual owner has perfect skill, but can trade only once, then the upper bound on DLOM is Longstaff's model, or with our notation $DLOM_{i,t} = \frac{L\hat{P}_t}{V_t}$ and $w_{DLOM,i,t} = 1$.

Finally, once the DLOM has been estimated with the appropriate weights for the inability to hedge and skill, then the weight applied for DLOM overall is simply $w_{DLOM,i,t} = \frac{DLOM_{i,t}}{\left[\frac{L\hat{P}_t}{V_t}\right]}$. Alternatively, if DLOM is estimated using some other non-option methodology, then the option-based weight can be calculated for an independent rationality check.

We now provide a few illustrative scenarios and draw some insights from prior empirical studies.

[19] More likely explanations for observed premiums include joint ventures, rapidly changing prices, and pricing date uncertainty. The author thanks John Stockdale for this insight.

III. Illustrations and Reinterpretations of Prior Empirical Evidence

A. Illustrative Scenarios

Table V presents a detailed table of DLOM based on Longstaff's framework and the weighting scheme of our model. Recall Longstaff's model is equivalent to assuming a floating strike lookback put option and $r = \delta = 0\%$. Longstaff's model is the specific case of $w_{\neg Hedge,t} = 100\%$, $w_{Skill,i,t} = 100\%$ in the table. Table V varies both weights where w(S) denotes $w_{Skill,i,t}$ and w(NH) denotes $w_{\neg Hedge,t}$. Note that for high volatility, longer maturities, and higher weights, the implied DLOM exceeds 100 percent. The implication is that the instrument that lacks marketability becomes a liability and not an asset. Clearly, in almost all cases, the DLOM implying a liability is erroneous. Therefore, applying this weighting scheme affords a degree of flexibility needed by valuation professionals.

Table V.
Illustration of the General Options-Based DLOM Assuming $r = \delta = 0\%$.

Volatililty	20%	40%	60%	80%	20%	40%	60%	80%	20%	40%	60%	80%	20%	40%	60%	80%	20%	40%	60%	80%
Years	1	1	1	1	2	2	2	2	3	3	3	3	5	5	5	5	10	10	10	10
PV Put	$7.97	$15.85	$23.58	$31.08	$11.25	$22.27	$32.86	$42.84	$13.75	$27.10	$39.67	$51.16	$17.69	$34.53	$49.77	$62.89	$24.82	$47.29	$65.72	$79.41
Residual LB	$9.02	$20.28	$34.01	$50.44	$13.40	$31.46	$54.85	$84.17	$17.03	$41.28	$73.93	$116.04	$23.29	$59.19	$110.13	$178.15	$36.48	$100.23	$197.41	$332.40
w(S)/w(NH)	1/20%	1/40%	1/60%	1/80%	2/20%	2/40%	2/60%	2/80%	3/20%	3/40%	3/60%	3/80%	5/20%	5/40%	5/60%	5/80%	10/20%	10/40%	10/60%	10/80%
0%/0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
0%/25%	2%	4%	6%	8%	3%	6%	8%	11%	3%	7%	10%	13%	4%	9%	12%	16%	6%	12%	16%	20%
0%/50%	4%	8%	12%	16%	6%	11%	16%	21%	7%	14%	20%	26%	9%	17%	25%	31%	12%	24%	33%	40%
0%/75%	6%	12%	18%	23%	8%	17%	25%	32%	10%	20%	30%	38%	13%	26%	37%	47%	19%	35%	49%	60%
0%/100%	8%	16%	24%	31%	11%	22%	33%	43%	14%	27%	40%	51%	18%	35%	50%	63%	25%	47%	66%	79%
25%/0%	2%	5%	9%	13%	3%	8%	14%	21%	4%	10%	18%	29%	6%	15%	28%	45%	9%	25%	49%	83%
25%/25%	4%	9%	14%	20%	6%	13%	22%	32%	8%	17%	28%	42%	10%	23%	40%	60%	15%	37%	66%	103%
25%/50%	6%	13%	20%	28%	9%	19%	30%	42%	11%	24%	38%	55%	15%	32%	52%	76%	22%	49%	82%	123%
25%/75%	8%	17%	26%	36%	12%	25%	38%	53%	15%	31%	48%	67%	19%	41%	65%	92%	28%	61%	99%	143%
25%/100%	10%	21%	32%	44%	15%	30%	47%	64%	18%	37%	58%	80%	24%	49%	77%	107%	34%	72%	115%	163%
50%/0%	5%	10%	17%	25%	7%	16%	27%	42%	9%	21%	37%	58%	12%	30%	55%	89%	18%	50%	99%	166%
50%/25%	7%	14%	23%	33%	10%	21%	36%	53%	12%	27%	47%	71%	16%	38%	68%	105%	24%	62%	115%	186%
50%/50%	8%	18%	29%	41%	12%	27%	44%	64%	15%	34%	57%	84%	20%	47%	80%	121%	31%	74%	132%	206%
50%/75%	10%	22%	35%	49%	15%	32%	52%	74%	19%	41%	67%	96%	25%	55%	92%	136%	37%	86%	148%	226%
50%/100%	12%	26%	41%	56%	18%	38%	60%	85%	22%	48%	77%	109%	29%	64%	105%	152%	43%	97%	164%	246%
75%/0%	7%	15%	26%	38%	10%	24%	41%	63%	13%	31%	55%	87%	17%	44%	83%	134%	27%	75%	148%	249%
75%/25%	9%	19%	31%	46%	13%	29%	49%	74%	16%	38%	65%	100%	22%	53%	95%	149%	34%	87%	164%	269%
75%/50%	11%	23%	37%	53%	16%	35%	58%	85%	20%	45%	75%	113%	26%	62%	107%	165%	40%	99%	181%	289%
75%/75%	13%	27%	43%	61%	18%	40%	66%	95%	23%	51%	85%	125%	31%	70%	120%	181%	46%	111%	197%	309%
75%/100%	15%	31%	49%	69%	21%	46%	74%	106%	27%	58%	95%	138%	35%	79%	132%	197%	52%	122%	214%	329%
100%/0%	9%	20%	34%	50%	13%	31%	55%	84%	17%	41%	74%	116%	23%	59%	110%	178%	36%	100%	197%	332%
100%/25%	11%	24%	40%	58%	16%	37%	63%	95%	20%	48%	84%	129%	28%	68%	123%	194%	43%	112%	214%	352%
100%/50%	13%	28%	46%	66%	19%	43%	71%	106%	24%	55%	94%	142%	32%	76%	135%	210%	49%	124%	230%	372%
100%/75%	15%	32%	52%	74%	22%	48%	80%	116%	27%	62%	104%	154%	37%	85%	147%	225%	55%	136%	247%	392%
100%/100%	17%	36%	58%	82%	25%	54%	88%	127%	31%	68%	114%	167%	41%	94%	160%	241%	61%	148%	263%	412%

Note: w(S) denotes $w_{Skill,i,t}$, w(NH) denotes $w_{\neg Hedge,t}$, PV Put denotes the plain vanilla put option model value, Residual LB denotes the residual lookback put option portion, and 1/20% denotes 1 year to maturity with volatility of 20%.

Table VI again presents a detailed table of DLOM based on our general options-based framework and the weighting scheme of our model. Table VI differs from Table V only by assuming a positive interest rate of 5 percent ($r = 5\%, \delta = 0\%$). Note that for longer maturities, the plain vanilla put option actually declines in value due to the effect of discounting on put options (e.g., compare 5/80% PV Put of $45.29 with 10/80% PV Put of $44.80). Also, even though the residual lookback portion is monotonically increasing with maturity, the discounting effect lowers the value (e.g., compare 5/80% Residual LB of $157.49 with 10/80% Residual LB of $261.35).

Table VI.
Illustration of the General Options-Based DLOM Assuming $r = 5\%, \delta = 0\%$.

Volatililty	20%	40%	60%	80%	20%	40%	60%	80%	20%	40%	60%	80%	20%	40%	60%	80%	20%	40%	60%	80%
Years	1	1	1	1	2	2	2	2	3	3	3	3	5	5	5	5	10	10	10	10
PV Put	$5.57	$13.15	$20.65	$27.94	$6.61	$16.77	$26.75	$36.19	$7.00	$18.81	$30.29	$40.87	$7.02	$20.76	$33.86	$45.29	$5.85	$20.81	$34.42	$44.80
Residual LB	$8.72	$19.74	$33.14	$49.18	$12.53	$29.84	$52.13	$80.05	$15.43	$38.15	$68.54	$107.68	$19.84	$52.01	$97.23	$157.49	$26.93	$78.06	$154.92	$261.35
w(S)/w(NH)	1/20%	1/40%	1/60%	1/80%	2/20%	2/40%	2/60%	2/80%	3/20%	3/40%	3/60%	3/80%	5/20%	5/40%	5/60%	5/80%	10/20%	10/40%	10/60%	10/80%
0%/0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
0%/25%	1%	3%	5%	7%	2%	4%	7%	9%	2%	5%	8%	10%	2%	5%	8%	11%	1%	5%	9%	11%
0%/50%	3%	7%	10%	14%	3%	8%	13%	18%	3%	9%	15%	20%	4%	10%	17%	23%	3%	10%	17%	22%
0%/75%	4%	10%	15%	21%	5%	13%	20%	27%	5%	14%	23%	31%	5%	16%	25%	34%	4%	16%	26%	34%
0%/100%	6%	13%	21%	28%	7%	17%	27%	36%	7%	19%	30%	41%	7%	21%	34%	45%	6%	21%	34%	45%
25%/0%	2%	5%	8%	12%	3%	7%	13%	20%	4%	10%	17%	27%	5%	13%	24%	39%	7%	20%	39%	65%
25%/25%	4%	8%	13%	19%	5%	12%	20%	29%	6%	14%	25%	37%	7%	18%	33%	51%	8%	25%	47%	77%
25%/50%	5%	12%	19%	26%	6%	16%	26%	38%	7%	19%	32%	47%	8%	23%	41%	62%	10%	30%	56%	88%
25%/75%	6%	15%	24%	33%	8%	20%	33%	47%	9%	24%	40%	58%	10%	29%	50%	73%	11%	35%	65%	99%
25%/100%	8%	18%	29%	40%	10%	24%	40%	56%	11%	28%	47%	68%	12%	34%	58%	85%	13%	40%	73%	110%
50%/0%	4%	10%	17%	25%	6%	15%	26%	40%	8%	19%	34%	54%	10%	26%	49%	79%	13%	39%	77%	131%
50%/25%	6%	13%	22%	32%	8%	19%	33%	49%	9%	24%	42%	64%	12%	31%	57%	90%	15%	44%	86%	142%
50%/50%	7%	16%	27%	39%	10%	23%	39%	58%	11%	28%	49%	74%	13%	36%	66%	101%	16%	49%	95%	153%
50%/75%	9%	20%	32%	46%	11%	27%	46%	67%	13%	33%	57%	84%	15%	42%	74%	113%	18%	55%	103%	164%
50%/100%	10%	23%	37%	53%	13%	32%	53%	76%	15%	38%	65%	95%	17%	47%	82%	124%	19%	60%	112%	175%
75%/0%	7%	15%	25%	37%	9%	22%	39%	60%	12%	29%	51%	81%	15%	39%	73%	118%	20%	59%	116%	196%
75%/25%	8%	18%	30%	44%	11%	27%	46%	69%	13%	33%	59%	91%	17%	44%	81%	129%	22%	64%	125%	207%
75%/50%	9%	21%	35%	51%	13%	31%	52%	78%	15%	38%	67%	101%	18%	49%	90%	141%	23%	69%	133%	218%
75%/75%	11%	25%	40%	58%	14%	35%	59%	87%	17%	43%	74%	111%	20%	55%	98%	152%	25%	74%	142%	230%
75%/100%	12%	28%	46%	65%	16%	39%	66%	96%	19%	47%	82%	122%	22%	60%	107%	163%	26%	79%	151%	241%
100%/0%	9%	20%	33%	49%	13%	30%	52%	80%	15%	38%	69%	108%	20%	52%	97%	157%	27%	78%	155%	261%
100%/25%	10%	23%	38%	56%	14%	34%	59%	89%	17%	43%	76%	118%	22%	57%	106%	169%	28%	83%	164%	273%
100%/50%	12%	26%	43%	63%	16%	38%	66%	98%	19%	48%	84%	128%	23%	62%	114%	180%	30%	88%	172%	284%
100%/75%	13%	30%	49%	70%	17%	42%	72%	107%	21%	52%	91%	138%	25%	68%	123%	191%	31%	94%	181%	295%
100%/100%	14%	33%	54%	77%	19%	47%	79%	116%	22%	57%	99%	149%	27%	73%	131%	203%	33%	99%	189%	306%

Note: w(S) denotes $w_{Skill,i,t}$, w(NH) denotes $w_{\neg Hedge,t}$, PV Put denotes the plain vanilla put option model value, Residual LB denotes the residual lookback put option portion, and 1/20% denotes 1 year to maturity with volatility of 20%.

Table VII differs from Table VI only by assuming a positive dividend yield of 5 percent and a zero interest rate ($r = 0\%, \delta = 5\%$). Note that the plain vanilla put option significantly increases in value due to the effects of the dividend yield on put options. Also, the residual lookback portion values are the same as Table VI. Dividends for DLOM are fundamentally different from dividends for exchange-traded put options. Exchange-traded options are not adjusted for cash dividends, but DLOM applied to valuations do include the dividend over the restricted period. One easy way to handle incorporating dividends is to treat each anticipated cash dividend payment over the restricted period as a separate valuation complete with its own DLOM estimation. Thus, a quarterly pay dividend policy for a two year restricted stock would involve possibly nine separate calculations (eight dividend payments and one terminal valuation) that could then be rolled up into one aggregate valuation and related DLOM. Thus, the DLOM presented in Table VII is biased high as interim cash dividend payments are ignored, and the DLOM for these shorter horizon cash flows would be lower.

Table VII.
Illustration of the General Options-Based DLOM Assuming $r = 0\%, \delta = 5\%$.

Volatililty	20%	40%	60%	80%	20%	40%	60%	80%	20%	40%	60%	80%	20%	40%	60%	80%	20%	40%	60%	80%
Years	1	1	1	1	2	2	2	2	3	3	3	3	5	5	5	5	10	10	10	10
PV Put	$10.45	$18.02	$25.52	$32.82	$16.13	$26.29	$36.26	$45.70	$20.92	$32.74	$44.22	$54.80	$29.14	$42.88	$55.98	$67.41	$45.19	$60.16	$73.77	$84.15
Residual LB	$8.72	$19.74	$33.14	$49.18	$12.53	$29.84	$52.13	$80.05	$15.43	$38.15	$68.54	$107.68	$19.84	$52.01	$97.23	$157.49	$26.93	$78.06	$154.92	$261.35
w(S)/w(NH)	1/20%	1/40%	1/60%	1/80%	2/20%	2/40%	2/60%	2/80%	3/20%	3/40%	3/60%	3/80%	5/20%	5/40%	5/60%	5/80%	10/20%	10/40%	10/60%	10/80%
0%/0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
0%/25%	3%	5%	6%	8%	4%	7%	9%	11%	5%	8%	11%	14%	7%	11%	14%	17%	11%	15%	18%	21%
0%/50%	5%	9%	13%	16%	8%	13%	18%	23%	10%	16%	22%	27%	15%	21%	28%	34%	23%	30%	37%	42%
0%/75%	8%	14%	19%	25%	12%	20%	27%	34%	16%	25%	33%	41%	22%	32%	42%	51%	34%	45%	55%	63%
0%/100%	10%	18%	26%	33%	16%	26%	36%	46%	21%	33%	44%	55%	29%	43%	56%	67%	45%	60%	74%	84%
25%/0%	2%	5%	8%	12%	3%	7%	13%	20%	4%	10%	17%	27%	5%	13%	24%	39%	7%	20%	39%	65%
25%/25%	5%	9%	15%	20%	7%	14%	22%	31%	9%	18%	28%	41%	12%	24%	38%	56%	18%	35%	57%	86%
25%/50%	7%	14%	21%	29%	11%	21%	31%	43%	14%	26%	39%	54%	20%	34%	52%	73%	29%	50%	76%	107%
25%/75%	10%	18%	27%	37%	15%	27%	40%	54%	20%	34%	50%	68%	27%	45%	66%	90%	41%	65%	94%	128%
25%/100%	13%	23%	34%	45%	19%	34%	49%	66%	25%	42%	61%	82%	34%	56%	80%	107%	52%	80%	113%	149%
50%/0%	4%	10%	17%	25%	6%	15%	26%	40%	8%	19%	34%	54%	10%	26%	49%	79%	13%	39%	77%	131%
50%/25%	7%	14%	23%	33%	10%	21%	35%	51%	13%	27%	45%	68%	17%	37%	63%	96%	25%	54%	96%	152%
50%/50%	10%	19%	29%	41%	14%	28%	44%	63%	18%	35%	56%	81%	24%	47%	77%	112%	36%	69%	114%	173%
50%/75%	12%	23%	36%	49%	18%	35%	53%	74%	23%	44%	67%	95%	32%	58%	91%	129%	47%	84%	133%	194%
50%/100%	15%	28%	42%	57%	22%	41%	62%	86%	29%	52%	78%	109%	39%	69%	105%	146%	59%	99%	151%	215%
75%/0%	7%	15%	25%	37%	9%	22%	39%	60%	12%	29%	51%	81%	15%	39%	73%	118%	20%	59%	116%	196%
75%/25%	9%	19%	31%	45%	13%	29%	48%	71%	17%	37%	62%	94%	22%	50%	87%	135%	31%	74%	135%	217%
75%/50%	12%	24%	38%	53%	17%	36%	57%	83%	22%	45%	74%	108%	29%	60%	101%	152%	43%	89%	153%	238%
75%/75%	14%	28%	44%	61%	21%	42%	66%	94%	27%	53%	85%	122%	37%	71%	115%	169%	54%	104%	172%	259%
75%/100%	17%	33%	50%	70%	26%	49%	75%	106%	32%	61%	96%	136%	44%	82%	129%	186%	65%	119%	190%	280%
100%/0%	9%	20%	33%	49%	13%	30%	52%	80%	15%	38%	69%	108%	20%	52%	97%	157%	27%	78%	155%	261%
100%/25%	11%	24%	40%	57%	17%	36%	61%	91%	21%	46%	80%	121%	27%	63%	111%	174%	38%	93%	173%	282%
100%/50%	14%	29%	46%	66%	21%	43%	70%	103%	26%	55%	91%	135%	34%	73%	125%	191%	50%	108%	192%	303%
100%/75%	17%	33%	52%	74%	25%	50%	79%	114%	31%	63%	102%	149%	42%	84%	139%	208%	61%	123%	210%	324%
100%/100%	19%	38%	59%	82%	29%	56%	88%	126%	36%	71%	113%	162%	49%	95%	153%	225%	72%	138%	229%	346%

Note: w(S) denotes $w_{Skill,i,t}$, w(NH) denotes $w_{\neg Hedge,t}$, PV Put denotes the plain vanilla put option model value, Residual LB denotes the residual lookback put option portion, and 1/20% denotes 1 year to maturity with volatility of 20%.

The general option-based DLOM method presented here is flexible enough to handle the rich diversity observed in actual practice while still remaining rather parsimonious. We turn now to reinterpreting some prior research.

B. Prior Research

Dyl and Jiang (2008) examine Longstaff's (1995) model for practical applications. In their paper, Dyl and Jiang (2008) present a specific case study where volatility is given as 60.5 percent and maturity is 1.375 years. Relying on a variety of empirical sources, they opine that the DLOM should be approximately 23 percent and, for the sake of argument, assume they are correct. As the case study involved an estate, one would assume $w_{Skill,i,t} = 0\%$. An alternative interpretation is that $w_{\neg Hedge,t} = 83\%$

would result in the same 23 percent DLOM.[20] Thus, the general option-based approach presented here provides a rational way to connect option-based DLOM approaches with other existing methodologies.

Table VIII presents alternative perspectives on Finnerty's (2012) Exhibit 6. Recall Finnerty's DLOM value cannot exceed 32.28 percent due to the volatility time parameter reaching a maximum of $\nu^2(T-t)=\ln\{2\}$, while actual DLOMs are often well in excess of 32.28 percent. Table VIII is based on assuming DLOM is first driven by the inability to hedge and only then driven by skill. Clearly, there are many other explanations possible. In only four cases, skill registered positive values, three of them occurring with low volatilities. Thus, Table VIII provides an illustration of the flexibility of the general option-based model presented here.

[20] We estimate the plain vanilla put option with $r=0\%,\delta=0\%$ is equal to \$4.21 (the stock price was give at \$15.1875). The residual lookback put portion was \$6.48 for a lookback put value of \$10.69.

Table VIII.
An Alternative Interpretation of Mean Implied DLOM

Volatility Range (%)	Mean Implied DLOM	Implied Volatility (Finnerty)	Implied Maturity (Finnerty)	Implied Weight W(S)	Implied Weight W(¬ H)	Implied Weight W(DLOM)
Panel A: Offerings Announced Prior to February 1997						
0.0-29.9	19.47%	24.3%	14.3	65.6%	100.0%	81.4%
30.0-44.9	--	--	--	--	--	--
45.0-59.9	10.51	55.3	0.7	0.0	64.0	24.6
60.0-74.9	13.82	62.0	1.0	0.0	40.8	15.1
75.0-89.9	24.15	81.1	2.2	0.0	55.7	18.7
90.0-104.9	34.97	96.6	$+\infty$	0.0	69.2	21.3
105.0-120.0	61.51	112.6	$+\infty$	2.81	100.0	30.2
>120.0	44.10	138.7	$+\infty$	0.0	65.5	44.1
Average	**24.50%**	**67.5%**	**3.4**	**0.0%**	**66.8%**	**24.1%**
Volatility Range (%)	**Mean Implied DLOM**	**Implied Volatility (Finnerty)**	**Implied Maturity (Finnerty)**	**Implied Weight W(S)**	**Implied Weight W(¬ H)**	**Implied Weight W(DLOM)**
Panel B: Offerings Announced After February 1997						
0.0-29.9	12.66%	21.7%	6.8	40.7%	100.0%	68.3
30.0-44.9	18.56	33.8	6.7	31.1	100.0	62.0
45.0-59.9	15.92	52.3	1.9	0.0	77.2	32.5
60.0-74.9	19.21	66.5	1.8	0.0	73.8	29.5
75.0-89.9	21.37	82.7	1.6	0.0	66.6	25.2
90.0-104.9	21.61	96.5	1.2	0.0	58.3	20.9
105.0-120.0	24.89	111.7	1.3	0.0	58.8	19.9
>120.0	29.71	146.8	1.4	0.0	55.3	16.4
Average	**21.31%**	**73.6%**	**2.0**	**0.0%**	**74.2%**	**29.0%**

Note: This table is based on Finnerty (2012), Exhibit 6. As I do not have access to the raw data, I assumed the dividend yield was zero. Finnerty denotes Finnerty's model assuming zero dividend yield. For each maturity (T) and Mean Model Predicted Discount, the implied volatility was estimated. The average implied volatility of the three maturities is reported above as the Implied Volatility (Finnerty). The Implied Maturity is solved such that the Mean Implied DLOM equals Finnerty's model based on the estimated implied volatility. W(S) denotes the weight applied to skill, W(¬H) denotes the weight applied for the portion not hedged, and W(DLOM) is the overall weight applied to the general model DLOM.

V. Conclusions

A general option-based approach to estimating the discount for lack of marketability is introduced. It was demonstrated to be general enough to address important DLOM challenges, including restriction period, volatility, hedging availability, and investor skill. The general option-based DLOM model was shown to contain the Chaffe model, Longstaff model, and the Finnerty model as special cases. The model also contains two weighting variables that provide valuation professionals much needed flexibility in addressing the unique challenges of each non-marketable valuation assignment. Several prior results were reinterpreted based on the model presented here.

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