*Business Valuation Discounts and Premiums, Second Edition*
By Shannon P. Pratt


# Appendix C

# How Much Can Marketability Affect Security Values?

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**Abstract**

How marketability affects security prices is one of the most important issues in finance. We derive a simple analytical upper bound on the value of marketability using option-pricing theory. We show that discounts for lack of marketability can potentially be large even when the illiquidity period is very short. This analysis also provides a benchmark for assessing the potential costs of exchange rules and regulatory requirements restricting the ability of investors to trade when desired. Furthermore, these results provide new insights into the relation between discounts for lack of marketability and the length of the marketability restriction.

The issue of how marketability affects the value of securities is of fundamental importance in finance. This has been dramatically illustrated by the recent collapse of several well-known financial institutions that were unable to sell investment assets quickly enough to meet unexpected cash flow needs. This issue has also become increasingly important to regulators, rating agencies, security exchanges, auditors, and institutional investors.

There are many situations in which the marketability of a security may be restricted. For example, when an investor lends securities under a reverse repurchase agreement, the investor foregoes the right to sell the securities until they are returned—a lesson painfully learned by Orange County. For many investors, the marketability of initial public offering (IPO) shares can be temporarily restricted. This is because underwriters often pressure investors who are allocated shares in an IPO to refrain from flipping or immediately reselling the shares. This implicit restriction on marketability may explain a portion of the underpricing of IPOs. Another example is letter stock. This is stock issued by firms under SEC Rule 144 that cannot be sold by an investor for a two-year period after it is acquired. As shown by Silber (1992), letter stock is typically placed privately at 30 to 35 percent discounts to the value of otherwise identical unrestricted stock.

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This article presents a simple analytical upper bound on the value of marketability. The intuition behind these results can best be conveyed by considering a hypothetical investor with perfect market timing ability who is restricted from selling a security for $T$ periods. If the marketability restriction were to be relaxed, the investor could then sell when the price of the security reached its maximum. Thus, if the marketability restriction were relaxed, the incremental cash flow to the investor would essentially be the same as if he swapped the time-$T$ value of the security for the maximum price attained by the security. The present value of this lookback or liquidity swap represents the value of marketability for this hypothetical investor, and provides an upper bound for any actual investor with imperfect market timing ability.

This analysis provides a number of new insights about how marketability restrictions affect security values. First, we show that discounts for lack of marketability can be large even when the length of the marketability restriction is very short. Second, the upper bound provides a benchmark for estimating the valuation effects of marketability restrictions such as circuit breakers, trading halts, and prohibitions on program trading. Finally, these results allow us to assess directly whether empirical estimates of discounts for lack of marketability are consistent with rational market pricing.

## I. THE FRAMEWORK

We first describe the framework in which we derive the upper bound on the value of marketability. An important advantage of this framework is that we do not need to make all of the assumptions about informational asymmetries, investor preferences, etc. that would be required in a full general equilibrium model. The cost of this, of course, is that we only obtain bounds, rather than an explicit model of the value of marketability.[1] To make the intuition more clear, we focus on the simplest possible framework in this section. This framework, however, could clearly be extended to provide tighter upper bounds.

Let $V$ denote the current or time-zero value of a security that is continuously traded in a frictionless market. We assume that the equilibrium dynamics of $V$ are given by the stochastic process

$$dV = \mu V dt + \sigma V dZ, \tag{1}$$

where $\mu$ and $\sigma$ are constants and $Z$ is a standard Wiener process. We also assume that the riskless interest rate $r$ is constant.

Consider a hypothetical investor who holds the security in his portfolio, but is restricted from selling the security prior to some fixed time $T$. The value of this security to this investor equals the present value of a cash flow of $V_T$ to be received at time $T$.[2] Now assume that this investor has perfect market timing ability that would allow him to sell the security and reinvest the proceeds in the riskless asset at the time $\tau$ that maximizes the value of his portfolio. Let $M_T$ denote the time-$T$ payoff to this investor if the sale could be timed optimally, where $M_T = \max_{0 \leq \tau \leq T}(e^r(T - \tau)V_\tau)$. As long as the investor cannot sell the security prior to time $T$, however, he cannot benefit from having perfect market timing ability.

This marketability restriction imposes an important opportunity cost on this hypothetical investor since the security position is only worth $V_T$ to the investor at time $T$ if he is restricted from selling, but would be worth $M_T$ if he were allowed to sell earlier.[3] Thus, using a standard dominance or no-arbitrage argument, the value of marketability to an investor with perfect market timing ability is simply the present value of the incremental cash flow $M_T - V_T$ that the investor would receive if the marketability restriction were relaxed. Clearly, the value of marketability would be less for an actual investor with imperfect market timing ability. Thus, the present value of the incremental cash flow $M_T - V_T$ represents an upper bound on the value of marketability.[4]

This incremental cash flow $M_T - V_T$ can also be viewed as the payoff from an option on the maximum value (including interest from reinvesting the sale proceeds) of the security $M_T$, where the strike price of the option $V_T$ is stochastic. Since $M_T \leq V_T$, this lookback option will always be in the money at expiration. Hence, max(0, $M_T - V_T$) $= M_T - V_T$. Alternatively, the cash flow $M_T - V_T$ can be viewed as the payoff of a liquidity swap in which $V_T$ is swapped for $M_T$ at time $T$.

## II. THE UPPER BOUND

The present value of $M_T - V_T$ can be determined using standard risk-neutral valuation techniques familiar from option-pricing theory. Let $F(V, T)$ denote the present value of $M_T - V_T$. This present value equals

$$F(V,\ T) = e^{-rT}E[M_T] = e^{-rT}E[V_T], \tag{2}$$

where the expectation is taken with respect to the risk-neutral dynamics for $V$. Using the well-known density function for the maximum of a Brownian motion process in Harrison (1985), the expectations in equation (2) can be evaluated directly to give the following closed-form solution for the upper bound,

$$F(V,\ T) = V\left(2 + \frac{\sigma^2 T}{2}\right)N\left(\frac{\sqrt{\sigma^2 T}}{2}\right) + V\sqrt{\frac{\sigma^2 T}{2\pi}}\exp\left(-\frac{\sigma^2 T}{8}\right) - V, \tag{3}$$

where $N(\cdot)$ is the cumulative normal distribution function.[5]

The upper bound $F(V, T)$ is proportional to the current value of the security $V$. Thus, bounds on the value of marketability, or equivalently, bounds on the size of the discount for lack of marketability, can easily be expressed as a percentage of the value of $V$. It is readily shown that the upper bound is an increasing function of length of the marketability restriction $T$. In addition, the upper bound is an increasing function of the variance of returns $\sigma^2$. This is intuitive, since the more volatile the price of the security, the higher is the opportunity cost of not being able to trade. Taking the limit of $F(V, T)$ shows that the upper bound converges smoothly to zero as $T \to 0$.

This upper bound represents the largest discount for lack of marketability that could be sustained in a market with rational investors. If illiquid securities could be acquired at prices less than $F(V,T)$ below those of otherwise identical liquid securities, then arbitrage profits could potentially be achieved by holding nonmarketable securities and synthesizing marketability using derivatives.

This upper bound is illustrated in Table I, which reports the percentage upper bounds for values of $\sigma^2$ comparable to those for 6-month, 1-year, and 2-year Treasury securities. The percentage bounds for a 1-day nonmarketability period range from 0.053 to 0.210. The percentage bounds for a 5-day nonmarketability period are only about twice as large. This shows that the per-unit-time-period effect of illiquidity is largest for relatively small values of $T$. These results have important implications for the overnight and term repo

**Table I** Upper Bounds for Percentage Discounts for Lack of Marketability

The standard deviations $\sigma = 0.0125,\ 0.0250,$ and $0.0500$ correspond to the approximate historic standard deviations of returns for 6 month, 1-year, and 2-year Treasury securities.

| Marketability Restriction Period | $\sigma = 0.0125$ | $\sigma = 0.0250$ | $\sigma = 0.0500$ |
|---|---|---|---|
| 1 Day | 0.053 | 0.105 | 0.210 |
| 5 Days | 0.118 | 0.235 | 0.471 |
| 10 Days | 0.166 | 0.333 | 0.667 |
| 20 Days | 0.235 | 0.471 | 0.944 |
| 30 Days | 0.288 | 0.577 | 1.157 |
| 60 Days | 0.408 | 0.817 | 1.639 |
| 90 Days | 0.500 | 1.001 | 2.010 |

markets, since the difference between the general and special collateral rates should reflect the value of the marketability foregone by lending the security.

Figure 1 graphs the upper bound as a function of the nonmarketability period for values of σ ranging from 0.10 to 0.30. This range of volatility is consistent with typical stock return volatilities. As shown, the upper bound is an increasing concave function of the length of the marketability restriction. In addition, Figure 1 shows that discounts for lack of marketability can be very large even when the duration of restricted marketability is fairly short. This can also be seen in Table II, which reports numerical values for the percentage upper bound using the same range of volatilities. Table II shows that the upper bound ranges from 0.421 to 1.268 percent for a 1-day marketability restriction. The upper bound ranges from 1.337 to 4.052 percent for a 10-day marketability restriction.

**Figure 1** Upper Bounds for Percentage Discounts for Lack of Marketability Graphed as a Function of the Length of the Marketability Restriction Period Measured in Days and for Varying Values of the Standard Deviation of Returns Denoted as Sigma

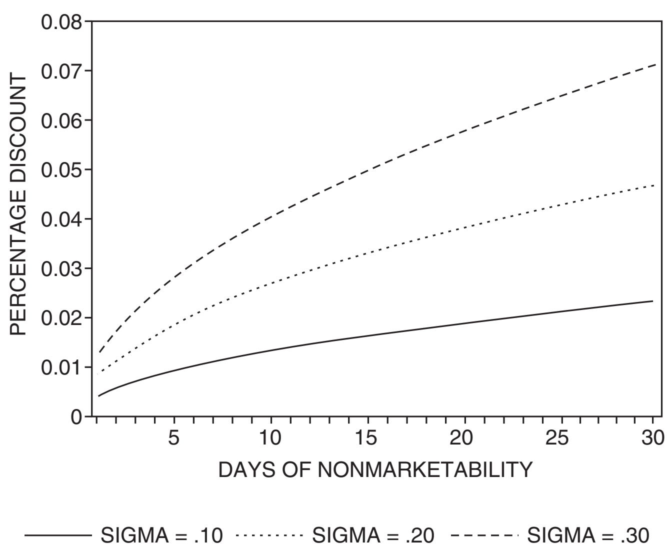


**Table II** Upper Bounds for Percentage Discounts for Lack of Marketability
(The standard deviations correspond to the range typically observed for equity securities.)

| Marketability Restriction Period | $\sigma = 0.10$ | $\sigma = 0.20$ | $\sigma = 0.30$ |
|---|---|---|---|
| 1 Day | 0.421 | 0.844 | 1.268 |
| 5 Days | 0.944 | 1.894 | 2.852 |
| 10 Days | 1.337 | 2.688 | 4.052 |
| 20 Days | 1.894 | 3.817 | 5.768 |
| 30 Days | 2.324 | 4.691 | 7.100 |
| 60 Days | 3.299 | 6.683 | 10.153 |
| 90 Days | 4.052 | 8.232 | 12.542 |
| 180 Days | 5.768 | 11.793 | 18.082 |
| 1 Year | 8.232 | 16.984 | 26.276 |
| 1 Years | 11.793 | 24.643 | 38.605 |
| 5 Years | 19.128 | 40.979 | 65.772 |

The magnitude of the upper bounds for restriction periods measured in days or weeks has important implications for equity markets, since there are many situations in which the marketability of shares is restricted for a short period of time. For example, IPO underwriters often allocate shares to investors with the implicit understanding that the shares will not be flipped or immediately resold in the aftermarket. Investors who violate this implicit understanding may be less likely to receive allocations in attractive future IPOs. This implicit restriction on marketability may only last for a few days or weeks, during which time the underwriter may engage in market stabilization efforts. Our results suggest that the cost to the investor of the temporary restriction on selling IPO shares could be fairly substantial given the fact that the volatility of returns may be particularly high during this period.

These results also provide some measure of the potential cost to investors of imposing market restrictions such as circuit breakers, trading halts, or prohibitions against program trading. Table II suggests that the potential cost of these restrictions could again be very sizable. An important implication of this is that prices of securities in markets where liquidity may be interrupted could be substantially lower than they otherwise might be because of the expected costs of nonmarketability. Our analysis provides a framework for evaluating the potential costs of different forms of exchange and regulatory requirements.

The upper bound can also be viewed as the maximum amount that any investor would be willing to pay in order to obtain immediacy in liquidating a security position. Thus, this upper bound provides an endogenous measure of the largest possible bid-ask spread or transaction cost for a security. In contrast, previous research on the valuation of illiquid securities by Amihud and Mendelson (1986) and Boudoukh and Whitelaw (1993) and on the valuation of securities in the presence of transaction costs by Constantinides (1986) and Vayanos and Vila (1992) takes the bid-ask spread or transaction costs for the security to be exogenous.

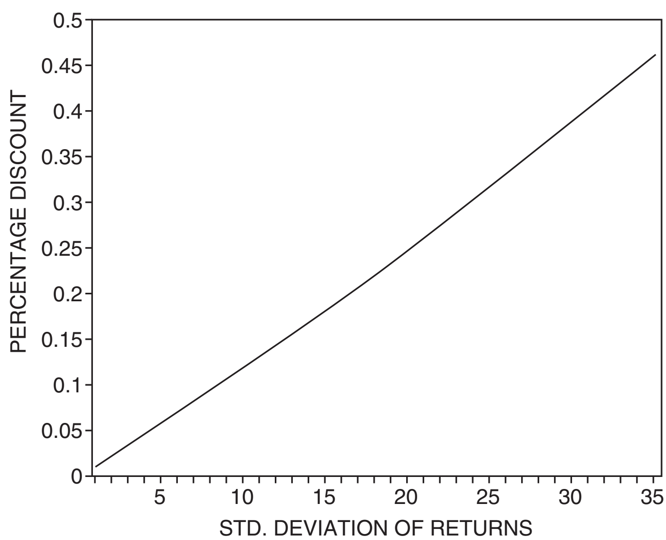
## III. A COMPARISON

It is also interesting to compare the upper bound to empirical estimates of discounts for lack of marketability. In particular, much of the empirical evidence about discounts for lack of marketability focuses on the pricing of SEC Rule 144 restricted stock. This is stock issued by a firm that is not registered for public trading, but is otherwise identical to publicly traded stock. The primary limitation of Rule 144 stock is that the recipient cannot sell the shares for a two-year period. After two years, the shares become marketable, subject to several minor trading-volume limitations. Restricted shares are typically issued by firms via private placements instead of the usual public offering mechanism. By comparing the price at which the restricted stock is privately placed to the market price for the firm's registered shares, the discount for lack of marketability can be directly measured.

Pratt (1989) summarizes the evidence from eight separate studies of restricted stock. The median percentage discount found in these studies is approximately 35 to 40 percent. This range is fairly consistent across all of the studies summarized by Pratt. This range is also consistent with the results of a recent study by Silber (1992) who finds that the mean discount for lack of marketability is 34 percent in a sample of private placements of stock during the 1981 to 1988 period.

In order to make comparisons, Figure 2 graphs the percentage upper bound on the discount for lack of marketability for a wide range of volatilities. Assuming that the average standard deviation of returns for the firms studied by Pratt (1989) and Silber (1992) is in the range of 0.25 to 0.35, Figure 2 suggests that empirical estimates of the discount for lack of marketability closely approximate the upper bound. In one sense, this is a surprising finding since the upper bound was derived from the perspective of a theoretical investor with perfect market timing ability. These results, however, suggest that the upper bound may actually be a tight bound. Thus, the analytical results in this article may

**Figure 2** Upper Bounds for Percentage Discounts for Lack of Marketability Graphed as a Function of the Standard Deviations of Returns. The Length of the Marketability Restriction Period Is Two Years, Corresponding to the Length of the Marketability Restriction for Letter Stock.



actually provide useful approximations of the value of marketability, rather than just serving as an upper bound.

## IV. CONCLUSION

This article provides a first step toward developing a practical model for valuing liquidity in financial markets. The results of this analysis can be used to provide rough order-of-magnitude estimates of the valuation effects of different types of marketability restrictions. In fact, the empirical evidence suggests that the upper bound may actually be a close approximation to observed discounts for lack of marketability. More importantly, however, these results illustrate that option-pricing techniques can be useful in understanding liquidity in financial markets and that liquidity derivatives have potential as tools for managing and controlling the risk of illiquidity.

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## NOTES

1. Mayers (1972, 1973, 1976), Brito (1977), Stapleton and Subrahmanyam (1979), and Boudoukh and Whitelaw (1993) present general equilibrium models of the returns on nonmarketable assets.

Their results suggest that the size of the equilibrium discount for lack of marketability depends critically on how closely the optimal strategy approximates the buy-and-hold strategy.

2. Observe that nonmarketability is investor-specific rather than security-specific in this framework. This differs from the equilibrium models presented in Amihud and Mendelson (1986) and Boudoukh and Whitelaw (1993). Since the other investors in this market are unrestricted, derivative claims on $V$ can be priced using standard no-arbitrage arguments.
3. Note that $M_T$ will generally be higher than the maximum value reached by the underlying asset price since it includes interest from reinvesting the proceeds of the sale.
4. We are implicitly making the standard no-arbitrage assumption that the price $V$ of the underlying asset is exogenous and is not affected by whether this hypothetical investor is restricted or not.
5. The first term in equation (2) equals $e^{-rT}$ times $e^{rT}$ times the expected maximum of the discounted process $e^{-rt}V_t$. This discounted process is a martingale with respect to the risk-neutral dynamics for $V$.