Analysis and Critique of the Average Strike Put Option Marketability Discount Model

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Abstract

In this paper I examine the results from Finnerty's average strike put option marketability discount model, and demonstrate some easy-to-compute lower and upper bounds for the average strike put option obtained using geometric average options for which exact closed-form formulas exist. The results obtained using Finnerty's formula appear to exceed the upper bound significantly. I investigate the source of the inconsistency and propose an adjusted closed-form formula for the price of the arithmetic average strike put option and the associated marketability discount. The adjusted formula is within the established lower and upper bounds, and agrees with numerical simulation results.

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Marketability Discount Model

Introduction

It is well documented in a number of empirical studies that the values of securities are affected by their relative marketability. For two otherwise identical stocks, investors will demand a significant discount to the price of the security which cannot be converted into cash instantly.

The use of option pricing models as a tool to quantify discounts for lack of marketability ("DLOM") is increasingly popular among valuation practitioners. The categorization of marketability discounts as options is well supported by empirical studies that demonstrate high correlation between the size of the observed marketability adjustment, on the one hand, and the volatility and length of restriction period, on the other (Finnerty, 2002).

One of the first option-based models of marketability discounts is the so called "protective put", which attempts to quantify the discount for lack of marketability as the value of an at-the-money put option. This model suffers from a number of conceptual fallacies, however, since it only measures the fair price of an "insurance policy" that would protect the stock holder against declines in the stock price. While it is true that the holder of restricted shares is subject to the risk of realizing losses, the cost of insuring the portfolio does not seem to be a good proxy for the discount for lack of marketability. As such, the protective put model appears to be theoretically mis-specified, despite its popularity due to its ease of implementation and agreement with empirical data.

A conceptually superior model developed by Longstaff (1995) quantifies the discount for lack of marketability as a lookback put option on the forward price process. The model focuses on measuring the value of the opportunity given up by the holder of restricted shares compared to the value that would otherwise be realized with the optimal behavior of an investor holding identical freely-traded stock. While this modeling framework is more realistic than the set of assumptions used in the protective put model, the lookback put option model assumes that the unrestricted investor will be able to sell his or her shares at the highest possible stock price, therefore implying perfect market timing. As such, the model might be more suitable for quantifying the marketability adjustments associated with restricted shares held by insiders and in the presence of asymmetrical information (Finnerty, 2002). Furthermore, the discounts predicted by the model tend to be significant for volatilities over 30 percent and longer holding periods, exceeding the reasonable boundary of 100 percent.

The assumption of perfect market timing in Longstaff's model is relaxed in the average strike put option model developed by Finnerty (2002). This model also focuses on quantifying the value foregone by an investor holding restricted shares. However, rather than assuming a perfect market timing ability, Finnerty's model assumes that the holder of freely-traded stock is equally likely to sell their shares at any given time, absent of any insider information. Consequently, the strike price of the put used to quantify the marketability adjustment is set as the arithmetic average of the forward stock prices (in contrast to the lookback put where the strike price is set to the maximum of the possible forward prices in a given stock price path). Finnerty provides the following formula for the value of the marketability discount:

$$DLOM = V(0)[e^{(r-q)T} N\left(\frac{r-q}{v}\sqrt{T} + \frac{v\sqrt{T}}{2}\right) - N\left(\frac{r-q}{v}\sqrt{T} - \frac{v\sqrt{T}}{2}\right)] \tag{1A}$$

$$v^2 = \sigma^2 T + \ln[2\{e^{\sigma^2 T} - \sigma^2 T - 1\}] - 2\ln[e^{\sigma^2 T} - 1] \tag{1B}$$

In this paper I examine the results from Finnerty's formula for the price of an average strike put option, and demonstrate some easy-to-compute lower and upper bounds for the average strike put option obtained using geometric average options for which exact closed-form formulas exist. The results obtained using Finnerty's formula appear to exceed the upper bound significantly. I investigate the source of the inconsistency and propose an adjusted closed-form formula for the price of the arithmetic average strike option and the associated marketability discount. The adjusted formula is within the established lower and upper bounds, and agrees with numerical simulation results.

Lower and Upper Bounds

Assume that the stock price $V(t)$ follows the geometric diffusion process:

$$dV = (r-q)Vdt + \sigma V dZ \tag{2}$$

where r is the continuously compounded risk-free rate, q is the continuous dividend yield, σ is the volatility of the stock price and Z is standard Wiener process. We further assume that r, q and σ are constant. Let $F(t) = V(t)e^{(r-q)(T-t)}$ represent the forward price at any given time t. Note that we have the boundaries $F(0) = V(0)e^{(r-q)T}$ and $F(T) = V(T)$. Applying Ito's lemma we obtain that the forward price $F(t)$ follows the diffusion process:

$$dF = \sigma F dZ \tag{3}$$

Let $A(T)$ denote the arithmetic average of the forward prices in continuous time:

$$A(T) = \frac{1}{T}\int_0^T F(u)du \tag{4}$$

Following Finnerty (2002), the size of the marketability discount is given by the price of an average strike put option, where $E[.]$ denotes the expectation operator in a risk-neutral setting:

$$\begin{aligned} DLOM &= e^{-rT}E[A(T) - V(T)]^+ = \\ &= e^{-rT}E\begin{bmatrix} A(T) - V(T) & if\ A(T) > V(T) \\ 0 & if\ A(T) \le V(T) \end{bmatrix} \end{aligned} \tag{5}$$

Note that since the average strike put option's payoff $Max[A(T) - V(T), 0] = [A(T) - V(T)]^+$ is dominated by the quantity $A(T)$, we have the following simple upper boundary for the size of the discount for lack of marketability:

$$DLOM = e^{-rT} E[A(T) - V(T)]^+ \leq e^{-rT} E[A(T)] = \\ = e^{-rT} F(0) = V(0) e^{-qT} \leq V(0) \tag{6}$$

It follows from the expression above that the size of the marketability discount, expressed as a percentage of the initial security price $V(0)$, will never exceed 100 percent. While this property is reasonably intuitive, it contrasts with the results from the expression for the price of the average strike put given by Finnerty. Finnerty's formula is known to produce discounts that exceed 100 percent, in particular for larger trading restriction periods ($T > 6$ years). I investigate the source of the inconsistency in the following sections.

A put-call parity relationship for average strike put and call options produces the following result:

$$e^{-rT} E[A(T) - V(T)]^+ - e^{-rT} E[V(T) - A(T)]^+ = \\ = e^{-rT} E[A(T)] - e^{-rT} E[V(T)] = e^{-rT} F(0) - e^{-rT} F(0) = 0 \tag{7}$$

Therefore, the price of the average strike put option $e^{-rT} E[A(T) - V(T)]^+$ is equal to the price of the average strike call option $e^{-rT} E[V(T) - A(T)]^+$.

Let $G(T)$ denote the geometric average of the forward prices, again in continuous time:

$$\ln G(T) = \frac{1}{T} \int_0^T \ln F(u)\, du \tag{8}$$

Since $G(T) \leq A(T)$, an arithmetic average strike put option is bounded from below by a geometric average strike put. Similarly, an arithmetic average strike call option is bounded from above by a geometric average strike call. We can use these inequalities, as well as the put-call parity result in (7) in order to derive lower and upper bounds for the size of the marketability discount in terms of geometric average strike options:

$$e^{-rT} E[G(T) - V(T)]^+ \leq e^{-rT} E[A(T) - V(T)]^+ = \\ = e^{-rT} E[V(T) - A(T)]^+ \leq e^{-rT} E[V(T) - G(T)]^+ \tag{9}$$

This is convenient, because prices of geometric average options can be computed analytically with a closed-form expression similar to the Black-Scholes formula. Unlike the arithmetic average, the geometric average of the forward prices is known to be log-normally distributed. While there are considerably more accurate bounds for the prices of arithmetic average options available in academic literature, the use of geometric average options here is, for practical purposes, acceptable since the goal is to get to a reasonable approximation for the size of the marketability discount.

Before I state the actual closed-form expressions for the price of the geometric average options above, I consider another interesting result. Since the "arithmetic average" (4) in the average strike put is the mean of the forward prices $F(t) = V(t)e^{(r-q)(T-t)}$, it is convenient to re-write the payoff function for the option as follows:

$$e^{-rT}E[A(T) - V(T)]^{+} = e^{-rT}E[A(T) - F(T)]^{+} \tag{10}$$

We are therefore considering the price of an arithmetic average put option where the underlying diffusion price process follows a geometric Brownian motion with no drift (3).

Henderson and Wojakowksi (2001) establish a symmetry relationship between the prices of floating and fixed strike average options. The price of a floating strike average put option is equal to the price of an at-the-money fixed strike average call option, where the roles of the risk-free rate r and the dividend yield q are reversed:

$$p_f\left(S(0), \frac{K}{S(0)}, r, q, 0, T\right) = c_x(K, S(0), q, r, 0, T) \tag{11}$$

The subscripts "f" and "x" denote floating and fixed strike options respectively. Using Henderson and Wojakowski's own notation, $A(T) = \frac{1}{T}\int_0^T S(u)du$, $p_f(S(0), \lambda, r, q, 0, T) = e^{-rT}E[A(T) - \lambda S(T)]^{+}$, and $c_x(K, S(0), r, q, 0, T) = e^{-rT}E[A(T) - K]^{+}$ (note the reversal of the risk-free rate r and the dividend yield q in the symmetry relationship (11), however).

When the underlying is a geometric Brownian motion with no drift, under a risk-neutral setting, the dividend yield is equal to the risk-free rate:

$$dF = \sigma F dZ = (r - q^{*})F dt + \sigma F dZ, \;\; q^{*} = r \tag{12}$$

We can therefore apply the symmetry relationship to the price of the average strike put, and express the discount for lack of marketability as a fixed-strike, average price call:

$$e^{-rT}E[A(T) - F(T)]^{+} = e^{-rT}E[A(T) - F(0)]^{+} \tag{13}$$

Actually, this symmetry relationship is also valid for geometric average options, which can also be verified directly since the prices of geometric average calls and puts have known-closed form solutions:

$$e^{-rT}E[G(T) - F(T)]^{+} = e^{-rT}E[G(T) - F(0)]^{+} \tag{14A}$$

$$e^{-rT}E[F(T) - G(T)]^{+} = e^{-rT}E[F(0) - G(T)]^{+} \tag{14B}$$

Therefore, using (14A) and (14B) in connection with (10), the inequality (9) for the size of the marketability discount can be re-written as:

$$e^{-rT}E[G(T)-F(0)]^{+} \le e^{-rT}E[A(T)-F(T)]^{+} = \\ = e^{-rT}E[F(T)-A(T)]^{+} \le e^{-rT}E[F(0)-G(T)]^{+} \tag{15}$$

Since the distribution of the geometric average $G(T)$ is lognormal with known parameters, the exact formula for the lower and upper bounds is known (Kemna & Vorst, 1990):

$$lower\ bound = e^{-rT}E[G(T)-F(0)]^{+} = \\ = F(0)\left[e^{(b-r)T}N(d_1) - e^{-rT}N(d_2)\right] = \\ = V(0)\left[e^{(b-q)T}N(d_1) - e^{-qT}N(d_2)\right] \tag{16A}$$

$$upper\ bound = e^{-rT}E[F(0)-G(T)]^{+} = \\ = F(0)\left[e^{-rT}N(-d_2) - e^{(b-r)T}N(-d_1)\right] = \\ = V(0)\left[e^{-qT}N(-d_2) - e^{(b-q)T}N(-d_1)\right] \tag{16B}$$

$$d_1 = \frac{\sigma\sqrt{T}}{4\sqrt{3}}, \qquad d_2 = -3d_1, \qquad b = -\frac{\sigma^2}{12} \tag{16C}$$

The expressions for the upper and lower bounds are proportional to the current share price $V(0)$, so that the discount for lack of marketability can be easily expressed as a percentage of the share price. I use the expressions to calculate the exact lower and upper bounds for the percentage marketability discount for a non-dividend paying stock (q=0%), and compare the results with the data published by Finnerty using formulas (1A) and (1B). Consistent with Finnerty (2002), the range of the restriction period is from 3 months to 5 years, and the range of volatility is from σ=10% to σ=80%. The risk-free rate is r=5%.

Table 1 – Lower Bound (16A)

Restriction Period	**Lower Bound – Marketability Discounts (q=0%)**							
	σ=10%	**σ=20%**	**σ=30%**	**σ=40%**	**σ=50%**	**σ=60%**	**σ=70%**	**σ=80%**
3 months	1.14%	2.26%	3.36%	4.43%	5.48%	6.51%	7.51%	8.48%
6 months	1.61%	3.17%	4.69%	6.16%	7.58%	8.95%	10.27%	11.53%
9 months	1.96%	3.86%	5.69%	7.44%	9.11%	10.71%	12.23%	13.66%
1 year	2.26%	4.43%	6.51%	8.48%	10.36%	12.12%	13.77%	15.31%
2 years	3.17%	6.16%	8.95%	11.53%	13.89%	16.01%	17.91%	19.56%
3 years	3.86%	7.44%	10.71%	13.66%	16.26%	18.51%	20.40%	21.94%
4 years	4.43%	8.48%	12.12%	15.31%	18.03%	20.28%	22.05%	23.35%
5 years	4.93%	9.38%	13.30%	16.65%	19.41%	21.58%	23.16%	24.17%

Table 2 – Upper Bound (16B)

Restriction Period	Upper Bound – Marketability Discounts (q=0%)							
	σ=10%	σ=20%	σ=30%	σ=40%	σ=50%	σ=60%	σ=70%	σ=80%
3 months	1.16%	2.34%	3.55%	4.76%	6.00%	7.26%	8.52%	9.81%
6 months	1.65%	3.34%	5.06%	6.82%	8.62%	10.44%	12.29%	14.16%
9 months	2.03%	4.11%	6.25%	8.43%	10.66%	12.94%	15.24%	17.58%
1 year	2.34%	4.76%	7.26%	9.81%	12.42%	15.08%	17.77%	20.50%
2 years	3.34%	6.82%	10.44%	14.16%	17.97%	21.84%	25.75%	29.68%
3 years	4.11%	8.43%	12.94%	17.58%	22.32%	27.12%	31.93%	36.72%
4 years	4.76%	9.81%	15.08%	20.50%	26.03%	31.59%	37.12%	42.56%
5 years	5.35%	11.03%	16.98%	23.10%	29.31%	35.51%	41.62%	47.58%

Table 3 – Finnerty's average strike put formula ((1A), (1B))

Restriction Period	Finnerty (2002) – Marketability Discounts (q=0%)							
	σ=10%	σ=20%	σ=30%	σ=40%	σ=50%	σ=60%	σ=70%	σ=80%
3 months	1.41%	1.89%	2.44%	2.99%	3.55%	4.11%	4.67%	5.23%
6 months	2.84%	3.81%	4.90%	6.01%	7.12%	8.22%	9.31%	10.38%
9 months	4.29%	5.75%	7.38%	9.04%	10.69%	12.32%	13.91%	15.45%
1 year	5.76%	7.71%	9.88%	12.09%	14.27%	16.40%	18.45%	20.42%
2 years	11.81%	15.76%	20.12%	24.43%	28.56%	32.42%	35.97%	39.15%
3 years	18.16%	24.18%	30.68%	36.93%	42.69%	47.83%	52.25%	55.93%
4 years	24.83%	32.95%	41.52%	49.52%	56.56%	62.49%	67.25%	70.88%
5 years	31.84%	42.08%	52.64%	62.13%	70.09%	76.39%	81.07%	84.32%

Note that the marketability discounts obtained using Finnerty's formula for the price of the arithmetic average strike put option significantly exceed the upper bound derived using the closed-form expression for the price of a geometric average price put option. It appears that Finnerty's formula tends to overprice the average strike puts, and correspondingly overstate the size of the implied marketability discounts. The pricing error seems to be severe for holding periods of 2 years or larger, with the predicted discounts exceeding the upper bound two to three times. Furthermore, for holding periods of 6 years of larger, Finnerty's formula results in discounts larger than 100 percent, which contradicts the bound established in (6).

Adjusted Formula for the Average Strike Option

In this section I investigate the source of the inconsistency in the results obtained by using Finnerty's formula and provide an adjusted formula for the price of the average strike put option, and the associated discount for lack of marketability.

In the framework of Finnerty's model, the discount for lack of marketability is given by the price of an arithmetic average strike put option written on the forward prices of the underlying, which by symmetry is equal to the price of an at-the-money fixed strike, average price call option on the forward prices:

$$e^{-rT}E[A(T)-F(T)]^{+} = e^{-rT}E[A(T)-F(0)]^{+} \tag{17}$$

Since the moments of the average $A(T)$ are known, following Levy (1992) one can reasonably assume that the average of the forward prices is log-normally distributed, and match the moments of the assumed log-normal distribution with the actual calculated moments of the average. This approach, known as the Fenton-Wilkinson method, offers a closed-form approximate solution for the price of the arithmetic average option, and is found to be reasonably accurate in practice (particularly for smaller volatilities or times to maturity). However, even for larger volatilities and timeframes to maturity of the option, the loss of accuracy is not that significant for the purposes of estimating the size of a marketability discount where precisions to several decimal places is not needed. Hence, Finnerty (2002) applies the Fenton-Wilkinson method to derive a closed-form solution for the price of the arithmetic average put, and the size of the corresponding marketability discount.

The approach taken by Finnerty is to assume that the pair $[A(T), F(T)]$ is jointly log-normal, and then use the known first and second moments of the average, $E[A(T)]$ and $E[A(T)^2]$, in addition to the joint expectation $E[A(T)F(T)]$, in order calculate the distribution parameters of the jointly log-normal pair. An application of Margrabe's formula for the option to exchange one asset for another yields a closed-form solution for the average strike put option price $e^{-rT}E[A(T) - F(T)]^+$. A more detailed outline of this approach can be found in Finnerty (2002) and is also sketched in Appendix II of this article.

An alternative approach is to utilize the symmetry relationship above in order to price the at-the-money fixed strike, average price call option $e^{-rT}E[A(T) - F(0)]^+$. Again, assuming that $A(T)$ is log-normal, one can use the known moments of the average $E[A(T)]$ and $E[A(T)^2]$ in order to establish the parameters of the distribution of the arithmetic average, and then apply the Black-Scholes formula in a straightforward manner. Note that the two approaches are only approximations for the true price of the arithmetic average option and would not satisfy the symmetry relationship (i.e. they result in slightly different approximate prices for the arithmetic average option). Furthermore, these types of closed-form approximations do not establish an error bound, and the accuracy of the prices obtained via the approximations is typically tested by comparison to simulation results. Using Monte Carlo simulations, I obtain results which indicate that the second approach appears to yield slightly more accurate approximations than the first approach which was taken by Finnerty (2002). Since the second approach is also "simpler" as one does not need to calculate the joint expectation $E[A(T)F(T)]$, I use the second approach to obtain the price of the arithmetic average price call and the corresponding discount for lack of marketability.

Assume that the distribution of $A(T)$ is log-normal, such that

$$\ln A(T) \sim N(\mu_T, v_T^2) \tag{18}$$

The moment-generating function of $\ln A(T)$ is given by the expression $E[A(T)^k] = e^{\mu_T k + \frac{1}{2} v_T^2 k^2}$. Therefore, we can match the calculated first two moments of the average against the moments of the assumed log-normal distribution, in order to obtain its parameters:

$$\left| \begin{array}{l} E[A(T)] = e^{\mu_T + \frac{1}{2} v_T^2} \\ E[A(T)^2] = e^{2\mu_T + 2v_T^2} \end{array} \right. \tag{19}$$

Solving the two equations simultaneously yields the following expressions for the parameters μ_T and ν_T:

$$\left|\begin{array}{l} \mu_T = 2\ln E[A(T)] - \frac{1}{2}\ln E[A(T)^2] \\ v_T^2 = \ln E[A(T)^2] - 2\ln E[A(T)] \end{array}\right. \tag{20}$$

The expressions for the first two moments are obtained by applying some tedious algebra, as demonstrated by Finnerty (2002) and also shown in Appendix I:

$$E[A(T)] = F(0) \tag{21A}$$

$$E[A(T)^2] = \frac{2F(0)^2}{(\sigma^2 T)^2}\{e^{\sigma^2 T} - \sigma^2 T - 1\} \tag{21B}$$

Substituting these expressions into the equations above, one finds

$$\mu_T = \ln F(0) - \frac{1}{2}v_T{}^2 \tag{22A}$$

$$v_T^2 = \ln[2\{e^{\sigma^2 T} - \sigma^2 T - 1\}] - 2\ln[\sigma^2 T] \tag{22B}$$

Applying the Black-Scholes formula for the price of a call option with strike $F(0)$, where the underlying $A(T)$ follows a lognormal price process with annualized volatility $v = \frac{v_T}{\sqrt{T}}$, one finds:

$$\begin{aligned} &e^{-rT}E[A(T) - F(0)]^+ = \\ &= e^{-rT}\{E[A(T)]N(d_1) - F(0)N(d_2)]\} = \\ &= e^{-rT}F(0)[\,N(d_1) - N(d_2)] \end{aligned} \tag{23A}$$

$$\begin{aligned} d_1 &= \frac{\ln\frac{F(0)}{F(0)} + \frac{1}{2}v^2 T}{v\sqrt{T}} = \frac{1}{2}v\sqrt{T} = \frac{1}{2}v_T \\ d_2 &= d_1 - v\sqrt{T} = -\frac{1}{2}v\sqrt{T} = -\frac{1}{2}v_T \end{aligned} \tag{23B}$$

Thus, the price of the average strike put option (equivalent to the price of the at-the-money average price call), and the associated discount for lack of marketability, are given by the following simple expression:

$$e^{-rT}E[A(T) - F(0)]^+ = e^{-qT}V(0)\left[\,2N\left(\frac{v_T}{2}\right) - 1\right] \tag{24A}$$

$$v_T^2 = \ln[2\{e^{\sigma^2 T} - \sigma^2 T - 1\}] - 2\ln[\sigma^2 T] \tag{24B}$$

In the table below, I show the marketability discounts implied from formulas (24A) and (24B) for volatilities ranging from 10% to 80%, and holding periods between 3 months and 5 years. The dividend yield is assumed to be zero, and the risk-free rate is 5%.

Table 4 – Adjusted average strike put formula ((24A), (24B))

Restriction Period	**Average Strike Put – Marketability Discounts (q=0%)**							
	σ=10%	**σ=20%**	**σ=30%**	**σ=40%**	**σ=50%**	**σ=60%**	**σ=70%**	**σ=80%**
3 months	1.15%	2.30%	3.46%	4.61%	5.77%	6.93%	8.09%	9.25%
6 months	1.63%	3.26%	4.89%	6.53%	8.17%	9.82%	11.48%	13.14%
9 months	2.00%	3.99%	6.00%	8.01%	10.03%	12.06%	14.10%	16.17%
1 year	2.30%	4.61%	6.93%	9.25%	11.60%	13.96%	16.34%	18.75%
2 years	3.26%	6.53%	9.82%	13.14%	16.51%	19.93%	23.40%	26.95%
3 years	3.99%	8.01%	12.06%	16.17%	20.35%	24.63%	29.01%	33.49%
4 years	4.61%	9.25%	13.96%	18.75%	23.65%	28.69%	33.87%	39.18%
5 years	5.16%	10.36%	15.64%	21.05%	26.61%	32.35%	38.25%	44.29%

The results are compared to prices obtained via Monte Carlo numerical simulations where the price of the corresponding geometric average option is used as a control variate, following Kemna & Vorst (1990).

Table 5 – Average strike put – numerical results

Restriction Period	**Monte Carlo simulation (N=100,000) – Marketability Discounts (q=0%)**							
	σ=10%	**σ=20%**	**σ=30%**	**σ=40%**	**σ=50%**	**σ=60%**	**σ=70%**	**σ=80%**
3 months	1.16%	2.32%	3.48%	4.65%	5.79%	6.94%	8.09%	9.25%
6 months	1.64%	3.27%	4.91%	6.53%	8.16%	9.78%	11.38%	12.95%
9 months	2.00%	4.00%	5.99%	7.99%	9.96%	11.95%	13.90%	15.79%
1 year	2.31%	4.61%	6.90%	9.20%	11.48%	13.72%	15.95%	18.20%
2 years	3.26%	6.52%	9.72%	13.01%	16.18%	19.20%	22.36%	25.20%
3 years	3.99%	7.96%	11.93%	15.73%	19.61%	23.42%	27.12%	30.47%
4 years	4.60%	9.17%	13.72%	18.17%	22.55%	26.70%	31.06%	34.79%
5 years	5.15%	10.25%	15.26%	20.20%	24.81%	29.75%	34.10%	38.36%

The results obtained using the adjusted formula in (24A) and (24B) from Table 4 agree reasonably well with the numerical results from Table 5, in contrast with the results form Finnerty's formula presented in Table 3. The results from the adjusted formula are within the lower and upper boundaries, presented in Tables 1 and 2 respectively. Furthermore, the adjusted formula suggests a simple relationship between the size of the marketability discount and the dividend yield of the security.

Conclusion

As the adjusted formula for the average strike put option demonstrates, the marketability discounts implied by Finnerty's model are in fact substantially lower than originally hypothesized. This is because the original formula by Finnerty ((1A) and (1B)) tends to overprice the average strike put, especially for larger holding periods. This is consistent with Finnerty's own conclusions that the marketability discounts implied by his model tend to overstate the empirical discounts across all volatilities for a two-year restriction period, but agree closely with the empirical discounts for a one-year restriction period. It is possible the adjusted formula for the average strike put option will correct for this overvaluation and provide a better empirical fit for the two-year holding period data. Further tests are necessary in order to confirm the reasonableness of the average strike put option marketability discount model with the adjusted formula for the price of the Asian option.

Appendix I

Below are derivations for the first two moments of the arithmetic average $A(T) = \frac{1}{T}\int_0^T F(u)du$, following an approach identical to the one in Finnerty (2002), who starts with a discrete average before passing onto a continuous limit.

$$E[A(T)] = E\left[\frac{1}{T}\int_0^T F(u)du\right] = \frac{1}{T}\int_0^T E[F(u)]du = \frac{1}{T}\int_0^T F(0)du = F(0) \qquad \text{(I-1)}$$

$$E[A(T)^2] = E\left[\frac{1}{T^2}\int_0^T F(u)du \int_0^T F(v)dv\right] =$$

$$= \frac{1}{T^2}\int_0^T\int_0^T E[F(u)F(v)]dudv = \frac{2}{T^2}\int_0^T\int_0^v E[F(u)F(v)]dudv =$$

$$= \frac{2F(0)^2}{T^2}\int_0^T\int_0^v e^{\rho\sigma^2\sqrt{uv}}dudv = \frac{2F(0)^2}{T^2}\int_0^T\int_0^v e^{\sqrt{\frac{u}{v}}\sigma^2\sqrt{uv}}dudv =$$

$$= \frac{2F(0)^2}{T^2}\int_0^T\int_0^v e^{\sigma^2 u}dudv = \frac{2F(0)^2}{\sigma^2 T^2}\int_0^T (e^{\sigma^2 v} - 1)dv =$$

$$= \frac{2F(0)^2}{(\sigma^2 T)^2}\{e^{\sigma^2 T} - \sigma^2 T - 1\} \qquad \text{(I-2)}$$

$$E[A(T)F(T)] = E\left[\frac{1}{T}\int_0^T F(u)du * F(T)\right] =$$

$$= \frac{1}{T}\int_0^T E[F(u)F(T)]du = \frac{F(0)^2}{T}\int_0^T e^{\rho\sigma^2\sqrt{uT}}du =$$

$$= \frac{F(0)^2}{T}\int_0^T e^{\sqrt{\frac{u}{T}}\sigma^2\sqrt{uT}}du = \frac{F(0)^2}{T}\int_0^T e^{\sigma^2 u}du =$$

$$= \frac{F(0)^2}{\sigma^2 T}\{e^{\sigma^2 T} - 1\} \qquad \text{(I-3)}$$

Appendix II

Assume that $[A(T), F(T)]$ is jointly-lognormal such that

$$\ln A(T) \sim N(\mu_1, {v_1}^2) \tag{II-1A}$$

$$\ln F(T) \sim N(\mu_2, {v_2}^2) \tag{II-1B}$$

The moment generating function of the bivariate normal $[\ln A(T), \ln F(T)]$ is given by the expression

$$E[A(T)^{k_1} F(T)^{k_2}] = e^{\mu_1 k_1 + \mu_2 k_2 + \frac{1}{2}({v_1}^2 {k_1}^2 + 2\rho v_1 v_2 k_1 k_2 + {v_2}^2 {k_2}^2)} \tag{II-2}$$

Therefore, one can use the expressions for the first two moments of $A(T)$, as well as the joint expectation $E[A(T)F(T)]$ (see Appendix I) in order to solve for μ_1, μ_2, v_1, v_2 and ρ. Details of the derivation can be found in Finnerty (2002):

$$\mu_1 = \ln F(0) - \frac{1}{2}{v_1}^2 \tag{II-3A}$$

$$ {v_1}^2 = \ln[2\{e^{\sigma^2 T} - \sigma^2 T - 1\}] - 2\ln[\sigma^2 T] \tag{II-3B}$$

$$\mu_2 = \ln F(0) - \frac{1}{2}{v_2}^2 \tag{II-3C}$$

$$ {v_2}^2 = \sigma^2 T \tag{II-3D}$$

$$\rho v_1 v_2 = \ln\left[e^{\sigma^2 T} - 1\right] - \ln[\sigma^2 T] \tag{II-3E}$$

Equipped with this information, one can use Margrabe's formula for the value of an option to exchange one asset for another. For two underlying price processes that follow correlated geometric Brownian motions

$$\frac{dS_i}{S_i} = \alpha_i dt + \sigma_i dZ_i \tag{II-4A}$$

$$dZ_1 dZ_2 = \rho dt \tag{II-4B}$$

The formula for an option to exchange one asset for another is given by the expression

$$\begin{aligned} &e^{-rT} E[S_1(T) - S_2(T)]^+ = \\ &= e^{-rT} E[S_1(T)] N(d_1) - e^{-rT} E[S_2(T)] N(d_2) \\ &= S_1(0) e^{(\alpha_1 - r)T} N(d_1) - S_2(0) e^{(\alpha_2 - r)T} N(d_2) \end{aligned} \tag{II-5A}$$

$$d_1 = \frac{\ln\frac{S_1(0)}{S_2(0)} + \left(\alpha_1 - \alpha_2 + \frac{1}{2}v^2\right)T}{v\sqrt{T}}, \qquad d_2 = d_1 - v\sqrt{T},$$

$$v^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \tag{II-5B}$$

Note, however, that v above is the annualized volatility, whereas the generalized volatility for a time horizon T is given by $v_T = v\sqrt{T}$. Therefore, we have the following relationship

$$v_T{}^2 = v^2 T = v_1{}^2 + v_2{}^2 - 2\rho v_1 v_2 \tag{II-6}$$

Substituting (II-3B), (II-3D) and (II-3E) into (II-6), we get

$$v_T{}^2 = v^2 T = \sigma^2 T + \ln[2\{ e^{\sigma^2 T} - \sigma^2 T - 1\}] - 2\ln[e^{\sigma^2 T} - 1] \tag{II-7}$$

This is the same as the expression given by Finnerty (2002) in (1B), however, it is the square of the "time-period" volatility v_T as opposed to the square of the annualized volatility v. Plugging (II-7) into (II-5A) and (II-5B), one obtains the price of the average strike put option as

$$e^{-rT}E[A(T) - F(T)]^+ = e^{-qT}V(0)\left[2N\left(\frac{v_T}{2}\right) - 1\right] \tag{II-8A}$$

$$v_T^2 = \sigma^2 T + \ln[2\{ e^{\sigma^2 T} - \sigma^2 T - 1\}] - 2\ln[e^{\sigma^2 T} - 1] \tag{II-8B}$$

Note that the formula for the price of the average strike put option in (II-8A) and (II-8B) is very similar to the formula for the price of the equivalent at-the-money average price call option given in (24A) and (24B), and only differs by the estimate for the time-period volatility v_T. While in theory both of these prices should agree exactly because of the symmetry relationship between floating strike puts and fixed strike, floating price calls – in practice, they yield slightly different answers for the size of the marketability discount because each of these pricing formulas is an approximation rather than an exact solution. However, for the purpose of estimating the size of a marketability discount where precision is not critical, using either one of them would provide a reasonably accurate result.

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