Stochastic processes 2WB20 (Modelling Project)

2WB20
Group 5

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1 | Introduction

Emergency Medical Services (EMS) providers are vital in ensuring quick response to emergency calls, with the primary goal of minimizing response times. This objective is particularly critical in areas with unique geographical constraints, such as Texel, the largest of the Wadden Islands. Texel spans an area of 170.00 km^2 and has a population of approximately 13,500 inhabitants Texel. Due to its size and isolated location, the island faces specific challenges in emergency medical services provision.

As reported on the website "moving dot" wad [n.d.], due to the small size of the islands that constitute the archipelago, there are no hospitals on the islands, and in the event of an emergency, patients must be transported to the nearest hospital by an helicopter, ensuring immediate and rapid transport.

This report examines the factors influencing ambulance deployment and response times on Texel, utilizing stochastic modeling and simulation. We aim to identify optimal strategies in order to guarantee an efficient service, especially given the island's unique geographical constraints. Given the critical nature of response time in emergency medical situations, especially in geographically constrained areas like Texel, it is important to explore optimal strategies for ambulance deployment.

By applying both discrete-time and continuous-time Markov chain models, we aim to simulate various scenarios to understand the behavior of the EMS system under different conditions. Markov chain models are powerful mathematical tools that can capture the probabilistic nature of ambulance movements and response times. These models allow us to create a framework where we can test different deployment strategies, predict response times, and identify potential areas for improvement in the EMS system.
Mathematics and modelling play a key role in addressing the complexities of emergency medical response scenarios, a problem that is still relevant today. By exploiting mathematical principles, we can build stochastic models that simulate the dynamics of ambulance distribution.

2 | Description of the problem

In this study, we focus on optimizing ambulance dispatch in Texel, the largest of the Wadden Islands, assuming the presence of two ambulance stations on the island. Texel's rural landscape presents unique challenges for EMS management, as emphasized in the research by van Barneveld [2017]. Unlike urban areas, rural regions like Texel typically operate with a smaller fleet of ambulances.

Consequently, the unavailability of even a single ambulance, whether due to occupancy or distance, significantly impacts response capabilities. Therefore, precise deployment and redistribution of ambulances in rural regions are important to ensure optimal coverage and response times.

Furthermore, rural EMS regions experience greater variability in demand across different areas. While some areas may experience minimal demand for emergency services, others, such as urban centers or tourist destinations, may exhibit consistently higher demand. As a result, ambulances traveling between areas of high demand often traverse areas with low demand, providing limited coverage during transit.

In emergency situations requiring transportation to the mainland hospital, helicopters play a critical role due to the absence of local hospital facilities on Texel. However, operational challenges such as poor weather conditions can impede helicopter operations, necessitating alternative means of transportation for patients.

Response time serves as a critical metric within EMS systems, representing the interval from receiving the emergency call to ambulance arrival at the scene. With reference to the study on EMS system repositioning by Alanis et al. [2013], EMS performance goals mainly concentrate on how quickly emergency responders arrive at the scene for calls marked as urgent by the dispatcher.
The primary objective is to ensure that a specified percentage of high-priority calls are within predetermined time thresholds. The National Reference Framework for Distribution & Availability established an optimal distribution to reach 95% of the population within fifteen minutes of receiving an emergency call.
Despite Texel's modest size, achieving optimal coverage poses challenges due to geographical constraints, diverse population distributions and costs of maintaining ambulances.

According to data from "Ambulancezorg Nederland" amb [2022], emergency crews in the Netherlands maintain a constant state of readiness, particularly during the daytime. Rather than returning to a fixed station after responding to an incident, ambulances remain mobile. This dynamic readiness ensures swift responses to calls, with no lag in response time.

To address this scenario, several mathematical methods and operational models have been proposed to solve the main problems such as optimizing resource allocation and management in emergency medical services (EMS). Some of the approaches proposed in the literature are mathematical programming models, Markov chains and stochastic programming, which have been extensively analyzed and successfully applied in the practical operations of EMS systems.

One approach involves using a Markov chain model to analyze and optimize the dynamic repositioning of ambulances, as described by Alanis et al. [2013]. Indeed, a two-dimensional Markov chain model of an emergency medical services system is proposed and analyzed that relocates ambulances using a tabular compliance policy, commonly used in practice.
This model allows for adjusting the positions of ambulances in response to changes in system conditions, enhancing service coverage and reducing response times.

Another interesting idea, which also involves Markov chains, is the optimization of ambulance dispatching, proposed by Jagtenberg et al. [2024]. These have developed algorithms to decide which ambulance to send to a specific incident, minimising response times and maximising resource availability.

Furthermore, the reliability model by Ball and Lin [1993] concentrates on strategically situating ambulance stations. It employs a stochastic programming approach that incorporates probabilistic constraints to address both the location and sizing issues of stations. This model takes into account the inherent uncertainty in emergency service demand.

The latest approach is the one proposed by Beraldi et al. [2024], which analyzes the problem, again, from a stochastic programming point of view. This problem deals with the uncertainty and variability of service demand, with the aim of providing a robust model capable of handling possible future scenarios.

3 | Methodology

The report is structured to provide a comprehensive analysis of EMS operations on Texel. The methodology involves constructing stochastic models, b applying discrete-time and continuous-time Markov chain models, to represent the dynamics of ambulance availability and dispatch. The first model is a simple one with a single ambulance post, which helps establish a basic understanding of the system. The second, more complex model includes two ambulance posts with overlapping coverage areas, allowing for a more detailed examination of interactions between multiple posts.
Simulation plays a crucial role in this analysis. By visualizing the results through heat maps, we can identify the stationary probabilities of having a certain number of ambulances available at any given time. These simulations help in understanding the impact of various parameters, such as the probability of emergency occurrence and the service completion rate, on the overall performance of the EMS system. Moreover, we make the following assumption: in each time step only one ambulance can leave/ come back to the post.

3.1 | Simple model: One ambulance post in discrete time

We start by considering there is only one post available for the ambulances on the island. Hence the state space is $S = \{i = 0, 1, 2, ..., n\}$, where i represents the number of ambulances available in the post in each time step and with $n \geq 1$.
Let p be the probability that an emergency happens and q the probability that the service is complete. Let assume that $0 < p < 1$ and $0 < q < 1$, indeed these two describe a probability and this means they are non-negative, but we set these values not equal to zero or one, because otherwise our model would lose its logical sense and end up in very particular cases.
Then, we define $p_{ij}(n) = \mathbb{P}\{X_n = j | X_{n-1} = i\}$ where X_n describe the number of ambulances in the post at the time n.
Now we need compute the transition probabilities. Let analyzed the different cases:

1. $p_{(0,0)} = (1 - q)$ this means that if we have 0 ambulances, the next time step we will have still zero ambulances only if no emergency is completed

2. $p_{(0,1)} = q$ this means that if we have 0 ambulances, the next time step we will have an ambulance only if an emergency is completed,

3. $p_{(0,i)} = 0 \quad \forall\, i \in \{2, ..., n\}$ this means that if we start with zero ambulances we cannot have 2 ambulances or more in the next time step

4. $p_{(i,i-1)} = p(1 - q) \quad \forall\, i \in \{2, ..., n - 1\}$, this means that if we have i ambulances, the next time step we will have one less ambulance only if no emergency is completed and one emergency occurs

5. $p_{(i,i)} = (1 - p)(1 - q) + pq \quad \forall\, i \in \{2, ..., n - 1\}$, this means that if we have i ambulances, the next time step we will have still i ambulances only if an emergency is completed and one occurs or no emergency occurs and none is completed

6. $p_{(i,i+1)} = (1 - p)q \quad \forall\, i \in \{2, ..., n - 1\}$, this means that if we have i ambulances, the next time step we will have one more ambulance only if an emergency is completed and no emergency occurs

7. $p_{(i,i\pm k)} = 0 \quad \forall\, k \in \{2, ..., n - 2 - i\}$,

8. $p_{(n,n-1)} = p$ this means that if we have all the ambulances, the next time step we will have one less ambulance only if an emergency occurs

9. $p_{(n,n)} = 1 - p$ this means that if we have all the ambulances, the next time step we still will have all of them only if no emergency occurs,

10. $p_{(n,i)} = 0 \quad \forall \quad i \in \{0, ..., n-2\}$

So to sum up:

$$P_{ij} = \begin{cases} 1-q & \text{if } i=0 \text{ and } j=0 \\ q & \text{if } i=0 \text{ and } j=1 \\ p(1-q) & \text{if } i=j-1 \text{ and } i \neq 0 \\ (1-p)(1-q)+pq & \text{if } i=j \text{ and } i\neq 0 \text{ and } i \neq n \\ (1-p)q & \text{if } i=j+1 \text{ and } i \neq 0 \text{ and } i \neq n \\ p & \text{if } i=n \text{ and } j=n-1 \\ 1-p & \text{if } i=n \text{ and } j=n \\ 0 & \text{otherwise} \end{cases}$$

Plugging it into a matrix form, it becomes:

$$P = \begin{pmatrix} 1-q & q & 0 & 0 & \cdots & 0 & 0 & 0 \\ p(1-q) & (1-p)(1-q)+pq & (1-p)q & 0 & \cdots & 0 & 0 & 0 \\ 0 & p(1-q) & (1-p)(1-q)+pq & (1-p)q & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p(1-q) & (1-p)(1-q)+pq & (1-p)q \\ 0 & 0 & 0 & 0 & \cdots & 0 & p & 1-p \end{pmatrix}$$

The model described above can be represented with the transition diagram in the following figure (3.1)

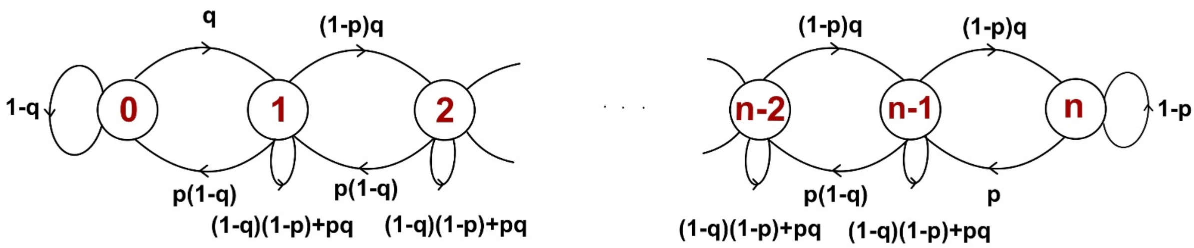

Figure 3.1: Transition diagram of the Markov chain

Having obtained this probability matrix, the following information can be computed:

1. the stationary probabilities $\Pi_j \quad \forall j \in \{0, ..., n\}$ which represent the probability that we have a number of ambulances equal to j in the post;
2. the limiting probability $p_{ij} = \lim_{n\to\infty} \mathbb{P}\{X_n = j | X_0 = i\}$ which represent the probability that for n that goes to infinity, we have j ambulances, if we start with i ambulances;
3. $\mathbb{E}[T_{i,j}]$ where $T_{i,j} = inf\{n \geq 1 \ : \ X_n = j | X_0 = i\}$ which represent the expected amount of time for going from i to j ambulances in the post.

3.1.1 | Stationary probabilities

Let find these values step by step. In order to find the stationary probabilities two different situations must be analyzed.
Firstly, we solve the following system of equations making the assumption that $p \neq q$:

$$\begin{cases} \Pi_0 = (1-q)\Pi_0 + p(1-q)\Pi_1 \\ \Pi_1 = q\Pi_0 + ((1-p)(1-q)+pq)\Pi_1 + p(1-q)\Pi_2 \\ \Pi_2 = (1-p)q\Pi_1 + ((1-p)(1-q)+pq)\Pi_2 + p(1-q)\Pi_3 \\ \vdots \\ \Pi_i = (1-p)q\Pi_{i-1} + ((1-p)(1-q)+pq)\Pi_i + p(1-q)\Pi_{i+1} \\ \vdots \\ \Pi_{n-2} = (1-p)q\Pi_{n-3} + ((1-p)(1-q)+pq)\Pi_{n-2} + p(1-q)\Pi_{n-1} \\ \Pi_{n-1} = (1-p)q\Pi_{n-2} + ((1-p)(1-q)+pq)\Pi_{n-1} + p\Pi_n \\ \Pi_n = (1-p)q\Pi_{n-1} + (1-p)\Pi_n \end{cases}$$

Let assume we have the relation:

$$\Pi_{i+1} = k\,\Pi_i$$

By substituting these values within the system we obtain the following relation:
∘ if $i = 0$:

$$\Pi_1 = \frac{q}{p(1-q)}\Pi_0$$

∘ if $i = 1$:
we substitute what we have found in the previous relation in the second equation of our system

$$\Pi_2 = \frac{q(1-p)}{p(1-q)}\Pi_1$$

∘ if $2 < i < n-2$:

$$\Pi_{i+1} = \frac{p+q-2pq \pm \sqrt{(2pq-p-q)^2 - 4(1-p)(1-q)pq}}{2p(1-q)}\Pi_i = \begin{cases} 1 \cdot \Pi_i \\ \frac{q(1-p)}{p(1-q)}\Pi_i \end{cases}$$

Obviously, the first solution is the trivial solution of the equation $\Pi = \Pi P$, so we have that

$$\Pi_{i+1} = \frac{q(1-p)}{p(1-q)}\Pi_i$$

∘ if $i = n$:

$$\Pi_n = \frac{q(1-p)}{p}\Pi_{n-1}$$

∘ if $i = n-1$: we substitute what we have found in the previous relation in the second-to-last equation of our system

$$\Pi_{n-1} = \frac{q(1-p)}{p(1-q)}\Pi_{n-2}$$

For simplicity of notation, we define the following parameters:

$$\alpha = \frac{q}{p(1-q)}$$

$$\beta = \frac{q(1-p)}{p(1-q)}$$

$$\gamma = \frac{q(1-p)}{p}$$

So now we have

$$\begin{cases} \Pi_1 = \alpha\Pi_0 \\ \Pi_2 = \beta\Pi_1 = \alpha\beta\Pi_0 \\ \Pi_3 = \beta\Pi_2 = \alpha\beta^2\Pi_0 \\ \vdots \\ \Pi_i = \alpha\beta^{i-1}\Pi_0 \\ \vdots \\ \Pi_{n-1} = \beta\Pi_{n-2} = \alpha\beta^{n-2}\Pi_0 \\ \Pi_n = \gamma\Pi_{n-1} = \gamma\alpha\beta^{n-2}\Pi_0 \end{cases}$$

But we also know that:

$$\sum_{i=0}^{n} \Pi_i = 1$$

So

$$\Pi_0 + \sum_{i=1}^{n-1} \alpha\beta^{i-1}\Pi_0 + \gamma\alpha\beta^{n-2}\Pi_0 = 1 \tag{3.1}$$

That is equal to:

$$\Pi_0 + \alpha\frac{\beta - \beta^n}{\beta - \beta^2}\Pi_0 + \gamma\alpha\beta^{n-2}\Pi_0 = 1$$

So we can conclude that:

$$\Pi_0 = \frac{1}{\omega}$$

where

$$\omega = 1 + \alpha\frac{\beta - \beta^n}{\beta - \beta^2} + \gamma\alpha\beta^{n-2}$$

So we have obtained that

$$\begin{cases} \Pi_0 = \frac{1}{\omega} \\ \Pi_1 = \frac{\alpha}{\omega} \\ \Pi_2 = \frac{\alpha\beta}{\omega} \\ \vdots \\ \Pi_i = \frac{\alpha\beta^{i-1}}{\omega} \\ \vdots \\ \Pi_{n-1} = \frac{\alpha\beta^{n-2}}{\omega} \\ \Pi_n = \frac{\gamma\alpha\beta^{n-2}}{\omega} \end{cases}$$

Whereas if $p = q$ then the previous system is simplified:

$$\begin{cases} \Pi_0 = (1-p)\Pi_0 + p(1-p)\Pi_1 \\ \Pi_1 = p\Pi_0 + (2p^2 - 2p + 1)\Pi_1 + p(1-p)\Pi_2 \\ \Pi_2 = (1-p)p\Pi_1 + (2p^2 - 2p + 1)\Pi_2 + p(1-p)\Pi_3 \\ \vdots \\ \Pi_i = (1-p)p\Pi_{i-1} + (2p^2 - 2p + 1)\Pi_i + p(1-p)\Pi_{i+1} \\ \vdots \\ \Pi_{n-2} = (1-p)p\Pi_{n-3} + (2p^2 - 2p + 1)\Pi_{n-2} + p(1-p)\Pi_{n-1} \\ \Pi_{n-1} = (1-p)p\Pi_{n-2} + (2p^2 - 2p + 1)\Pi_{n-1} + p\Pi_n \\ \Pi_n = (1-p)p\Pi_{n-1} + (1-p)\Pi_n \end{cases}$$

By adopting the previous method, it can be concluded that

$$\alpha = \frac{1}{1-p}$$

$$\beta = 1$$
$$\gamma = 1 - p$$

and

$$\begin{cases} \Pi_0 = \frac{1}{\omega} \\ \Pi_1 = \frac{\alpha}{\omega} \\ \Pi_2 = \frac{\alpha}{\omega} \\ \vdots \\ \Pi_i = \frac{\alpha}{\omega} \\ \vdots \\ \Pi_{n-1} = \frac{\alpha}{\omega} \\ \Pi_n = \frac{\gamma\alpha}{\omega} \end{cases}$$

where

$$\omega = 1 + (n-1)\alpha + \gamma\alpha = 2 + \frac{n-1}{1-p}.$$

3.1.2 | Limiting probabilities

In order to compute the stationary probabilities we have to make 3 different assumptions:

- if $p < q$, in this way we ensure that our chain is irreducible and ergodic. In this way we have all the hypotheses to apply the Theorem 2 of the Handout 1 of this course, which let us to conclude that

$$\lim_{n\to\infty} \mathbb{P}\{X_n = j | X_0 = i\} = \Pi_j \quad \forall j \in S;$$

- if $p = q$ the arrival rate of events (service requests) is equal to the completion rate. This implies that the system has a dynamic equilibrium where service requests are fulfilled at exactly the same rate as they arrive. The Markov chain will be recurrent, which means that each state will be visited infinite times in time. However, the chain will be classified as null-recurrent, because the average time to return to a specific state will be infinite. This is due to the nature of the perfect equilibrium between arrival and completion of emergence. So can conclude that:

$$\lim_{n\to\infty} \mathbb{P}\{X_n = j | X_0 = i\} = 0 \quad \forall j \in S;$$

- if $p > q$, the arrival rate of events p is greater than the completion rate q. This means that the service demands (emergency arrivals) exceed the completion capacity (available ambulances) and ambulances are unable to meet the demand for emergencies. In this case the Markov chain is transient so we have that $\mathbb{E}[T_j] = \infty$ for all $j \in S$, and thus

$$\lim_{n\to\infty} \mathbb{P}\{X_n = j | X_0 = i\} = 0 = 0 \quad \forall j \in S.$$

3.1.3 | Expected time step

For this last point we compute the expected value of the two most relevant cases. First of all it must be observed that the Theorem 1 of the Handout 1 holds if $p < q$ so we can state that

$$\mathbb{E}[T_i] = \mathbb{E}[T_{i,i}] = \frac{1}{\Pi_i} \quad \forall i \in S,$$

otherwise $\mathbb{E}[T_i] = \infty$. These values describe the expected amount of time needed to start with i ambulances and end with the same amount. Then, the other relevant cases are $\mathbb{E}[T_{0,n}]$ and $\mathbb{E}[T_{n,0}]$. These two value represent the expected amount of time that it is needed to start with no ambulance and end with all of

them, and vice versa.
In order to find the value of $\mathbb{E}[T_{0,n}]$ we need to solve the following system:

$$\begin{cases} E_0 = 1 + (1-q)E_0 + qE_1 \\ E_1 = 1 + p(1-q)E_0 + ((1-p)(1-q)+pq)E_1 + (1-p)qE_2 \\ \vdots \\ E_i = 1 + p(1-q)E_{i-1} + ((1-p)(1-q)+pq)E_i + (1-p)qE_{i+1} \\ \vdots \\ E_{n-1} = 1 + p(1-q)E_{n-2} + ((1-p)(1-q)+pq)E_{n-1} + (1-p)qE_n \\ E_n = \frac{1}{\Pi_n} \end{cases}$$

For the sake of simplicity, we have denoted $E_i = \mathbb{E}[T_{i,n}]$. Since solving this system computationally complex, we solved the linear system of equations by simulation. The following result is given by setting $n = 5$ and we reported this expression for $\mathbb{E}[T_{0,n}] = E_0$:

$$\begin{aligned} E_0 = \frac{1}{(-1+p)^4 q^5 \Pi_n} \big(& q^5 - 4pq^5 + 6p^2q^5 - 4p^3q^5 + p^4q^5 \\ & + p^4\Pi_n + 2p^3q\Pi_n - 5p^4q\Pi_n + 3p^2q^2\Pi_n \\ & - 10p^3q^2\Pi_n + 10p^4q^2\Pi_n + 4pq^3\Pi_n - 15p^2q^3\Pi_n \\ & + 20p^3q^3\Pi_n - 10p^4q^3\Pi_n + 5q^4\Pi_n - 20pq^4\Pi_n \\ & + 30p^2q^4\Pi_n - 20p^3q^4\Pi_n + 5p^4q^4\Pi_n \big) \end{aligned}$$

Similarly, we solved the system to find $\mathbb{E}[T_{n,0}]$:

$$\begin{cases} E'_n = 1 + pE'_{n-1} + (1-p)E'_n \\ E'_{n-1} = 1 + p(1-q)E'_{n-2} + ((1-p)(1-q)+pq)E'_{n-1} + (1-p)qE'_n \\ \vdots \\ E'_i = 1 + p(1-q)E'_{i-1} + ((1-p)(1-q)+pq)E'_i + (1-p)qE'_{i+1} \\ \vdots \\ E'_1 = 1 + p(1-q)E'_0 + ((1-p)(1-q)+pq)E'_1 + (1-p)qE'_2 \\ E'_0 = \frac{1}{\Pi_0} \end{cases}$$

For the sake of simplicity, we have denoted $E'_i = \mathbb{E}[T_{i,0}]$. As for the previous case, we solved the linear system of equations by simulation and we set $n = 5$

$$\begin{aligned} & (4\Pi_0p^5q^5 - 8\Pi_0p^5q^4 + 6\Pi_0p^5q^3 - 2\Pi_0p^5q^2 - 10\Pi_0p^4q^5 + 21\Pi_0p^4q^4 - 17\Pi_0p^4q^3 + 11\Pi_0p^4q^2 \\ & - 4\Pi_0p^4q + \Pi_0p^4 + 10\Pi_0p^3q^5 - 22\Pi_0p^3q^4 + 9\Pi_0p^3q^3 - 4\Pi_0p^3q^2 - 5\Pi_0p^2q^5 + 15\Pi_0p^2q^4 \\ & - 2\Pi_0p^2q^3 + \Pi_0p^2q^2 + \Pi_0pq^5 - 6\Pi_0pq^4 + \Pi_0q^4 - p^5q^4 + 4p^5q^3 - 6p^5q^2 + 4p^5q - p^5)/ \\ & (\Pi_0p^6q^5 - 4\Pi_0p^6q^4 + 6\Pi_0p^6q^3 - 4\Pi_0p^6q^2 + \Pi_0p^6q - \Pi_0p^5q^5 + 5\Pi_0p^5q^4 - 10\Pi_0p^5q^3 \\ & + 10\Pi_0p^5q^2 - 5\Pi_0p^5q + \Pi_0p^5) \end{aligned}$$

3.2 | Simple model: One ambulance post in continuous time

In this subsection we analysed the same problem, but with a continuous Markov chain. The relationship between discrete-time and continuous-time models can be observed through the parameters λ and μ, which essentially represent the continuous counterparts to the probabilities p and q. While p and q in the discrete time represent the probabilities of emergencies happening and service completion respectively, λ and μ define the rates of these events by representing the leaving and arriving of ambulances at the station.

We can describe the continuous Markov chain with a birth-death process. We keep the same state space $S = \{i = 0, 1, 2, ..., n\}$, and let set the $\lambda_i = \lambda \;\; \forall i \in S$ as the birth rates and $\mu_i = \mu \;\;\; \forall i \in S$ as the death rates, such that $\lambda > 0$ and $\mu > 0$.
We can represent the model described above with the transition diagram in the following figure (3.2)

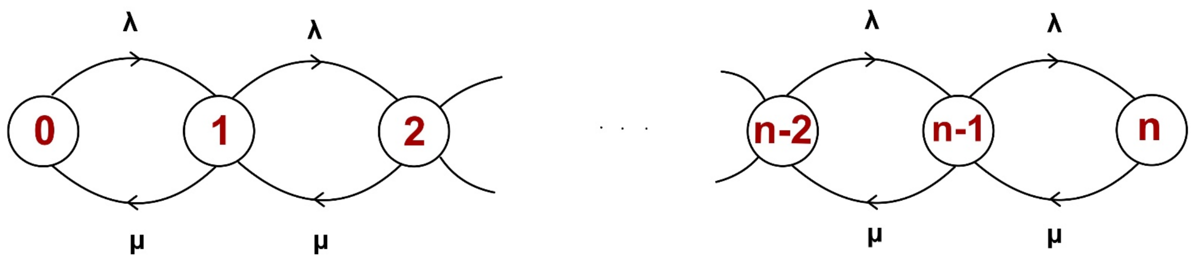

Figure 3.2: Transition diagram of the birth-death process

The generator $G = (g_{ij} : i, j \geq 0)$ is given by:

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu & -(\lambda+\mu) & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu & -\mu \end{pmatrix}$$

From this matrix, it is easy to see that $\nu_i \equiv \nu = \lambda + \mu \ \forall i \in S$ and so the transition probabilities are $p_{ij} = \frac{g_{ij}}{\nu}$ $, i \neq j$. Hence, the transition matrix P for the associated embedded Markov chain is:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

3.2.1 | Stationary probabilities for G

Now, it is possible to compute the stationary distribution by solving the system $\Pi G = 0$:

$$\begin{cases} -\lambda\Pi_0 + \mu\Pi_1 = 0 \\ \lambda\Pi_0 - (\lambda+\mu)\Pi_1 + \mu\Pi_2 = 0 \\ \vdots \\ \lambda\Pi_{i-1} - (\lambda+\mu)\Pi_i + \mu\Pi_{i+1} = 0 \\ \vdots \\ \lambda\Pi_{n-2} - (\lambda+\mu)\Pi_{n-1} + \mu\Pi_n = 0 \\ \lambda\Pi_{n-1} - \mu\Pi_n = 0 \end{cases}$$

From this we find the following relation:

$$\Pi_i = \left(\frac{\lambda}{\mu}\right)^i \Pi_0$$

But we also know that:

$$\sum_{i=0}^{n} \Pi_i = 1$$

So we have:

$$\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i \Pi_0 = 1$$

Which allows us to state that

$$\Pi_0 = \frac{1}{\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i} = \frac{(\lambda-\mu)\mu^n}{\lambda^{n+1} - \mu^{n+1}}$$

and

$$\begin{cases} \Pi_0 = \frac{(\lambda-\mu)\mu^n}{\lambda^{n+1}-\mu^{n+1}} \\ \Pi_1 = \frac{\lambda}{\mu}\frac{(\lambda-\mu)\mu^n}{\lambda^{n+1}-\mu^{n+1}} \\ \vdots \\ \Pi_i = \frac{\lambda^i}{\mu^i}\frac{(\lambda-\mu)\mu^n}{\lambda^{n+1}-\mu^{n+1}} \\ \vdots \\ \Pi_{n-1} = \frac{\lambda^{n-1}}{\mu^{n-1}}\frac{(\lambda-\mu)\mu^n}{\lambda^{n+1}-\mu^{n+1}} \\ \Pi_n = \frac{\lambda^n}{\mu^n}\frac{(\lambda-\mu)\mu^n}{\lambda^{n+1}-\mu^{n+1}} \end{cases}$$

Whereas if $\lambda = \mu$ then the previous system is simplified:

$$\begin{cases} -\lambda\Pi_0 + \lambda\Pi_1 = 0 \\ \lambda\Pi_0 - 2\lambda\Pi_1 + \lambda\Pi_2 = 0 \\ \vdots \\ \lambda\Pi_{i-1} - 2\lambda\Pi_i + \lambda\Pi_{i+1} = 0 \\ \vdots \\ \lambda\Pi_{n-2} - 2\lambda\Pi_{n-1} + \lambda\Pi_n = 0 \\ \lambda\Pi_{n-1} - \lambda\Pi_n = 0 \end{cases}$$

Which means that all the Π_i are equal to each other. So we have

$$\sum_{i=0}^{n} \Pi_0 = (n+1)\Pi_0 = 1$$

Which allows us to state that if $\lambda = \mu$

$$\Pi_i = \frac{1}{n+1} \quad \forall i \in S$$

3.2.2 | Stationary probabilities for P

In order to have the stationary distribution $\hat{\Pi}$, from the associated embedded Markov chain, we have to compute $\hat{\Pi} = \hat{\Pi}P$

$$\begin{cases} \hat{\Pi}_0 = \frac{\mu}{\lambda+\mu}\hat{\Pi}_1 \\ \hat{\Pi}_1 = \hat{\Pi}_0 + \frac{\mu}{\lambda+\mu}\hat{\Pi}_2 \\ \hat{\Pi}_2 = \frac{\lambda}{\lambda+\mu}\hat{\Pi}_1 + \frac{\mu}{\lambda+\mu}\hat{\Pi}_3 \\ \vdots \\ \hat{\Pi}_i = \frac{\lambda}{\lambda+\mu}\hat{\Pi}_{i-1} + \frac{\mu}{\lambda+\mu}\hat{\Pi}_{i+1} \\ \vdots \\ \hat{\Pi}_{n-1} = \frac{\lambda}{\lambda+\mu}\hat{\Pi}_{n-2} + \hat{\Pi}_n \\ \hat{\Pi}_n = \frac{\lambda}{\lambda+\mu}\hat{\Pi}_{n-1} \end{cases}$$

From this we find the following relation:

$$\hat{\Pi}_i = \frac{\lambda}{\mu}\hat{\Pi}_{i-1} \quad \forall\ 2 \le i \le n-1$$

$$\hat{\Pi}_i = \left(\frac{\lambda}{\mu}\right)^{i-1}\frac{\lambda+\mu}{\mu}\hat{\Pi}_0 \quad \forall\ 1 \le i \le n-1$$

and

$$\hat{\Pi}_n = \frac{\lambda}{\lambda+\mu}\hat{\Pi}_{n-1} = \frac{\lambda}{\lambda+\mu}\left(\frac{\lambda}{\mu}\right)^{n-1}\frac{\lambda+\mu}{\mu}\hat{\Pi}_0 = \left(\frac{\lambda}{\mu}\right)^{n}\hat{\Pi}_0$$

But we also know that:

$$\sum_{i=0}^{n} \hat{\Pi}_i = 1$$

So we have:

$$\hat{\Pi}_0 + \sum_{i=1}^{n-1} \left(\frac{\lambda}{\mu}\right)^{i-1} \frac{\lambda+\mu}{\mu} \hat{\Pi}_0 + \left(\frac{\lambda}{\mu}\right)^n \hat{\Pi}_0 = 1$$

So we can conclude that:

$$\hat{\Pi}_0 = \frac{1}{\omega}$$

where

$$\omega = 1 + \sum_{i=1}^{n-1} \left(\frac{\lambda}{\mu}\right)^{i-1} \frac{\lambda+\mu}{\mu} + \left(\frac{\lambda}{\mu}\right)^n = 1 + \left(\frac{\lambda}{\mu}\right)^n - \frac{(\lambda+\mu)\left(\lambda - \frac{\lambda^n}{\mu^{n-1}}\right)}{\lambda(\lambda-\mu)}$$

and

$$\begin{cases} \hat{\Pi}_0 = \frac{1}{\omega} \\ \hat{\Pi}_1 = \frac{\lambda}{\mu} \frac{\lambda+\mu}{\mu} \frac{1}{\omega} \\ \hat{\Pi}_2 = \frac{\lambda}{\mu} \frac{\lambda+\mu}{\mu} \frac{1}{\omega} \\ \vdots \\ \hat{\Pi}_i = \left(\frac{\lambda}{\mu}\right)^{i-1} \frac{\lambda+\mu}{\mu} \frac{1}{\omega} \\ \vdots \\ \hat{\Pi}_{n-1} = \left(\frac{\lambda}{\mu}\right)^{n-2} \frac{\lambda+\mu}{\mu} \frac{1}{\omega} \\ \hat{\Pi}_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{\omega} \end{cases}$$

3.3 | Complex model: two ambulance posts with overlap

In this section, we are going to design a stochastic model that analyzes emergency medical service response on Texel, the largest of the Wadden islands with two ambulance posts having overlapping coverage areas. We start using a Markov chain approach to determine the probability of timely response and the optimal number of ambulances. We begin by defining the state space of our Markov chain, S. $S = \{(i, j)|i, j = 0, 1, 2, ..., n\}$, where i and j represent the number of ambulances available at post 1 and 2 respectively. Transition between states occur due to two main types of events: ambulances being dispatched to emergencies and ambulances returning from emergencies. Thus, their probabilities can be defined based on the rates of these events: λ, the rate of medical emergencies; μ, the rate at which a single ambulance completes its task and returns to its post. Moreover we can assume that λ and μ are constants. We finally assume that there is no driving time. Then the transition probabilities are:

1. $P((i,j) \to (i-1,j)) = \frac{i}{i+j}\lambda$ if $i > 0$ (dispatching an ambulance from post 1)
2. $P((i,j) \to (i,j-1)) = \frac{j}{i+j}\lambda$ if $j > 0$ (dispatching an ambulance from post 2)
3. $P((i,j) \to (i+1,j)) = \mu$ if $i < n$ (an ambulance returns to post 1)
4. $P((i,j) \to (i,j+1)) = \mu$ if $j < n$ (an ambulance return to post 2)

However, this is a simplified version of the model, where emergencies are divided by ambulances places on two different posts but what if there is an overlapping coverage areas on the island? In that case, an ambulance from either post could respond to an emergency and we assume that it will depend on the relative availability of ambulances at each post.

Assume that fraction α of the emergencies occurs in overlapping area and 1-α be the fraction of emergencies occurring outside the overlapping area, where closest ambulance responds.
Let (x,y) be the coordinate where the emergency happens and let $d_1(x,y)$, $d_2(x,y)$ be the distances from (x,y) and post 1 and 2 respectively. Then the probabilities of dispatching from post 1 and 2 are respectively:

$$p_1(i,j,x,y) = \frac{\frac{1}{d_1(x,y)}}{\frac{1}{d_1(x,y)} + \frac{1}{d_2(x,y)}}, p_2(i,j,x,y) = \frac{\frac{1}{d_2(x,y)}}{\frac{1}{d_1(x,y)} + \frac{1}{d_2(x,y)}}$$

We assume that the island has a shape of a rectangle. The aim is to calculate the expected dispatched probabilities $E(p_1)$ and $E(p_2)$ which represent the average probability of dispatch from post1 and post2 respectively.

Define ρ_1 the proportion of the island's total area that is covered exclusively by post 1.
Define ρ_2 as the proportion of the island's total area where the two posts overlap.
Define ρ_3 as the proportion of the island's total area that is covered exclusively by post 2.

Then we have that :
$\alpha = \frac{\rho_2}{\rho_1+\rho_2+\rho_3}$
$\mathbb{E}[p_1] = \frac{\rho_1+\frac{1}{2}\rho_2}{\rho_1+\rho_2+\rho_3}$
$\mathbb{E}[p_2] = \frac{\rho_3+\frac{1}{2}\rho_2}{\rho_1+\rho_2+\rho_3}$

Then the transition probabilities become:

1. $P((i,j) \to (i-1,j)) = \lambda((1-\alpha) + \alpha E(p_1))$ if $i > 0$ (dispatching an ambulance from post 1)
2. $P((i,j) \to (i,j-1)) = \lambda((1-\alpha) + \alpha E(p_2))$ if $j > 0$ (dispatching an ambulance from post 2)
3. $P((i,j) \to (i+1,j)) = \mu$ if $i < n$ (an ambulance returns to post 1)
4. $P((i,j) \to (i,j+1)) = \mu$ if $j < n$ (an ambulance return to post 2)

We consider this another interesting approach to our problem, but due to lack of time we were unable to conduct further analysis and derive results from this model. So from now on the results in the following sections concern the models from subsection 3.1 and subsection 3.2.

4 | Simulation

4.1 | Heat Map Analysis of Discrete Time Stationary Probabilities

To analyze the performance and behavior of our EMS system on Texel Island, we conducted simulations to generate heat maps for the stationary probabilities Π_0 and Π_n. The probabilities were calculated over a grid of p and q values ranging from 0 to 1, and the results were visualized as heat maps.

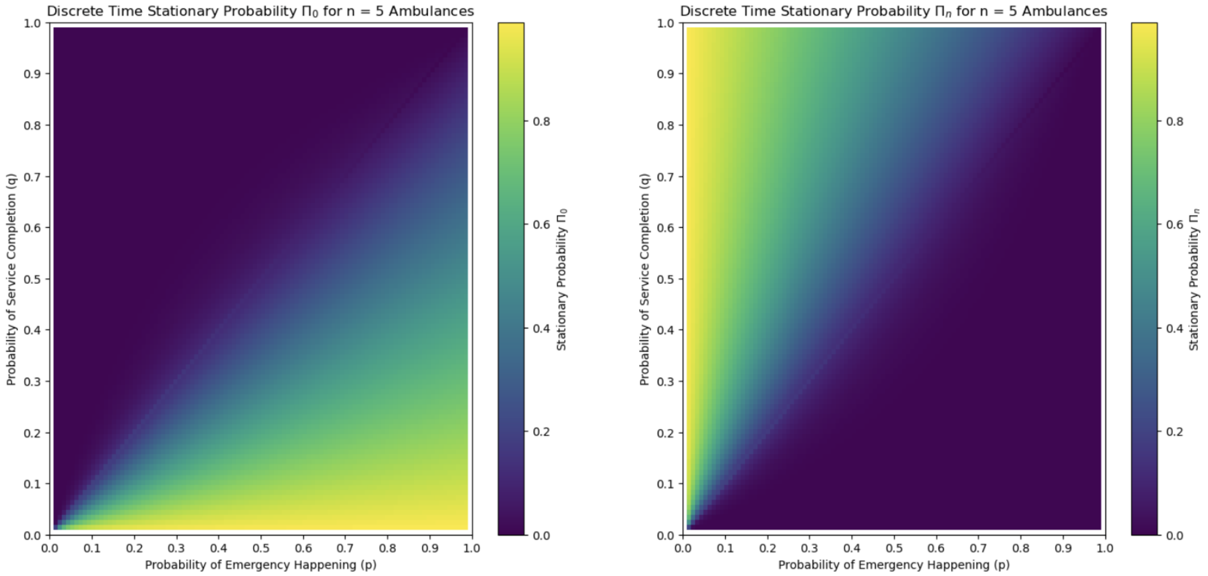

Figure 4.1: Heat Maps of Discrete Time Stationary Probability Π_0 and Π_n for $n = 5$ Ambulances

Heat Map of Stationary Probability Π_0 (Left of Figure 4.1): This heat map represents the likelihood of having zero ambulances available.

- **Low p and q values**: At low p and q values, Π_0 can be low if services are completed faster than emergencies occur, or high if there are too many emergencies without enough completions.
- **Increasing p with low q**: As p increases with low q, Π_0 increases. This suggests a higher likelihood of having no ambulances available because the high emergency rate leads to more ambulances being dispatched, while the low service completion rate means that ambulances are slow to return to availability.
- **Increasing p proportionally with q**: When p increases proportionally with q, the probability Π_0 remains low. This indicates an optimal allocation of ambulances where the rate of emergencies and the rate of service completions are balanced, ensuring that ambulances are available even as they are frequently dispatched.
- **High p and q values**: At high p and q values, Π_0 can be low if services are completed faster than emergencies occur, or high if there are too many emergencies without enough completions.

Heat Map of Stationary Probability Π_n (Right of Figure 4.1): This heat map represents the likelihood of having all n ambulances available.

- **Low p and q values**: At low p and q values, Π_n is high if services are completed faster than emergencies occur, allowing more ambulances to return to the station. Π_n is low if there are too many emergencies without enough completions, causing many ambulances to remain outside.
- **Increasing q with low p**: When the probability of service completion (q) increases faster than the probability of emergencies happening (p), Π_n increases. This is because the few emergencies that do occur are quickly resolved, allowing ambulances to return to availability more rapidly. As a result, the likelihood of having all ambulances available at the station increases.

- **Increasing p proportionally with q**: When p and q increase proportionally, Π_n can be moderate to high, indicating that an optimal balance is maintained where ambulances are frequently dispatched and returned to availability in a balanced manner.
- **High p and q values**: At high p and q values, Π_n can be high if the service completion rate keeps pace with the emergency rate, ensuring that ambulances are quickly returned to availability after being dispatched, or low otherwise.

4.2 | Heat Map Analysis of Continuous Time Stationary Probabilities

To analyze the performance and behavior of our EMS system with continuous time assumptions, we conducted simulations to generate heat maps for the stationary probabilities Π_0 and Π_n. The probabilities were calculated over a grid of λ and μ values ranging from 0 to 1, and the results were visualized as heat maps.

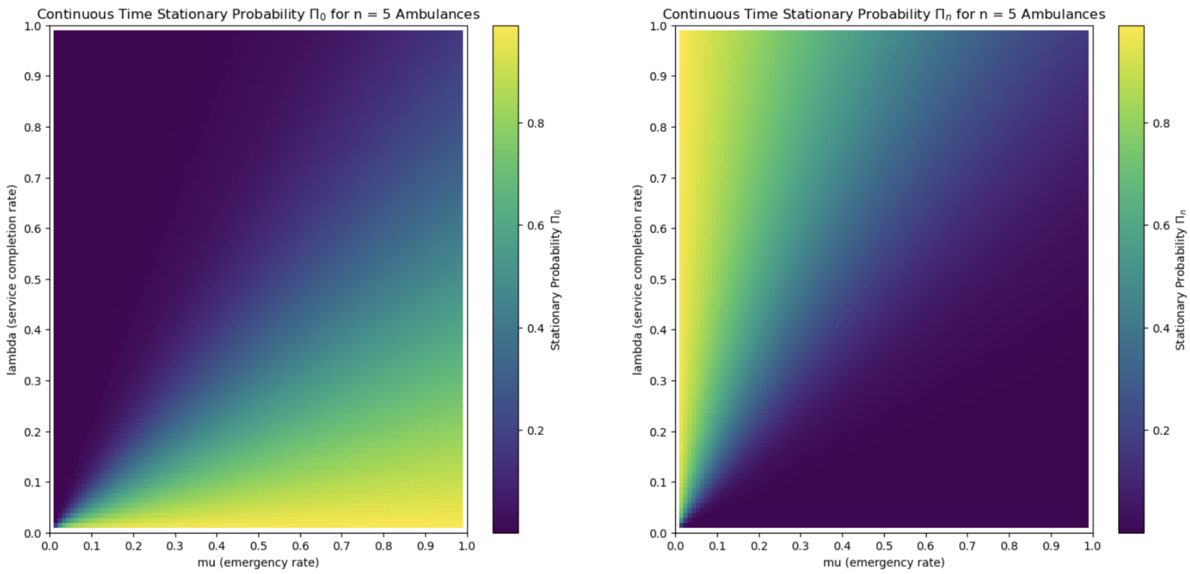

Figure 4.2: Heat Maps of Continuous Time Stationary Probability Π_0 and Π_n for $n = 5$ Ambulances.

Heat Map of Stationary Probability Π_0 (Left of Figure 4.2): This heat map represents the likelihood of having zero ambulances available.

- **Low λ and μ values:** At low λ and μ values, Π_0 can be low if ambulances return to the station faster than they leave, or high otherwise.
- **Increasing μ with low λ:** As μ increases with low λ, Π_0 increases and is high. This suggests a higher likelihood of having no ambulances available because ambulances are leaving at a faster rate than they are returning to the station.
- **Increasing μ proportionally with λ:** When λ and μ increase proportionally, Π_n remains low to moderate. This indicates an optimal allocation of ambulances where the rate of emergencies and the rate of service completions are balanced, ensuring that ambulances are available even as they are frequently dispatched.
- **High λ and μ values:** At high λ and μ values, Π_0 can be low if ambulances return faster than leave the station, or high if there are too many emergencies without enough completions.

Heat Map of Stationary Probability Π_n (Right of Figure 4.2): This heat map represents the likelihood of having all n ambulances available.

- **Low λ and μ values:** At low λ and μ values, Π_n can be high if ambulances return to the station faster than they leave, or low otherwise.

- **Increasing λ with low μ:** As λ increases with low μ, Π_n increases and is high. This suggests a higher likelihood of having all ambulances available because ambulances are returning at a faster rate than they are leaving the station.
- **Increasing μ proportionally with λ:** When λ and μ increase proportionally, Π_n remains low to moderate. This indicates that an optimal balance is maintained between dispatching ambulances frequently and returning to the station.
- **High λ and μ values:** At high λ and μ values, Π_n can be high if the service completion rate keeps pace with the emergency rate, ensuring that ambulances are quickly returned to availability after being dispatched, or low otherwise.

4.3 | Heat Map Analysis of Embedded Chain Stationary Probabilities

In this section, we explore the stationary probabilities for the embedded Markov chain associated with our EMS system. The embedded Markov chain is derived from the continuous-time Markov chain (CTMC) by observing the system at the instances of state transitions.

The relationship between the embedded chain and the continuous-time chain lies in the fact that the embedded chain captures the sequence of state transitions, while the CTMC captures the timing of these transitions. Therefore, the stationary probabilities of the embedded chain provide insights into the relative frequencies of states, which align with the probabilities calculated for the CTMC in continuous time when normalized by the transition rates.

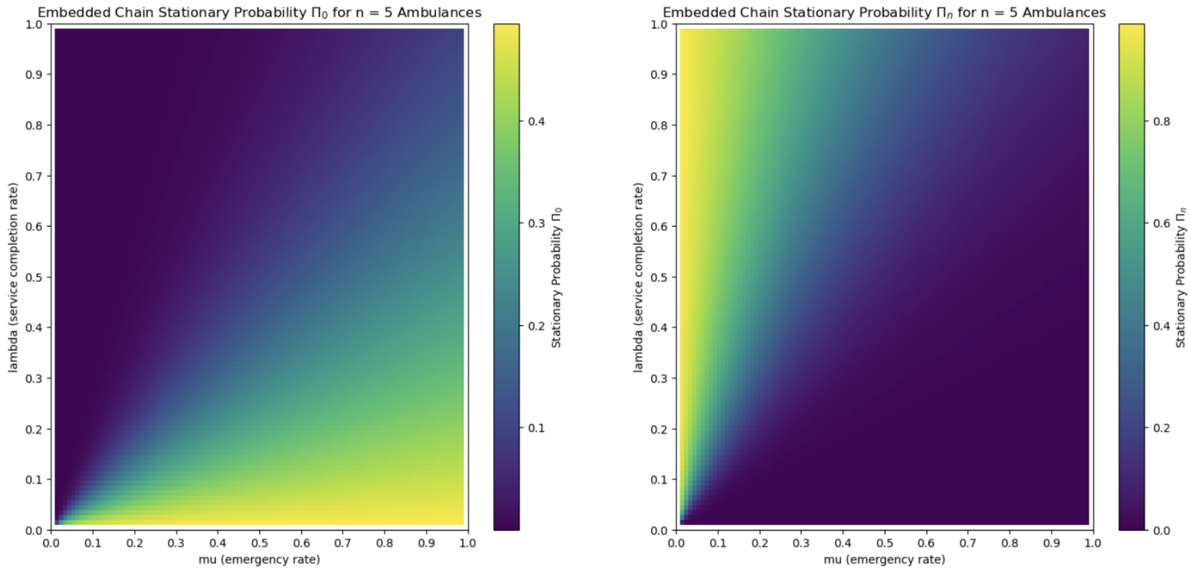

Figure 4.3: Heat Maps of Stationary Probability Π_0 and Π_n for $n = 5$ Ambulances (same as Figure 4.2).

We will not delve into a detailed analysis of the heat maps again, as the explanations provided in Section 4.2 are applicable, except in the case where μ is increasing proportionally with λ. In this case for the embedded chain, Π_n is lower compared to the continuous-time chain. This difference arises because, in the embedded chain, we only observe the system at the points of transitions, which might not fully capture the dynamics of service completions and emergency arrivals that occur in continuous time.

Thus, the heat maps for the embedded chain stationary probabilities Π_0 and Π_n are effectively the same as those presented in Figure 4.2, with the noted exception. Figure 4.3 reaffirms the consistency between the embedded chain and the continuous-time analysis, with the slight difference for Π_n as mentioned.

In summary, the stationary probabilities for the embedded Markov chain confirm the findings from the continuous-time analysis, providing a robust understanding of the system's behavior across different emergency and service completion rates.

5 | Conclusion

In this research, stochastic models were analyzed to optimize emergency medical services in the Wadden Islands, focusing specifically on Texel Island.

A discrete-time Markov chain was selected as a first scenario where one post is available in the Island. Results from Section 4.1 highlight the crucial role played by probabilities of an emergency occurring (p) and being resolved (q) in determining ambulance availability at the post.

The robustness and adaptability of our approach are additional strengths. The model is extremely flexible, allowing easy integration of new variables or adaptation to changes in operating conditions. For example, we are able to quickly reflect new ambulance allocation policies by modifying the model as needed. In addition, the simplicity of the data required for our discrete-time model is a significant advantage, simplifying data collection and management and making the process more efficient and accessible.

The second approach used is the use of a birth death process, keeping the same state space as in the previous case. In this context, the arrival of an ambulance occurs with a λ rate and its departure with a μ rate: again, these two variables play a key role in the value assumed by the stationary probabilities (as shown in Section 4.2).
This model allows very detailed temporal solution, accurately representing events occurring in real time. This is particularly useful for the purpose of capturing rapid changes in ambulance availability, improving accuracy in resource management. This approach can more realistically represent the complexity and variability of the events and continuous variations in resource availability, providing a more accurate and reliable view of the system.

Both approaches have common limitations. Firstly, they assumes that at each time instant there is at most one island-wide emergency and that at most one emergency is resolved. Of course, in real life it could be the case that at the same time instant multiple emergencies occur.
In addition, these models do not consider the time it takes for ambulances to reach the scene of the emergency, provide aid, and return back to post.

Through an analysis of historical data and considering adjustments in ambulance availability, our study provides insights into how the optimal number could be identified to minimize the likelihood of ambulance unavailability. While theoretically this problem could be addressed by infinitely increasing the number of ambulances, such a solution is impractical in real-world settings due to resource constraints. Future models should incorporate considerations of the possible expenses in providing ambulance services, ensuring a balance between availability and cost-effectiveness.

6 | Bibliography

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7 | Appendix 1

7.1 | Group plan

Plan Duration | Actual Start | % Complete | Actual (beyond plan) | % Complete (beyond plan)

ACTIVITY	PEOPLE	PLAN START	PLAN DURATION	ACTUAL START	ACTUAL DURATION	PERCENT COMPLETE
Research	Giorgia - Britney	1	1	1	2	100%
Brainstorm	All	1	4	1	5	100%
Define Parameters	Giulia - Stella	2	1	2	1	100%
Define state space	Giulia-Stella	2	1	2	4	100%
Base Model Markov chain	Giulia - Giorgia	4	3	4	4	100%
Start Simulation	Stella - Britney	4	3	4	3	100%
Apply feedback from base model	All	4	1	4	2	100%
Advanced model	Giulia - Stella	2	2	2	3	100%
Result Analysis-Simulation	Britney - Giorgia	7	2	7	2	100%
Conclusion	Giorgia - Britney	8	1	8	1	100%
Final Report Tasks	All	8	2	8	2	98%

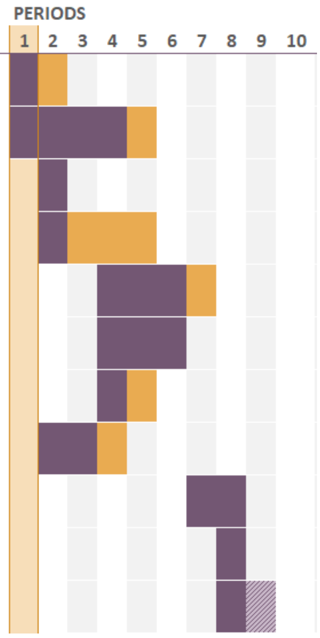


Figure 7.1: Final Group Planning

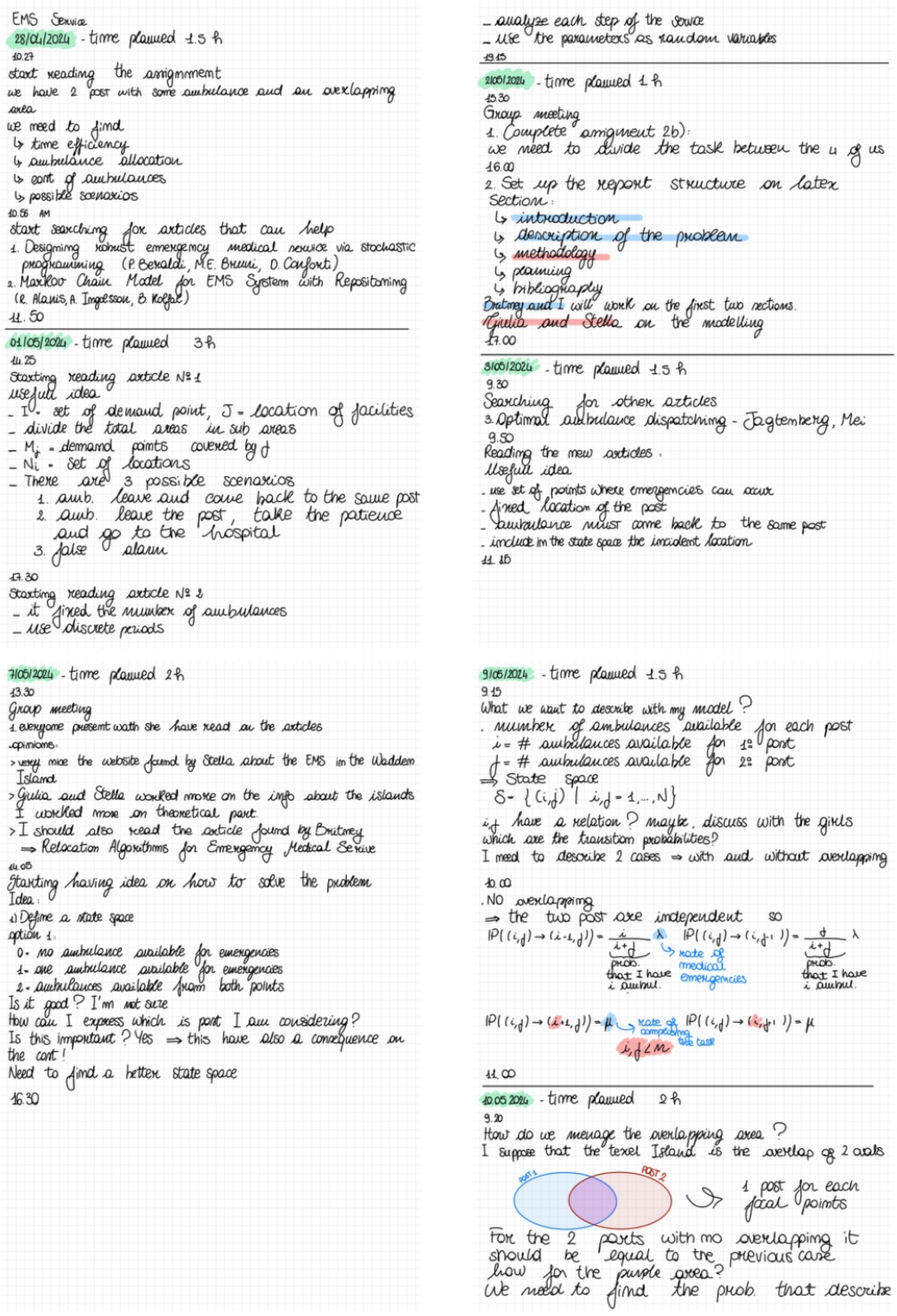
7.2 | Log books

7.2.1 | Styliani Tsilimidou

Task name	Time planned	Time spend	Task description	Outcome
Ideas for the model based on the research for Wadden Islands	1.5 hours	1.5 hours	Make assumptions for the shape of the island to use in the geographical aspect of our model	I read carefully the findings from the research and managed to come up with ideas on how to treat distance factor in our model
Model parameters	30 minutes	30 minutes	Define model parameters and then transfer them into Latex	I wrote down the parameters that we decided to focus on during the meeting with the superivsor
Research for transition probabilities	1 hours	2 hours	Find transition probabilities from state to state based on the distances and the state space	Me and Giulia found relations that could work. It took longer than expected because we had to make sure that all the rates and aspects were defined correctly
Finalized the first draft for the meeting with the supervisor	2 hours	2 hours	To be able to get feedback on our progress we have to complete a first version of our model	I wrote down my part on the report explaining the model. We managed to have a draft completed
Meeting with the professor to ask questions	30 minutes	30 minutes	We got stuck with the transition probabilities because of the distances and the statespace so we have to seek out for help to the professor during the week	The meeting was productive. We got some ideas on how to move forward and defining the probabilities without dependencies.
Define new ratios based on the feedback	1 hour	1 hour	Implement the feedback to get rid of dependencies	I managed to define some new ratios in order to get read of dependencies of (x,y) and it worked. It became too advanced to be able to compute further results so then we got stuck.
Defined new transition probabilities-transition matrix	1 hours	3 hours	Meet to discuss the new version of the model and update version of the matrix in Latex	I met with Giulia and Giorgia to discuss how to proceed with one ambulance post. We managed to find a simpler verison and we also defined which questions we wanted to answer. Then I wrote results in Latex
Revised Plan	1 hour	1 hour	Revise the planning based on our progress	I had to make adjustments on who is going to work on what based on our needs. I managed to see what is feasible
Solving new stationary distribution	2 hours	3 hours	Solve the linear system of equations to obtain the stationary distribution	We got stuck on how to solve the system because it got really complicated. Then we saw which cases to take. I redid Giorgia's calculations to make sure it's right
Solving new expected time	2 hours	3 hours	After finding the stationary distribution solve the linear system to obtain the expected time.	After trying to solve it by hand I used simulation to solve the system of linear equations
Finalizing the new model	4 hours	4 hours	Answer all the remaining questions/tasks we have to for the model and check the results on simulation	We had to make sure the final model is right. Making all the necessary calculations and adjustments took a long time. We also weren't sure for the heatmaps but then we manage to have the right output
Redifining the advanced model section in the report and finilizing	2 hours	2 hours	Update the report sections based on the final model	We decided to keep our previous idea in the report. So I had to explain some more things which I did and then I worked on some other parts on the report as well
Preparing the appendix for the report	30 minutes	30 minutes	Implementing the simulation in appendices	I implemented the code used to solve the linear system of equations

Figure 7.2: Individual Logbook - Stella

7.2.2 | Giorgia Barra



EMS Service
28/04/2024 - time planned 1.5 h
10.27
start reading the assignment
we have 2 post with some ambulance and an overlapping area
we need to find
↳ time efficiency
↳ ambulance allocation
↳ cost of ambulances
↳ possible scenarios
10.55 AM
start searching for articles that can help
1. Designing robust emergency medical service via stochastic programming (P. Beraldi, M.E. Bruni, D. Conforti)
2. Markov Chain Model for EMS System with Repositioning (R. Alanis, A. Ingolfsson, B. Kolfal)
11.50

01/05/2024 - time planned 3 h
14.25
Starting reading article № 1
usefull idea
- I - set of demand point, J - location of facilities
- divide the total areas in sub areas
- M_j - demand points covered by j
- N_i - set of locations
- There are 3 possible scenarios
  1. amb. leave and come back to the same post
  2. amb. leave the post, take the patience and go to the hospital
  3. false alarm

17.30
Starting reading article № 2
- it fixed the number of ambulances
- use discrete periods
- analyze each step of the service
- use the parameters as random variables
19.15

2/05/2024 - time planned 1 h
15.30
Group meeting
1. Complete assignment 2b):
we need to divide the task between the 4 of us
16.00
2. Set up the report structure on latex
Section:
↳ introduction
↳ description of the problem
↳ methodology
↳ planning
↳ bibliography
Britney and I will work on the first two sections.
Giulia and Stella on the modelling
17.00

3/05/2024 - time planned 1.5 h
9.30
Searching for other articles
3. Optimal ambulance dispatching - Jagtenberg, Mei
9.50
Reading the new articles.
Usefull idea
- use set of points where emergencies can occur
- fixed location of the post
- ambulance must come back to the same post
- include in the state space the incident location
11.15

7/05/2024 - time planned 2 h
13.30
Group meeting
1 everyone present wath she have read on the articles
opinions:
> very nice the website found by Stella about the EMS in the Wadden Island
> Giulia and Stella worked more on the info about the islands
I worked more on theoretical part.
> I should also read the article found by Britney
⇒ Relocation Algorithms for Emergency Medical Service
14.05
Starting having idea on how to solve the problem
Idea:
1) Define a state space
option 1:
0 - no ambulance available for emergencies
1 - one ambulance available for emergencies
2 - ambulances available from both points
Is it good? I'm not sure
How can I express which is post I am considering?
Is this important? Yes ⇒ this have also a consequence on the cost!
Need to find a better state space
16.30

9/05/2024 - time planned 1.5 h
9.15
What we want to describe with my model?
. number of ambulances available for each post
i = # ambulances available for 1° post
j = # ambulances available for 2° post
⇒ State space
$S = \{(i,j) \mid i,j = 1,\dots,N\}$
i,j have a relation? maybe, discuss with the girls
which are the transition probabilities?
I need to describe 2 cases ⇒ with and without overlapping
10.00
. NO overlapping
⇒ the two post are independent so

$$\mathbb{P}((i,j)\to(i-1,j)) = \underbrace{\frac{i}{i+j}}_{\substack{\text{prob.}\\ \text{that I have}\\ i \text{ ambul.}}}\ \lambda \qquad \mathbb{P}((i,j)\to(i,j-1)) = \underbrace{\frac{j}{i+j}}_{\substack{\text{prob.}\\ \text{that I have}\\ i \text{ ambul.}}}\ \lambda$$

↳ rate of medical emergencies

$$\mathbb{P}((i,j)\to(i+1,j)) = \mu \qquad \mathbb{P}((i,j)\to(i,j+1)) = \mu$$

↳ rate of completing the task
$i,j < n$
11.00

10.05.2024 - time planned 2 h
9.20
How do we menage the overlapping area?
I suppose that the texel Island is the overlap of 2 ovals
POST 1
POST 2
1 post for each focal points
For the 2 parts with no overlapping it should be equal to the previous case
how for the purple area?
We need to find the prob. that describe

Figure 7.3: Individual Logbook - Giorgia

if an ambulance is dispatched from post 1 or from post 2.
Which parameters do I need? i, j, x, y geographic coord. of the emerg.

$p_1 = \frac{1/d_1(x,y)}{1/d_1(x,y)+1/d_2(x,y)}$

$p_2 = \frac{1/d_2(x,y)}{1/d_1(x,y)+1/d_2(x,y)}$

11.45

14/05/2024
Meeting with the supervisor
. λ, μ must be constant
. describe the Island as a rectangle, easier for calculation
19.30
Writing the report
21.30

15/05/2024 - time planned 1.5 h
10.25
Writing the report
12.30

18/05/2024 - time planned 1h
13.45
What we need to compute?
. stationary probabilities π_i
↳ meaning = prob of having i ambulances
. limiting probabilities
↳ meaning - prob that I will have j ambul. if I start with i ambul. for $n \to \infty$
. $\mathbb{E}[T_{i,j}]$
↳ meaning - expected time that we need to go from i to j ambulances
14.30

30/05/2024 - time planned 2 h
16.30
We will start with a simpler situation
1 post:
The state space is $S = \{m \mid m = 1, \dots, N\}$
I define $\mathbb{P}\{\text{emergency happen}\} = p$ $\mathbb{P}\{\text{emergency solved}\} = q$
p and $q \neq 0$ why?
Zero doesn't make any sense for our problem
Transition probabilities

$p(0,0) = (1-q)$
$p(0,1) = q$
$p(i,i) = (1-p)(1-q) + pq$
$p(i,i-1) = p(1-q)$
$p(i,i+1) = (1-p)q$
$p(N,N) = 1-p$
$p(N,N-1) = p$

18.15

31/05/2024 - time planned 2h
14.30
Meeting with the girls
- Stella and I will work on stationary probabilities
- Giulia and Britney one the other part
16.00
Computing the stationary probabilities seems very hard because $|S| = \infty$
We can compute it for a simple case with just 5 ambulances
Better ask to the supervisor
16.30

21/05/2024 - time planned 1.5 h
19.30
We need to find a relation between i and j in order to compute P
$i + j + X = n$
↳ number ambulances that are out = number of current emergencies
How can I describe X?
. We can decide a number
3 different number
↳ for area 1
↳ for area 2
↳ for the overlap
. we can describe it with a stochastic process
How can we handle that case?
Better to ask to the supervisor
20.50

23/05/2024 - time planned 1h
15.45
Meeting with the girls
We need to find another way, our model is too complex
Britney and I will ask to the supervisor
16.40

4/06/2024
Meeting with the supervisor
10.45
Note:
for find the stat. prob we use the relation in our system $\pi_i = \alpha \pi_{i+1}$
Try to find eigenvalues and eigenvector
> adapt ex 13 to our problem

6/06/2024 - time planned 2 h
14.30
Solving the system for st. prob.

$$\begin{cases} \pi_0 = (1-q)\pi_0 + p(1-q)\pi_1 \\ \pi_1 = q\pi_0 + [(1-p)(1-q)+pq]\pi_1 + p(1-q)\pi_2 \\ \pi_2 = (1-p)q\pi_1 + [(1-p)(1-q)+pq]\pi_2 + p(1-q)\pi_3 \\ \pi_i = (1-p)q\pi_{i-1} + [(1-p)(1-q)+pq]\pi_i + p(1-q)\pi_{i+1} \\ \pi_m = (1-p)q\pi_{m-1} + (1-p)\pi_m \end{cases}$$

& Remember to add $p, q \neq 0, 1$

$\Rightarrow \pi_1 = \frac{q}{p(1-q)}\pi_0 = \alpha\pi_0$

$\Rightarrow \pi_m = \frac{p}{q(1-p)}\pi_{m-1} = \gamma\pi_{m-1}$

$\Rightarrow \pi_{i+1} = \begin{cases} 1\pi_i \\ \frac{q(1-p)}{p(1-q)}\pi_i = \beta\pi_i \end{cases}$

$\pi_0 + \alpha\pi_0 + \alpha\beta\pi_0 + \dots + \gamma\alpha\beta^{m-2}\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{\omega}$

17.30

8/06/2024 - time planned 2h
15.20
Computing the expected time.

$\mathbb{E}[T_{0,N}] = 1 + p_{0,0}\mathbb{E}[T_{0,N}] + p_{0,1}\mathbb{E}[T_{1,N}]$
$\mathbb{E}[T_{1,N}] = 1 + p_{1,0}\mathbb{E}[T_{0,N}] + p_{1,1}\mathbb{E}[T_{1,N}] + p_{1,2}\mathbb{E}[T_{2,N}]$
$\mathbb{E}[T_{2,N}] = 1 + p_{2,1}\mathbb{E}[T_{1,N}] + p_{2,2}\mathbb{E}[T_{2,N}] + p_{2,3}\mathbb{E}[T_{3,N}]$
$\vdots$

too complex to solve
16.30

Figure 7.4: Individual Logbook - Giorgia

try with Py
17.20

12/06/2024 - time planned 1.5 h
9.15
New idea: use heatmaps to compute the relation between π_i, q, p
start coding
10.30
! Problem: the heatmap do not match with the logic
Checking previous parts and find the error
11.15
Correcting the code

14/06/2024
Last meeting
Use birth-death process to use continous Markov chain

16/06/2024 - time planned 1.5 h
14.30
Writing the report
16.00

17/06/2024 - time planned 2 h
11.30
computing π_i and $\hat{\pi}_i$ for bd process
12.45
13.30
Write this part in the report and find the heatmap
15.00

19/06/2024 - time planned 1.5 h
15.20
Writing the conclusion of the report
16.50

Figure 7.5: Individual Logbook - Giorgia

7.2.3 | Saw Yu Xuan (Britney)

Period/ Week	Date	Time Planned	Actual Time	Description of Planned Activity & Actual Achievement
1	23 April 2024	1 hour	30 min	Attended modelling briefing
1	26 April 2024	5 min	5 min	Contacted group members to form Whatsapp group
2	29 April 2024	2 hours	2 hours	Research for first project meeting - Understanding the question - Listing possible research questions - Brainstorming the factors: 1. Number of ambulances at each post 2. Demand/distribution of the population 3. Determining shape of the island - Analysis: To decide on optimal parameters, can do cost/benefit analysis based on balancing probability of reaching the accident in 5 minutes
2	30 April 2024	1 hour	1 hour	First meeting with supervisor (Sanne) - Shared our initial ideas with Sanne Group meeting - Consolidate our ideas - Set up next meeting on 2 May
2	2 May 2024	3 hours	3 hours	Read planning and organizing documents. Group meeting to work on planning - Plan timeline for plan 1b - Set up next meeting on 7 May Work on plan 1a individually. - Analysed my own time management strategy using the Mindtools questionnaire - Completed plan 1a
3	6 May 2024	1 hour	1 hour	Research - Look for papers that we could cite in our bibliography
3	7 May 2024	3 hours	2.5 hours	Brainstorm Group meeting to kickstart project - Set up Latex document on Overleaf - Discussed our discrete time model - Looked for papers to cite in our bibliography - Read through the papers to see if they are relevant to our model

Figure 7.6: Individual Logbook - Britney

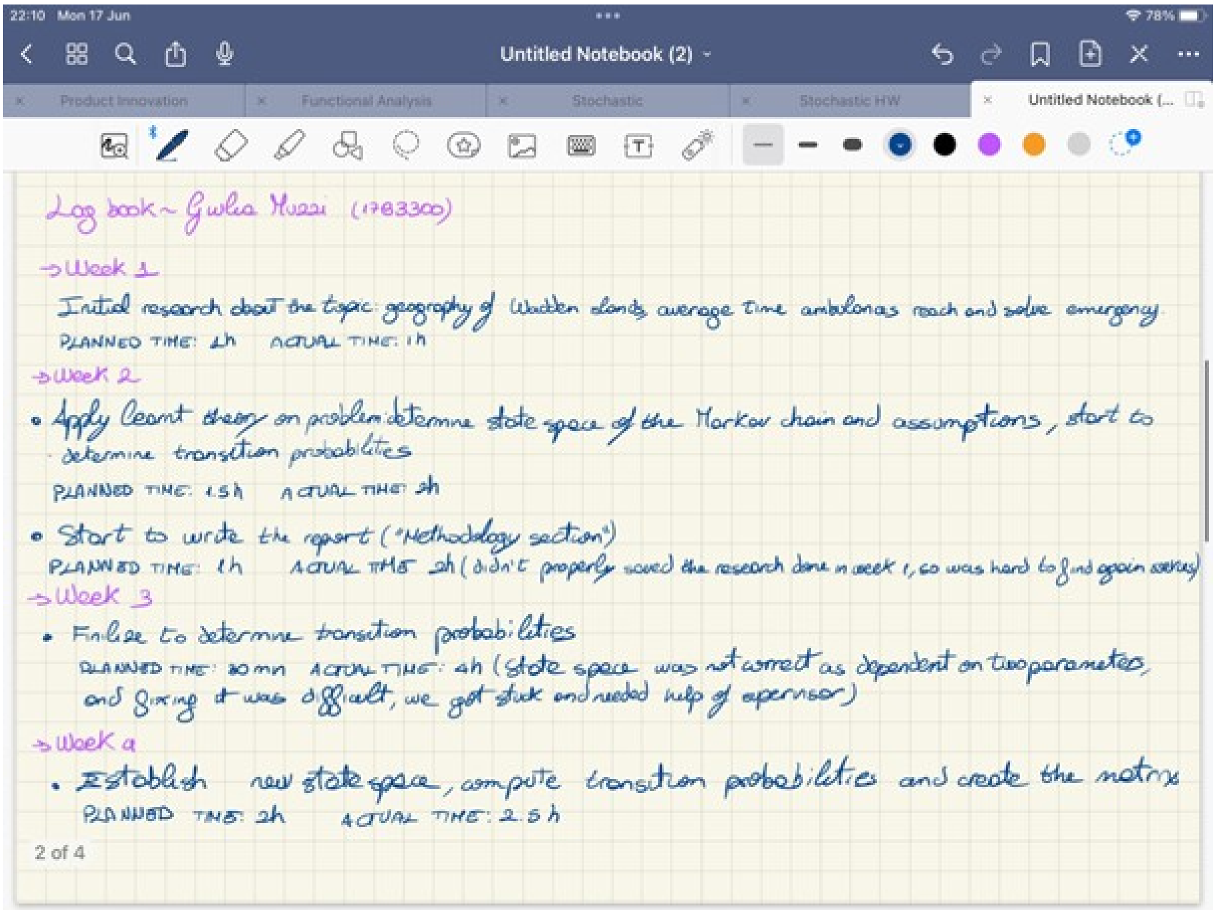
3	8 May 2024	2 hours	2 hours	Start working on the Latex document - Review introduction section 1 - Worked on the "Description of the problem" section 2 using papers
3	9 May 2024	2 hours	2 hours	Review the latex document so far - Refined "Description of the problem" section 2 and checked for any errors
4	14 May 2024	1 hour	2 hours	Second meeting with supervisor (Sanne) - Discussed our state space which we are stuck on and Sanne gave us another suggestion to consider Group meeting (Base model Markov chain) - Discussed how we can refine our methodology and clarified with Professor Sem Borst
4	18 May 2024	1 hour	1.5 hours	Read 2 project proposals to complete peer review - Reviewed and gave feedback for Call Center project - Reviewed and gave feedback for Manchester United project
5	20 May 2024	1 hour	1 hour	Work on plan 2 - Complete and submitted plan 2
5	21 May 2024	30 min	30 min	Complete peer review for plan 2 - Gave feedback for 2 peer reviews
5	23 May 2024	1 hour	1 hour	Base model Markov chain 3rd meeting with supervisor (Sanne) - Discussed our methodology to see whether it is more feasible - Revised our model based on new parameters that are simpler to understand than our previous model, which we now keep as a new section called "complex model"
7	4 June 2024	1 hour	1 hour	Base model Markov chain 4th meeting with supervisor (Sanne) - Discussed how we can calculate our new stationary probabilities and expected values - Discussed the calculation eigenvalues/eigenvectors for our new model
7	7 June 2024	3 hours	3 hours	Calculate eigenvalues for the discrete time model - Attempted calculation by hand, which was unsuccessful for unknown n - Used Python to calculate the eigenvalues and found that the values are reasonable only when n < 5.
8	11 June 2024	2 hours	2 hours	Start looking at how to conduct simulations - Checked through the stationary probabilities to ensure that they are reasonable - Decided on heat maps to vary p and q
8	13 June 2024	4 hours	5 hours	Continue conducting simulations

Figure 7.7: Individual Logbook - Britney

				- Produced 2 heatmaps to vary p and q and find the stationary probabilities of Pi_0 and Pi_n - Discovered that there was an error due to division of 0 in a fraction, so the case where p = q was added into the methodology writing - However, the heatmap for Pi_n appears to be incorrect
8	14 June 2024	1 hour	1 hour	5th meeting with supervisor (Sanne) - Showed our calculated stationary probabilities and expected values - Checked that our expected values are within reasonable values - Discuss if heatmaps are reasonable - Discussed the possibility of calculating continuous time stationary probabilities for more analysis and comparison Group meeting - Decided to work on continuous time stationary probabilities (birth death process) rather than do cost/benefit analysis as we initially planned. - Allocated workload for next steps
8	15 June 2024	2 hours	2 hours	Result analysis (section 4) for the simulation - Completed section 4.1 analysis with the heatmaps produced earlier - The "incorrect" heatmap for Pi_n is still unresolved
9	17 June 2024	1 hour	1 hour	Group meeting - Discussed our progress since our meeting on 14 June - Analyzed the heatmaps - Allocated the rest of the workload
9	17 June 2024	2 hours	2 hours	Continue result analysis (section 4) for the simulation - Wrote code for the simulations of continuous time stationary probabilities - Completed section 4.2 and 4.3 analysis
9	19 June 2024	1 hour	1 hour	Review the final report - Checked through all sections and made minor revisions

Figure 7.8: Individual Logbook - Britney

7.2.4 | Giulia Muzzi



Log book ~ Giulia Muzzi (1783300)

→ Week 1

Initial research about the topic: geography of Wadden islands average time ambulances reach and solve emergency

PLANNED TIME: 1h ACTUAL TIME: 1h

→ Week 2

- Apply learnt theory on problem: determine state space of the Markov chain and assumptions, start to determine transition probabilities

PLANNED TIME: 1.5h ACTUAL TIME: 3h

- Start to write the report ("Methodology section")

PLANNED TIME: 1h ACTUAL TIME: 2h (didn't properly saved the research done in week 1, so was hard to find again sources)

→ Week 3

- Finalize to determine transition probabilities

PLANNED TIME: 30 min ACTUAL TIME: 4h (state space was not correct as dependent on two parameters, and fixing it was difficult, we got stuck and needed help of supervisor)

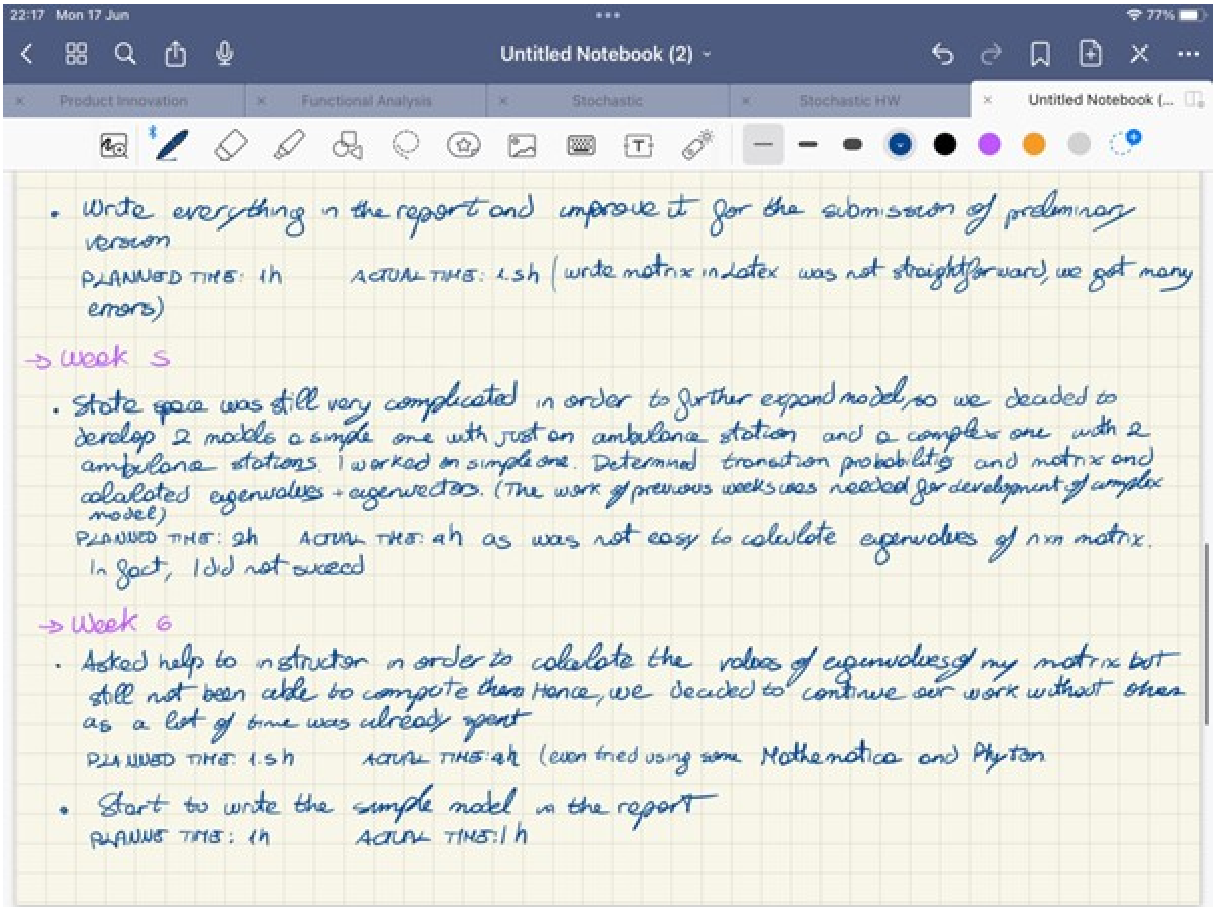
→ Week 4

- Establish new state space, compute transition probabilities and create the matrix

PLANNED TIME: 2h ACTUAL TIME: 2.5h

2 of 4

Figure 7.9: Individual Logbook - Giulia



- Write everything in the report and improve it for the submission of preliminary version
  PLANNED TIME: 1h ACTUAL TIME: 1.5h (write matrix in latex was not straightforward, we got many errors)

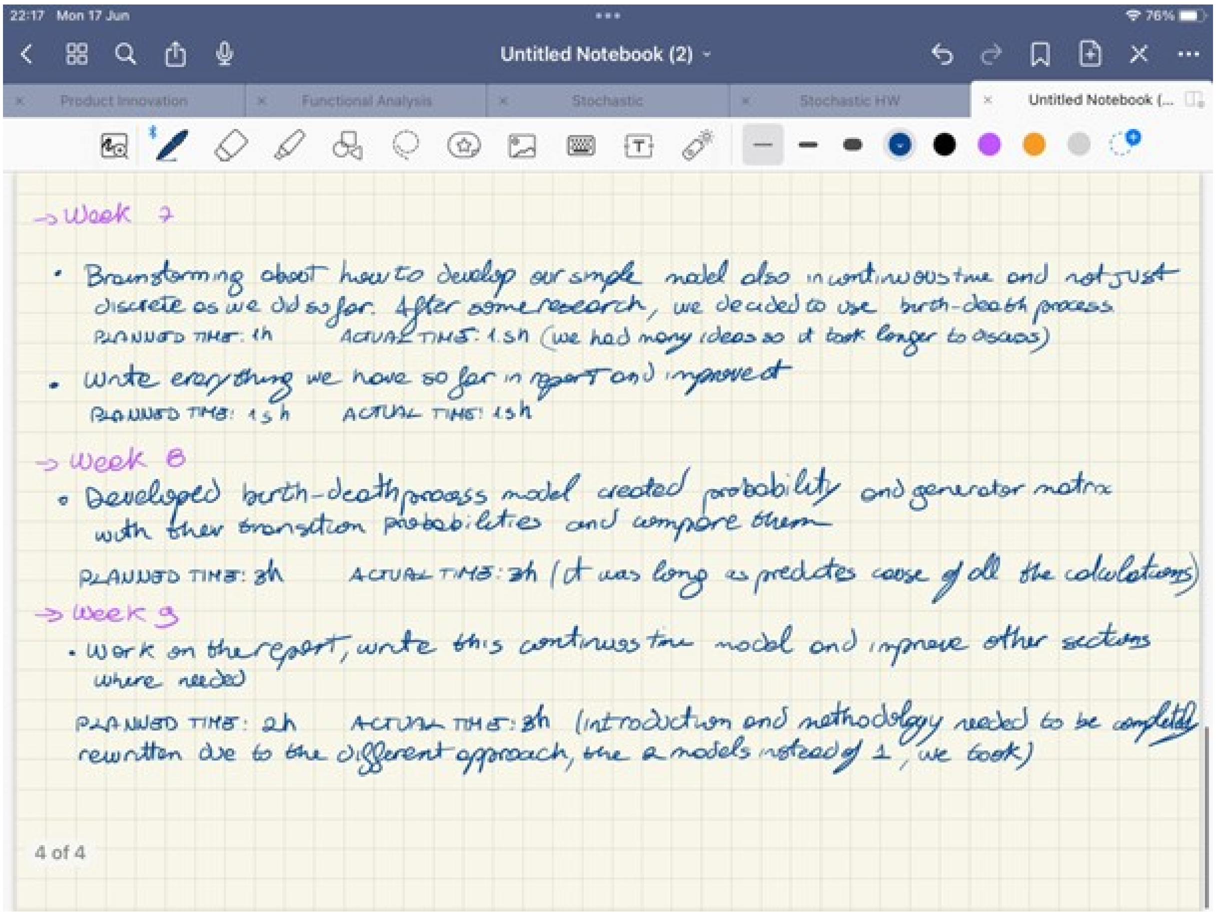
→ Week 5

- State space was still very complicated in order to further expand model, so we decided to develop 2 models a simple one with just an ambulance station and a complex one with 2 ambulance stations. I worked on simple one. Determined transition probabilities and matrix and calculated eigenvalues + eigenvectors. (The work of previous weeks was needed for development of complex model)
  PLANNED TIME: 2h ACTUAL TIME: 4h as was not easy to calculate eigenvalues of n×n matrix. In fact, I did not succeed

→ Week 6

- Asked help to instructor in order to calculate the values of eigenvalues of my matrix but still not been able to compute them. Hence, we decided to continue our work without them as a lot of time was already spent
  PLANNED TIME: 1.5h ACTUAL TIME: 4h (even tried using some Mathematica and Phyton
- Start to write the simple model in the report
  PLANNED TIME: 1h ACTUAL TIME: 1h

Figure 7.10: Individual Logbook - Giulia



→ Week 7

- Brainstorming about how to develop our simple model also in continuous time and not just discrete as we did so far. After some research, we decided to use birth-death process
  PLANNED TIME: 1h ACTUAL TIME: 1.5h (we had many ideas so it took longer to discuss)
- Write everything we have so far in report and improve it
  PLANNED TIME: 1.5 h ACTUAL TIME: 1.5 h

→ Week 8

- Developed birth-death process model created probability and generator matrix with their transition probabilities and compare them
  PLANNED TIME: 3h ACTUAL TIME: 3h (it was long as predicted cause of all the calculations)

→ Week 9

- Work on the report, write this continuous time model and improve other sections where needed
  PLANNED TIME: 2h ACTUAL TIME: 3h (introduction and methodology needed to be completely rewritten due to the different approach, the 2 models instead of 1, we took)

4 of 4

Figure 7.11: Individual Logbook - Giulia

8 | Code

The following code was used to generate the heat maps for the various stationary probabilities. This includes the calculation of the parameters and the generation of the heat map using Python.

```
### Simulation (heat map) for discrete time

import numpy as np
import matplotlib.pyplot as plt

# Define the parameters
def calculate_parameters(p, q):
    if p == q:
        alpha = 1 / (1 - p)
        beta = 1
        gamma = 1 - p
    else:
        alpha = q / (p * (1 - q))
        beta = q * (1 - p) / (p * (1 - q))
        gamma = (q * (1 - p))/ p
    return alpha, beta, gamma

# Calculate omega
def calculate_omega(alpha, beta, gamma, n):
    if beta == 1: # p = q
        return 1 + (n-1) * alpha + gamma * alpha

    else:
        return 1 + (alpha * (beta - beta**n) / (beta - beta**2)) + gamma * alpha * beta**(n-2)

# Calculate stationary probabilities
def stationary_probabilities(p, q, n):
    alpha, beta, gamma = calculate_parameters(p, q)
    omega = calculate_omega(alpha, beta, gamma, n)
    Pi_0 = 1 / omega
    Pi_n = gamma * alpha * beta**(n-2) / omega
    return Pi_0, Pi_n

# Generate evenly spaced range of p and q values
p_values = np.linspace(0.01, 0.99, 100)
q_values = np.linspace(0.01, 0.99, 100)

# Initialize matrices
n = 5  # Number of ambulances
pi_0_values = np.zeros((len(p_values), len(q_values)))
pi_n_values = np.zeros((len(p_values), len(q_values)))

# Compute stationary probabilities pi_0 and pi_n for each pair of p and q
for i, p in enumerate(p_values):
    for j, q in enumerate(q_values):
        Pi_0, Pi_n = stationary_probabilities(p, q, n)
        pi_0_values[i, j] = Pi_0
        pi_n_values[i, j] = Pi_n

# Plot the heat maps
fig, axs = plt.subplots(1, 2, figsize=(18, 8))
ticks = np.linspace(0, 1, 11)  # Define tick marks

# Heat map for Pi_0
```

```
im1 = axs[0].imshow(pi_0_values.T, extent=[0.01, 0.99, 0.01, 0.99], origin='lower',
                    aspect='auto')
fig.colorbar(im1, ax=axs[0], label='Stationary Probability $\Pi_0$')
axs[0].set_xlabel('Probability of Emergency Happening (p)')
axs[0].set_ylabel('Probability of Service Completion (q)')
axs[0].set_title('Discrete Time Stationary Probability $\Pi_0$ for n = 5 Ambulances')
axs[0].set_xticks(ticks)
axs[0].set_yticks(ticks)

# Heat map for Pi_n
im2 = axs[1].imshow(pi_n_values.T, extent=[0.01, 0.99, 0.01, 0.99], origin='lower',
                    aspect='auto')
fig.colorbar(im2, ax=axs[1], label='Stationary Probability $\Pi_n$')
axs[1].set_xlabel('Probability of Emergency Happening (p)')
axs[1].set_ylabel('Probability of Service Completion (q)')
axs[1].set_title('Discrete Time Stationary Probability $\Pi_n$ for n = 5 Ambulances')
axs[1].set_xticks(ticks)
axs[1].set_yticks(ticks)

plt.show()



### Simulation (heat map) for continuous time
# number of ambulances
n = 5

# Calculate stationary probabilities
def stationary_probabilities_G(l, m, n):
    if l == m:
        Pi_0 = 1/(n+1)
        Pi_n = 1/(n+1)
    else:
        Pi_0 = ((l-m)*m**n) / (l**(n+1) - m**(n+1))
        Pi_n = ((l**n)/(m**n))* (((l-m)*m**n)/ (l**(n+1) - m**(n+1)))
    return Pi_0, Pi_n

# Generate evenly spaced range of lambda and miu values
l_values = np.linspace(0.01, 0.99, 100)
m_values = np.linspace(0.01, 0.99, 100)

# Initialize matrices
n = 5  # Number of ambulances
pi_0_values = np.zeros((len(l_values), len(m_values)))
pi_n_values = np.zeros((len(l_values), len(m_values)))

# Compute stationary probabilities pi_0 and pi_n for each pair of p and q
for i, l in enumerate(l_values):
    for j, m in enumerate(m_values):
        Pi_0, Pi_n = stationary_probabilities_G(l, m, n)
        pi_0_values[i, j] = Pi_0
        pi_n_values[i, j] = Pi_n

# Plot the heat maps
fig, axs = plt.subplots(1, 2, figsize=(18, 8))
ticks = np.linspace(0, 1, 11)  # Define tick marks

# Heat map for Pi_0
```

```
im1 = axs[0].imshow(pi_0_values, extent=[0.01, 0.99, 0.01, 0.99], origin='lower',
                    aspect='auto')
fig.colorbar(im1, ax=axs[0], label='Stationary Probability $\Pi_0$')
axs[0].set_xlabel('mu (emergency rate)')
axs[0].set_ylabel('lambda (service completion rate)')
axs[0].set_title('Continuous Time Stationary Probability $\Pi_0$ for n = 5 Ambulances')
axs[0].set_xticks(ticks)
axs[0].set_yticks(ticks)

# Heat map for Pi_n
im2 = axs[1].imshow(pi_n_values, extent=[0.01, 0.99, 0.01, 0.99], origin='lower',
                    aspect='auto')
fig.colorbar(im2, ax=axs[1], label='Stationary Probability $\Pi_n$')
axs[1].set_xlabel('mu (emergency rate)')
axs[1].set_ylabel('lambda (service completion rate)')
axs[1].set_title('Continuous Time Stationary Probability $\Pi_n$ for n = 5 Ambulances')
axs[1].set_xticks(ticks)
axs[1].set_yticks(ticks)

plt.show()



### Simulation (heat map) for embedded chain
# number of ambulances
n = 5

# Calculate stationary probabilities
def stationary_probabilities_G(l, m, n):
    sum_term = (l+m)/m * sum((l / m)**(i-1) for i in range(1, n ))
    Pi_0 = 1 / (1 + sum_term + (l / m)**n)
    Pi_n = ((l / m)**n) * Pi_0
    return Pi_0, Pi_n

# Generate evenly spaced range of lambda and miu values
l_values = np.linspace(0.01, 0.99, 100)
m_values = np.linspace(0.01, 0.99, 100)

# Initialize matrices
n = 5  # Number of ambulances
pi_0_values = np.zeros((len(l_values), len(m_values)))
pi_n_values = np.zeros((len(l_values), len(m_values)))

# Compute stationary probabilities pi_0 and pi_n for each pair of p and q
for i, l in enumerate(l_values):
    for j, m in enumerate(m_values):
        Pi_0, Pi_n = stationary_probabilities_G(l, m, n)
        pi_0_values[i, j] = Pi_0
        pi_n_values[i, j] = Pi_n

# Plot the heat maps
fig, axs = plt.subplots(1, 2, figsize=(18, 8))
ticks = np.linspace(0, 1, 11)  # Define tick marks

# Heat map for Pi_0
im1 = axs[0].imshow(pi_0_values, extent=[0.01, 0.99, 0.01, 0.99], origin='lower',
                    aspect='auto')
fig.colorbar(im1, ax=axs[0], label='Stationary Probability $\Pi_0$')
```

```
axs[0].set_xlabel('mu (emergency rate)')
axs[0].set_ylabel('lambda (service completion rate)')
axs[0].set_title('Embedded Chain Stationary Probability $\Pi_0$ for n = 5 Ambulances')
axs[0].set_xticks(ticks)
axs[0].set_yticks(ticks)

# Heat map for Pi_n
im2 = axs[1].imshow(pi_n_values, extent=[0.01, 0.99, 0.01, 0.99], origin='lower',
                    aspect='auto')
fig.colorbar(im2, ax=axs[1], label='Stationary Probability $\Pi_n$')
axs[1].set_xlabel('mu (emergency rate)')
axs[1].set_ylabel('lambda (service completion rate)')
axs[1].set_title('Embedded Chain Stationary Probability $\Pi_n$ for n = 5 Ambulances')
axs[1].set_xticks(ticks)
axs[1].set_yticks(ticks)

plt.show()
```

This is the code used for computing the results from the linear set of equations for the expected time.

```
import sympy as sp

def solve_symbolic_system(n):
    # Define symbolic variables
    p, q, pi_n = sp.symbols('p q pi_n')
    E = sp.symbols(f'E0:{n+1}')

    # Initialize the system of equations
    equations = []

    # Define the equations based on the given system
    equations.append(sp.Eq(E[0], 1 + (1 - q) * E[0] + q * E[1]))

    for i in range(1, n):
        eq = sp.Eq(E[i], 1 + p * (1 - q) * E[i - 1] + ((1 - p) * (1 - q) + p * q) * E[i] + (1 - p) *
        equations.append(eq)

    equations.append(sp.Eq(E[n], 1 / pi_n))

    # Solve the system of equations
    solution = sp.solve(equations, E)

    return solution

# Number of equations
n = 5

# Solve the symbolic system
symbolic_solution = solve_symbolic_system(n)
```