Generalizations and Hierarchies

A whirlwind tour of some advanced topics

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The power of CT stems from its generality

- CT is the mathematics of pattern, structure and interaction
  ↳ Emphasizes concepts + systems in context

- Each category is like a "mathematical universe": a collection of similar things + their relationships

- Each category has an "internal language": adopting it makes our statements 'well-typed', so it's harder to say things that don't make sense + easier to 'carve nature at its joints". (+ observe sources of non-compositionality / emergence...)
  ↳ And often this language can be written graphically

- CT is even able to describe itself ⇝ 'synthetic perspective'...
  (We will see more of this shortly)

Overview

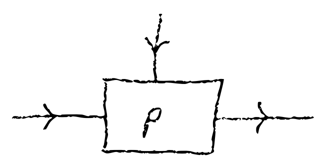
Aims: (1) introduce 'synthetic' approach
(2) build up to well-typed science
(3) emphasize importance of perspective / context

- 2-categories + beyond

- Enriched categories

- Plenty of examples (though sadly still quite 'mathematical'!)
  ↳ diagrammatic thinking (monoidal cat.s, etc.), open systems

- Universality + perspective
  ↳ Yoneda lemma



The archetypal 2-category $\underline{\underline{\mathrm{Cat}}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathcal{D} \end{tikzcd}$$

For any $f : X \to Y$ in $\mathcal{C}$,

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$

commutes.

0-cells : categories

1-cells : functors

2-cells : natural transformations

+ axioms to make sense of

Horizontal

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathcal{D} \arrow[r, bend left=50, "F'"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\alpha' \Downarrow"] & \mathcal{E} \end{tikzcd}$$

$\shortparallel$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left=40, "F' \circ F"] \arrow[rr, bend right=40, "G' \circ G"'] \arrow[rr, phantom, "\alpha' \circ \alpha \Downarrow"] & & \mathcal{E} \end{tikzcd}$$

Vertical

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \beta"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow \beta \circ \alpha"] & \mathcal{D} \end{tikzcd}$$

($\underline{\underline{\text{NB}}}$ (n,r)-category notation ...)

Enriched categories: another perspective on higher cat.s

A category $\mathcal{C}$ has a set of objects, $\mathcal{C}_0$, and
for each pair of objects A, B,
a set $\mathcal{C}(A, B)$ of morphisms $A \longrightarrow B$.

But what if we replace the <u>set</u> $\mathcal{C}(A, B)$
with another kind of object - eg an object in $\mathcal{V}$?

↳ We get a $\mathcal{V}$-enriched category, or $\mathcal{V}$-category.

(of course, there are some axioms to ensure the two kinds of composition play well together.)

'Normal' categories (as above) are Set-categories.

2-categories are Cat-categories!

(<u>NB</u> Bicategories and 'weak' enrichment...)

Example: a Cat-category, $\mathcal{K}$

If $\mathcal{K}$ is a Cat-category, then it has

a collection of objects / 0-cells, $\mathcal{K}_0$; and

for each pair of objects A, B,

a <u>category</u> $\mathcal{K}(A, B)$ of morphisms.

This means that $\mathcal{K}(A, B)$ itself has

a collection of objects $\mathcal{K}(A, B)_0$ — the 1-cells $A \to B$ of $\mathcal{K}$;

and for each pair $F, G : A \to B$,

a set $\mathcal{K}(A, B)(F, G)$ — the 2-cells $F \Rightarrow G$ of $\mathcal{K}$

(subject to axioms of composition...).

We can use these ideas to add extra data to morphisms...

Example: categories with extra data

Note: every set is a discrete category,
so every category is 'trivially' a $\mathbf{Cat}$-category.

If we want to record whether / how a diagram commutes,
we can do this by promoting the hom-sets to hom-categories:

eg
$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd} \longmapsto \begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\checkmark"] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd} \text{ or } \begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\swArrow"] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$

$\mathcal{D}(FX, GY)(\alpha_Y \circ Ff, Gf \circ \alpha_X) = \emptyset$ $\qquad$ $= \{\text{true}\}$ $\qquad$ $= \{\text{proofs}\}$

This path leads to ∞-categories and homotopy type theory...

Or we could measure 'how far' a diagram is from commuting...
$\hookrightarrow$ emergent properties!

Example: metric spaces

We can make the extended positive reals $[0, \infty]$ into a category:
the objects are the points of the interval, and
there is a morphism $a \to b$ whenever $a \geq b$.

('tensor' or 'parallel' product...)

This category also has a 'monoidal' product, addition:
$a + b \mapsto (a+b)$ and if $a \geq a'$ and $b \geq b'$, then $a + b \geq a' + b'$.

This means we can enrich in $([0, \infty], \geq, +)$!

A category S enriched in $[0, \infty]$ has:
a set S_0 of objects ('points'), and
for each pair x, y of points, a number $S(x, y)$ ⇝ the distance!...
such that $0 \geq S(x, x)$ and

$$S(x, y) + S(y, z) \geq S(x, z).$$

So S is a (generalized / 'quasi') metric space!

(And we can enrich in the category of these!)

Monoidal categories and open systems

Often we think of the morphisms of a category as transformations or processes,
with the objects representing boundaries or interfaces.

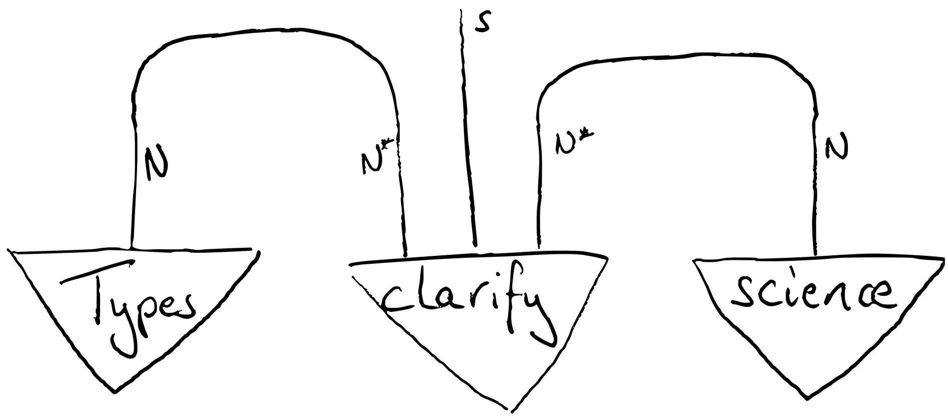
<u>Examples</u>:
- functions as computations
- wiring diagrams
- generative models
- utterances of natural language

And as the examples suggest, we may often want to connect ('wire') such "open systems" together, composing them sequentially or in parallel, perhaps considering these composites as new systems in their own right.

We can do this using the structures of monoidal categories, which give rise to powerful graphical calculi ('string diagrams').

Monoidal categories and open systems

Example from natural language:



grammar (~wiring) "information flow" ...

words / meanings

Nouns: elements of type N

Sentences: elements of type S

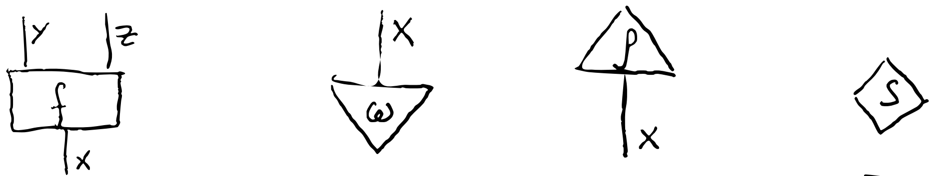
Verbs: processes $N \otimes N \to S$

or elements of $N^* \otimes S \otimes N^*$

cf Coecke et al (2010) etc...

(We will see other examples later...)

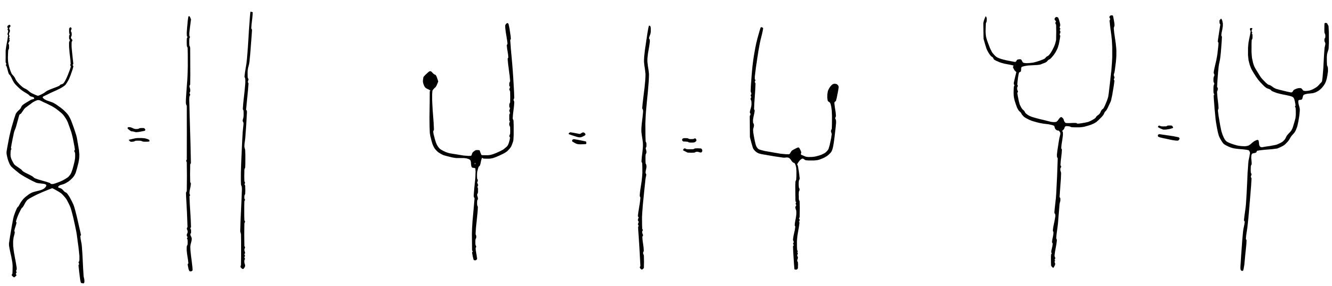
Monoidal categories and open systems



$f : X \to Y \otimes Z$ $\omega : I \to X$ $\rho : X \to I$ $s : I \to I$

~ a closed system!

+ symmetry and 'copy-delete' axioms:



Monoidal categories are bicategories

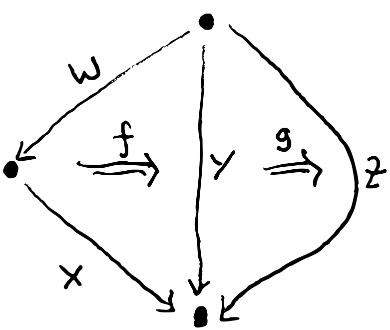
We can turn any monoidal category $\mathcal{C}$ into a bicategory, $B\mathcal{C}$, called its 'delooping'.

This makes the 2 dimensions explicit.

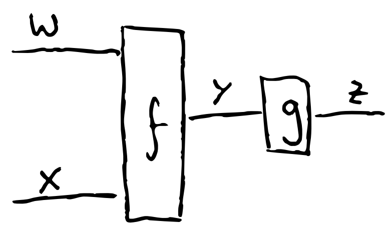
$B\mathcal{C}$ has a single 0-cell, $\bullet$.

Its 1-cells are the objects of $\mathcal{C}$, composed by $\otimes$.

The 2-cells are the morphisms of $\mathcal{C}$, composed by $\circ$.



$\simeq$



$$W \otimes X \xrightarrow{f} Y \xrightarrow{g} Z$$

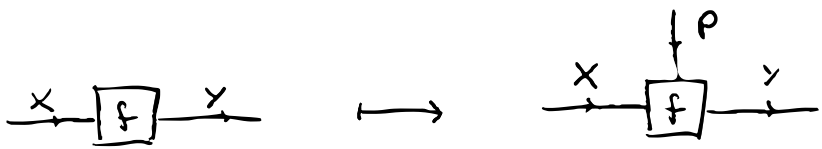
"Poincaré duality"

Adding more dimensions with parameterized morphisms

Sometimes, we want the choice of morphism to be controlled by another process.

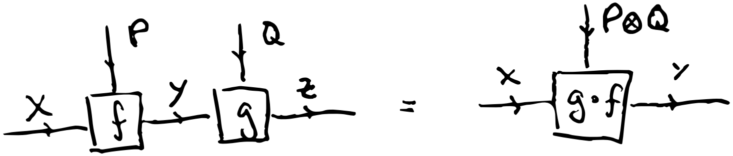
└ Consider: neural net weights, synaptic plasticity.

This means adding another dimension to the diagrams:

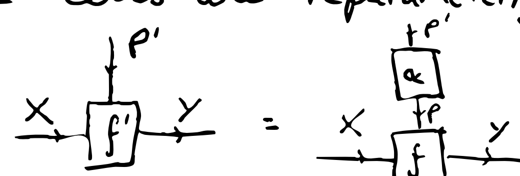


We can do this in any monoidal category, $\mathbb{C}$, turning $\mathbb{C}$ into a monoidal bicategory, $\mathbf{Para}(\mathbb{C})$.

$$\mathbf{Para}(\mathbb{C})_0 := \mathbb{C}_0. \qquad \mathbf{Para}(\mathbb{C})(X, Y) := \sum_{P : \mathbb{C}_0} \mathbb{C}(P \otimes X, Y)$$



2-cells are 'reparameterizations':



Recap

- 'Higher' categorical structures let us express clearly and precisely the compositional structure of systems / concepts / processes.

  In particular, they emphasize openness + interaction.

- Categorical tools enforce 'type discipline', making it harder to say things that don't make sense.

... but there's a 3rd magic ingredient: perspective / context.

* How you see the world depends on where you view it from!

  ↳ The blind men and the elephant ...

The 'Yoneda' perspective

Dual to the statement that one's view depends on one's perspective is the statement that to know a thing is to see it from all perspectives. (This is the blind men's moral.)

Formalizing this idea gives us the Yoneda Lemma: the fundamental theorem of category theory.

Slightly more precisely, the Yoneda Lemma says that you can identify an object with the collection of ways of looking into (or out of) it.

The morphisms are important — less so the objects.

The Yoneda Lemma is true for all categories, and is familiar to us:

- We know a place by how to reach it;
- we know a word by the company it keeps;
- we know a concept by its connotations.

(We will leave the formal statement + proof to another day, however!)

Universal constructions

Much of the power of CT is in its ability to supply structural explanations for observed phenomena: to see how function follows form.

(So far, most success has been in mathematics + comp. sci., but it's spreading!)

This power is most potent when 'perspective-independent': when the explanation only depends on ambient structures in the cat.

Such constructions are called 'universal', and they correspond to "things you can do anywhere" or "things you can always say in the internal language".

(And the Yoneda Lemma is always at the heart of a universality proof.)

Universal constructions often have a flavour of 'optimality', but they can be quite simple, too:

(product example
+ proof-relevance!)

Bonus: Functorial semantics + "proofs as paths"

[+ (lax) monoidal functors]

Conclusion: well-typed science, in context

My hope is that categorical tools will help us think clearly about all kinds of systems.

In particular, to understand the structure of and interaction between mind and world:

the mind in the world, and
the world in the mind.

Later, we will see some tentative first steps towards a categorical account of the mechanisms of this interaction.