

Category theory for cognitive science

Some formal and conceptual analogies
(abridged version)

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Categories and cognition

1 Introduction

- Objectives: What/Why/How of category theory
- Perspective: Category theory as theory of formal analogies
- Where category theory *meets* cognitive science: *compositionality*

2 Basic correspondences

- Categories and composition
- Functors and representation
- Natural transformations and re-representation (computation)

3 Universal cognition

- Universal morphisms (limits) and compositionality
- Universals and *systematicity*

4 Discussion

- The *universal mapping principle* for cognitive science

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Objectives: What/Why/How of categories

Objectives: basic answers to three basic questions:

- ① What is category theory?
 - ▶ A theory of structure
 - ▶ A theory of analogy
 - ▶ A theory of universal (mapping) properties
- ② Why is category theory important (to cognitive science)?
 - ▶ A *lingua franca* for cognitive science
 - ▶ Explanations without *ad hoc* assumptions
- ③ How is category theory applied (in cognitive science)?
 - ▶ A *universal mapping principle* for cognitive science (Phillips, 2021a)

Learning objectives: schedule

Schedule (category theory concepts):

- ① Session 1: Basics – categories, functors, natural transformations
 - ▶ categories and compositionality
 - ▶ functors and representation
 - ▶ natural transformations and re-representation (computation)
- ② Session 2: Universal constructions
 - ▶ universal morphisms
 - ▶ limits/colimits
 - ▶ recursion/corecursion (iteration)

 Linking concept: (commutative) square of relations, \square

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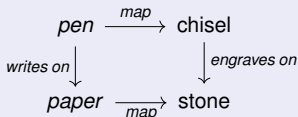
Perspective: Category theory as formal analogies

Conceptual analogies:

- Pen is to paper as chisel is to stone
- Category Theory is to Mathematics as Analogy Theory is to Cognitive Science

Compare:

- Structure Mapping Theory (Gentner, 1983) – analogy ($map : Source \rightarrow Target$)



- Category Theory (Eilenberg & Mac Lane, 1945) – natural transformation ($\eta : F \rightrightarrows G$)

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ F(f) \downarrow & & \downarrow G(g) \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

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Where category theory *meets* cognitive science

Compositionality

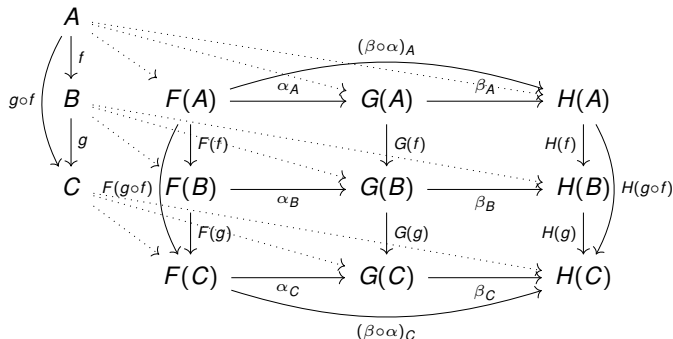
Category theory as a formal framework for compositionality, e.g.,

$$\begin{array}{ccc} \textit{John} & \xrightarrow{\rho} & \textit{John} \\ \textit{loves} \downarrow & & \downarrow \textit{loves} \\ \textit{Mary} & \xrightarrow{\rho} & \textit{Mary} \end{array}$$

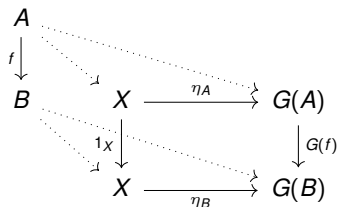
Dimensions of compositionality

Three dimensions of composition:

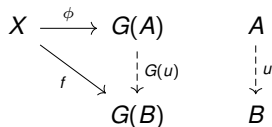
- vertical: composition of arrows (within category)
- horizontal: composition of natural transformations
- out-of-plane (not shown): composition of functors



Commutative triangles and universal structures



triangles are special “squares” – cf. diagram for a *universal mapping property*:



every object X has an identity arrow 1_X (usually omitted from diagrams)

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Categories and compositionality

Compositionality (basic): method for putting two things together to form something

In cognitive science (classical version):

- Principle: *composite symbol* built up from *constituent symbols* and their *relations*
- Example: *red circle* built from *red* and *circle* and their order (\neq *circle red*)
- Motivation: *systematicity* and *productivity* properties of cognition

In category theory (basic form):

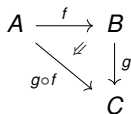
- Principle: *composite arrow* built up from *arrows* and *composition operation* (\circ)
- Example: $g \circ f$ built from f and g applied to \circ ($\neq f \circ g$)
- Motivation: complex “structure” as composition of arrows

Category (definition)

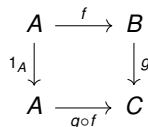
A category $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1, \text{dom}, \text{cod}, \text{id}, \circ)$ consists of:

- *objects*, $\mathbf{C}_0 = \{A, B, C, \dots\}$
- *arrows*, $\mathbf{C}_1 = \{f, g, h, \dots\}$ —arrow $f : A \rightarrow B$ from *domain* A to *codomain* B
 - ▶ including *identity arrow* for each object: $\text{id} : A \mapsto (1_A : A \rightarrow A)$
- *domain/codomain maps*: $\text{dom}(f) = A$, $\text{cod}(f) = B$
- *composition operation*, i.e. $f : A \rightarrow B$ composed with $g : B \rightarrow C$ is $g \circ f : A \rightarrow C$
 - ▶ $1_B \circ f = f = f \circ 1_A$ — compare $1 \cdot x = x = x \cdot 1$
 - ▶ $h \circ (g \circ f) = (h \circ g) \circ f$ — compare $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

 Identity arrows are usually omitted from diagrams



same as



Categories and composition (example: orders)

An ordered set is a category:

- *Ann is shorter (not taller) than Ben*—transitive



- *Ann is not taller than Ann*—reflexive: $\text{Ann} \leq \text{Ann}$

Formally, an ordered set (P, \leq) corresponds to a category:

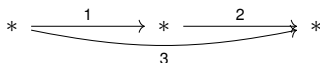
- objects: $p \in P$
- arrows: $p \rightarrow q$ whenever $p \leq q$
 - ▶ identities: $p \leq p$, i.e. $\text{id} : p \mapsto (\leq_p : p \rightarrow p)$
- domain, codomain: $\text{dom}(\leq_{pq}) = p$, $\text{cod}(\leq_{pq}) = q$
- composition: $p \leq q$ composed with $q \leq r$ is $p \leq r$

Natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ constitute category (\mathbb{N}, \leq) .

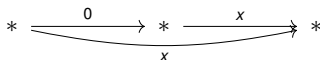
Categories and composition (example: operations)

Addition of natural numbers $(\mathbb{N}, +, 0)$ —called a *monoid*—as a category

- $1 + 2 = 3$ (composition)



- $0 + x = x$ (identity)




The category has

- one object: $*$ (name unimportant)
- one arrow: $x : * \rightarrow *$ for each number $x \in \mathbb{N}$ (identity arrow is $0 : * \rightarrow *$)
- domain, codomain: $dom(x) = *$, $cod(x) = *$
- composition is addition: $n \circ m$ is $m + n$, which is
 - ▶ *associative*: $x + (y + z) = (x + y) + z$
 - ▶ *unital*: $x + 0 = x = 0 + x$

Categories and composition (e.g., sets and functions)


Sets and functions make up a category: **Set**

- objects: sets, $A = \{a, a', \dots\}$, $B = \{b, b', \dots\}$
- arrows: functions, $f : A \rightarrow B$; $a \mapsto b$, $a' \mapsto b'$, ...
 - ▶ identities are identity functions, $1_A : a \mapsto a$
- domain, codomain: $\text{dom}(f) = A$, $\text{cod}(f) = B$
- composition: composition of functions, $g \circ f : a \mapsto g(f(a))$

 Objects and arrows have *internal* structure: the actions on elements, $a \mapsto f(a)$

Contrast categories: set (S) as category vs. category of sets

- set (S): objects are elements $a \in S$; arrows are (only) identities $1_a : a \rightarrow a$
- **Set**: objects are sets; arrows are functions, $f : S \rightarrow T$; $a \mapsto f(a)$

 Notational difference: \rightarrow (arrow) vs. \mapsto (action)

Look ahead: monoid as category vs. category of monoids

Categories and structures as arrows (e.g., monoids)

Equivalently, a monoid (M, μ, η) is a set M and a pair of arrows:

- a *binary* function, $\mu : M \times M \rightarrow M$ and
- a *nullary* function $\eta : 1 \rightarrow M$ picking out the unit

expressed as the diagram

$$M \times M \xrightarrow{\mu} M \xleftarrow{\eta} 1$$

or, the single arrow

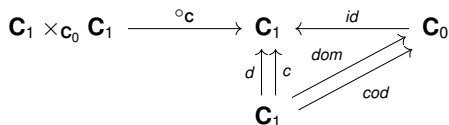
$$M \times M + 1 \xrightarrow{\mu + \eta} M$$

For example, addition of natural numbers is the monoid (\mathbb{N}, μ, η) , where

- $\mu : (m, n) \mapsto m + n$
- $\eta : * \mapsto 0$.

Category structure as arrow

A category as the diagram of arrows ($d = id \circ dom$, $c = id \circ cod$):



Category structure as one arrow ($\mathbf{C}_{\bullet} = \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 + 2\mathbf{C}_1 + \mathbf{C}_0$):

$$\mathbf{C}_{\bullet} \xrightarrow{\circ_{\mathbf{C}} + d + c + id} \mathbf{C}_1$$

Category of arrows as objects

A category of arrows and squares:

- objects are arrows: $\alpha : A \rightarrow A'$
- arrows are squares: pairs (f, f') making the squares commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha \downarrow & & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array}$$

- composition is “pasting” of squares:

$$\begin{array}{ccccc} & & g \circ f & & \\ & \frown & & \smile & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \alpha \downarrow & & \downarrow \beta & & \downarrow \gamma \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \\ & \smile & & \frown & \\ & g' \circ f' & & & \end{array}$$

Categories of structures (e.g., monoids)

Recall monoid structure as arrow (now an object):

$$M \times M + 1 \xrightarrow{\mu + \eta} M$$

object is arrow (structure), arrow is square (*structure-preserving* map):

$$\begin{array}{ccc} M \times M + 1 & \xrightarrow{h \times h + 1} & N \times N + 1 \\ \mu + \eta \downarrow & & \downarrow \mu + \eta \\ M & \xrightarrow{h} & N \end{array}$$

i.e. a monoid homomorphism $h : M \rightarrow N$ (cf. algebraic definition).

A *monoid homomorphism* is a function $h : M \rightarrow N$ such that for all $a, b \in M$

- $h(a \cdot b) = h(a) \cdot h(b)$
- $h(e_M) = e_N$

Look ahead: category as structure

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Functors and representation

Representation (basic): one thing standing in for another thing

In cognitive science (classical version):

- Principle: semantic relations between parts “mirrored” by syntactic relations between corresponding symbols—*partial algebra homomorphism*
- Example: *circle is left of square* corresponds to *left-of(circle, square)*
- Motivation: *systematicity of representation* properties

In category theory (functor version):

- Principle: structure-preserving map (functor), *category homomorphism*,
 $F(g \circ f) = F(g) \circ F(f)$
- Example: composite arrow $g \circ f$ is represented as $F(g \circ f)$ by the composite of constituent arrow representations $F(f)$ and $F(g)$
- Motivation: represent objects A and arrows $f : A \rightarrow B$ in one domain as objects $F(A)$ and arrows $F(f) : F(A) \rightarrow F(B)$ in another (co)domain

Functor (definition)

A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is a (structure-preserving) map sending the objects and arrows in \mathbf{C} to objects and arrows in \mathbf{D} that preserves:

- domains, codomains: $F(\text{dom}(f)) = \text{dom}(F(f))$, $F(\text{cod}(f)) = \text{cod}(F(f))$
- identities: $F(1_A) = 1_{F(A)}$
- composition: $F(g \circ f) = F(g) \circ F(f)$

$$\begin{array}{ccc} A & \xrightarrow{\quad \quad \quad} & F(A) \\ \downarrow f & & \downarrow F(f) \\ g \circ f \swarrow B & \xrightarrow{\quad \quad \quad} & F(B) \quad \searrow F(g \circ f) \\ \downarrow g & & \downarrow F(g) \\ C & \xrightarrow{\quad \quad \quad} & F(C) \end{array}$$

Functors are structure-preserving maps

A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is a pair of maps $(F_0, F_1) : (\mathbf{C}_0, \mathbf{C}_1) \rightarrow (\mathbf{D}_0, \mathbf{D}_1)$ preserving domains/codomains:

$$\begin{array}{ccc} \mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1 \\ \text{dom} \downarrow \downarrow \text{cod} & & \text{dom} \downarrow \downarrow \text{cod} \\ \mathbf{C}_0 & \xrightarrow{F_0} & \mathbf{D}_0 \end{array}$$

identities:

$$\begin{array}{ccc} \mathbf{C}_0 & \xrightarrow{F_0} & \mathbf{D}_0 \\ \text{id} \downarrow & & \downarrow \text{id} \\ \mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1 \end{array}$$

compositions:

$$\begin{array}{ccc} \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 & \xrightarrow{F_1 \times F_1} & \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 \\ \circ_{\mathbf{C}} \downarrow & & \downarrow \circ_{\mathbf{D}} \\ \mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1 \end{array}$$

Functors are category homomorphisms

Recall category structure as arrow (now an object):

$$\mathbf{C}_\bullet \xrightarrow{\circ_{\mathbf{C}} + d + c + id} \mathbf{C}_1$$

object is arrow (structure), arrow is square (*structure-preserving* map):

$$\begin{array}{ccc} \mathbf{C}_\bullet & \xrightarrow{F_1 \times F_1 + 2F_1 + F_0} & \mathbf{D}_\bullet \\ \circ_{\mathbf{C}} + d + c + id \downarrow & & \downarrow \circ_{\mathbf{D}} + d + c + id \\ \mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1 \end{array}$$

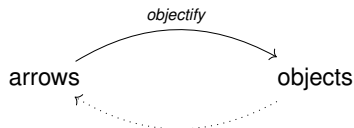
i.e. a category homomorphism $F : \mathbf{C} \rightarrow \mathbf{D}$ (cf. algebraic definition).

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- domains, codomains: $F(dom(f)) = dom(F(f))$, $F(cod(f)) = cod(F(f))$
- identities: $F(1_A) = 1_{F(A)}$
- composition: $F(g \circ f) = F(g) \circ F(f)$

Interim: arrow \rightarrow object

Internalization/chunking: enclosing (hiding) structure



Compare:

- arrows as objects
- verbs as nouns
- actions as states
- lines as points
- squares as lines (between lines)

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Natural transformations and re-representation

Computation (basic): transformations of representations

In cognitive science (classical version):

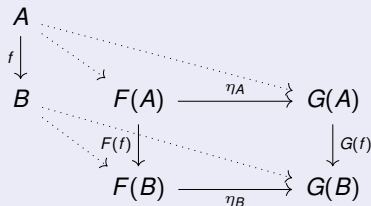
- Principle: symbolic representations and processes
- Example: What is to the left of circle and square? circle
- Motivation: *systematicity of inference* properties

In category theory (natural transformation version):

- Principle: commutative square (representation homomorphism)
- Example: composites to constituents, $A \times B \rightarrow A$
- Motivation: naturality, generality (transcend specifics)

Natural transformation (definition)

A natural transformation $\eta : F \rightrightarrows G$ is a family of arrows, $\eta_A : F(A) \rightarrow G(A)$, making the following diagram (square) commute:



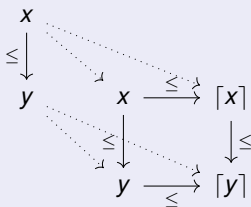
 Functors $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{C} \rightarrow \mathbf{D}$ are from the *same domain* to the *same codomain*

Nat. trans. (example): approximation

The *ceiling function* sends each real number x to the smallest integer not less than x : as a functor, $\lceil \cdot \rceil : \mathbb{R} \rightarrow \mathbb{Z}; x \mapsto \lceil x \rceil, (x \leq y) \mapsto (\lceil x \rceil \leq \lceil y \rceil)$.

Example: $2.1 \mapsto 3$ and $(2.9 \leq 3.1) \mapsto (3 \leq 4)$

The ceiling function as a natural transformation: $1_{\mathbb{R}} \rightrightarrows \iota \circ \lceil \cdot \rceil$, where $1_{\mathbb{R}}$ is the identity functor on the category of ordered reals and ι is the injection of integers, $\iota : \mathbb{Z} \rightarrow \mathbb{R}$



Nat. trans. (example): parts and wholes

Functors: product – $\Pi : (A, B) \mapsto A \times B$; projection – $\dot{\Pi} : (A, B) \mapsto A$

Natural transformation, $\acute{\pi} : \Pi \rightarrow \dot{\Pi}$ (cf. $A \wedge B \Rightarrow A$)

$$\begin{array}{ccc} (A, B) & A \times B & \xrightarrow{\acute{\pi}_{A,B}} A \\ (f,g) \downarrow & f \times g \downarrow & \downarrow f \\ (C, D) & C \times D & \xrightarrow{\acute{\pi}_{C,D}} C \end{array}$$

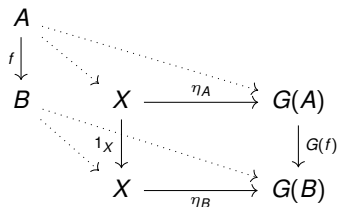
 Compare with *injection*; coproduct – $\Pi : (A, B) \mapsto A + B$

Natural transformation, $\acute{\iota} : \dot{\Pi} \rightarrow \Pi$ (cf. $A \Rightarrow A \vee B$)

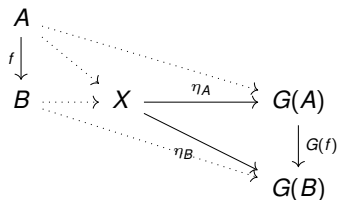
$$\begin{array}{ccc} (A, B) & A & \xrightarrow{\acute{\iota}_{A,B}} A + B \\ (f,g) \downarrow & f \downarrow & \downarrow f+g \\ (C, D) & C & \xrightarrow{\acute{\iota}_{C,D}} C + D \end{array}$$

Natural transformation: commutative triangles

Constant functor – $X : A \mapsto X, f \mapsto 1_X$



simplifies to



Natural transformation (example): least element

Zero (0) is the *least* natural number (\mathbb{N}) – as natural transformation between functors:

- Zero – $0 : \mathbb{N} \rightarrow \mathbb{N}; x \mapsto 0$
- Identity – $1_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}; x \mapsto x$

Natural transformation $0 \leq 1_{\mathbb{N}}$ (cf. $0 \leq x$ for all $x \in \mathbb{N}$)

$$\begin{array}{ccc} 0 & \xrightarrow{\leq} & x \\ & \searrow \leq & \downarrow \leq \\ & & y \end{array}$$

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Universal morphisms and limits

Category theory constructions are typically defined in terms of *universal (mapping) properties*, i.e. a property shared by all instances of some type.

A *universal mapping property*:

- a map that is *common (universal)* to *all* instances in given context and
- a map that is *unique (specific)* to *each* instance

Some intuition: travelling to the conference

Universal morphism (definitions)

Primal form: A *universal morphism* from object X to functor $F : \mathbf{B} \leftarrow \mathbf{A}$ is a pair (A, α) making the following diagram (triangle) commute (i.e. the *unique-existence* condition):

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & F(A) \\ & \searrow g & \downarrow F(u) \\ & & F(Y) \end{array} \qquad \begin{array}{c} A \\ \downarrow u \\ Y \end{array}$$

 unique-existence: for every Y and every g there exists a unique (dashed) arrow u

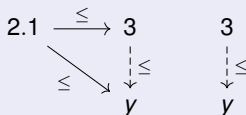
Dual form: A *universal morphism* from functor $F : \mathbf{B} \rightarrow \mathbf{A}$ to object Y is a pair (B, β) making the following diagram (triangle) commute (i.e. the *unique-existence* condition):

$$\begin{array}{ccc} X & & F(X) \\ \downarrow u & & \downarrow F(u) \\ B & & F(B) \end{array} \qquad \begin{array}{ccc} & & \\ & \searrow f & \\ & & Y \end{array} \qquad \begin{array}{ccc} & & \\ & \xrightarrow{\beta} & \end{array}$$

Universal morphism (examples): closest element

Primal form:

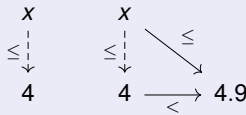
The *ceiling* of x (e.g., 2.1) is the smallest integer not less than x (i.e. 3)



i.e. the pair $(3, \leq)$ from 2.1 to the inclusion functor $\iota : (\mathbb{R}, \leq) \leftarrow (\mathbb{Z}, \leq)$.

Dual form:

The *floor* of y (e.g., 4.9) is the largest integer not greater than y (i.e. 4)



i.e. the pair $(4, \leq)$ from the inclusion functor $\iota : (\mathbb{Z}, \leq) \rightarrow (\mathbb{R}, \leq)$ to 4.9.

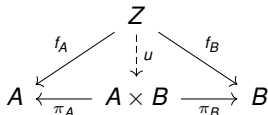
Limits and composition (of arrows *and* objects)

Limits constitute a general class of universal morphisms, seen here as another form of compositionality, e.g. a limit of A and B is their product, $A \times B$, and two maps retrieving A and B from the product, i.e. $A \times B \rightarrow A$ and $A \times B \rightarrow B$

A *limit* is the “best” way to pick out (reference) an arrangement of objects and arrows in some category:

- a map (cone) that is *common (universal)* to *all* references in given context and
- a map that is *unique (specific)* to *each* reference

Example: all pairs of maps $f_A : Z \rightarrow A$ and $f_B : Z \rightarrow B$ pass through product P (written $A \times B$) and maps $\pi_A : A \times B \rightarrow A$ and $\pi_B : A \times B \rightarrow B$, i.e. $f = \pi \circ u$, where $f = (f_A, f_B)$



Limits: Compositionality as universal construction

A *limit* is like an optimal focus of attention: the “best” way to pick out a collection of objects and morphisms of shape J in a category \mathbf{C} :

Example: a *product* is the best way to pick out a *pair* of objects A and B in \mathbf{C}

Construction	Category theory	Concept (attention)	Product
shape	category	window of attention	$2 = (\cdot, \cdot)$
diagram	functor	contents of attention	$(A, B) : 2 \rightarrow \mathbf{C}$
cone	(vertex, legs/nt.)	spotlight of attention	$(Z, (f_A, f_B))$
cone homomorph.	map (of cones)	shift of perspective	$u : Z \rightarrow A \times B$
limit (univ. cone)	univ. morphism	optimal perspective	$(A \times B, (\pi_A, \pi_B))$

Some intuition: spotlight of attention

 All limits have this form: universal cone to a J -shaped diagram in \mathbf{C}

Universals and limits (diagrams)

Diagrams are formal constructions used to reference a part of a category

A *diagram* D of *shape* J in a category \mathbf{C} is a functor $D : J \rightarrow \mathbf{C}$.

Some examples of diagrams are:

- *empty*, $D_\emptyset : 0 \rightarrow \mathbf{C}$, i.e. no objects or arrows
- *point*, $X : (\cdot) \rightarrow \mathbf{C}$, the object X and its identity 1_X
- *pair*, $(A, B) : (\cdot, \cdot) \rightarrow \mathbf{C}$, the pair of objects A and B (and their identities)
- *arrow*, $f : (\downarrow) \rightarrow \mathbf{C}$, the arrow $f : A \rightarrow B$

\mathbf{C}^J is the category of J -shaped diagrams; e.g., \mathbf{C}^\downarrow is the category of arrows

The diagonal functor $\Delta : \mathbf{C} \rightarrow \mathbf{C}^J$ sends each object A to a J -shaped diagram of A

Example: $\Delta : \mathbf{C} \rightarrow \mathbf{C}^\downarrow$ sends A to its identity arrow, i.e. $\Delta(A) = (1_A : A \rightarrow A)$

Universals and limits (definition)

Diagrams (functors, $D : J \rightarrow \mathbf{C}$), cones and cone homomorphisms:

A *cone* from a vertex V to a base D (i.e. a J -shaped *diagram*) is a pair (V, ϕ) , where ϕ is a natural transformation; a *cone homomorphism* is a map $h : V \rightarrow W$ such that

$$\begin{array}{ccc} V & \xrightarrow{\phi_i} & D_i \\ & \searrow \phi_j & \downarrow \delta_{ij} \\ & & D_j \\ & \nearrow \psi_i & \\ W & \xrightarrow{\psi_j} & D_j \end{array}$$

Limits to diagrams (D) are *universal* cones:

A *limit* to D is a universal morphism (L, ϕ) from *diagonal* functor $\Delta : \mathbf{C} \rightarrow \mathbf{C}^J$ to D :

$$\begin{array}{ccc} V & \Delta(V) & \\ \downarrow u & \downarrow \Delta(u) & \searrow \phi \\ L & \Delta(L) & \xrightarrow{\psi} D \end{array}$$

Universals and limits (example: products)

A product of objects A and B is a limit to a pair diagram, $(A, B) : 2 \rightarrow \mathbf{C}$

A *product* of objects A and B is a product object $A \times B$ and two arrows retrieving A and B , i.e. $\pi_A : A \times B \rightarrow A$ and $\pi_B : A \times B \rightarrow B$

$$\begin{array}{ccc} V & \Delta(V) & \\ \langle f, g \rangle \downarrow & \Delta \langle f, g \rangle \downarrow & \searrow (f, g) \\ A \times B & \Delta(A \times B) & \xrightarrow{(\pi_A, \pi_B)} (A, B) \end{array}$$

 Specific products depend on specific categories

Specific products (examples)

In **Set** (category of sets and functions), a product of sets A and B is their Cartesian product and two projections:

- Cartesian product: $A \times B = \{(a, b) | a \in A, b \in B\}$
- projections: $\pi_A : (a, b) \mapsto a$ and $\pi_B : (a, b) \mapsto b$

In **Set**[⊆] (category of sets and inclusions), a product of sets A and B is their intersection and two insertions:


- Intersection: $A \times B = \{c | c \in A, c \in B\}$
- insertions: $\iota_A : c \mapsto c$ and $\iota_B : c \mapsto c$ (also written $A \cap B \subseteq A$ and $A \cap B \subseteq B$)

Universals and limits (example: terminals)

A terminal (final) object is a limit to an *empty diagram*, $D_\emptyset : 0 \rightarrow \mathbf{C}$

A *terminal object* (denoted 1) has unique arrow to it from every object in \mathbf{C} :

$$\begin{array}{ccc} V & \Delta(V) = D_\emptyset & \\ \downarrow u & \downarrow 1 & \searrow 1 \\ 1 & \Delta(1) = D_\emptyset & \xrightarrow{1} D_\emptyset \end{array}$$

 $D_\emptyset \in \mathbf{C}^0 \cong \mathbf{1}$ (one-object category); unique arrow 1 is empty family of arrows (\emptyset); terminal object is pair $(1, \emptyset)$, usually just denoted 1 with triangle ignored

Specific terminal objects depend on specific categories. Contrast

- sets and functions (**Set**): a terminal object is any *singleton set*, $\{*\}$, where $*$ is the only (unnamed) element; the unique arrow is the constant function
- (P, \leq) : the terminal object is the *top element* (\top), i.e. $p \leq \top$ for all $p \in P$

Universals and colimits (definition)

Diagrams ($D : J \rightarrow \mathbf{C}$), cocones and cocone homomorphisms:

A *cocone* from a vertex V to a base D is a pair (V, ϕ) , where ϕ is a natural transformation; a *cocone homomorphism* is a map $h : W \rightarrow V$ such that

$$\begin{array}{ccc}
 D_i & \xrightarrow{\psi_i} & W \\
 & \searrow \psi_j & \downarrow h \\
 \delta_{ij} \downarrow & & \\
 D_j & \xrightarrow{\phi_i} & V \\
 & \nwarrow \phi_j &
 \end{array}$$

Colimits to diagrams (D) are *universal* cocones:

A *colimit* to D is a universal morphism (L, ϕ) from D to diagonal functor $\Delta : \mathbf{C} \rightarrow \mathbf{C}^J$:

$$\begin{array}{ccc}
 D & \xrightarrow{\phi} & \Delta(L) \\
 & \searrow \psi & \downarrow \Delta(u) \\
 & & \Delta(V)
 \end{array}
 \qquad
 \begin{array}{c}
 L \\
 \downarrow u \\
 V
 \end{array}$$

Universals and colimits (example: coproducts)

A coproduct (sum) of objects A and B is a colimit to a pair diagram, $(A, B) : 2 \rightarrow \mathbf{C}$

A coproduct of objects A and B is a coproduct object $A + B$ and two arrows inserting A and B , i.e. $\iota_A : A \rightarrow A + B$ and $\iota_B : B \rightarrow A + B$

$$\begin{array}{ccc} (A, B) & \xrightarrow{(\iota_A, \iota_B)} & \Delta(A + B) \\ & \searrow (f, g) & \downarrow \Delta[f, g] \\ & & \Delta(V) \end{array} \qquad \begin{array}{c} A + B \\ \downarrow [f, g] \\ V \end{array}$$

Specific coproducts depend on specific categories. Contrast

- sets and functions (**Set**): *disjoint union*, i.e. $\{(1, a) | a \in A\} \cup \{(2, b) | b \in B\}$, and *injections*, i.e. $\iota_A : a \mapsto (1, a)$ and $\iota_B : b \mapsto (2, b)$
- sets and inclusions: set union, i.e. $A \cup B$, and *insertions*, i.e. $A \subseteq A \cup B$ and $B \subseteq A \cup B$

Universals and colimits (example: initials)

An initial (cofinal) object is a colimit to an *empty diagram*, $D_\emptyset : \emptyset \rightarrow \mathbf{C}$

A *initial object* (denoted 0) has unique arrow from it to every object in \mathbf{C} :

$$\begin{array}{c} 0 \\ \downarrow u \\ V \end{array}$$

Specific initial objects depend on specific categories. Contrast

- sets and functions (**Set**): the initial object is the empty set (\emptyset); the unique arrow is the empty function
- (P, \leq) : the initial object is the *bottom element* (\perp), i.e. $\perp \leq p$ for all $p \in P$

Outline

1 Introduction

- Objectives: What/Why/How of category theory
- Perspective: Category theory as theory of formal analogies
- Where category theory *meets* cognitive science: *compositionality*

2 Basic correspondences

- Categories and composition
- Functors and representation
- Natural transformations and re-representation (computation)

3 Universal cognition

- Universal morphisms (limits) and compositionality
- **Universals and *systematicity***

4 Discussion

- The *universal mapping principle* for cognitive science

Universals and *systematicity*

Systematicity problem: explain *why* having certain cognitive abilities implies having certain other cognitive abilities, e.g., the ability to understand the expression

- “John loves Mary” *if and only if*
- “Mary loves John”

or, conversely, *why not* the case of having the ability to understand the expression

- “John loves Mary” *but not*
- “Mary loves John”

i.e. why cognitive abilities cluster as *equivalence classes* in a certain way.

Classical explanation: combinatorial syntax and semantics (Fodor & Pylyshyn, 1988):

- a *grammar* common to all instances
- an *instantiation* specific to each instance

But, why *that* particular grammar (beyond fitting data)?


- explains how (possible), but not why (necessarily follows)
- grammar chosen to fit data—*ad hoc* (Aizawa, 2003)

Systematicity and universal morphism

Categorical explanation: underlying a systematicity property is a universal morphism (Phillips & Wilson, 2010). Each capacity is composed from

- a *mediating arrow* common to all instances and
- a *unique arrow* specific to each instance

Some intuition: capacities cluster around a universal morphism (common structure)

 Every universal morphism is an initial/terminal object in a *comma category*—the “best” one can do for the given situation

Systematicity (example: coloured shapes)

Inferring colours and shapes:

- Colours: red, green, blue; Shapes: circle, triangle, square
- Coloured shapes: \circ , \circ , \triangle , \square , ...
- Colour projection: $\circ \mapsto \text{red}$, $\circ \mapsto \text{green}$, $\triangle \mapsto \text{blue}$, $\square \mapsto \text{blue}$, ...
- Shape projection: $\circ \mapsto \text{circle}$, $\circ \mapsto \text{circle}$, $\triangle \mapsto \text{triangle}$, $\square \mapsto \text{square}$, ...

In **Set**:

- Colours: $C = \{\text{red}, \text{green}, \text{blue}\}$; Shapes: $S = \{\text{circle}, \text{triangle}, \text{square}\}$
- Coloured-shapes: $CS = C \times S$
- Colour projection: $\pi_C : CS \rightarrow C$
- Shape projection: $\pi_S : CS \rightarrow S$
- Product: $(C \times S, (\pi_C, \pi_S))$

Systematicity (empirical test: product map)

Product map ($f \times g$): dashed arrow in the following commutative diagram

$$\begin{array}{ccccc} A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \\ f \downarrow & & \downarrow f \times g & & \downarrow g \\ C & \xleftarrow{\pi_C} & C \times D & \xrightarrow{\pi_D} & D \end{array} \qquad \begin{array}{c} \text{Cue} \\ \downarrow \text{map} \\ \text{Target} \end{array}$$

Empirical test: product of cue-target maps;

- Alphabet (letters): $A = \{k, m, p\}$; $B = \{g, r, v\}$
- Cues (letter pairs): $A \times B$; Targets (coloured shapes): $C \times S$
- Maps: AlphaA-to-Colour: $a2c : A \rightarrow C$; AlphaB-to-Shape: $b2s : B \rightarrow S$
- Product map: $a2c \times b2s : A \times B \rightarrow C \times S$
- Systematicity (?): training map implies testing map (generalization)

Some intuition: from partial map to complete map (Phillips, Takeda, & Sugimoto, 2016)

Categorical vs classical products

Classical compositionality: tokening principle (Fodor & Pylyshyn, 1988)


Constituent tokened whenever complex host is tokened: e.g.,

- $A \times B = \{(a, b) | a \in A, b \in B\}, \pi_A : (a, b) \mapsto a; \pi_B : (a, b) \mapsto b$

Categorical compositionality: unique-existence condition

Constituent symbol need not be tokened whenever complex host symbol is tokened: e.g.,

- $P = \{n | 1 \leq n \leq |A| \cdot |B|\}, \pi'_A : n \mapsto a_i, \dots; \pi'_B : n \mapsto b_j, \dots$

 classical (canonical) product as a special case of categorical product

Comma category: linking naturals and universals

Comma categories are constructed from functors with the same *codomain* category

Suppose functors $S : \mathbf{A} \rightarrow \mathbf{C}$ and $T : \mathbf{B} \rightarrow \mathbf{C}$. A comma category $(S \downarrow T)$ has:

- objects – a triple (A, B, ϕ) for each object A in \mathbf{A} , each object B in \mathbf{B} and each arrow $\phi : S(A) \rightarrow T(B)$ in \mathbf{C}
- arrows – a pair (f, g) for each arrow $f : A \rightarrow A'$ in \mathbf{A} and each arrow $g : B \rightarrow B'$ in \mathbf{B} such that the diagram

$$\begin{array}{ccc} S(A) & \xrightarrow{\phi} & T(B) & (A, B, \phi) \\ S(f) \downarrow & & \downarrow T(g) & \downarrow (f, g) \\ S(A') & \xrightarrow{\phi'} & T(B') & (A', B', \phi') \end{array}$$

in \mathbf{C} is a commutative square, and

- composition – (vertical) pasting of compatible commutative squares

Univ. morphism: from X to F is *initial* in $(X \downarrow F)$; from F to X is *terminal* in $(F \downarrow X)$

Comma category (example) – products


A product of A and B is the terminal object $(A \times B, \pi)$ in the comma category $(\Delta \downarrow (A, B))$:

$$\begin{array}{ccc} \Delta(V) & & (V, (f, g)) \\ \Delta \langle f, g \rangle \downarrow & \searrow (f, g) & \downarrow \langle f, g \rangle \\ \Delta(A \times B) & \xrightarrow{\pi} & (A, B) \end{array} \quad \begin{array}{c} (A \times B, \pi) \end{array}$$

$$\pi = (\pi_A : A \times B \rightarrow A, \pi_B : A \times B \rightarrow B)$$

For initials/terminals (hence, universal constructions generally):

- *All roads lead to Rome!* – “global optima”

 All universals are *unique up to unique isomorphism*: e.g., $(B \times A, \pi')$ is also a product, but only one arrow $\phi : (A \times B, \pi) \cong (B \times A, \pi') : \phi^{-1}$

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Universals and cognition

A *universal mapping principle* for cognitive science (Phillips, 2021a):

- the “best” one can do in the given context
- compositionality, systematicity, productivity as universal (mapping) properties

Some other applications of principle:

- cognitive complexity (Phillips, Wilson, & Halford, 2009)
- learning and generalization (Phillips et al., 2016; Phillips, 2018)
- compositionality (Phillips, 2020)
- relational schema induction (Phillips, 2021b)

Further reading

Some introductions to category theory:

- conceptual (Lawvere & Schanuel, 2009; Simmons, 2011)
- formal (Leinster, 2014; Mac Lane, 1998)
- applied (Fong & Spivak, 2018; Spivak, 2014)
- philosophical (Kromer, 2007; Marquis, 2009)
- computational (Bird & De Moor, 1997; Walters, 1991) and many others ...

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