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The Logic of Public Announcements, Common Knowledge, and Private Suspicions (extended abstract)

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Abstract

This paper presents a logical system in which various group-level epistemic actions are incorporated into the object language. That is, we consider the standard modeling of knowledge among a set of agents by multi-modal Kripke structures. One might want to consider actions that take place, such as announcements to groups privately, announcements with suspicious outsiders, etc. In our system, such actions correspond to additional modalities in the object language. That is, we do not add machinery on top of models (as in Fagin et al [1], but we reify aspects of the machinery in the logical language.

The first case of the kind of logics we consider appears in Gerbrandy and Groeneveld [3]. They introduced a language in which one can faithfully represent all of the reasoning in examples such as the Muddy Children scenario. In that paper we find operators for updating worlds via announcements to groups of agents who are isolated from all others. We advance this by considering many more actions; and by using a more general semantics.

Our logic contains the infinitary operators used in the standard modeling of common knowledge. We present a sound and complete logical system for the logic, and we study its expressive power.

1 Introduction

One of the interesting developments in epistemic logic in recent years is the importation of ideas from dynamic logic. This development promises to enrich our formal accounts of knowledge by incorporating *knowledge change* and *actions leading to knowledge change* into existing frameworks. This paper continues the work on epistemic logics with group updates initiated in Gerbrandy and Groeneveld [3]. The idea is to meld the approaches of multi-agent epistemic logic as appears in Fagin et al [1]) with the work of Veltman [5] on update semantics. Veltman's goal is to give a "dynamic" account of meaning which replaces truth-conditional semantics by notions having to do with changes of information state.

An example of an announcement To see what the subject is about, consider the case of a set \mathcal{A} of three agents, say A , B , and C ; two atomic propositions, p and q , and a Kripke model with four worlds w , x , y , and z depicted below. We have written out the accessibility relations

w		x		y		z	
p	$A : w, x$	p	$A : w, x$	$\neg p$	$A : y, z$	p	$A : y, z$
q	$B : w, y, z$	$\neg q$	$B : x$	q	$B : w, y, z$	$\neg q$	$B : w, y, z$
	$C : w, y$		$C : x, z$		$C : w, y$		$C : x, z$

Figure 1: The Kripke model W

in tabular form. In other notation, we have $w \rightarrow_A w$, $w \rightarrow_A x$, $x \rightarrow_C z$, $w \models p$, etc. We have some standard semantic facts, such as $w \models \neg \Box_B p$ (in w , B does not know p , since p is false in y), and $w \models \Box_A \neg \Box_B p$, etc.

Now suppose someone comes to each world $v \in W$ where p holds, takes A and B off to the side, and tells them (together) that indeed, p holds there. We want to *update* the worlds so that A and B 's accessibility relations only include worlds where p is true. On the other hand, C was excluded from the announcement (and in fact at this point, we want to assume that C did not even know about it.) So C 's accessibility relation should not change. We want to represent the updated version v' in a way that captures the epistemic alternatives available to each of the agents. Our proposal is that the worlds below represent the updated versions of the corresponding worlds above. Note first that in the updated worlds, we have kept C 's accessibility relations the way they were, since C was not party to the communication.

w'		x'		z'	
p	$A : w', x'$	p	$A : w', x'$	p	$A : z'$
q	$B : w', z'$	$\neg q$	$B : x'$	$\neg q$	$B : w', z'$
	$C : w, y$		$C : x, z$		$C : x, z$

Figure 2: The updated model W' adds worlds w' , x' , and z' to W

Further, consider the update of w , and focus on the worlds accessible to B . Before the update, B used to think that w , y , and z were possible. It should be clear why there is no trace of y in the worlds accessible to B after the update of w . So we need to update w to *some* new world; this is why w' is needed. And w' should have the same propositional content, since announcements do not change facts. The main question might be: why in w' do A 's accessibility relations point to w' and x' (and not w and x)? And why do B 's point to w' and z' (and not w and z)? The reason is that announcing p to A and B should mean that not only do A and B think $\neg p$ is impossible, but also that $\neg p$ should be impossible *from all the worlds they think possible*, etc. This leads to a *fixed point* semantics for updates via actions. It generalizes the “conscious” updates of [3, 2] to arbitrary Kripke models. (The presentation of [3, 2] uses non-wellfounded sets; these are essentially model-world pairs modulo bisimulation.)

The main justification for our proposal on how to model actions comes from looking at semantic facts that hold in the updated worlds. For example, $\langle W', w' \rangle \models \Box_{\{A, B\}}^* p$. That is, in W' at the world in w' , it is common knowledge among A and B that p holds. (We use the standard modeling of common knowledge via infinite iteration. See, e.g., Fagin et al [1] for this.) On the other hand, one can check that the updates do not change any knowledge facts

for C : for all $v \in W$, $\langle W, v \rangle \models \Box_C \varphi$ iff $\langle W', v' \rangle \models \Box_C \varphi$. Both of these consequences seem right.

Suspicion Recall that we introduced the announcement to A and B by saying that someone takes them off and announces a fact. Our modeling of this has a feature that might not be appropriate in a lot of applications: C was not only excluded from the announcement, but after the announcement C has the mistaken view that nothing whatsoever happened. It was as if the announcement were completely private. This is a very strong assumption to make, and one might well want to model situations where C knows that *some* announcement is made, or (as in our example) that C suspects that p was announced to A and B , or that C listens in to the announcement while A and B are unaware of this, etc. We'll see how these effects can be modeled in Examples 3.3–3.5 below.

Contents of this paper What we want to do in this paper is to study two logical systems that have *action operators* in addition to propositional and epistemic operators. For example, we will have an *announcement operator* $[p]_{\{A,B\}}$. For each proposition φ , $[p]_{\{A,B\}}\varphi$ will again be a proposition. We'll define the syntax and semantics in Section 2. But as an example, we'll want to say that $\langle W, w \rangle \models [p]_{\{A,B\}}\Box_{A,B}^* p$, since $\langle W', w' \rangle \models \Box_{\{A,B\}}^* p$. Examples are presented in Section 3.

The two logical systems of the paper are called $\mathcal{L}([\alpha])$ and $\mathcal{L}([\alpha], \Box^*)$. The second system extends the first by including infinitary modalities \Box_B^* for groups of agents. These are familiar from propositional dynamic logic (PDL) and from the standard modeling of common knowledge. The main technical work of the paper is a set of results on those languages. First, we show in Section 4 that $\mathcal{L}([\alpha])$ is no more expressive than ordinary modal logic. This leads fairly easily to a completeness result for $\mathcal{L}([\alpha])$. For the larger language, we have a contrasting result: $\mathcal{L}([\alpha], \Box^*)$ is more expressive than modal logic with the operators \Box_B^* . We axiomatize the validities of $\mathcal{L}([\alpha], \Box^*)$ in Section 5.

For reasons of space, we are not presenting any proofs in this extended abstract, and we also are omitting several sections of technical material.

2 A Logic of Epistemic Actions

2.1 Syntax

We begin with a set AtProp of atomic propositions, and we define two sets simultaneously: the language $\mathcal{L}([\alpha])$, and a sets of *actions* (over $\mathcal{L}([\alpha])$).

$\mathcal{L}([\alpha])$ is the smallest collection which includes the set AtProp of atomic propositions, and which is closed under \neg , \wedge , \Box_A for $A \in \mathcal{A}$, and $[\alpha]\varphi$, where α is an action over $\mathcal{L}([\alpha])$, and $\varphi \in \mathcal{L}([\alpha])$.

An *action structure* (over $\mathcal{L}([\alpha])$) is a finite Kripke frame K over the set \mathcal{A} of agents, together with a map $\text{PRE} : K \rightarrow \mathcal{L}$. (PRE stands for “precondition,” and we discuss this below when we turn to the semantics.) We call the worlds of K *action tokens*. An *action* is a pair $\alpha = \langle K, k \rangle$ consisting of an action structure K and some $k \in K$. We usually suppress K and k in our notation and use α instead. For example, we write $\text{PRE}(\alpha)$ instead of $\text{PRE}(k)$. Each action α thus is just like a finite model-world pair with an additional function PRE.

The actions constitute a Kripke frame Actions in the natural way, by setting $\langle K, k \rangle \rightarrow_A \langle L, l \rangle$ iff $K = L$ and $k \rightarrow_A l$ in K .

This completes the definitions of $\mathcal{L}([\alpha])$ and of the classes of actions (and action structures) over $\mathcal{L}([\alpha])$.

2.2 Semantics

The ideas There are a few leading ideas behind the semantics. First, our action structures are Kripke frames. We therefore think of the actions in the same way that we think of frames. That is, our actions are viewed differently by different agents. So to say that $k \rightarrow_A l$ means, intuitively, that if k is the action token that really happens, then A thinks it is possible that l happens. The actions themselves are pointed frames, so they come with an additional piece of information, namely a fixed action token which really does take place.

Second, we'll approach the semantics of $\langle W, w \rangle \models [\alpha]\varphi$ by defining an action function $(\langle W, w \rangle, \alpha) \mapsto \langle W, w \rangle^\alpha$ taking model-world pairs and actions and returning another model-world pair. Then we'll set $\langle W, w \rangle \models [\alpha]\varphi$ iff $\langle W, w \rangle^\alpha \models \varphi$.

Third, as the example in Section 1 suggests, the effect of an action α on $\langle W, w \rangle$ is to introduce *copies of parts of W* . We say that a world w *survives* $k \in K$ if $\langle W, w \rangle \models \text{PRE}(k)$. That is, we think of $\text{PRE}(k)$ as the presupposition or precondition necessary for k to happen. We want preconditions because some epistemic actions cannot intuitively happen in some worlds. The easiest example is in S5-models with *truthful announcements*: a person cannot get a truthful announcement that φ unless φ really is true in the actual world. We introduce (formal) copies (w, α) , whenever w survives α . This copy (w, α) is supposed to represent the effect of the action α on $\langle W, w \rangle$. We want to make these copies into a Kripke model. All of our work rests on the assumption that our actions do not change facts, so the atomic propositions true at (w, α) are going to be the same as those true at w in W . The accessibility relations are harder. The idea is that we want to have a fixed-point characterization:

$$\langle W, w \rangle^\alpha \rightarrow_A \langle W, x \rangle^\beta \quad \text{iff} \quad w \rightarrow_A x \text{ in } W, \text{ and } \alpha \rightarrow_A \beta.$$

To get a feeling for this, think in the S5 case, where \rightarrow_A is an equivalence relation of indistinguishability. If A cannot distinguish between $\langle W, w \rangle$ and $\langle W, x \rangle$ and also cannot distinguish between actions α and β , then A should not be able to distinguish $\langle W, w \rangle^\alpha$ and $\langle W, x \rangle^\beta$. So the thrust of the formal definition is to show how to solve this fixed-point characterization explicitly.

The details As with the syntax, we define two things simultaneously: the semantic relation $\langle W, w \rangle \models \varphi$, and a partial operation $(\langle W, w \rangle, \alpha) \mapsto \langle W, w \rangle^\alpha$.

Given a model W and an action structure K , we define the model W^K as follows:

1. The worlds of W^K are the pairs $(w, k) \in W \times K$ such that $\langle W, w \rangle \models \text{PRE}(k)$.
2. For such pairs, $(w, k) \rightarrow_A (w', k')$ iff $w \rightarrow_W w'$ and $k \rightarrow_A k'$.
3. We interpret the atomic propositions by setting $v_{W^K}((w, k)) = v_W(w)$. That is, p is true at (w, k) in W^K iff p is true at w in W .

Given an action $\alpha = \langle K, k \rangle$ and a model-world pair $\langle W, w \rangle$, we say that $\langle W, w \rangle^\alpha$ is defined iff $\langle W, w \rangle \models \text{PRE}(k)$, and in that case we set $\langle W, w \rangle^\alpha = \langle W, w \rangle^{\langle K, k \rangle} = \langle W^K, (w, k) \rangle$.

The semantics of our language is given by extending the usual clauses for modal logic by one for actions:

$$\langle W, w \rangle \models [\alpha]\varphi \quad \text{iff} \quad \langle W, w \rangle^\alpha \text{ is defined implies } \langle W, w \rangle^\alpha \models \varphi.$$

As is customary, we abbreviate $\neg[\alpha]\neg\varphi$ by $\langle \alpha \rangle \varphi$. Then we have

$$\langle W, w \rangle \models \langle \alpha \rangle \varphi \quad \text{iff} \quad \langle W, w \rangle^\alpha \text{ is defined and } \langle W, w \rangle^\alpha \models \varphi.$$

The larger language $\mathcal{L}([\alpha], \Box^*)$ We also consider a larger language $\mathcal{L}([\alpha], \Box^*)$. This is defined by adding operators \Box_B^* for all subsets $B \subseteq \mathcal{A}$. (When we do this, of course we get more actions as well.) The semantics works exactly as in PDL.

Logics determined by sets of actions Throughout this paper, we restrict attention to $\mathcal{L}([\alpha])$ and $\mathcal{L}([\alpha], \Box^*)$ as we have defined them. However, if one is only interested in a special class C of actions, then it makes sense to consider the logics restricted to C . For $\mathcal{L}([\alpha])$, this would not change things very much. For $\mathcal{L}([\alpha], \Box^*)$, on the other hand, the restriction can give simpler logical systems. We will not explore any of these issues in this paper.

3 Examples

Example 3.1 *The trivial announcement τ .* Here we use a Kripke frame with just one world l , with $l \rightarrow_A l$ for all A , and with $\text{PRE}(\tau) = \text{true}$. We call $\tau = \langle K, l \rangle$ the *trivial announcement*. For all W , $W \cong W^\tau$ via $w \mapsto (w, \tau)$. It follows by a trivial induction on φ that $\models [\tau]\varphi \leftrightarrow \varphi$.

Example 3.2 *Secure group-announcements a la Gerbrandy-Groeneveld.* Let $\mathcal{A} = \{A, B, C\}$, and suppose that the group $B = \{A, B\}$ gets together and announces φ publicly. We model this using a two-world frame $K = \{k, l\}$ with $k \rightarrow_A k$, $k \rightarrow_B k$, $k \rightarrow_C l$, and $l \rightarrow_D l$ for all $D \in \mathcal{A}$. We set $\text{PRE}(k) = \varphi$ and $\text{PRE}(l) = \text{true}$. Let $\alpha = \langle K, k \rangle$ and $\beta = \langle K, l \rangle$. If φ is atomic, then $\langle W, w \rangle \models [\alpha]\Box_B^*\varphi$. That is, the announcement of φ created common knowledge. However if φ is a sentence like $\Diamond p$, then we might well have $\langle W, w \rangle \models \varphi \wedge \neg[\alpha]\varphi$. This is a desirable feature of any semantics for the updating of information: an announcement of a negative fact does not in general create knowledge of that fact. In any case, we also have a modeling that C does not know, or even suspect, that the announcement happened. That is, $\models \Box_C\varphi \leftrightarrow [\alpha]\Box_C\varphi$. Moreover, A and B know this. There is no way to say this explicitly in our language (because our language cannot refer directly to actions themselves), but we do have the following:

$$\models \Box_A\Box_C\varphi \leftrightarrow [\alpha]\Box_A\Box_C\varphi \quad \text{and also} \quad \models \Box_B^*\Box_C\varphi \leftrightarrow \Box_B^*\Box_C\varphi.$$

Example 3.3 *Group announcements with a suspicious outsider.* Consider Example 3.2. In a world (w, α) after the announcement, C does not consider it possible that A and B know anything which they did not know at the outset. Put slightly differently,

$$\text{If } \langle W, w \rangle^\alpha \models \Diamond_C\Box_A\varphi, \text{ then } \langle W, w \rangle \models \Box_C\Box_A\varphi.$$

In order to model suspicion, we would add the arrow $k \rightarrow_C k$ to K . Call the resulting actions α' and β' . Then if φ is atomic (so that an announcement of φ results in the knowledge that φ holds), and if $\langle W, w \rangle \models \Diamond_C \Box_A \neg \varphi$, then

$$\langle W, w \rangle^{\alpha'} \models \Diamond_C \Box_A \varphi, \text{ but } \langle W, w \rangle \models \Box_C \Box_A \neg \varphi.$$

Note that this action structure is an S5 structure, and so are $W^{\alpha'}$ and $W^{\beta'}$. This is important because in many situation we want to restrict attention to models and actions which are S5.

Example 3.4 *Group announcements with a secure wiretap.* In the last example, C was not aware that A and B got the announcement φ , and indeed A and B knew this. We'll modify this example to allow C to listen in and learn that φ was announced. In this example, C will not learn φ , and also A and B will be unaware that they are being wiretapped. (For this reason, it is best to think of \Box here as referring to *belief* rather than *knowledge*.)

We again use K with three worlds, say k , l , and m . The worlds k and l are exactly as in Example 3.2. We put $m \rightarrow_A k$, $m \rightarrow_B k$, and $m \rightarrow_C m$; also $\text{PRE}(m) = \varphi$. Let k , l , and m determine actions α , β , and γ , respectively. Then γ is the wiretapped action, for the following reasons: For all ψ ,

$$\models [\gamma] \Box_A \Box_C \psi \leftrightarrow \Box_C \psi.$$

That is, after A 's beliefs about C 's beliefs are not affected by γ . On the other hand, if φ is an atomic proposition such that $\langle W, w \rangle \models \varphi \wedge \neg \Box_A \varphi$, then

$$\models [\gamma] \Box_C \Box_B^* \varphi \wedge \neg \Box_C \Box_B^* \varphi.$$

That is, after γ , C believes that φ is common knowledge for A and B , but C was not aware of this before. It can also be checked that A and B will not know that C learned anything by γ , as in Example 3.2.

Example 3.5 *Announcements with suspicion of a wiretap.* In the action γ of Example 3.4, A and B did not suspect that they were wiretapped. We can model an action where A and B do suspect, but not know for sure, that they are wiretapped (which in fact they are). For this, we just add two more arrows $m \rightarrow_A m$ and $m \rightarrow_B m$ to the action structure K of Example 3.4. Then the action γ' determined by m in this new structure models the announcement with suspicion of a wiretap. Indeed, action γ' is exactly like γ (in the previous example) with respect to wiretapping: C does listen in and learns that φ was announced. But now A and B are conscious of this possibility: they consider as a possible action the very action γ' that takes place. In other words, they suspect they are being wiretapped.

In this action, we assumed C knows that A and B suspect but not know about a wiretap. Of course, one could model other possibilities, by appropriately changing the action .

Example 3.6 *Message-passing.* It is possible to represent the information content of message-passing on (possibly faulty, or wiretapped) channels. This is not a special case of an announcement with suspicion, since the sender and receiver might disagree as to the security of the channel.

Many other types of examples are possible. We can represent misleading epistemic actions, e.g. lying, or more generally acting such that some people do not suspect that your action is possible. We can also represent gratuitous suspicion (“paranoia”): maybe no “real” action has taken place, except that some people start suspecting some action (e.g., some private communication) has taken place. We are also not restricted to S5-type actions: we can model a situation where an agent comes to believe that φ , without believing that he believes φ .

4 The Logic for $\mathcal{L}([\alpha])$

In this section we present a sound and complete logic for $\mathcal{L}([\alpha])$. Here is the key axiom:

Proposition 4.1 *The following Action Axiom is sound:*

$$[\alpha]\Box_A\varphi \leftrightarrow (\text{PRE}(\alpha) \rightarrow \bigwedge \{\Box_A[\beta]\varphi : \alpha \rightarrow_A \beta\}).$$

The completeness result for $\mathcal{L}([\alpha])$ is based on a translation of $\mathcal{L}([\alpha])$ to ordinary modal logic \mathcal{L} . And this translation is based on the following *term rewriting system* \mathcal{R} :

$$\begin{array}{ll} [\alpha]p & \rightsquigarrow \text{PRE}(\alpha) \rightarrow p \\ [\alpha]\neg\psi & \rightsquigarrow \text{PRE}(\alpha) \rightarrow \neg[\alpha]\psi \\ [\alpha](\psi \wedge \chi) & \rightsquigarrow [\alpha]\psi \wedge [\alpha]\chi \\ [\alpha]\Box_A\psi & \rightsquigarrow \text{PRE}(\alpha) \rightarrow \bigwedge \{\Box_A[\beta]\psi : \alpha \rightarrow_A \beta\} \end{array}$$

In all of our work on term rewriting, we want to consider $\mathcal{L}([\alpha])$ as a *two-sorted* set, consisting of sentences and actions. We regard sentences as terms in the usual way. Actions can be regarded as terms, too: Let K be a Kripke frame K with n worlds, and let there be m agents in \mathcal{A} . Then K can be regarded as an $n \times m$ -ary function symbol f_K , using the PRE functions in the natural way. Given $n \times m$ sentences $\bar{\varphi}$, $f_K(\bar{\varphi})$ is an action. The point of all of this is to allow rewriting to go on *inside of actions*.

Proposition 4.2 *There is a wellfounded relation $<$ on the sentences of $\mathcal{L}([\alpha])$ and the actions over $\mathcal{L}([\alpha])$ such that for all rules $\varphi \rightsquigarrow \psi$ of \mathcal{R} , then $\psi < \varphi$. Moreover, if $\alpha \rightarrow^* \beta$, then $[\alpha]\Box_C^*\psi > [\beta]\psi$.*

This takes some work, and because of space limitations we must omit the entire discussion except to say that we use the *lexicographic path order*, first studied by Kamin and Levy, and by Dershowitz. At this point, we want to use the rewrite system \mathcal{R} above, applying the rules at arbitrary points inside of sentences. (For example, consider what happens with something like $[\alpha][\beta]\varphi$. We might rewrite $[\beta]\varphi$ using some rule, say to ψ ; and then rewrite $[\alpha]\psi$ to $[\gamma]\psi$.)

Lemma 4.3 *A sentence $\varphi \in \mathcal{L}([\alpha])$ is a normal form iff it is a modal sentence (that is, iff φ contains no actions).*

A sentence $\varphi \in \mathcal{L}([\alpha], \Box^)$ is a normal form iff it is built from atomic propositions using \neg , \wedge , \Box_A , \Box_B^* , or if it is of the form $[\alpha]\Box_B^*\psi$, where α is an action in normal form, and ψ , too, is in normal form.*

An action α is a normal form action if each $\text{PRE}(\beta)$ is a normal form sentence for all β such that $\alpha \rightarrow^ \beta$.*

In the next result, we let \mathcal{L} be ordinary modal logic over AtProp (where of course there are no actions), and \mathcal{L}_∞ is the *infinitary* version of modal logic, where we have conjunctions of arbitrary sets of sentences.

Proposition 4.4 *Concerning our languages \mathcal{L} , $\mathcal{L}([\alpha])$, $\mathcal{L}([\alpha], \Box^*)$, and \mathcal{L}_∞ :*

1. *There is a translation $t : \mathcal{L}([\alpha]) \rightarrow \mathcal{L}$ such that every sentence $\varphi \in \mathcal{L}([\alpha])$ is semantically equivalent to $\varphi^t \in \mathcal{L}$.*
2. *The map t extends to a map from $\mathcal{L}([\alpha], \Box^*)$ to \mathcal{L}_∞ . In fact, each φ^t may be taken to be a recursive sentence of ordinal height $< \omega^\omega$.*

4.1 A Completeness Result for $\mathcal{L}([\alpha])$

Based on our translations above, we can get a short completeness result for $\mathcal{L}([\alpha])$. First, we consider a logical system whose axioms are any complete set of axioms of the modal logic K together with more axioms corresponding to the rewriting system, and with necessitation for \Box and for actions $[\alpha]$. The axioms and rules are listed in the box in Figure 3. (But we only use the axioms and rules which do not contain the $*$ or \circ symbols. The remaining material is used for the larger language $\mathcal{L}([\alpha], \Box^*)$.)

Proposition 4.5 *This logical system for $\mathcal{L}([\alpha])$ is strongly complete: $\Sigma \vdash \varphi$ iff $\Sigma \models \varphi$.*

The first strong completeness result of this kind is due to Gerbrandy and Groeneveld [3]. Actually, they only worked with the language determined by the actions of Example 3.2 rather than the full $\mathcal{L}([\alpha])$. Their proof did not use translation to modal logic, but instead studied the strongly extensional quotient of the canonical model of (multi-agent) K.

5 $\mathcal{L}([\alpha], \Box^*)$

5.1 $\mathcal{L}([\alpha], \Box^*)$ is More Expressive than Modal Logic with \Box^*

In contrast to our translations results for $\mathcal{L}([\alpha])$, the larger language $\mathcal{L}([\alpha], \Box^*)$ cannot be translated into \mathcal{L} or even to $\mathcal{L}(\Box^*)$ (modal logic with extra modalities \Box_B^*). So completeness results for $\mathcal{L}([\alpha], \Box^*)$ cannot simply be based on translations the way we saw for $\mathcal{L}([\alpha])$.

In the result below, we assume that the set \mathcal{A} is a singleton, and so we omit it from the notation. We use an action α defined as follows. There is one point, say k , and $k \rightarrow k$. Also, we take $\text{PRE}(k)$ to be some atomic proposition p . We also let q be some atomic proposition distinct from p .

Theorem 5.1 *The sentence $\langle \alpha \rangle \Diamond^* q$ of $\mathcal{L}([\alpha], \Box^*)$ cannot be expressed by any sentence of $\mathcal{L}(\Box^*)$. In fact, $\langle \alpha \rangle \Diamond^* q$ cannot be expressed by any set of sentences of $\mathcal{L}(\Box^*)$.*

The proof uses an adaptation of Ehrenfeucht games on specific finite models.

At this point, we turn to the completeness result for $\mathcal{L}([\alpha], \Box^*)$. It is easy to check that there is no hope of getting a *strong* completeness result (where one has arbitrary sets of hypotheses).

The best one can hope for is a system where $\vdash \varphi$ if and only if $\models \varphi$. We achieve this with a logical system which is listed in Figure 3 below. The key semantic result is a reduction of the truth in some model W of sentences of the form $\langle \alpha \rangle \Diamond_C^* \varphi$ to the existence of certain paths in W .

Proposition 5.2 $\langle W, w \rangle \models \langle \alpha \rangle \Diamond_C^* \varphi$ iff there is a sequence of worlds from W

$$w = w_0 \rightarrow_{A_1} w_1 \rightarrow_{A_2} \cdots \rightarrow_{A_{k-1}} w_{k-1} \rightarrow_{A_k} w_k$$

where $k \geq 0$, and also a sequence of actions of the same length k ,

$$\alpha = \alpha_0 \rightarrow_{A_1} \alpha_1 \rightarrow_{A_2} \cdots \rightarrow_{A_{k-1}} \alpha_{k-1} \rightarrow_{A_k} \alpha_k$$

such that $A_i \in \mathcal{C}$ and $\langle W, w_i \rangle \models \text{PRE}(\alpha_i)$ for all $0 \leq i \leq k$, and $\langle W, w_k \rangle \models \langle \alpha_k \rangle \varphi$.

Remark The case $k = 0$ just says that $\langle W, w \rangle \models \langle \alpha \rangle \Diamond_C^* \varphi$ is implied by $\langle W, w \rangle \models \langle \alpha \rangle \varphi$.

This result underlies the soundness of the main rule of our logical system, Action Rule. We restate it below.

The Action Rule Let ψ be sentence, and let \mathcal{C} be a set of agents. Consider sentences χ_β for all β such that $\alpha \rightarrow_C^* \beta$ (including α itself). Assume that:

1. $\vdash \chi_\beta \rightarrow [\beta]\psi$.
2. If $A \in \mathcal{C}$ and $\beta \rightarrow_A \gamma$, then $\vdash \chi_\beta \wedge \text{PRE}(\beta) \rightarrow \Box_A \chi_\gamma$.

From these assumptions, infer $\vdash \chi_\alpha \rightarrow [\alpha]\Box_C^* \psi$.

Remark Recall that there are only finitely many β such that $\alpha \rightarrow_C^* \beta$, since each is determined by a world of the same Kripke frame that determines α . So even though the Action Rule might look like it takes infinitely many premises, it really only takes finitely many.

Another point: if one so desires, both the Action Rule and the Induction Rule could be replaced by (more complicated) axiom schemas.

Lemma 5.3 (Soundness) *The Action Rule is sound.*

At this point, we have already discussed the Action Axiom and the Action Rule. The Mix Axiom and the Induction Rule are familiar from the Segerberg axiomatization of PDL. The Composition Axiom takes a definition.

Definition Let $\alpha = \langle K, k \rangle$ and $\beta = \langle L, l \rangle$ be actions. Then the *action composition* $\alpha \circ \beta$ is the action defined as follows. Consider the product set $K \times L$. We turn this into a Kripke frame using the restriction of the product arrows. We get an action structure by setting

$$\text{PRE}((k', l')) = \text{PRE}(k') \wedge [\langle K, k' \rangle] \text{PRE}(l').$$

Finally, we set $\alpha \circ \beta = \langle K \times L, (k, l) \rangle$.

Basic Axioms	
All propositional validities	
($[\alpha]$ -normality)	$\vdash [\alpha](\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)$
(\Box_A -normality)	$\vdash \Box_A(\varphi \rightarrow \psi) \rightarrow (\Box_A\varphi \rightarrow \Box_A\psi)$
* (\Box_B^* -normality)	$\vdash \Box_B^*(\varphi \rightarrow \psi) \rightarrow (\Box_B^*\varphi \rightarrow \Box_B^*\psi)$
Additional Axioms	
(Atomic Permanence)	$\vdash [\alpha]p \leftrightarrow (\text{PRE}(\alpha) \rightarrow p)$
(Partial Functionality)	$\vdash [\alpha]\neg\chi \leftrightarrow (\text{PRE}(\alpha) \rightarrow \neg[\alpha]\chi)$
(Action Axiom)	$\vdash [\alpha]\Box_A\varphi \leftrightarrow (\text{PRE}(\alpha) \rightarrow \bigwedge\{\Box_A[\beta]\varphi : \alpha \rightarrow_A \beta\})$
* Composition Axiom	$\vdash [\alpha][\beta]\varphi \leftrightarrow [\alpha \circ \beta]\varphi$
* Mix Axiom	$\vdash \Box_B^*\varphi \rightarrow \varphi \wedge \bigwedge\{\Box_A\Box_B^*\varphi : A \in \mathcal{B}\}$
Modal Rules	
(Modus Ponens)	From $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, infer $\vdash \psi$
($[\alpha]$ -necessitation)	From $\vdash \varphi$, infer $\vdash [\alpha]\varphi$
(\Box_A -necessitation)	From $\vdash \varphi$, infer $\vdash \Box_A\varphi$
* (\Box_B^* -necessitation)	From $\vdash \varphi$, infer $\vdash \Box_B^*\varphi$
* Induction Rule	From $\vdash \chi \rightarrow \psi$ and $\vdash \chi \rightarrow \Box_A\chi$ for all $A \in \mathcal{B}$, infer $\vdash \chi \rightarrow \Box_B^*\psi$
* Action Rule	
Let ψ be sentence, and let \mathcal{C} be a set of agents. Consider sentences χ_β for all β such that $\alpha \rightarrow_{\mathcal{C}}^* \beta$ (including α itself). Assume that:	
1. $\vdash \chi_\beta \rightarrow [\beta]\psi$.	
2. If $A \in \mathcal{C}$ and $\beta \rightarrow_A \gamma$, then $\vdash \chi_\beta \wedge \text{PRE}(\beta) \rightarrow \Box_A\chi_\gamma$.	
; From these assumptions, infer $\vdash \chi_\alpha \rightarrow [\alpha]\Box_{\mathcal{C}}^*\psi$.	

Figure 3: The logical system for $\mathcal{L}([\alpha], \Box^*)$. For $\mathcal{L}([\alpha])$, we drop the * axioms and rules.

Proposition 5.4 *Concerning the composition operation:*

1. $(W^\alpha)^\beta \cong W^{\alpha \circ \beta}$ via the restriction of $((w, k'), l') \mapsto (w, (k', l'))$ to $(W^\alpha)^\beta$.
2. The Composition Axiom is sound.
3. $\alpha \circ (\beta \circ \gamma) \cong (\alpha \circ \beta) \circ \gamma$.

We also want to add a rule to our rewriting system \mathcal{R} corresponding to the Composition Axiom: $[\alpha][\beta]\varphi \rightsquigarrow [\alpha \circ \beta]\varphi$. This leads to normal forms discussed in Lemma 4.3.

The remaining rule of the system is necessitation for \Box_B^* modalities, and the soundness of this rule is trivial. This completes the discussion of the axioms and rules of our logical system.

Remark It is possible to drop the Composition Axiom in favor of a more involved version of the Action Rule. The point is the Composition Axiom simplifies the normal forms of the $\mathcal{L}([\alpha], \Box^*)$: Without the Composition Axiom, the normal forms of sentences of $\mathcal{L}([\alpha], \Box^*)$ would be of the form $[\alpha_1][\alpha_2] \cdots [\alpha_r]\psi$, where each α_i is a normal form action and ψ is a normal form sentence. The Composition Axiom insures that the normal forms are of the form $[\alpha]\psi$. So if we were to drop the Composition Axiom, we would need a formulation of the Action Rule which involved *sequences* of actions. It is not terribly difficult to formulate such a rule, and completeness can be obtained by an elaboration of the work which we shall do. We did not present this work, mostly because adding the Composition Axiom leads to shorter proofs.

5.2 Some Syntactic Results

Proposition 5.5 *For all $A \in \mathcal{C}$ and all β such that $\alpha \rightarrow_A \beta$,*

1. $\vdash [\alpha]\Box_C^*\psi \rightarrow [\alpha]\psi$.
2. $\vdash [\alpha]\Box_C^*\psi \wedge \text{PRE}(\alpha) \rightarrow \Box_A[\beta]\Box_C^*\psi$.

Proposition 5.6 *Let τ be the action from Example 3.1. Then for all φ , $\vdash [\tau]\varphi \leftrightarrow \varphi$.*

Definition Let α and α' be actions. We write $\vdash \alpha \leftrightarrow \alpha'$ if α and α' are based on the same Kripke frame W and the same world w , and if for all $v \in W$, $\vdash \text{PRE}(v) \leftrightarrow \text{PRE}'(v)$, where PRE is the announcement function for α , and PRE' for α' .

We note the following bisimulation-like properties:

1. If $\vdash \alpha \leftrightarrow \alpha'$, then also $\vdash \text{PRE}(\alpha) \leftrightarrow \text{PRE}(\alpha')$.
2. Whenever β' is such that $\alpha' \rightarrow_C^* \beta'$, then there is some β such that $\vdash \beta \leftrightarrow \beta'$ and $\alpha \rightarrow_C^* \beta$.

These follow easily from the way we defined PRE on actions in terms of functions on frames.

Proposition 5.7 *If $\vdash \alpha \leftrightarrow \alpha'$, then for all ψ , $\vdash [\alpha]\psi \leftrightarrow [\alpha']\psi$.*

Lemma 5.8 (Substitution) *Let φ be any sentence, and let $\vdash \chi \leftrightarrow \chi'$. Suppose that ψ comes from φ by replacing the atomic sentence p by χ at some point, and suppose that ψ comes from φ by replacing the atomic sentence p by χ' at the same point. Then $\vdash \psi \leftrightarrow \psi'$.*

Lemma 5.9 *Consider our languages $\mathcal{L}([\alpha])$ and $\mathcal{L}([\alpha], \Box^*)$. For every sentence φ there is some normal form $nf(\varphi) \leq \varphi$ such that $\vdash \varphi \leftrightarrow nf(\varphi)$.*

5.3 Completeness of the Logical System for $\mathcal{L}([\alpha], \Box^*)$

Theorem 5.10 (Completeness) *For all φ , $\vdash \varphi$ iff $\models \varphi$. Moreover, this relation is decidable.*

The proof is based on the filtration argument for completeness of PDL due to Kozen and Parikh [4]. We show that every consistent φ has a finite model, and that the size of the model is recursive in φ .

We shall need to use some results concerning the rewriting system \mathcal{R} which we introduced in Section 4. In particular, recall that we have a wellfounded relation $<$ on $\mathcal{L}([\alpha], \Box^*)$, and normal forms $nf(\varphi)$ for each φ . The results there are used in this section.

Definition Let $s(\varphi)$ be the set of subsentences of φ , including φ itself. We define a function $f : \mathcal{L}([\alpha], \Box^*) \rightarrow \mathcal{P}(\mathcal{L}([\alpha], \Box^*))$ by recursion on the wellfounded relation $<$ as follows: For normal forms, f works as follows:

$$\begin{aligned} f(p) &= \{p\} \\ f(\neg\varphi) &= \{\neg\varphi\} \cup f(\varphi) \\ f(\varphi \wedge \psi) &= \{\varphi \wedge \psi\} \cup f(\varphi) \cup f(\psi) \\ f(\Box_A \varphi) &= \{\Box_A \varphi\} \cup f(\varphi) \\ f(\Box_B^* \varphi) &= \{\Box_B^* \varphi\} \cup \{\Box_A \Box_B^* \varphi : A \in \mathcal{B}\} \cup f(\varphi) \\ f([\alpha] \Box_C^* \psi) &= \bigcup \{s(\Box_A [\beta] \Box_C^* \psi) : \alpha \rightarrow_C^* \beta \text{ \& } A \in \mathcal{C}\} \\ &\quad \cup \bigcup \{f([\beta] \psi) : \alpha \rightarrow_C^* \beta\} \end{aligned}$$

For φ not in normal form, let $f(\varphi) = f(nf(\varphi))$. (Note that we need to define f on sentences which are not normal forms, because $f([\beta] \psi)$ figures in $f([\alpha] \Box_C^* \varphi)$. Also, the definition makes sense because $[\beta] \psi < [\alpha] \Box_C^* \psi$, and because $nf(\varphi) \leq \varphi$ for all φ .)

Lemma 5.11 *For all φ :*

1. $f(\varphi)$ is a finite set of normal form sentences.
2. $nf(\varphi) \in f(\varphi)$.
3. If $\psi \in f(\varphi)$, then $s(\psi) \subseteq f(\varphi)$.
4. If $[\gamma] \Box_C^* \chi \in f(\varphi)$ and $\gamma \rightarrow_C^* \delta$, then $f(\varphi)$ also contains $\Box_A [\delta] \Box_C^* \chi$, $[\delta] \Box_C^* \chi$, $\text{PRE}(\delta)$, and $nf([\delta] \chi)$.

The set Δ Now fix a consistent φ , and set $\Delta = f(\varphi)$. This set Δ is the version for our logic of the Fischer-Ladner closure of φ . Let $\Delta = \{\psi_1, \dots, \psi_n\}$. Given a maximal consistent set U of $\mathcal{L}([\alpha], \Box^*)$, let

$$\llbracket U \rrbracket = \pm \psi_1 \wedge \dots \wedge \pm \psi_n,$$

where the signs are taken in accordance with U . That is, if $\psi_i \in U$, then ψ_i is a conjunct of $\llbracket U \rrbracket$; but if $\psi_i \notin U$, then $\neg \psi_i$ is a conjunct. We write $U \equiv V$ iff $\llbracket U \rrbracket = \llbracket V \rrbracket$.

We now define a filtration to be called \mathcal{F} . The worlds of \mathcal{F} will be the equivalence classes $[U]$ under \equiv , where U is a maximal consistent set in the logic. We set the atomic propositions true at $[U]$ to be $\text{AtProp} \cap U$. Furthermore, we also set

$$[U] \rightarrow_A [V] \text{ in } \mathcal{F} \quad \text{iff} \quad \llbracket U \rrbracket \wedge \Diamond_A \llbracket V \rrbracket \text{ is consistent.} \quad (1)$$

Definition Consider a sentence $[\alpha] \Diamond_C^* \psi$ and a path in \mathcal{F}

$$[V_0] \rightarrow_{A_1} [V_1] \rightarrow_{A_2} \cdots \rightarrow_{A_{k-1}} [V_{k-1}] \rightarrow_{A_k} [V_k]$$

where $k \geq 0$ such that each $A_i \in \mathcal{C}$, and for which there exists a sequence of actions length k ,

$$\alpha = \alpha_0 \rightarrow_{A_1} \alpha_1 \rightarrow_{A_2} \cdots \rightarrow_{A_{k-1}} \alpha_{k-1} \rightarrow_{A_k} \alpha_k$$

such that $\text{PRE}(\alpha_i) \in V_i$ for all $0 \leq i \leq k$, and $\langle \alpha_k \rangle \psi \in V_k$. We call such a sequence a *good path from $[V_0]$ for $\langle \alpha \rangle \Diamond_C^* \psi$* .

Lemma 5.12 *Let $[\alpha] \Box_C^* \psi \in \Delta$. If there is a good path from $[V_0]$ for $\langle \alpha \rangle \Diamond_C^* \neg \psi$, then $\langle \alpha \rangle \Diamond_C^* \neg \psi \in V_0$.*

Lemma 5.13 *If $\llbracket V_0 \rrbracket \wedge \langle \alpha \rangle \Diamond_C^* \psi$ is consistent, then there is a good path from $[V_0]$ for $\langle \alpha \rangle \Diamond_C^* \psi$.*

Lemma 5.14 (Truth Lemma) *Consider a sentence φ , and also the set $\Delta = \Delta(\varphi)$. For all $\psi \in \Delta$ and $U \in \mathcal{F}$, $\psi \in U$ iff $\langle \mathcal{F}, [U] \rangle \models \psi$.*

Now we prove completeness and decidability of the logic for $\mathcal{L}([\alpha], \Box^*)$. Let φ be consistent. By Lemma 5.9, there is a normal form φ' such that $\vdash \varphi \leftrightarrow \varphi'$. By the Truth Lemma, φ' holds in the filtration \mathcal{F} . So φ' has a model; thus φ has one, too. This establishes completeness. For decidability, note that the size of the filtration is computable in the size of the original φ .

6 Conclusions and Future Research

In this paper, we have continued the program begun in Gerbrandy and Groeneveld [3] and Gerbrandy [2] of adding epistemic update operators to modal logic. We believe that the following are our main contributions:

1. We formulated a logical system with new modalities corresponding to intuitive group-level epistemic actions. These actions include natural formalizations of announcements to groups with, and without, suspicion by outsiders.
2. The original semantics from [3] and [2] used non-wellfounded sets. Our semantics works with arbitrary Kripke models. The advantages of doing this are that the logic can be used by those who do not know about non-wellfounded sets, and also that completeness results are slightly stronger with a more general semantics.
3. Without infinitary operations, the logic may be translated into standard modal logic. Therefore, we obtained a completeness proof for this fragment, just by adding to standard modal logic whatever axioms are needed to do the translation.

4. But adding concepts common knowledge leads to a much more expressive language.
5. The validities of $\mathcal{L}([\alpha], \Box^*)$ can be axiomatized along the lines of propositional dynamic logic. As a result, we get the finite model property and hence decidability.

There are a number of questions which we are pursuing at this time. The relation between communications and our actions needs to be investigated. It would be interesting to prove that certain natural actions cannot be expressed in our framework, and then to expand it to encompass them. We are investigating a more “dynamic” versions of the logic, in which carrying out an action is one sort of program, and in addition in which the Kleene star operation on programs adds further power. We have studied the expressive power of various fragments when the actions are secure public announcements. We would like to get a completeness theorem for this kind of dynamic logic. We want to investigate completeness for certain natural classes of actions and models, such as when all are S4 or S5. This would be useful for getting logics of what might be called “knowledge actions” (what we have are closer to doxastic actions). We would like to investigate whether our languages and logics can help in formalizing and studying the knowledge programs of Fagin et al [1]. We think it would be interesting to take other notions of common knowledge, ones which do not reduce the notion to iterated knowledge, and carry out the same kind of investigations that we have done in this paper. Finally, note that all of our actions presuppose truth in a certain sense: the precondition functions PRE must be satisfied for a world to be updated. This assumption fails in many applications, and we think it would be useful to think about incorporating ideas from belief revision into our framework.

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