Fast Economic Model Predictive Control Based on NLP-Sensitivities

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Abstract

This study adapts the advanced step NMPC framework to Economic NMPC. Here, sufficient conditions for nominal stability are derived for NMPC controllers that incorporate economic stage costs with appropriate regularization. To guarantee these conditions, we derive a constructive strategy to calculate the regularization term directly. Moreover, we extend the sensitivity components in the advanced step NMPC framework to consider a rigorous path-following algorithm. This approach accounts for active set changes and allows much weaker constraint qualifications. Moreover, using an ℓ_1 formulation of the NMPC problem satisfies these constraint qualifications and allows more reliable solution of the moving horizon optimization problem, even in the presence of noise. Finally, all of these concepts are demonstrated on a detailed case study with a continuously stirred tank reactor.

Keywords: Economic model predictive control, Advanced step nonlinear model predictive control, Nonlinear programming, Sensitivity, Path-following method

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1. Introduction

The ultimate goal of any operation strategy for a process plant is to make profit. Traditionally the approach for achieving this goal has been to translate the economic objectives into control objectives [1], and then to design a control system which meets the control objectives. This approach leads to a hierarchical two-layer structure, where the upper layer performs the economic optimization

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and calculates the optimal setpoints for the controlled variables in the layer below. The lower layer then keeps the controlled variables at their given setpoints. In principle, any kind of controller may be used in the lower layer, but in the context of controlling complex nonlinear plants, model predictive controllers (MPC) have proved to be very successful because of their ability to handle interactive plants with input and output constraints in an optimized manner.

However, simply controlling some measurements at their setpoints generally does not maximize the economic potential, especially in complex plants. Instead, this can be achieved by incorporating economic information into the lower layers, by finding a set of self-optimizing controlled variables [2]. Here the controlled variables are chosen systematically such that keeping them at their setpoints gives an acceptable economic loss in spite of disturbances. Alternatively one may attempt to select the controlled variables as the necessary optimality conditions as done in NCO-tracking [3]. These approaches work well when the active constraints do not change too frequently, and may be combined [4].

An alternative option, which is available when an optimization-based controller such as MPC is used, is to include the economic criterion directly into the cost function of the controller. This approach is often referred to as Economic MPC (eMPC), and has been gaining increased interest in recent years. The idea of using an economic objective in nonlinear MPC (NMPC), instead of a tracking objective has been applied in e.g. [5, 6]. This approach was shown to work well, but at that time there were no fundamental results on the stability and robustness for model predictive control with an economic objective.

While Lyapunov stability properties have been well-studied for tracking NMPC problems [7], extending these results to Economic NMPC is not straightforward, because the stage cost of the Economic NMPC controller generally does not attain a minimum at the steady state optimal operation point. This poses challenges for Lyapunov based stability proofs, which assume that the stage cost is minimal at the desired equilibrium point, such that the value function can be used as a Lyapunov function for the closed loop system. Results on the stability of MPC for convex linear systems with an economic cost were obtained in [8], and a first Lyapunov stability proof based on strong duality arguments was published in [9], and later extended in [10] to dissipative systems. An easier to verify stability condition based on convexity of Hessian of the steady state problem was given in [11, 12], which also considered robustness properties for Economic MPC based on Input-to-State (ISS) stability.

Another important issue for economic model predictive control is to obtain optimization results fast enough for use in a real-time optimization feedback scheme.

Beside the development of more efficient hardware and software for solving nonlinear programming problems (NLP), this challenge has been met by calculating fast approximate solutions, which are typically based on solution point sensitivity [13, 14], and have proved to be very successful. In previous work [15] we developed the advanced step NMPC (asNMPC) algorithm, where the NLP associated with the moving horizon controller is solved in background (between sampling times) and a fast sensitivity update of the NLP solution is made once the actual state is known. While these sensitivity approaches usually work well, they rely on strong regularity assumptions (in the form of constraint qualifications and second order conditions) that may not always hold in practice, and the problem of changing active constraints is approached in a rather heuristic way. Further, as the sensitivity based approach is inherently local, it is limited to small parameter changes in the NLP.

In this paper we present a collection of recent advances in this area. One of the main contributions of this paper is to extend Lyapunov stability results based on convexity arguments to the case of Economic NMPC. We establish stability results for three different terminal criteria: (1) fixed terminal point with finite cost function, (2) fixed terminal point with infinite horizon cost function, and (3) terminal set and a terminal cost for a finite horizon cost function. Moreover, we provide a simple constructive method based on Gershgorin bounds for regularizing the stage cost such that stability is guaranteed.

Another contribution of this paper is an extension of asNMPC based on applying multiple sensitivity steps in a path-following fashion. Our new path-following method is designed to handle non-unique multiplier values and changing active constraints; it works especially well with ℓ_1 penalty formulations of the NMPC problem. We review a very general sensitivity result, where a linear program (LP) is solved to obtain the multiplier values, and a quadratic program (QP) is solved for obtaining the directional derivative for a given perturbation. This sensitivity result is then used as a basis for a new path-following method for obtaining fast approximations of the optimal solution of the NMPC problem. Finally, a selection of these results is demonstrated by a detailed case study based on a CSTR model.

We use the convention that the ∇-operator, when used without subscript, is taken with respect to the optimization variables. We add a subscript, e.g. ∇_p to indicate that we consider the derivative with respect to p; this avoids ambiguity when it is not clear from the context.

2. Stability properties of Economic NMPC

Nonlinear model predictive control with tracking type objective functions has seen considerable analytical development of properties, which guarantee both nominal and robust stability, see e.g. [16] . When NMPC with economic objective is considered, the conditions on the stage and terminal costs must be modified in order to guarantee Lyapunov stability. This section discusses these modifications and shows conditions for regularization of economic-based stage costs that lead to the stability guarantee.

2.1. The NMPC formulation

We define the NMPC problem,

$$
\begin{aligned}
\min_{z_l, v_l} \quad & \Psi(z_N) + \sum_{l=0}^{N-1} \psi(z_l, v_l) && \text{(1a)}\\
\text{s. t.} \quad z_{l+1} &= f(z_l, v_l) \quad l = 0, \ldots N-1 && \text{(1b)}\\
z_0 &= x_k && \text{(1c)}\\
z_l &\in \mathbb{X}, v_l \in \mathbb{U}, z_N \in \mathbb{X}_f && \text{(1d)}
\end{aligned}
$$

where we assume, without loss of generality, that the actual and predicted states $x_k, z_l \in \Re^{n_x}$, and actual and predicted controls $u_k, v_l \in \Re^{n_u}$, are restricted to the convex domains $\mathbb{X}$ and $\mathbb{U}$, respectively, and the terminal state is restricted to a convex terminal region $\mathbb{X}_f$. The stage cost is given by $\psi(\cdot,\cdot) : \Re^{n_x+n_u} \to \Re$, while the terminal cost is denoted by $\Psi(\cdot) : \Re^{n_x+n_u} \to \Re$. For tracking problems, we can assume that the state and control variables can be defined with respect to setpoint and reference values, and that the dynamic model and costs are Lipschitz continuous and satisfy $f(0,0) = 0$, $\psi(0,0) = 0$ and $\Psi(0) = 0$.

After solution of (1) the control action is extracted from the optimal trajectory $\{z_0^* \ldots z_N^* \, v_0^*, \ldots, v_{N-1}^*\}$ as $u_k = v_0^*$. The plant then evolves as,

$$x_{k+1} = f(x_k, u_k) \tag{2}$$

and we shift the measurement sequence one step forward, set $k = k+1$ and use the new state estimate x_k to solve the next NMPC problem (1).

Ideally the NMPC problem is solved instantly, and the plant inputs are updated without any time delay as soon as new measurements are available. This hypothetical case is referred to as *ideal NMPC* (iNMPC). In practice however, there will always be computational delay to solve Problem (1).

To show stability, we need the following:

Definition 1. *A continuous function* $\alpha(\cdot) : \Re \rightarrow \Re$ *is a* $\mathscr{K}_\infty$ *function if* $\alpha(0) = 0, \alpha(s) > 0, \forall s > 0$*, it is nondecreasing with s and* $\alpha(s) \rightarrow \infty$ *when* $s \rightarrow \infty$*.*

NMPC has well-known stability properties (see [17, 7]) with the following assumptions.

Assumption 1. *(Nominal Stability Assumptions of NMPC)*

1. *The terminal cost* $\Psi(\cdot)$ *satisfies* $\Psi(x) > 0, \forall x \in \mathbb{X}_f \setminus \{0\}$*, where* $\mathbb{X}_f$ *is the terminal region.*
2. *There exits a local control law* $u = h_f(x)$ *such that* $f(x, h_f(x)) \in \mathbb{X}_f, \forall x \in \mathbb{X}_f$*.*
3. $\Psi(f(x, h_f(x))) - \Psi(x) \leq -\psi(x, h_f(x)), \forall x \in \mathbb{X}_f$*.*
4. $\psi(x,u)$ *satisfy* $\alpha_p(|x|) \leq \psi(x,u) \leq \alpha_q(|x|)$ *where* $\alpha_p(\cdot)$ *and* $\alpha_q(\cdot)$ *are* $\mathscr{K}_\infty$ *functions* [1].

Problem (1) and Assumption 1 extend to consider three particular formulations:

P1 $N \rightarrow \infty$, $\mathbb{X}_f = \{0\}$ and $\Psi(x,u) = 0$. From an analytical perspective, this is the desired case, but difficult to implement with nonlinear dynamic models in (1).

P2 N is finite, $\mathbb{X}_f = \{0\}$ and $\Psi(x,u) = \psi(z_N, 0)$. This case mirrors the infinite horizon but specification of N sufficiently large is difficult in general.

P3 N is finite, a suitable terminal region $\mathbb{X}_f$ has been calculated for (1) and a controller $h_f(x)$ is known with a suitable terminal cost $\Psi(x,u)$. In practice, these conditions can be hard to determine although they are widely assumed.

Under the assumptions above we can state the following theorem.

Theorem 1. *(Nominal Stability of Ideal NMPC) Consider the moving horizon problem (1) and associated control law* $u = h_f(x)$ *that satisfies Assumption 1. Then, the objective function of (1) is a Lyapunov function and the closed-loop system is asymptotically stable.*

[1] Boundedness of u by x follows when we make the assumption of *weak controllability*, i.e., $\sum_{k=0}^{N-1} |u_k| \leq \gamma(|x|)$, where $\gamma(\cdot)$ is a $\mathscr{K}_\infty$ function.

Proof. See e.g [7]. □

Robust stability properties for NMPC are developed in [7]. In particular, ideal NMPC satisfies the *Input to State Stability (ISS)* property for plants that have bounded uncertainties.

2.2. *Economic NMPC and dynamic real-time optimization*

As noted in the introduction, Economic NMPC is not as simple as just replacing the stage costs in (1) from a tracking type to an economic type, because a number of challenges to stability and robustness must be considered. For Economic NMPC, the setpoint (normalized to 'zero' in Theorem 1) is unknown [8], so it is not clear how to reach a steady state equilibrium point, and Assumption 1(4) is generally not satisfied by the economic stage cost, as required by Lyapunov stability. In fact, Angeli et al. [10] note that stability cannot be expected in general, and they show that, if the initial condition is feasible and there is at least one admissible control sequence to an equilibrium point, then the asymptotically average performance is no worse than that of the best admissible steady state. That is, asymptotically the controller may perform better by *not* going to a steady state at all.

To satisfy Assumption 1(4) for the Lyapunov stability requirement, the Economic NMPC Controller must satisfy a dissipativity condition on the stage cost and dynamic model [10]. As discussed in [9, 10], a sufficient condition for dissipativity is satisfied when the stage cost and dynamic model form a strongly dual problem. Moreover, as shown in [11, 12], these strongly dual problems arise when the steady state optimization problem has a uniformly strong convex Lagrange function. This sufficient condition is used in the current study to develop an Economic NMPC Controller that satisfies nominal stability.

To define implicit reference values for the states and controls, we consider the steady state optimization problem given by:

$$\min_{z,v} \psi(z,v), \text{ s.t. } z = f(z,v), z \in \mathbb{X}, v \in \mathbb{U} \tag{3}$$

with the solution given by (z^*, v^*). We introduce a transformed system by subtracting the optimal steady state from the predicted values as follows:

$$\bar{z}_l = z_l - z^*, \quad \bar{v}_l = v_l - v^* \tag{4}$$

and the transformed state evolves according to

$$\bar{z}_{l+1} = \bar{f}(\bar{z}_l, \bar{v}_l) = f(\bar{z}_l + z^*, \bar{v}_l + v^*) - z^* \tag{5}$$

and $\bar{z}_l \in \bar{\mathbb{X}}$ and $\bar{u}_l \in \bar{\mathbb{U}}$, where $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ are the corresponding sets for the transformed system. From equation (5), we see that when $(\bar{z}_l, \bar{v}_l) = (0,0)$ we have $z^* = f(z^*, v^*)$. Similarly, we define the transformed stage and terminal costs as:

$$\begin{aligned} \bar{\psi}(\bar{z}_l, \bar{v}_l) &= \psi(\bar{z}_l + z^*, \bar{v}_l + v^*) - \psi(z^*, v^*) && (6) \\ \bar{\Psi}(\bar{z}_N) &= \Psi(\bar{z}_N + z^*) - \Psi(z^*) && (7) \end{aligned}$$

so that $\bar{\Psi}(0) = \bar{\psi}(0,0) = 0$, and the transformed NMPC subproblem is now given by:

$$\begin{aligned} \min \quad & \bar{\Psi}(\bar{z}_N) + \sum_{l=0}^{N-1} \bar{\psi}(\bar{z}_l, \bar{v}_l) && (8) \\ \text{s.t.} \quad \bar{z}_{l+1} &= \bar{f}(\bar{z}_l, \bar{v}_l), \quad l = 0, \ldots, N \\ \bar{z}_0 &= x_k - x_k^*,\ \bar{z}_l \in \bar{\mathbb{X}},\ \bar{v}_l \in \bar{\mathbb{U}},\ \bar{z}_N \in \bar{\mathbb{X}}_f \end{aligned}$$

A key concern is that Assumption 1(4) generally does not hold for transformed economic stage costs $\bar{\psi}$ and, hence, the optimal objective of problem (8) is not a Lyapunov function, which can be used to prove stability. Instead, as suggested in [9, 12] we augment the transformed cost to form the *rotated* stage cost given by:

$$\begin{aligned} \phi(\bar{z}_l, \bar{v}_l) &= \bar{\psi}(\bar{z}_l, \bar{v}_l) + \bar{\lambda}^T(\bar{z}_l - \bar{f}(\bar{z}_l, \bar{v}_l)) && (9) \\ \Phi(\bar{z}_N) &= \bar{\Psi}(\bar{z}_N) + \bar{\lambda}^T \bar{z}_N && (10) \end{aligned}$$

where $\bar{\lambda}$ is the multiplier from the equality constraints in (3) (determined by equation (20) in Section 3). Note that the functions ϕ and Φ are zero at the origin ($\phi(0,0) = \Phi(0) = 0$). Moreover, the stage cost ϕ satisfies Assumption 1(4) if it is strongly convex. As shown next, the origin is also the unique, global minimum if ϕ and Φ are strongly convex functions.

From $\bar{f}(0,0) = 0$, $\phi(\bar{z}, \bar{v}) \geq 0$, $\phi(0,0) = 0$, and $[\bar{z}^T \bar{v}^T] \nabla \phi(0,0) \geq 0$ we can write:

$$\begin{aligned}\phi(\bar{z},\bar{v}) &= \nabla\phi(0,0)^T\begin{bmatrix}\bar{z}\\ \bar{v}\end{bmatrix}+\frac{1}{2}\int_0^1[\bar{z}^T\ \bar{v}^T]\nabla^2\phi(\tau\bar{z},\tau\bar{v})\begin{bmatrix}\bar{z}\\ \bar{v}\end{bmatrix}d\tau\\ &\geq \frac{1}{2}\int_0^1[\bar{z}^T\ \bar{v}^T]\nabla^2\phi(\tau\bar{z},\tau\bar{v})\begin{bmatrix}\bar{z}\\ \bar{v}\end{bmatrix}d\tau. \end{aligned} \tag{11}$$

If $\phi(\bar{z}_i,\bar{v}_i)$ is strongly convex and bounded above by a $\mathscr{K}_\infty$ function, we have for some $\gamma > 0$,

$$\begin{aligned}\alpha_q(|\bar{z}|)\geq\phi(\bar{z},\bar{v}) &\geq \frac{1}{2}\int_0^1[\bar{z}^T\ \bar{v}^T]\nabla^2\phi(\tau\bar{z},\tau\bar{v})\begin{bmatrix}\bar{z}\\ \bar{v}\end{bmatrix}d\tau\\ &\geq \gamma(|\bar{z}|^2+|\bar{v}|^2)\geq\alpha_p(|\bar{z}|). \end{aligned} \tag{12}$$

and therefore Assumption 1(4) holds. A similar property holds for $\Phi(\bar{z})$ if it is strongly convex and bounded above by a $\mathscr{K}_\infty$ function. The resulting transformed and rotated NMPC problem is given by:

$$\begin{aligned} &\min && \Phi(\bar{z}_N)+\sum_{l=0}^{N-1}\phi(\bar{z}_l,\bar{v}_l) \\ &\text{s.t.} & \bar{z}_{l+1} &= \bar{f}(\bar{z}_l,\bar{v}_l),\quad l=0,\dots,N\\ && \bar{z}_0 &= x_k-x_k^*,\ \bar{z}_l\in\bar{\mathbb{X}},\ \bar{v}_l\in\bar{\mathbb{U}},\ \bar{z}_N\in\bar{\mathbb{X}}_f\end{aligned} \tag{13}$$

Now, if ϕ and Φ are not strongly convex functions, the *original* stage costs can always be augmented with regularization terms in (1) as follows:

$$\psi(z_l,v_l) := \psi(z_l,v_l)+\frac{1}{2}\left\|\begin{bmatrix}z_l-z^*\\ v_l-v^*\end{bmatrix}\right\|_Q^2 \tag{14}$$

$$\Psi(z_l) := \Psi(z_l)+\frac{1}{2}\|z_l-z^*\|_{Q_1}^2 \tag{15}$$

where $Q=\begin{bmatrix}Q_1 & \\ & Q_2\end{bmatrix}$ is a suitably defined matrix containing the weightings for the states (Q_1) and the inputs (Q_2). Such a regularization leads to the desired convexity property for the *rotated* stage costs and ensures that Assumption 1(4) holds.

In addition, from the definitions of ϕ and Φ it is easy to see that

$$\Phi(\bar{z}_{N+1}) - \Phi(\bar{z}_N) \leq -\phi(\bar{z}_N, \bar{v}_N)$$

holds when Assumption 1(3) holds. On the other hand, note that existence of a terminal region in (13) must still be assumed.

To obtain nominal stability for Economic NMPC under Assumption 1, we prove the following theorem.

Theorem 2 (Nominal Stability of Economic NMPC)**.** *Consider the moving horizon problem (13) that satisfies Assumption 1 for the problem formulations described by P1, P2 or P3. Then,*

1. *the objective function of (13) is a Lyapunov function and the closed-loop system is asymptotically stable,*
2. *the minimizer of problem (13) is the same minimizer as in (8) for problem formulations P1, P2 and P3, and hence, is equivalent to the minimizer of problem (1) with the original economic stage and terminal costs.*

Proof: The first part of the theorem follows directly from Assumptions 1 and Theorem 1. For the second part of the theorem, we start with formulation P1 and note that the objective at the unique, optimal solution of (13) is given by:

$$\begin{aligned}
\sum_{l=0}^{\infty} \phi(\bar{z}_l, \bar{v}_l) &= \sum_{l=0}^{\infty} [\psi(\bar{z}_l, \bar{v}_l) + \bar{\lambda}^T(\bar{z}_l - \bar{f}(\bar{z}_l, \bar{v}_l))] \\
&= \sum_{l=0}^{\infty} [\psi(\bar{z}_l, \bar{v}_l) + \bar{\lambda}^T(\bar{z}_l - \bar{z}_{l+1})] \\
&= \bar{\lambda}^T \bar{x}_k + \sum_{l=0}^{\infty} \psi(\bar{z}_l, \bar{v}_l). \qquad (16)
\end{aligned}$$

Other than the constant term $\bar{\lambda}^T \bar{x}_k$ in the objective of (13) we see that both (13) and (8) are identical and consequently have the same values. To show that $(\bar{z}_l, \bar{v}_l)$ is also the optimal solution of (8), assume that this is not the case and that the optimum of (8) is given by $(\hat{z}_l, \hat{v}_l) \neq (\bar{z}_l, \bar{v}_l)$. Then from (16):

$$\sum_{l=0}^{\infty} \phi(\bar{z}_l, \bar{v}_l) = \bar{\lambda}^T \bar{x}_k + \sum_{l=0}^{\infty} \psi(\bar{z}_l, \bar{v}_l) > \bar{\lambda}^T \bar{x}_k + \sum_{l=0}^{\infty} \psi(\hat{z}_l, \hat{v}_l) = \sum_{l=0}^{\infty} \phi(\hat{z}_l, \hat{v}_l)$$

which contradicts the optimality of $(\bar{z}_l, \bar{v}_l)$ for (13). Hence $(\bar{z}_l, \bar{v}_l)$ is also the optimal solution of (8).

For P2, with finite N, $\Psi(\bar{z}_N) = \psi(\bar{z}_N, 0)$ and $\bar{z}_N = 0$, we have:

$$\begin{aligned}
\sum_{l=0}^{N-1} \phi(\bar{z}_l, \bar{v}_l) + \Phi(\bar{z}_N) &= \sum_{l=0}^{N-1} [\psi(\bar{z}_l, \bar{v}_l) + \bar{\lambda}^T(\bar{z}_l - \bar{f}(\bar{z}_l, \bar{v}_l))] + \psi(0,0) + \bar{\lambda}^T \bar{z}_N \\
&= \bar{\lambda}^T \bar{x}_k + \sum_{l=0}^{N-1} \psi(\bar{z}_l, \bar{v}_l).
\end{aligned} \tag{17}$$

Again, other than the constant term $\bar{\lambda}^T \bar{x}_k$ in the objective of (13) we see that both (13) and (8) are identical and consequently have the same solutions. Using the same argument as for P1, it is clear that this is also the optimal solution of (8).

Finally for P3, with finite N we have:

$$\begin{aligned}
\sum_{l=0}^{N-1} \phi(\bar{z}_l, \bar{v}_l) + \Phi(\bar{z}_N) &= \sum_{l=0}^{N-1} [\psi(\bar{z}_l, \bar{v}_l) + \bar{\lambda}^T(\bar{z}_l - \bar{f}(\bar{z}_l, \bar{v}_l))] + \Psi(\bar{z}_N) + \bar{\lambda}^T \bar{z}_N \\
&= \bar{\lambda}^T \bar{x}_k + \sum_{l=0}^{N-1} \psi(\bar{z}_l, \bar{v}_l) + \Psi(\bar{z}_N).
\end{aligned} \tag{18}$$

Again, we see that both (13) and (8) are essentially identical and consequently have the same solutions. Using the same argument as for P1, it is clear that this is also the optimal solution of (8). □

Remark 1. *Note that the transformation of the optimization variables and the rotation of the cost function are not necessary for implementing the NMPC. They are only needed to show the stability properties.*

A similar, but more general stability result is presented in [10] for strictly dissipative systems, where they also show that strong duality of the steady state problem (19) is a sufficient condition for strict dissipativity. Moreover, strong convexity of rotated stage costs is a sufficient condition for strong duality. The next section shows that strong convexity can always be ensured through the addition of quadratic regularization terms, which can be determined in a straightforward way.

3. Regularization of non-convex stage costs

Regularization of the stage cost is often necessary to satisfy Assumption 1(4), whether by dissipativity, strong duality or strong convexity. While an unregularized controller may perform asymptotically better by not going to a steady state

[10], stable performance in reaching a steady state equilibrium point is generally desired in most practical applications. Finding a sufficiently large regularization parameter for Lyapunov stability is difficult, however. A regularization procedure that leads to a sufficient condition for strict dissipativity is proposed in [10]. However, it requires global solution of a sequence of nonconvex maximization problems.

In this section we adopt a simpler approach and derive a regularization term for the original stage cost, as in (14), (15), that makes the rotated stage cost strongly convex. For this, we present a simple approach to find a set of weightings, which guarantee closed loop stability of the Economic MPC controller.

To facilitate the presentation, we consider the steady state minimization problem (3),

$$\begin{aligned} \min_{z,v} \quad & \psi(z,v) \\ s.t. \quad & z = f(z,v), z \in \mathbb{X}, v \in \mathbb{U} \end{aligned} \tag{19}$$

where the vector $[z^T v^T]^T \in \Re^{n_x+n_u}$ denotes the steady state input and control vector. At a local steady state solution z^*, v^*, we have a KKT point given by:

$$\begin{aligned} &\nabla_z \psi(z^*,v^*) + (I - \nabla_z f(z^*,v^*))\bar{\lambda} + \nabla_z g(z^*,v^*)\eta = 0 \\ &\nabla_v \psi(z^*,v^*) - \nabla_v f(z^*,v^*)\bar{\lambda} + \nabla_v g(z^*,v^*)\eta = 0 \\ &\eta \geq 0, \quad g(z^*,v^*) \leq 0, \quad \eta^T g(z^*,v^*) = 0. \end{aligned} \tag{20}$$

where we denote the regions $\mathbb{X}$ and $\mathbb{U}$ by the convex constraints $g(z,v) \leq 0$. The next result establishes a condition for strong convexity.

Lemma 3. *Consider the modified steady state problem:*

$$\begin{aligned} \min_{z,v} V(z,v) \equiv \psi(z,v) + \bar{\lambda}^T(z - f(z,v)) + 1/2||(z,v) - (z^*,v^*)||_Q^2 \\ s.t. \quad g(z,v) \leq 0, \end{aligned} \tag{21}$$

where Q is positive definite weighting matrix. If Q is selected sufficiently large, then z^*, v^* , the local solution of (19), is also the unique global solution of (21).

Proof: The point z^*, v^* is already feasible, as it satisfies $z^* - f(z^*,v^*) = 0$, $z^* \in$

$\mathbb{X}, v^* \in \mathbb{U}$ (19). The KKT conditions for (21) are

$$\nabla_z \psi(z^*, v^*) + Q_1(z^* - z^*) + (I - \nabla_z f(z^*, v^*))\bar{\lambda} + \nabla_z g(z^*, v^*)\eta = 0 \quad (22)$$
$$\nabla_v \psi(z^*, v^*) + Q_2(v^* - v^*) - \nabla_v f(z^*, v^*)\bar{\lambda} + \nabla_v g_v(z^*, v^*)\eta = 0$$
$$\eta \geq 0, \quad g(z^*, v^*) \leq 0, \quad \eta^T g(z^*, v^*) = 0. \quad (23)$$

and we find that z^*, v^* satisfies the KKT conditions of both (19) and (21). We now choose Q such that

$$\nabla^2 V(z, v; \bar{\lambda}) = \left(\nabla^2 \psi(z, v) - \sum_{i=1}^{n_z} \nabla^2 f_i(z, v)\bar{\lambda}_i + Q \right) \quad (24)$$

is positive definite for all $z \in \mathbb{X}, v \in \mathbb{U}$. Since $V(z, v)$ is a strongly convex function and $\mathbb{X}$ and $\mathbb{U}$ are convex regions, z^*, v^* is the unique global solution. □

Note that the rotated, regularized stage costs satisfy Assumption 1(4) when $\nabla^2 V$ is positive definite.

3.1. Finding the regularization Q

To find a valid diagonal regularization matrix Q we propose to use the Gershgorin property for a matrix $A = (a_{i,j})$, which states that

$$a_{i,i} - \sum_{i \neq j} |a_{i,j}| \leq \mu_i \leq a_{i,i} + \sum_{i \neq j} |a_{i,j}|. \quad (25)$$

where μ_i are the eigenvalues of A. This property can be used to systematically find a regularization such that the rotated stage cost is strongly convex. Let $a_{i,j}$ denote the elements of the matrix $A = \nabla^2 \psi(z^*, v^*) - \sum_{i=1}^{n_z} \nabla^2 f_i(z^*, v^*)\bar{\lambda}_i$ and let the diagonal elements of Q be denoted as q_i. To ensure that all eigenvalues of $\nabla^2 V = A + Q$ are positive, we require

$$0 < q_i + a_{i,i} - \sum_{i \neq j} |a_{i,j}| \leq \mu_i + q_i. \quad (26)$$

Expressed as a simple condition on the elements of the diagonal matrix Q, this becomes

$$q_i > \sum_{i \neq j} |a_{i,j}| - a_{i,i}, \quad (27)$$

which nevertheless must be satisfied for all z, v satisfying $z \in \mathbb{X}, v \in \mathbb{U}$. This simple condition determines a "minimal" regularization that guarantees stability of

Economic NMPC.

4. Sensitivity computations for fast NMPC

The central idea of advanced-step nonlinear model predictive control (asNMPC) [15] is to solve the nonlinear optimization one sample time in advance, but with a *predicted* initial state. The asNMPC framework consists of three basic steps:

1. At time k solve the (possibly regularized) problem (1) with the predicted state value ($z_{k+1} = f(x_k, u_k)$) at time $k+1$.
2. At sample time $k+1$, when the actual state x_{k+1} is known, compute an approximation to the optimal solution using sensitivity analysis of (1).
3. Set $k = k+1$ and go to Step 1.

To compute the approximation in Step 2, the behavior of the solution of (1) is analyzed in the neighborhood around the optimum, and based on solution point sensitivity, a correction is determined to compensate for the error between the prediction and the actual state. As the sensitivity updates can often be computed very quickly compared to the solution of the NLP, the plant inputs can be adapted to the new conditions with minimal delay.

The sensitivity corrections from the predicted optimal input can generally be computed by two approaches. The first is to apply a single sensitivity step, as proposed in [15]. This single step correction gives good results when the state prediction is close to the actual state at time $k+1$, and care is taken to ensure feasible inputs, e.g. through clipping [18]. The second approach, which we propose in this paper is to apply several subsequent sensitivity updates in a path-following manner. Let $p_0 = z_{k+1}$ be the predicted state at time $k+1$, and let $p_f = x_{k+1}$ be the actual (measured) state at time $k+1$. The idea is now to track the path of optimal solutions along $p(t) = p_0 + t(p_f - p_0)$ for $t = 0 \to 1$. Such path-following or homotopy methods have been also been studied by [19, 20].

Especially when the prediction p_0 is far from the actual state p_f, we can expect better results by applying this second approach.

4.1. NLP sensitivity formulation

Before presenting the path-following approach, we present some established sensitivity results in this section. To this end, we rewrite problem (1) as:

$$\begin{aligned} &\min \; F(\mathbf{x}, p) \\ \text{s. t. } \; &c(\mathbf{x}, p) = 0 \\ &g(\mathbf{x}, p) \leq 0, \end{aligned} \tag{28}$$

where $\mathbf{x} \in \Re^{n_\mathbf{x}}$ denotes the optimization variables, and $p \in \Re^{n_p}$ denotes a vector of parameters, which for problem (1), represents the initial state of the system. Further, $F : \Re^{n_\mathbf{x}} \times \Re^{n_p} \to \Re$ denotes the scalar cost function, $c : \Re^{n_\mathbf{x}} \times \Re^{p} \to \Re^{n_c}$ denotes the equality constraints, while $g : \Re^{n_\mathbf{x}} \times \Re^{p} \to \Re^{n_g}$ denotes the inequality constraints. An optimizer of (28) is denoted $\mathbf{x}^*(p)$, or simply $\mathbf{x}^*$, if the parameter p is clear from the context.

Ideally, problem (28) is re-solved for a new set of parameter values (initial states), i.e. at every sample time, and the optimal inputs are injected into the plant. However, solving a large NLP may take a long time and this time delay may cause instability [21] or suboptimal operation. So instead of obtaining the updated inputs by solving the NLP every time we receive new state measurements, an alternative approach is to compute fast approximated solutions based on solution point sensitivity. Key properties and results for calculating the sensitivity of the optimal solution of (28) can be found in [22, 23, 24]. To make this paper self-contained and to highlight some important aspects which we use in our path-following approach, we summarize these results below.

Associated with problem (28), we define the Lagrange function as

$$L(\mathbf{x}, \lambda, \eta, p) = F(\mathbf{x}, p) + \sum_{i=1}^{n_c} \lambda_i c_i(\mathbf{x}, p) + \sum_{j=1}^{n_g} \eta_j g_j(\mathbf{x}, p), \tag{29}$$

where λ and η are multiplier vectors of appropriate dimension. A point $\mathbf{x}^*$ is called a KKT-point if there exist multipliers λ, η, which satisfy

$$\begin{aligned} \nabla_x L(\mathbf{x}^*, \lambda, \eta, p) &= 0 \\ c(\mathbf{x}^*, p) &= 0 \\ g(\mathbf{x}^*, p) &\leq 0 \\ \eta^T g(\mathbf{x}^*, p) &= 0 \\ \eta &\geq 0. \end{aligned} \tag{30}$$

The set of all multipliers λ and η, which satisfy the KKT conditions (30) for a parameter p, is denoted by $\mathscr{M}(p)$. To simplify notation, we omit the argument when it is clear from the context and simply write $\mathscr{M}$. The set of active inequality indices is $J = \left\{ j \,|\, g_j(\mathbf{x}^*, p) = 0 \right\}$. The active set is defined as $\mathscr{A} = \{1, \ldots, n_c\} \cup J$, and for a given multiplier pair (λ, η), we define the sets $K_+ = \left\{ j \in J : \eta_j > 0 \right\}$, and its complement $K_0 = \left\{ j \in J : \eta_j = 0 \right\}$.

Definition 2 (Strict complementarity (SC) [25])**.** *Given a vector p, a local solution $\mathbf{x}^*$ of* (28) *and vectors $(\lambda, \eta) \in \mathscr{M}$, we say that the strict complementarity condition (SC) holds for λ, η only if $\eta_j - g_j(\mathbf{x}^*, p) > 0$ for each $j = 1, \ldots, n_g$.*

A constraint qualification is required for a local minimizer of (28) to be a KKT point [25].

Definition 3 (LICQ)**.** *Given a vector p and a point $\mathbf{x}^*$, the linear independence constraint qualification (LICQ) holds at $\mathbf{x}^*$ if the vectors*

$$\begin{aligned} &\nabla c_i(\mathbf{x}^*, p) \quad i = 1, \ldots, n_c, \\ &\nabla g_j(\mathbf{x}^*, p) \quad j \in J \end{aligned} \tag{31}$$

are linearly independent.

The LICQ implies that the multiplier set $\mathscr{M}$ is a singleton, i.e. the multipliers λ, η are unique. For $\mathbf{x}^*$ to be a minimum, a second order condition is required. In this context, we use the strong second order sufficient condition

Definition 4 (SSOSC)**.** *The strong second order sufficient condition (SSOSC) holds at $\mathbf{x}^*$ with multipliers λ and η if*

$$q^T \nabla_{\mathbf{xx}} L(\mathbf{x}^*, \lambda, \eta, p) q > 0 \quad \textit{for all } q \neq 0 \tag{32}$$

such that

$$\begin{aligned} \nabla_{\mathbf{x}} c_i(\mathbf{x}^*, p)^T q = 0, &\quad i = 1, .., n_c \\ \nabla_{\mathbf{x}} g_j(\mathbf{x}^*, p)^T q = 0, &\quad j \in K_+. \end{aligned} \tag{33}$$

Theorem 4 (Implicit function theorem applied to optimality conditions)**.** *Let $\mathbf{x}^*(p)$ be a KKT point that satisfies* (30)*, and assume that SC, LICQ and SSOSC hold at $\mathbf{x}^*$. Further let the functions F, c, g be at least $k+1$ times differentiable in $\mathbf{x}$ and k times differentiable in p. Then*

- $\mathbf{x}^*$ *is an isolated minimizer, and the associated multipliers* λ *and* η *are unique.*

- *for* p *in a neighborhood of* p_0 *the set of active constraints remains unchanged,*

- *for* p *in a neighborhood of* p_0 *there exists a* k *times differentiable function* $\sigma(p) = \left[\mathbf{x}^*(p)^T,\ \lambda(p)^T,\ \eta(p)^T\right]$, *that corresponds to a locally unique minimum for* (28).

Proof. See Fiacco [22]. □

For the case of Theorem 4, the sensitivity can be calculated by applying the implicit function theorem to the optimality conditions of (28), which results in

$$\begin{bmatrix} \nabla_{\mathbf{xx}}L & \nabla_{\mathbf{x}}c & \nabla_{\mathbf{x}}\bar{g} \\ \nabla_{\mathbf{x}}c^T & 0 & 0 \\ \nabla_{\mathbf{x}}\bar{g}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \nabla_p\mathbf{x} \\ \nabla_p\lambda \\ \nabla_p\bar{\eta} \end{bmatrix} = - \begin{bmatrix} \nabla_{p\mathbf{x}}L \\ \nabla_p c \\ \nabla_p\bar{g} \end{bmatrix}, \tag{34}$$

where the derivatives of L, c and g are evaluated at $\mathbf{x}^*, p_0$, $(\lambda, \eta) \in \mathscr{M}(p_0)$ and $\bar{g}$ and $\bar{\eta}$ are vectors with elements $\bar{g}_j$ and $\bar{\eta}_j$, $j \in J$, respectively.

The solution of the linear system (34) gives the optimal change of the solution with respect to a small perturbation in p around p_0. This approach is currently used in the asNMPC framework, and is also implemented in the software sIPOPT [26]. It has further been used by e.g. [27].

Theorem 4 requires strict complementarity (i.e. constant active set in the neighborhood of p_0) and LICQ (unique multipliers). In the cases where the active set can change, it is important to determine the right active set, as incorrect active sets can and generally will affect the accuracy and stability of the NMPC controller; therefore further treatment is needed. A fast and simple way is to apply a strategy named 'clipping in first interval' [18]. The basic idea is that instead of adding the full value of the change calculated by (34) to the nominal optimum $\mathbf{x}^*(p_0)$, only part of the change is added to ensure that the bounds on the first manipulated variable (v_0 in Problem (1)) are not violated. Although the clipping approach works well with input constraints, in more general cases, where the active constraints on outputs may change (strict complementarity does not hold) and where the multipliers are non-unique, a different approach is required. Here we need more general constraint qualifications and a more general second order condition to guarantee a locally unique minimizer, and to compute the corresponding sensitivity.

Definition 5 (MFCQ)**.** *The Mangasarian-Fromovitz constraint qualification (MFCQ) holds at the optimal point* $\mathbf{x}^*(p)$ *if and only if*

a) *the vectors* $\nabla_{\mathbf{x}} c_i(\mathbf{x}^*, p)$ *are linearly independent for all* $i = 1, \ldots, n_c$
b) *there exists a vector* w *such that*

$$\begin{aligned} \nabla_{\mathbf{x}} c_i(\mathbf{x}^*, p)^T w = 0 \quad & \textit{for } i = 1, \ldots, n_c \\ \nabla_{\mathbf{x}} g_j(\mathbf{x}^*, p)^T w < 0 \quad & \textit{for } j \in J. \end{aligned} \tag{35}$$

The MFCQ implies that the set of KKT multipliers, $\mathscr{M}(p)$ is a closed convex polytope [28]. Another constraint qualification we need is the constant rank constraint qualification.

Definition 6 (CRCQ [29])**.** *The constant rank constraint qualification holds at* $(\mathbf{x}^*, p_0)$*, if for any subset* $S \subset J$ *of active constraints the family*

$$\left\{ \nabla_{\mathbf{x}} g_j(\mathbf{x}, p) \; j \in S, \quad \nabla_{\mathbf{x}} c_i(\mathbf{x}, p) \; i = 1, .., n_c \right\} \tag{36}$$

remains of constant rank near the point $(\mathbf{x}^*, p_0)$.

Note that the CRCQ is neither stronger nor weaker than MFCQ in the sense that one implies the other [29].

Definition 7 (GSSOSC)**.** *The general strong second order sufficient condition (GSSOSC) is said to hold at* $\mathbf{x}^*$ *if the SSOSC holds for all multipliers* $\lambda, \eta \in \mathscr{M}$.

It has been shown by Kojima [30] that the conditions KKT-point, MFCQ, and GSSOSC are the weakest ones under which the perturbed solution of the (28) is locally unique. Under these general conditions we cannot expect the solution $\mathbf{x}^*(p)$ to be differentiable any longer (because of active set changes). However, it can be shown that the solution $\mathbf{x}^*(p)$ is directionally differentiable, and for obtaining sensitivity updates in an NMPC context, directional differentiability is sufficient.

We begin by presenting a very general sensitivity result for the optimization problem (28) due to [31, 23, 24].

Theorem 5. *Let* F, c, g *be twice continuously differentiable in* p *and* $\mathbf{x}$ *near* $(\mathbf{x}^*, p_0)$*, and let MFCQ and GSSOSC hold at* $(\mathbf{x}^*, p_0)$*. Then the solution* $\mathbf{x}^*(p)$ *is continuous in a neighborhood of* $(\mathbf{x}^*, p_0)$*, and the solution function* $\mathbf{x}^*(p)$ *is directionally differentiable.*

Moreover, for each p in a neighborhood of p_0, and direction $s \in \Re^{n_p}$ there exists a multiplier pair $(\lambda, \eta) \in \mathscr{M}$ such that the directional derivative uniquely solves the following convex quadratic program

$$\begin{aligned} \min_y \; & y^T \nabla_{\mathbf{x}p} L(\mathbf{x}^*, \lambda, \eta, p_0)s + \frac{1}{2} y^T \nabla_{\mathbf{xx}} L(\mathbf{x}^*, \lambda, \eta, p_0) y \\ s.t. \quad & \nabla_{\mathbf{x}} c_i(\mathbf{x}^*, p_0)^T y + \nabla_p c_i(\mathbf{x}^*, p_0)^T s = 0 \quad i = 1 \ldots n_c \\ & \nabla_{\mathbf{x}} g_j(\mathbf{x}^*, p_0)^T y + \nabla_p g_j(\mathbf{x}^*, p_0)^T s = 0 \quad j \in K_+ \\ & \nabla_{\mathbf{x}} g_j(\mathbf{x}^*, p_0)^T y + \nabla_p g_j(\mathbf{x}^*, p_0)^T s \le 0 \quad j \in K_0. \end{aligned} \tag{37}$$

If in addition the CRCQ holds, then the multiplier values (λ, η) at which the quadratic program (37) *must be evaluated, can be found as a solution of the following linear program:*

$$\begin{aligned} \max_{\lambda, \eta} \; & \lambda^T \nabla_p c(\mathbf{x}^*, p_0)^T s + \eta^T \nabla_p g(\mathbf{x}^*, p_0)^T s \\ & s.t. \quad (\lambda, \eta) \in \mathscr{M}(p_0) \end{aligned} \tag{38}$$

Proof. See [31, 23, 24]. □

The first part of Theorem 5 says there exists some element in $\mathscr{M}(p_0)$ which, when used to set up QP (37), lets us compute the directional derivative in any direction s. The second part of Theorem 5 provides a constructive way (under the CRCQ assumption) to determine the multipliers that must be used to set up QP (37).

The solution of the LP (38) is not always unique [24], i.e. there may be several multiplier vectors, which may be used to compute the sensitivity. On the other hand, by Theorem 5 the QP solution, i.e. the directional derivative in direction s is unique, even if the solution of (38) is not a singleton.

4.2. Multiple sensitivity step corrections (Path-following algorithm)

4.2.1. General concept of the path-following approach

Strictly speaking, the sensitivity results of the previous section are valid only for an infinitesimal perturbation around p_0, and as the perturbation becomes larger, the approximation becomes worse. This can have a significant impact on the performance of the closed loop system.

Therefore, instead of doing a single sensitivity step for the full perturbation $\Delta p = p_f - p_0$, as done in [15], we parameterize the change Δp by a new parameter

$t \in [0,1]$, such that $p(t) = p_0 + t(p_f - p_0)$. The idea is to use subsequent sensitivity updates to track the path of optimal solutions from $t = 0$ (corresponding to p_0) to $t = 1$ (corresponding to p_f).

For a sequence of scalars $t^{(m)} \in [0, 1]$, $m = 0, 1, 2, \ldots, n_m$ such that $t^{(m)} < t^{(m+1)}$, and with $t^{(0)} = 0$ and $t^{(n_m)} = 1$, the change in p between two instances of t can be expressed as

$$s^{(m)} = \left(t^{(m)} - t^{(m-1)}\right)\left(p_f - p_0\right), \quad m = 1, \ldots, n_m, \tag{39}$$

and the value of p at $t^{(m)}$ is

$$p(t^{(m)}) = p_0 + \sum_{l=1}^{m} s^{(l)}, \tag{40}$$

which is denoted $p^{(m)}$. Then we apply subsequent sensitivity updates to follow the path of optimal solutions for each $t^{(m)}$ from $p^{(0)} = p_0$ to $p^{(n_m)} = p_f$.

A path-following method in our context will compute estimates of the states $(\mathbf{x}_p)$ and the multipliers (λ_p, η_p) along the path, and will generally contain the main steps given below:

1. Initialization: Set $t = 0$, choose Δt, solve the NLP for $t = 0$, set $\mathbf{x}_p = \mathbf{x}^*(p_0)$, set $\lambda_p = \lambda^*(p_0), \eta_p = \eta^*(p_0)$

While $t < 1$ do:

2. For a parameter change corresponding to Δt, calculate sensitivity updates y for the states $\mathbf{x}_p$
3. Update states: $\mathbf{x}_p \leftarrow \mathbf{x}_p + y$
4. Update parameters: $t \leftarrow t + \Delta t$
5. Calculate sensitivity updates $\delta\lambda, \delta\eta$ for the multipliers λ_p, η_p
6. Update multipliers:
   $\lambda_p \leftarrow \lambda_p + \delta\lambda$
   $\eta_p \leftarrow \eta_p + \delta\eta$

End do;

The concept is similar to applying an explicit Euler integration method for finding the solution of an ordinary differential equation on a discretized time interval, and applying multiple sensitivity steps in such a way results in a smaller approximation error, as illustrated in Figure 1. Moreover, especially for large per-

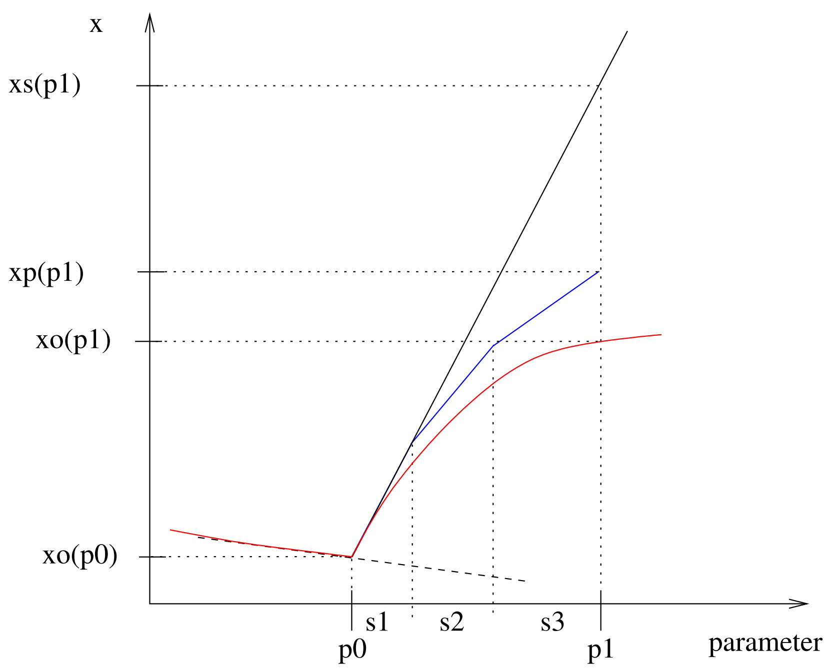


Figure 1: Path following approach and sensitivity update: For a large perturbation the single step sensitivity update (black) results in a large error between the true solution $\mathbf{x}^*(p_f)$ and the approximated solution $\mathbf{x}_s(p_f)$. The path-following approach (blue) with three segments results in a better approximation $\mathbf{x}_p(p_f)$ of the optimal solution $\mathbf{x}^*(p_f)$.

turbations $p_f - p_0$, which cause the active set to change, this approach can be expected to give better performance.

In the next sections, we describe how the sensitivity updates for the states and the multipliers can be computed.

4.2.2. Sensitivity updates for states **x** *along the path*

At each $t^{(m)}$, the approximated optimal solution $\mathbf{x}_p^{(m)} = \mathbf{x}_p(t^{(m)})$ is calculated as

$$\mathbf{x}_p^{(m)} = \mathbf{x}_p^{(m-1)} + y^{(m)}, \tag{41}$$

where $y^{(m)}$ is sensitivity update for perturbation segment $s^{(m)}$. At $t^{(0)}$ the state $\mathbf{x}^{(0)}$ is computed by solving NLP (28) and the corresponding (non-unique) multipliers are found by solving LP (38) for $s^{(1)}$. Along the path, for $t^{(m)}$, with $m \geq 1$, the state $\mathbf{x}_p^{(m)} = \mathbf{x}_p^{(m-1)} + y^{(m)}$ is computed by solving the following QP (using $\mathbf{x}_p^{(m-1)}$, $p^{(m-1)}$, $\lambda_p^{(m-1)}$, $\eta_p^{(m-1)}$ from the previous iteration at $t^{(m-1)}$):

$$\begin{aligned}
\min_{y} \quad & (y^{(m)})^T \nabla_{\mathbf{x}p} L^{(m-1)} s^{(m)} + \frac{1}{2} (y^{(m)})^T \nabla_{\mathbf{xx}} L^{(m-1)} y^{(m)} \\
\text{s.t} \quad & \\
& \nabla_{\mathbf{x}} (c_i^{(m-1)})^T y^{(m)} + \nabla_p (c_i^{(m-1)})^T s^{(m)} = 0 \quad \text{for } i = 1, \ldots, n_c \\
& \nabla_{\mathbf{x}} (g_j^{(m-1)})^T y^{(m)} + \nabla_p (g_j^{(m-1)})^T s^{(m)} = 0 \quad \text{for } j \in \tilde{K}_+^{(m-1)} \\
g_j^{(m-1)} + \quad & \nabla_{\mathbf{x}} (g_j^{(m-1)})^T y^{(m)} + \nabla_p (g_j^{(m-1)})^T s^{(m)} \leq 0 \quad \text{for } j \in \tilde{K}_0^{(m-1)},
\end{aligned} \tag{42}$$

where

$$\tilde{K}_+^{(m-1)} = \left\{ j \,|\quad \eta_j^{(m-1)} > 0 \right\} \tag{43}$$

$$\tilde{K}_0^{(m-1)} = \left\{ j \,|\quad \eta_j^{(m-1)} = 0 \right\}, \tag{44}$$

and where $c_i^{(m-1)}$ and $g_j^{(m-1)}$ denote the value of the constraint i and j, respectively, at the previous step $m-1$. Note that for NLP constraints that are active $g_j^{(m-1)} = 0$.

Apart from the last constraint of QP (42), which represents the linearized inactive inequality constraints, QP (42) is the same as the sensitivity QP (37). This means that as long as no previously inactive constraint becomes active, the QPs have the same solution. Whenever the optimal solution path leads to a constraint that becomes active, QP (42) will attempt to satisfy this constraint to first order.

If this is not possible, the QP will be infeasible, and the step size $s^{(m)}$ has to be reduced by assigning a lower value to $t^{(m)}$ and re-solving the QP.

4.2.3. Sensitivity updates for the multipliers (λ, η) along the path

The multipliers estimates λ_p and η_p are updated along the path according to

$$\lambda_p^{(m)} = \lambda_p^{(m-1)} + \delta\lambda^{(m)} \tag{45}$$

$$\eta_p^{(m)} = \eta_p^{(m-1)} + \delta\eta^{(m)}, \tag{46}$$

where $\delta\lambda$ and $\delta\eta$ correspond to the sensitivity updates of the multipliers.

Unfortunately it is not possible to directly use the results from Section 4.1 for calculating the sensitivity updates for the multipliers, because the LP (38) is infeasible unless evaluated at an optimal point, which generally requires the solution of the full NLP. Under the assumption of LICQ and strict complementarity of the active constraints of the sensitivity QP (37), the multipliers of the QP (37) correspond to the sensitivity of the multipliers of the original NLP. Therefore, one approach to determine the multiplier updates is to use the multipliers of QP (42). However, if the sensitivity updates along the path lead to a point where MFCQ holds, then the multipliers of the QP become non-unique, and it is not clear which QP multiplier values to use for the sensitivity updates.

To obtain estimates of the multiplier sensitivities $\delta\lambda$ and $\delta\eta$ along the path, we therefore propose a new method, which is inspired by LP (38), and where the updates $\delta\lambda^{(m)}$ and $\delta\eta^{(m)}$ are obtained from the solution of

$$\max_{\delta\lambda^{(m)}, \delta\eta^{(m)}} \left(\delta\lambda^{(m)}\right)^T \nabla_p c^{(m)T} s^{(m)} + \left(\delta\eta^{(m)}\right)^T \nabla_p g^{(m)T} s^{(m)} \tag{47}$$

s.t.

$$\begin{aligned} &\nabla_{\mathbf{x}p} L^{(m-1)} s^{(m)} + \nabla_{\mathbf{xx}} L^{(m-1)} y^{(m)} \\ &+ \sum_{i=1}^{n_c} (\nabla_{\mathbf{x}} c_i^{(m-1)})^T \delta\lambda_i^{(m)} + \sum_{j=1}^{n_g} (\nabla_{\mathbf{x}} g_j^{(m-1)})^T \delta\eta_j^{(m)} = 0 \end{aligned} \tag{48}$$

$$\delta\eta_j^{(m)} \geq 0, \quad j \in \tilde{K}_0^{(m-1)} \setminus \hat{K}_0^{(m-1)}, \quad \delta\eta_j^{(m)} = 0, \quad j \in \hat{K}_0^{(m-1)} \tag{49}$$

$$\eta_j^{(m-1)} + \delta\eta_j^{(m)} \geq 0, \quad j = 1..n_g. \tag{50}$$

where

$$\hat{K}_0^{(m-1)} = \{j | j \in \tilde{K}_0^{(m-1)}, g_j^{(m-1)} + \nabla_{\mathbf{x}}(g_j^{(m-1)})^T y^{(m)} + \nabla_p(g_j^{(m-1)})^T s^{(m)} < 0\}.$$

Whenever QP (42) is feasible the constraints (48)-(49) are satisfied. This is because (48)-(49) are the optimality conditions of the QP (42). The only constraint that can make the LP infeasible is (50), which becomes violated when a perturbation $s^{(m)}$ is so large that the active set changes (a constraint becomes inactive). In this case, the size of $s^{(m)}$ must be reduced by assigning a lower value to $t^{(m)}$, and both QP (42) and LP (47)-(50) must be re-solved until the LP becomes feasible. The largest value of $t^{(m)}$ at which constraint (50) is feasible corresponds to the parameter value at which a previously active constraint becomes inactive.

4.2.4. Some implementation issues in the context of asNMPC

The path-following concepts described above can be included in the asNMPC framework in several ways. A straightforward extension of the existing asNMPC method would be to initialize the path-following algorithm with the full perturbation, i.e. $n_m = 1$ such that $t^{(1)} = 1$ and $s = p_f - p_0$. Here the path-following algorithm is used only when active set changes occur. If QP (42) is feasible for $s = p_f - p_0$ (which is always the case for small perturbations without active set changes), this gives the same result as conventional asNMPC based on the implicit function theorem (assuming that LICQ is satisfied). However, if the active set changes, the step-size reduction based on feasibility of QP and LP will detect the points along the path where the active set changes.

An example path-following procedure using the building blocks presented above is presented in Algorithm 1. This algorithm repeatedly solves QP (42) and the corresponding LP (47-50) to update the states, constraints, multipliers and parameters until $t^{(m)} = 1$.

An alternative approach would be to initialize the path-following algorithm with a finer discretization with $\Delta t < 1$ (i.e. $n_m > 1$). This will follow the optimal path more closely, and adapt the step-size where necessary to detect the points where the active set changes. This is especially interesting for problems with very nonlinear behavior.

4.3. Modified NMPC optimization problem

Theorem 5 is based on very weak assumptions, and to the authors' knowledge this is at present the most general result for a constructive way to calculate NLP sensitivity. Moreover, we can consider an equivalent reformulation of problem

Algorithm 1 Pathfollowing Algorithm

1: Solve NLP for $t = 0$ ▷ Initialize algorithm
2: Set $\Delta t = 1$, $t \leftarrow \Delta t$ ▷ Initially try to make only one step
3: Set constants $0 < \alpha_1, \alpha_2 < 1$ ▷ Constants for reducing step size
4: **while** $t \leq 1$ **do**
5:     Solve QP (42) ▷ Calculate sensitivity step
6:     **if** QP(42) is feasible **then**
7:         Solve LP (47-50) ▷ Resolve nonunique multipliers
8:         **if** LP (47-50) feasible **then**
9:             $\mathbf{x}_p \leftarrow \mathbf{x}_p + y$ ▷ Update states
10:             $\lambda_p \leftarrow \lambda_p + \delta\lambda$ ▷ Update multipliers
11:             $\eta_p \leftarrow \eta_p + \delta\eta$
12:             $t \leftarrow t + \Delta t$ ▷ Update parameter
13:             $\Delta t \leftarrow 1 - t$ ▷ Update time step
14:         **else**
15:             $\Delta t \leftarrow \alpha_1 \Delta t$, $t \leftarrow t - \alpha_1 \Delta t$ ▷ Shorten step if LP is infeasible
16:             Go to Step 5
17:         **end if**
18:     **else**
19:         $\Delta t \leftarrow \alpha_2 \Delta t$, $t \leftarrow t - \alpha_2 \Delta t$ ▷ Shorten step if QP is infeasible
20:         Go to Step 5
21:     **end if**
22: **end while**

(1) that always satisfies CRCQ and MFCQ at its solution. Here we replace $\mathbb{X}$ and $\mathbb{X}_f$ in (1) with ℓ_1 penalties and assume, without loss of generality, that $\mathbb{U}$ can be represented by simple upper and lower bounds on v_l. The reformulated NLP (1) takes the following form:

$$\begin{aligned}
\min_{z_l, v_l, \varepsilon_l, s_l} \quad & \Psi(z_N) + \sum_{l=0}^{N-1} \bar{\psi}(z_l, v_l) + \rho \sum_{l=0}^{N-1} \varepsilon_l^T e \\
\text{s.t.} \quad & z_{l+1} = f(z_l, v_l), \quad l = 0, ..., N-1 \\
& z_0 = x_k \\
& g_{z,l}(z_l) + s_l = \varepsilon_l, v_a \le v_l \le v_b, \quad l = 0, ..., N-1 \\
& \varepsilon_l, s_l \ge 0,
\end{aligned} \tag{51}$$

where s_l and ε_l are slack variables and $e = [1, 1, \ldots, 1]^T$. Selecting ρ larger then a certain threshold, $\rho > \bar{\rho}$, will drive ε_l to zero. As proved in [25] when $\varepsilon_l = 0$, the stability properties of the mixed constrained problem (51) are identical to the hard constrained problem (1). Since $\rho > \bar{\rho} > 0$ and $\varepsilon_l \neq 0$, implies problem (1) is locally infeasible, we assume that a finite number ρ can be found[2]. Therefore we solve Problem (51) everywhere instead of Problem (1).

Note that since the inequality constraints in (51) are now simple bounds with constant constraint gradients, the CRCQ follows directly for this problem. Moreover, it is easy to show that since z_l are uniquely determined once x_k and v_l are fixed, the gradients for the equality constraints have full rank. Finally, at the solution of (51), it is easy to find feasible directions into the interior of the feasible region for those variables (v_l, s_l, ε_l) that are active at their bounds. As a result, MFCQ holds at the solution of (51) as well.

5. Economic nonlinear model predictive control on a CSTR case study

We illustrate some of the above concepts on a case study of a CSTR from [9] with a first order reaction $A \to B$. From a mass balance, we derive the following

[2] This corresponds to the common assumption that there exists a feasible input sequence, which steers the system to the terminal set. Among other considerations, this requires that the horizon N is long enough to satisfy the terminal conditions.

dynamic model

$$\begin{aligned}\frac{dc_A}{dt} &= \frac{\dot{m}}{V}(c_{Af} - c_A) - kc_A \\ \frac{dc_B}{dt} &= \frac{\dot{m}}{V}(-c_B) + kc_A.\end{aligned} \tag{52}$$

Here c_A and c_B denote the concentrations of components A and B, respectively. The reactor volume is $V = 10$ l, and the rate constant $k = 1.2$ l/(mol·min). Further, $\dot{m}$ denotes the manipulated input in l/min, and $c_{Af} = 1$mol/l denotes the feed concentration. Using state feedback NMPC, the economic stage cost in (1) is selected as

$$\psi(c_A, c_B, \dot{m}) = -\dot{m}\left(2c_B - \frac{1}{2}\right). \tag{53}$$

Different from [9], we set the variable bounds as:

$$10 \leq \dot{m} \leq 20 \tag{54}$$

$$0.45 \leq c_B \leq 1, \tag{55}$$

When large disturbances occur, variable bounds might be violated. Due to safety reasons, bounds on manipulated variables cannot be violated (hard constraints); while the violation of state variable bounds may be tolerated when necessary. Therefore we treat the bounds of c_B as soft constraints. As described in Section 4.3 and [32], we add a non-negative slack variable ε to the lower and upper bounds of c_B,

$$0.45 - \varepsilon \leq c_B \leq 1 + \varepsilon, \tag{56}$$

and we add an exact ℓ_1 penalty function that contains the slack variable to the stage cost:

$$\psi(c_A, c_B, \dot{m}) = -\dot{m}\left(2c_B - \frac{1}{2}\right) + \rho\varepsilon \tag{57}$$

where ρ is a number large enough to drive ε to zero. In our case we set $\rho = 1000$. Also, we note that the optimal steady states (from Problem (3)), are $c_A^* = 0.5, c_B^* = 0.5, \dot{m}^* = 12$.

5.1. *Regularization of the stage cost*

To ensure that the rotated stage cost is strongly convex, regularization terms $\frac{1}{2}[q_A(c_A - c_A^*)^2 + q_B(c_B - c_B^*)^2 + q_{\dot{m}}(\dot{m} - \dot{m}^*)^2]$ are added to the stage cost and the

Hessian matrix of the steady state optimization problem becomes:

$$\nabla^2 V = A + Q = \begin{bmatrix} q_A & 0 & 1 \\ 0 & q_B & 0 \\ 1 & 0 & q_{\dot{m}} \end{bmatrix} \tag{58}$$

and from the approach described in Section 3 we find the following Gershgorin bounds:

$$q_A > 1,\ q_B > 0,\ q_{\dot{m}} > 1 \tag{59}$$

Moreover, through an eigenvalue calculation, we can find smallest positive weights so that $q_A + q_B + q_{\dot{m}}$ is minimized, while ensuring that $A + Q$ is positive definite. These weights correspond exactly to the bounds determined in (58).

5.2. Simulation results

Our simulation results are organized such that we first demonstrate the effect of regularization on the CSTR with ideal NMPC with different levels of measurement noise and regularization weights. Then we proceed to show results for asNMPC where measurement noise causes the active set to change.

5.2.1. Effect of regularization

From the Gershgorin bounds, we set $q_A = 1 + \delta$, $q_{\dot{m}} = 1 + \delta$, $q_B = \delta > 0$ to ensure strong convexity. The prediction horizon is chosen as $N = 30$ and we simulate for 50 sample times.

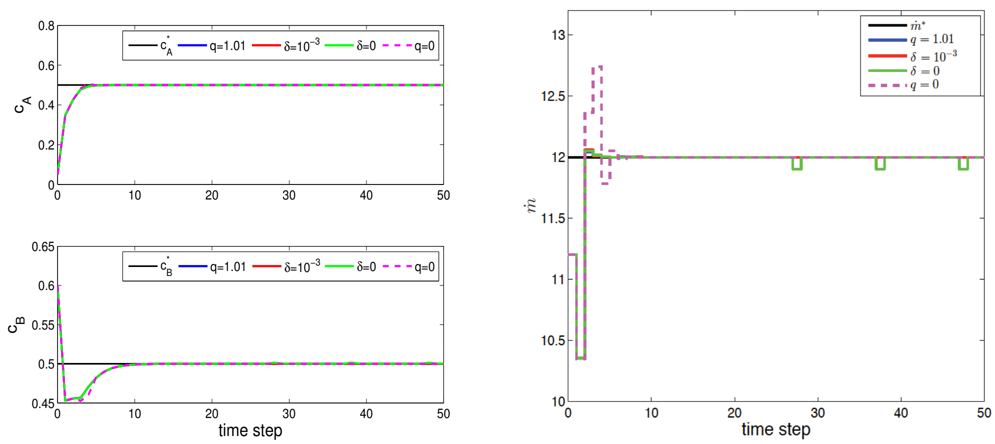
Perfect case, no measurement noise

We assume that all the states are known exactly, and start with the scenario where the model is known completely and there is no measurement noise. We consider different regularizations, starting with the limit, where we set the value of $\delta = 0$, and compare it with $\delta = 10^{-3}$ and with the original weighting factors $q_{i=A,B,\dot{m}} = 1.1$ used in [9]. Finally, we consider the case without any regularization, i.e. $q_{i=A,B,\dot{m}} = 0$.

Figure 2 shows the state profiles and control profiles obtained by simulating the CSTR in closed loop with different weights on regularization terms.

From Figure 2 it can be seen that when we regularize, where $\delta > 0$, the Gershgorin weights are large enough so that their effects are the same as with the weights in [9], that is the system is stable.

When $\delta = 0$, Equation (59) is at the stability bound. Here the state profile is identical as with $q_i = 1.1$, but a few oscillations are observed in the control profile.



(a) State profiles (b) Control profiles

Figure 2: Ideal NMPC, no measurement noise. The results with $q_{i=A,B,\dot{m}} = 1.1$ are shown in blue; with $q_A = q_{\dot{m}} = 1 + 10^{-3}, q_B = 10^{-3}$ is shown in red; with $q_A = q_{\dot{m}} = 1, q_B = 0$ is shown in green; and with $q_{i=A,B,\dot{m}} = 0$ is shown in magenta.

These oscillations disappear as we increase the time horizon.

In the case without any regularization at all, $q_{i=A,B,\dot{m}} = 0$ we have an oscillatory control action at the beginning, and it takes time for the manipulated variable to converge to their steady state optimal values. Note that convergence to the steady state optimal values is not a general property of unregularized Economic NMPC. As Angeli et al. [10] have shown, there may also be cases, where a lower cost can be observed by not converging to the steady state optimal values.

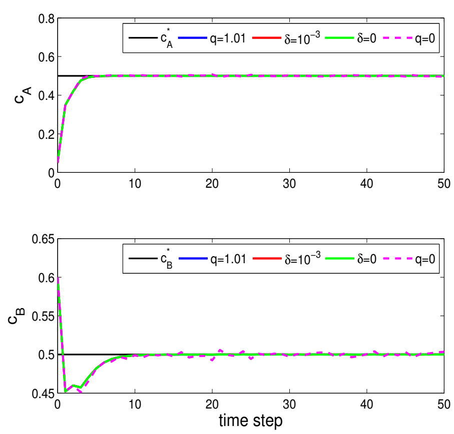
The third column in Table 1 shows the accumulated stage cost $\sum_{k=1}^{50}[-\dot{m}_k(2c_{B,k} - \frac{1}{2})]$. It can be observed that the costs of all the cases are identical. Without noise, slack variables ε of all cases are zero, and the state and control profiles with all weighting factors except $q_i = 0$ are the same, so the stage costs should be the same. Without any regularization $q_i = 0$, the state profiles are identical to the other cases, while the control profile above $\dot{m}^* = 12$ cancels with the profile below $\dot{m}^* = 12$ in order to yield the same accumulated cost.

Cases with measurement noise at different levels

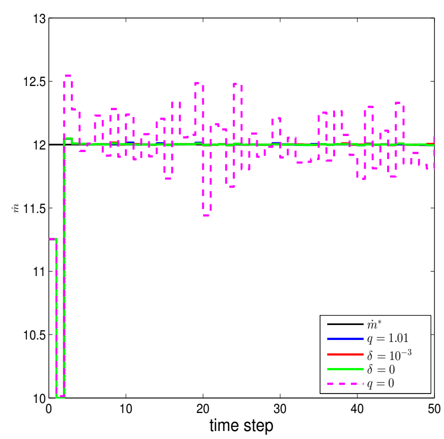
For the two states we add measurement noise with standard deviations at 1% of their equilibrium points. Figure 3 shows the state profiles and control profiles. From Figure 3 we observe that because of the measurement noise, control profiles

Table 1: Cost of ideal NMPC with different noise levels

Cost with		No noise	1% noise	5% noise
No regularization	$q_{i=A,B,\dot{m}} = 0$	-147.35	-146.90	-144.95
Marginal regularization	$\delta = 0$	-147.35	-146.90	-145.20
Small regularization	$\delta = 10^{-3}$	-147.35	-146.90	-145.20
Large regularization	$q_{i=A,B,\dot{m}} = 1.1$	-147.35	-146.90	-145.20



(a) State profiles

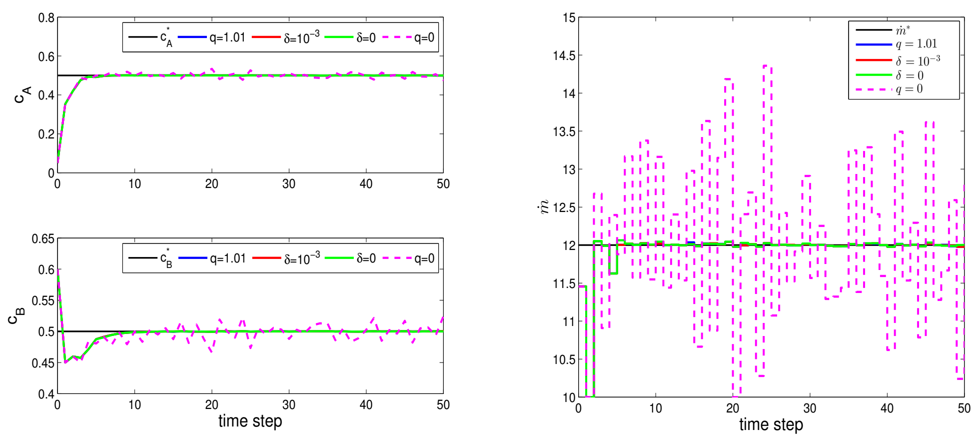


(b) Control profiles

Figure 3: Ideal NMPC, 1% of measurement noise. The results with $q_{i=A,B,\dot{m}} = 1.1$ are shown in blue; with $q_A = q_{\dot{m}} = 1+10^{-3}, q_B = 10^{-3}$ is shown in red; with $q_A = q_{\dot{m}} = 1, q_B = 0$ is shown in green; and with $q_{i=A,B,\dot{m}} = 0$ is shown in magenta.

vary with different weights. But the difference is very small when (59) is satisfied. Without any regularization terms, $q_{i=A,B,\dot{m}} = 0$, there are oscillations in the control profile, which leads to small oscillations in the state profiles. Table 1 shows the accumulated stage costs. It can be seen that all of the accumulated stage costs are essentially the same for all weighting factors. However, even with a small level of measurement noise (1%), the lack of regularization ($q_{i=A,B,\dot{m}} = 0$) leads to significant oscillations in its control profile, which is unacceptable.

We then increase noise level to 5% of the equilibrium points. Figure 4 shows the state and control profiles. As the noise level increases, oscillations are larger in the control profile, which lead to larger oscillations in the state profiles compared with Figure 3. Table 1 shows the accumulated stage costs for this case, too. One



(a) State profiles

(b) Control profiles

Figure 4: Ideal NMPC, 5% of measurement noise. The results with $q_{i=A,B,\dot{m}} = 1.1$ are shown in blue; with $q_A = q_{\dot{m}} = 1+10^{-3}, q_B = 10^{-3}$ is shown in red; with $q_A = q_{\dot{m}} = 1, q_B = 0$ is shown in green; and with $q_{i=A,B,\dot{m}} = 0$ is shown in magenta.

would expect that these costs would decrease with decreasing weights on regularization terms. However, it seems that regularization makes a positive contribution in the presence of measurement noise. Without regularization, i.e. $q_{i=A,B,\dot{m}} = 0$, the controller is not stabilizing, and we observe that its accumulated stage cost is the highest.

Moreover, from Table 1 we observe that accumulated stage costs tend to increase with increased noise levels. This is because the controller is optimizing based on incorrect information (without knowledge of the noise), so performance deteriorates with increased noise.

5.2.2. Advanced-step NMPC with economic stage costs

In this section we study the performance of asNMPC, where the noise level of 5% is chosen so that the active sets differ for the predicted problem and the actual problem, for which a sensitivity based approximated solution is found. We set $q_{i=A,B,\dot{m}} = 1.1$ so that the controller is stable if optimal manipulated variables when injected. To better demonstrate the effect, we zoom into the first 12 sample times of closed loop simulations, and apply the advanced-step NMPC strategy to the CSTR example. We show results for four cases:

Case 1 Ideal NMPC, as a benchmark.

Case 2 asNMPC, using the sensitivity calculation using sIPOPT [26], based on the implicit function theorem (34). (Since this controller may violate bounds on manipulated variables, it should not be implemented in practice.)

Case 3 asNMPC, as in Case 2, but with the manipulated variables v_0 outside the bounds "clipped" to remain within bounds[18].

Case 4 Path-following asNMPC (pasNMPC) with our new path-following approach applied to handle active set changes.

Figure 5 shows the state and input trajectories of the different cases. In particular, the lower bound of $\dot{m}$ becomes active at $time = 2$, and it is violated for Case 2 when (34) is applied directly. This follows because the lower bound is *inactive* at $time = 1$ and the sensitivity prediction from (34) leads to a large (and inaccurate) step that does not include the lower bound, and consequently violates it. This violation is corrected either by clipping (Case 3) or pasNMPC (Case 4). Moreover, note that the control profile of pasNMPC is more accurate, i.e., closer to Ideal NMPC (Case 1).

However, the heuristic clipping approach gives good results in this case-study, too, and has been shown to perform well also in other contexts with input constraints [18].

Table 2 shows accumulated stage costs of Cases 1, 3 and 4. The cost of Case 2 is not given as it is infeasible. Interestingly, the ideal NMPC has the highest cost. Here the noise is not predictable, and its effect on the cost may be positive or negative. For this example it turns out that the effect of noise makes Ideal NMPC peform slightly worse. Note however, that the absolute difference in costs is very small. Finally, when these simulations are performed without measurement noise (no active set changes), we observe no differences between Cases 1 to 4.

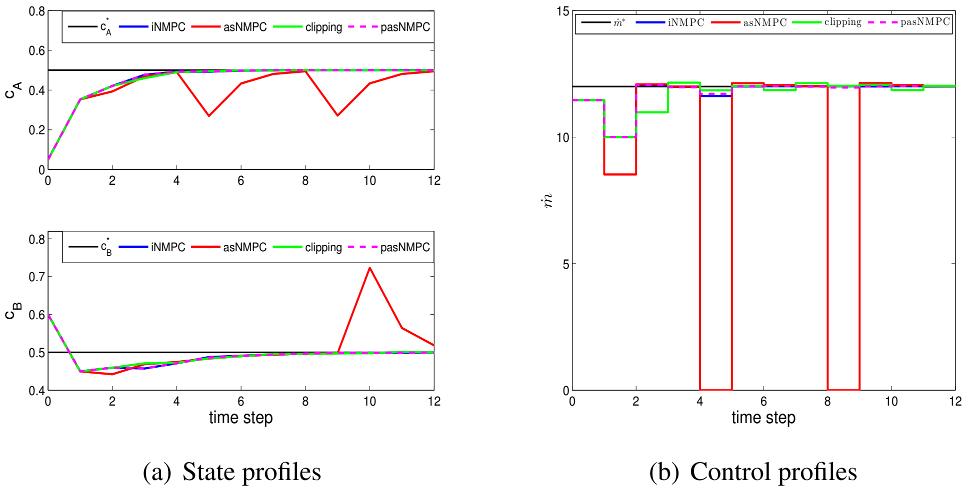


Figure 5: Comparison of the four cases, 5% of measurement noise. Case 1 is plotted in blue; Case 2 is plotted in red; Case 3 is plotted in green and Case 4 is plotted in magenta.

Table 2: Cost of Economic asNMPC with 5% measurement noise and $q_{i=A,B,\dot{m}} = 1.1$

Controller	Cost
Ideal NMPC	-33.00
Clipping asNMPC	-33.15
Pathfollowing NMPC	-33.05

6. Conclusions

This study considers a number of important open issues in economic nonlinear model predictive control, and our results lead to making NMPC a more viable technology in industrial practice.

An important part of this paper has been devoted to extending the Lyapunov stability proof using convexity arguments for the case with terminal costs and conditions. Moreover, we have presented a constructive way of calculating a "minimal" stabilizing regularization. These tools are easy to use for imposing and verifying stability on NMPC applications of large scale.

Further, we have presented and applied very general sensitivity results in order to develop a path-following algorithm, which can be used to obtain fast approximate solutions in asNMPC. Our algorithm is designed to handle active set changes and non-unique multiplier values along the path, and the step-size is adjusted based on the feasibility of the sensitivity QP and the corresponding LP. This enables us to closely follow the active constraint set. Moreover, we have shown that under a suitable problem reformulation, using ℓ_1 penalties within a suitably regularized cost function, the NMPC problem satisfies the required conditions for sensitivity calculation in the presence of active set changes. Our path-following approach therefore deals with this issue in a rigorous manner, especially when previously active constraints become inactive.

On the other hand, these stability results and the regularization procedure guarantee only nominal stability. In future studies we plan to extend these results to robust stability, and therefore develop a rigorous framework that guarantees stability for Economic NMPC in the presence of noise and model mismatch.

As the regularization based on Gershgorin bounds still leaves some degrees of freedom, another direction for future work is to examine if it is possible to exploit these degrees of freedom in such a way that the economic cost is influenced minimally, while stability is still guaranteed. In future work, we will extend our predictor path-following algorithm further with a corrector element. We will also analyze this algorithm with respect to convergence properties and error bounds.

Finally, although there has been much progress in these areas there are still open issues for the application of eMPC that need further research. These include further relaxing the conditions for stability proofs, numeric algorithms for fast approximate solutions, and elucidating the relationship between the Economic NMPC layer and the lower-level regulatory layer.

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