Bank Leverage Restrictions in General Equilibrium: Solving for Sectoral Value Functions

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Abstract

This paper examines welfare effects of bank leverage restrictions. The Global Financial Crisis sparked rigorous debate on the appropriate leverage restriction to stabilize the banking sector without stunting it. I provide a methodological contribution for analyzing welfare effects of bank leverage ratio restrictions over the business cycle in a general equilibrium model with bank runs. This paper's contribution simplifies the model solution process, which is valuable given the model's realistic yet highly nonlinear and complex structure. This framework facilitates the analysis of the tradeoffs associated with bank capital requirements - while unlimited leverage allows capital to flow most freely to its most efficient users, limiting leverage through capital requirements reduces the probability of a bank run.

Keywords: leverage, bank runs
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1 Introduction

The Global Financial Crisis triggered rigorous debate on the bank regulation necessary to reduce bank leverage and increase stability in the financial sector. For example, the Basel Committee on Banking Supervision agreed upon reforms, including a leverage ratio requirement for financial institutions, to be introduced in 2013 in the Third Basel Accord (Basel III).[1] Basel III scheduled a phase in of these leverage requirements, which would reach 4.5% equity to risk weighted assets in 2015 and Basel Endgame marks the final phase in beginning in 2025. In July 2013 several regulators including the U.S. Federal Reserve, Federal Deposit Insurance Corp. (FDIC), and the Office of the Comptroller of the Currency proposed a leverage ratio cap for eight globally systemically important banks of 6%.[2,3] However the question remains what will be the effects of these leverage caps on the real economy?

The contribution of this article is to provide a technical note that facilitates testing whether leverage ratio restrictions balance the trade-off between increased stability in the banking sector and allocational efficiency losses in a welfare improving way. The article builds on Gertler and Kiyotaki (2015), which interprets systemic risk in the economy as an economy wide run on the banking sector. I add an exogenous leverage restriction to the infinite horizon general equilibrium model of the macroeconomy with a banking sector, bank runs, and an endogenous price of capital in the bank run state. I study the effects generated by these leverage caps. The banks in the economy need to be highly levered following a crisis in order to buy back their efficient level of capital holdings, however the probability of a bank run is increasing in bank leverage. Bank crises occur endogenously in the model in the form of an economy wide run on the banking sector. Immediately following a crisis, returns on bank assets are high. These high returns loosen the banks' collateral constraint and allow them to take on very high leverage. The probability of a bank run occurring in the model is an increasing function of bank leverage as long as bank assets in a bank run are not sufficient to cover its liabilities. Thus it is during these high leverage periods immediately after a crisis, where the economy is most fragile and susceptible to another crisis.

The main contribution of this paper is to dynamically calculate the bank and household's value functions, which allows internal comparison of a leverage restriction relative to the Laissez-faire amount of leverage that the household would allow the bank to take on. I write the steady state values of both the household lifetime expected utility and the bankers' value function so that each value function can be iterated backward using

[1] Press Release, Basel Committee on Banking Supervision, Group of Governors and Heads of Supervision announces higher global minimum capital standards (Sept. 12, 2010), http://www.bis.org/press/p100912.pdf.

[2] Evan Weinberger, "Leverage Cap Leaves Big Banks with Unpalatable Choices," Law360, Jul. 9, 2013. A leverage ratio cap of 5% was proposed for the eight GSIBs insured bank holding companies, with additional surcharges to be phased in

[3] Fed and FDIC agree 6% leverage ratio for US Sifis, Central Banking Newsdesk, Jul. 10, 2013.

the specified laws of motion. This facilitates solving the model by guessing one value, the value of the household's value function in the bank run state, and following the laws of motion to reach the steady state; and then iterating until convergence to a steady state that indeed yields the same value in the bank run. Christiano et al. (2022) find that this model is highly non-linear and difficult to solve, thus the contribution I provide offers a meaningful improvement in the ability to solve this model quantitatively.

This technical innovation is important because it enables the banks' value function to be directly computed by backward induction from the steady state. This direct computation of the evolution of banks' value function facilitates calculating the maximum amount of leverage that the household is willing to lend to the bank. With this value in hand, the system compares the market value of leverage that arises from the bank's participation constraint to an externally specified leverage restriction (or macroprudential regulation). The system then decides each period whether the restricted value of leverage will be used or the maximum leverage given by the bank's participation constraint.

I study leverage restrictions in a model that utilizes the household's marginal utility in the banks' value function, however I do not include equity injections from households into banks, which simplifies the exposition of the leverage cap. I use the model to understand the dynamics at play under the leverage cap and when it is removed, which is instructive to understand how economic variables respond and the implications for the size of banks under each system. A leverage restriction that is time varying and more lenient in the periods where households have a higher marginal utility of consumption offers a welfare improvement relative to the Laissez-faire system, consistent with the finding in Gertler et al. (2020).

2 Model

This is an infinite horizon model of the macroeconomy with a banking sector where I enhance the model of Gertler and Kiyotaki (2015) to account for bank regulation in the form of leverage ratio restrictions, similar to Gertler et al. (2020).

Each period, there are two possible states of the world: a bank run state and a no bank run state, and the bank runs are anticipated. There are two types of agents, households and bankers, each type of agent has a continuum of measure unity. The productive technology in the economy is $f(K_t) = ZK_t$. In the bank run state, all of the households run on the entire banking sector. I will focus on the case where if a bank run materializes, the banks do not have sufficient assets to cover their liabilities. This means that the households will receive a fraction of their original deposits and the price of capital during the bank run, Q^*, drops as banks sell their capital at fire sale prices to the inefficient households. These price changes for both deposits and capital affect the household's budget constraint.

This is a two good economy, there is capital, the durable good, and there is the consumption good which is a non-durable good. The paper abstracts from capital accumulation so here there is a fixed supply of capital each period and it does not depreciate:

$$K_t^b + K_t^h = 1$$

Both bankers and households have production functions (f^B and f^H respectively). Households require both capital and units of the consumption good inputs in order to produce more units of the consumption good. In other words, the households pay a cost in consumption goods for operating capital. I will suppose that this cost is a convex, increasing function their capital holdings:

$$f^H(K_t^h, f(K_t^h)) = ZK_t^h$$

Where I assume:

$$f(K_t^h) = \frac{\alpha}{2}(K_t^h)^2$$

K_t^h units of capital remain.

The bankers are the efficient users of capital, they only require capital good inputs in order to produce more units of the consumption good.

$$f^B(K_t^b) = ZK_t^b$$

K_t^b units of capital remain.

When households sell more capital to the banks, the amount of consumption goods in the economy increases since the banks are more efficient at producing capital. Therefore, in the absence of financial frictions, banks would intermediate all of the capital stock. However, when the banks are constrained in their ability to borrow funds to purchase the capital, the households will directly hold some of the capital. When the financial constraints tighten on the bank the households will be forced to hold an elevated supply of capital.

However lending to the bank is risky because there is a probability of an economy wide bank run each period. The probability of a bank run depends on the amount of leverage that the banks have. The probability of a bank run p_t impacts the price of both capital and deposits. It also affects the banker's value function, which is calculated as the banker's return from operating honestly each period in the future, given that there is no bank run. When a bank run occurs, banks are liquidated and due to borrowing constraints, once they have zero net worth, they will never be able to take deposits again.

2.1 Households

The households both consume and save. The households can save either by lending funds to the competitive financial institutions, the banks, or by holding the capital directly. Every period, households receive a return on their asset holdings as well as an endowment of the consumption good, ZW^h. This setup allows the household endowment to vary proportionally with the aggregate productivity Z.

Deposits held by the banks are one period bonds. In the no bank run state, these bonds yield a non-contingent rate of return R_t. However, in the bank run state, these assets receive only a fraction x_{t+1} of the promised return. Where x_{t+1} is the total liquidation value of bank asset per unit of promised deposit. So that, the household's return on deposits can be expressed as:

$$R_t = \begin{cases} \bar{R}_t & \text{if no bank run,} \\ x_{t+1}\bar{R}_t & \text{if bank run occurs} \end{cases}$$

where $0 \leq x_t < 1$. In the run state, all depositors receive the same pro rata share of liquidated assets. Unlike in Diamond and Dybvig (1983), there is no sequential service constraint on depositor contract that links payoffs in the run state to depositors place in line.

Household utility U_t is given by:

$$U_t = E_t \left(\sum_{i=0}^{\infty} \beta^i \ln C^h_{t+i} \right)$$

Where C^h_t is household consumption, $0 < \beta < 1$, and Q_t is the market price of capital. The household chooses consumption, bank deposits D_t, and direct capital holdings K^h_t to maximize expected utility subject to the budget constraint:

$$C^h_t + D_t + Q_t K^h_t + f(K^h_t) = Z_t W^h + R_t D_{t-1} + (Z_t + Q_t) K^h_{t-1} + (1-\sigma) N_t$$

Suppose that p_t is the probability that households assign to an economy wide bank run occurring at time $t+1$. (A discussion of how p_t is determined will follow.) Since the households anticipate that a bank run will occur with positive probability, the rate of return promised on deposits, R_{t+1}, must satisfy the household's first order condition for deposits:

$$1 = R_{t+1} E_t \left[(1-p_t)\Lambda_{t,t+1} + p_t \Lambda^*_{t,t+1} x_{t+1} \right]$$

where $\Lambda^*_{t,t+1} = \beta \frac{C^h_t}{C^{h*}_{t+1}}$ is the household's intertemporal marginal rate of substitution conditional on a bank run at $t+1$. The depositor recovery rate, x_{t+1} in the event of a run depends on the rate of return promised on deposits R_{t+1}.

$$x_{t+1} = \min \left[1, \frac{(Q^*_{t+1} + Z_{t+1})k^b_t}{R_{t+1}d_t} \right]$$

In the spirit of the global games approach developed by Morris and Shin (1998) and applied to banks by Goldstein and Pauzner (2005), I postulate a reduced form that relates the probability of a bank run, p_t, to the aggregate recovery rate x_{t+1}. In this way, the probability p_t of the "sunspot" bank run outcome depends in a natural way on the fundamental x_{t+1}. In general, the probability that depositors assign to a bank run occurring in the following period is a decreasing function of the recovery rate:

$$p_t = \begin{cases} g(E_t(x_{t+1})) & \text{with} \quad g'(\cdot) < 0 \\ 0 & \text{if } E_t(x_{t+1}) = 1 \end{cases}$$

Where g follows the simple linear form:

$$g(\cdot) = 1 - E_t(x_{t+1})$$

Higher leverage chosen by banks today will decrease the recovery rate tomorrow, which increases the probability of a bank run occurring tomorrow. This increases R_{t+1}, the rate of return households require to hold assets from today until tomorrow. Therefore when the bank is choosing leverage to maximize its value function, the cost of deposits owed at $t+1$, R_{t+1}, will affect the bank's decision on how much leverage to take on. So banks internalize the impact that their choice of leverage has on p_t only indirectly through its affect on R_{t+1}.

2.2 Banks

Banks in this paper correspond to lightly regulated "shadow" banks or net borrowing banks in the unsecured interbank market, similar to the setting described in Lewis (2023) and Lewis (2021). These banks hold long-term securities and issue short-term debt, which makes them vulnerable to bank runs. Each banker manages a financial intermediary. Bankers fund their capital investments by issuing deposits to households as well as by investing their own net worth, n_t.

Bankers may be constrained in their ability to borrow deposits and they will attempt to save their way out of the financial constraints by accumulating their retained earnings. To limit this possibility that bankers will try to move towards one hundred percent equity financing, bankers have a finite expected lifetime and each banker has an i.i.d. probability σ of surviving until the next period and a probability $1-\sigma$ of exiting at the end of the current period. The expected lifetime of a banker is then $\frac{1}{1-\sigma}$.

Each period, new bankers enter with an endowment w^b which is received only in their first period of life. The number of entering bankers is equal to the number who exit, keeping the total number of bankers constant. Bankers are risk neutral and they will rebate their entire net worth to the households, as I discuss in detail later in the paper, in the period that they exit so that the expected utility of a continuing banker at the end of period t is given by:

$$V_t = E_t \left[\sum_{i=1}^{\infty} \beta^i (1-\sigma)\sigma^{i-1} \Pi_{t+i} n_{t+i} \right]$$

where $(1-\sigma)\sigma^{i-1}$ is the probability of a banker exiting at date $t+i$, n_{t+i} is the banker's terminal net worth upon exiting in period $t+i$, and Π_{t+i} is the household's marginal utility of consumption in period $t+i$. The bankers take the household's marginal utility of consumption a given.

Conditional on the productivity Z, the net worth of the "surviving" bankers is the gross return on assets net the cost of deposits. Banks can only increase net worth using their retained earnings. This friction is a reasonable approximation of banks in reality. In this paper however, I keep Z constant across time, an area for future analysis would be to explore the effects of shocking productivity Z.

$$n_{t+1} = (Z + Q_{t+1})\, k_t^b - R_{t+1} d_t$$

Exiting bankers no longer operate their banks and they rebate their net worth to the households in the period that they exit. Each period t, new and surviving bankers finance their asset holdings $Q_t k_t^b$ with newly issued deposits and net worth:

$$Q_t k_t^b = n_t + d_t$$

There is a limit to the amount of deposits that bankers can borrow in a given period. This constraint can be motivated by assuming that a moral hazard problem exists. In time t, after accepting the deposits, but still during the same period, the banker chooses whether to operate "honestly" or to divert the assets for his personal use. Operating honestly requires the banker to invest the deposits, wait until the next period, realize the returns on deposits and meet all deposit obligations. If the banker chooses to divert the assets, he will only be able to liquidate up to the fraction θ of the assets and he will only be able to do so slowly, in order to remain undetected. Therefore the banker must decide whether to divert at time t, before the resolution of uncertainty at time $t+1$. The cost of diverting assets is that the depositors are able to force the banker into bankruptcy in the next period. Therefore at time t, the bankers decide whether or not to divert the assets by comparing the franchise value of the financial intermediaries that they operate, V_t, to the potential gains from diverting funds $\theta_t \Pi_t Q_t k_t^b$. Where V_t is calculated as the present discounted value of the future payouts from operating the bank honestly every period. Any rational depositor will not lend deposits to a banker who has an incentive to divert funds. Therefore the following incentive constraint on the banker must hold.

$$\theta_t \Pi_t Q_t k_t^b \leq V_t$$

Given that bankers consume their net worth in the period that they exit, their franchise value can be restated recursively as the expected discounted value of the sum of their net worth conditional on exiting in the following period plus their franchise value conditional on continuing in the following period.

$$V_t = E_t \left[\beta(1-\sigma)\Pi_{t+1} n_{t+1} + \beta\sigma V_{t+1} \right]$$

So that the banker's optimization problem is to choose (k_t^b, d_t) each period to maximize the franchise value subject to the incentive constraint and the balance sheet constraints. As long as the return on bank capital is greater than banks' cost of deposits, banks will have incentive to take on the maximum amount of leverage available to them.

$$\phi_t = \frac{\psi_t}{\Pi_t \theta}$$

Since both the banker objective function and constraints are constant returns to scale, the optimization problem can be reduced to choosing the leverage multiple, ϕ_t to maximize the bank's "Tobin's q ratio," $\frac{V_t}{n_t} \equiv \psi_t$.

2.3 Aggregation

Given that the leverage multiple ϕ_t is independent of individual bank-specific factors and given a parameterization where the banker incentive constraint is binding in equilibrium, then the banks can be aggregated to yield the following relationship between total assets held by the banking system and total net worth:

$$\theta_t \Pi_t Q_t K_t^b = V_t.$$

The evolution of N_t is given by the sum of surviving and entering bankers as:

$$N_{t+1} = \sigma \left[(Z + Q_{t+1}) K_t^b - R_{t+1} D_t \right] + W^b.$$

Where $W^b = (1 - \sigma) w^b$ is the total endowment across all entering bankers and the first term is the accumulated net worth of bankers that were operating at period t and survived until period $t + 1$. Conversely, exiting bankers rebate the fraction $(1 - \sigma)$ of accumulated net worth back to the households:

Total output Y_t is equal to the sum of output from capital Z, household endowment ZW^h, and W^b.

$$Y_t = Z + ZW^h + W^b$$

The output is either used to pay capital management costs or for household consumption:

$$Y_t = f(K_t^h) + C_t^h.$$

The household marginal utility of consumption can be defined:

$$\Pi_t = \frac{1}{C_t^h}$$

2.4 Adding an Exogenous Leverage Constraint to the Model

I add an exogenous leverage restriction that is more restrictive than then endogenous leverage restriction which arises due to the agency problem that bankers face. Introducing the exogenous leverage restriction to the model adds an additional constraint to the banker's optimization problem. Now bankers must maximize their normalized value function subject to both their endogenous participation constraint as well as the exogenous leverage restriction ($\bar{\phi}$).

$$\psi_t = \max_{\phi_t} \mathbb{E}_t \left\{ [\beta(1 - \sigma)\Pi_t + \beta\sigma\psi_{t+1}] \frac{N_{t+1}}{N_t} \right\} \quad \text{s.t.}$$

$$\begin{aligned} \theta \Pi_t \phi_t &\leq \psi_t \\ \phi_t &\leq \bar{\phi}, \quad \text{for each } t \end{aligned}$$

Taking expectations over the probability that there is no bank run each period and given that the return on bank capital holdings is greater than the cost of deposits,

$$\frac{Z+Q_{t+1}}{Q_t} - R_t \geq 0$$

bankers maximize their value function by choosing the maximum amount of leverage so that, at the optimum, their value function can be written:

$$\psi_t = \min\left\{1, \frac{(Z+Q^*_{t+1})K^b_t}{(\min\{\frac{\psi_t}{\theta\Pi_t},\bar{\phi}\}-1)N_t R_t}\right\} \times \beta\left((1-\sigma)\Pi_t + \sigma\psi_{t+1}\right)$$
$$\times\left[\min\{\frac{\psi_t}{\theta\Pi_t},\bar{\phi}\}\frac{(Z+Q_{t+1})}{Q_t} - (\min\{\frac{\psi_t}{\theta\Pi_t},\bar{\phi}\}-1)R_t\right]$$

Which means that optimal choice of leverage is no longer equal to $\phi_t = \frac{\psi}{\theta\Pi_t}$, it is now equal to the minimum of this value and the exogenous leverage restriction. Therefore, in order to determine leverage, I must first model the banker's value function in order to know which constraint will bind. The bankers' value function is:

$$V_t = \mathbb{E}_t\left\{\sum_{i=1}^{\infty}\beta^i(1-\sigma)\sigma^{i-1}\Pi_{t+i}N_{t+i}\right\}$$

Given the law of motion of n_t

$$n_{t+1} = n_t\left[\phi_t\frac{Z+Q_{t+1}}{Q_t} - (\phi_t-1)R_t\right] \quad \text{and}$$
$$n_{t+i} = n_t\prod_{a=1}^{i}\left[\phi_{t+a-1}\frac{Z+Q_{t+a}}{Q_{t+a-1}} - (\phi_{t+a-1}-1)R_{t+a-1})\right]$$

which means that in the aggregate, the banker's normalized value function can be written as

$$\begin{aligned}
\psi_t &= \frac{V_t}{N_t} \\
\psi_t &= \mathbb{E}_t\left\{\beta(1-\sigma)\Pi_{t+1}\left[\phi_t\frac{(Z+Q_{t+1})}{Q_t} - (\phi_t-1)R_t\right]\right. \\
&+\ldots+ \left.\beta^\infty(1-\sigma)\sigma^{\infty-1}\Pi_{t+\infty}\prod_{a=1}^{\infty}\left[\phi_{t+a-1}\frac{(Z+Q_{t+a})}{Q_{t+a-1}} - (\phi_{t+a-1}-1)R_{t+a-1}\right]\right\}
\end{aligned}$$

$$\phi_t = \min\left\{\frac{\psi_t}{\theta\Pi_t}, \bar{\phi}\right\}$$

I solve for the path that the normalized value function follows to recover from a bank run numerically. Once I have the path for the banker's value function, I can determine the path for capped leverage as a function of the normalized value function.

3 Equation for the Bank's Value Function (Ψ_t) Under Capped Leverage

If there is no bank run, the law of motion of the net worth of individual bankers in business is

$$n_{t+1} = \left[k_t^b(Z_{t+1} + Q_{t+1}) - d_t R_t\right]$$

which can be written

$$n_{t+1} = n_t\left[\phi_t\frac{(Z_{t+1} + Q_{t+1})}{Q_t} - (\phi_t - 1)R_t\right]$$

where $\phi_t = \frac{Q_t k_t^b}{n_t}$.

Therefore, in the aggregate,

$$N_{t+1} = N_t\left[\Phi_t\frac{(Z_{t+1} + Q_{t+1})}{Q_t} - (\Phi_t - 1)R_t\right]$$

then

$$\begin{aligned} N_{t+2} &= N_{t+1}\left[\Phi_{t+1}\frac{(Z_{t+2} + Q_{t+2})}{Q_{t+1}} - (\Phi_{t+1} - 1)R_{t+1}\right] \\ &= N_t\left[\Phi_t\frac{(Z_{t+1} + Q_{t+1})}{Q_t} - (\Phi_t - 1)R_t\right]\left[\Phi_{t+1}\frac{(Z_{t+2} + Q_{t+2})}{Q_{t+1}} - (\Phi_{t+1} - 1)R_{t+1}\right] \end{aligned}$$

Following the recursion:

$$
\begin{aligned}
N_{t+i} &= N_t\left[\Phi_t\frac{(Z_{t+1}+Q_{t+1})}{Q_t}-(\Phi_t-1)R_t\right]\\
&\times\ \left[\Phi_{t+1}\frac{(Z_{t+2}+Q_{t+2})}{Q_{t+1}}-(\Phi_{t+1}-1)R_{t+1}\right]\\
&\times\ ...\\
&\times\ ...\\
&\times\ \left[\Phi_{t+i}\frac{(Z_{t+i}+Q_{t+i})}{Q_{t+i-1}}-(\Phi_{t+i}-1)R_{t+i-1}\right]
\end{aligned}
$$

The value function of an individual banker is calculated as:

$$
V_t=\mathbb{E}_t\left[\sum_{i=1}^{\infty}\beta^i(1-\sigma)\sigma^{i-1}c_{t+i}^b\right]
$$

where $c_{t+i}^b=n_{t+i}=(Z_{t+i}+Q_{t+i})k_{t+i-1}^b-R_{t+i-1}d_{t+i-1}$

So in the aggregate, the value function should be calculated:

$$
\begin{aligned}
V_t &= \mathbb{E}_t\left\{\sum_{i=1}^{\infty}\beta^i(1-\sigma)\sigma^{i-1}N_{t+i}\right\}\\
&= \mathbb{E}_t\left\{\beta(1-\sigma)N_{t+1}+\beta^2(1-\sigma)\sigma N_{t+2}+...+\beta^i(1-\sigma)\sigma^{i-1}N_{t+i}\right\}\\
&= N_t\mathbb{E}_t\Bigg\{\beta(1-\sigma)\left[\Phi_t\frac{(Z_{t+1}+Q_{t+1})}{Q_t}-(\Phi_t-1)R_t\right]\\
&+\ \beta^2(1-\sigma)\sigma\left[\Phi_t\frac{(Z_{t+1}+Q_{t+1})}{Q_t}-(\Phi_t-1)R_t\right]\left[\Phi_{t+1}\frac{(Z_{t+2}+Q_{t+2})}{Q_{t+1}}-(\Phi_{t+1}-1)R_{t+1}\right]\\
&\cdots\\
&\cdots\\
&\cdots\\
&+\ \beta^i(1-\sigma)\sigma^{i-1}\left[\Phi_t\frac{(Z_{t+1}+Q_{t+1})}{Q_t}-(\Phi_t-1)R_t\right]\left[\Phi_{t+1}\frac{(Z_{t+2}+Q_{t+2})}{Q_{t+1}}-(\Phi_{t+1}-1)R_{t+1}\right]\\
&\times\ ...\ \times\left[\Phi_{t+i}\frac{(Z_{t+i}+Q_{t+i})}{Q_{t+i-1}}-(\Phi_{t+i-1}-1)R_{t+i-1}\right]\Bigg\}
\end{aligned}
$$

Let I represent the maximum number of periods forward from period t. Replacing the expectation with $\prod_{i=1}^{I}(1-p_{t+i})$ for each period i in the summation, since bankers only consume each period in the state that there is no bank run, yields the following:

$$
\begin{aligned}
V_t &= N_t \Bigg\{ \beta(1-\sigma)(1-p_{t+1}) \left[\Phi_t \frac{(Z_{t+1}+Q_{t+1})}{Q_t} - (\Phi_t - 1)R_t \right] \\
&+ \beta^2(1-\sigma)\sigma(1-p_{t+1})(1-p_{t+2}) \left[\Phi_t \frac{(Z_{t+1}+Q_{t+1})}{Q_t} - (\Phi_t - 1)R_t \right] \\
&\times \left[\Phi_{t+1} \frac{(Z_{t+2}+Q_{t+2})}{Q_{t+1}} - (\Phi_{t+1} - 1)R_{t+1} \right] + \\
&\dots \\
&\dots \\
&+ \beta^I(1-\sigma)\sigma^{I-1} \prod_{i=1}^{I}(1-p_{t+i}) \left[\Phi_{t+i-1} \frac{(Z_{t+i}+Q_{t+i})}{Q_{t+i-1}} - (\Phi_{t+i-1} - 1)R_{t+i-1} \right] \Bigg\} \\
V_t &= N_t \Psi_t
\end{aligned}
$$

So that $N_t = 0$ implies $V_t = 0$.

W^b does not enter into the recursion because Ψ is the growth rate of net worth and the endowment, W^b, does not grow. The law of motion of net worth, or one period growth rate of net worth only depends on leverage, the price of capital, the price of deposits, and the amount of capital that the bank holds. I solve for the value function today as a function of the one period ahead variables. I set the current Ψ_t equal to the probability weighted average of the marginal return on unit of net worth to exiting bankers and to continuing bankers, multiplied by the current law of motion of net worth:

$$
\Psi_t = (1-p_t)\beta(1-\sigma+\sigma\Psi_{t+1}) \left[\Phi_t \frac{(Z_{t+1}+Q_{t+1})}{Q_t} - (\Phi_t - 1)R_t \right].
$$

The value function is the value function of the continuing banker, so at each period, the banker projects his value function out using the probability of a bank run, price of capital, etc. Bankers at each period compare their value of continuing with their value of going into autarky. They calculate their value function without considering bankers who will enter next period. Because if the banker were to abscond with capital and go into autarky, they would not receive the W^b from entering bankers next period. Furthermore, surviving bankers never know when a bank run will happen and considering an extreme case, in a bank run period, all bankers will be eliminated, no new bankers will enter for a period.

With the variable values along the equilibrium path, I calculate Ψ_t and check that the bankers' participation constraint is satisfied each period.

It can be verified numerically that this value function calculated after running the model with no leverage restriction equals $V_t = \theta \Phi_t N_t$. This holds when I use both $\frac{\left[K_t^b(Z_{t+1}+Q_{t+1})-D_t R_t\right]}{N_t}$ and $\left[\Phi_t \frac{(Z_{t+1}+Q_{t+1})}{Q_t} - (\Phi_t - 1)R_t\right]$ as the multiplier for the law of motion of net worth. Algebraically this is because:

If

$$N_{t+1} = N_t \left[\Phi_t \frac{(Z_{t+1}+Q_{t+1})}{Q_t} - (\Phi_t - 1)R_t\right] = \left[K_t^b(Z_{t+1}+Q_{t+1}) - D_t R_t\right]$$

where $\Phi_t = \frac{Q_t K_t^b}{N_t}$, then the law of motion of net worth is:

$$\left[\Phi_t \frac{(Z_{t+1}+Q_{t+1})}{Q_t} - (\Phi_t - 1)R_t\right] = \frac{\left[K_t^b(Z_{t+1}+Q_{t+1}) - D_t R_t\right]}{N_t}$$

4 Single Family Assumption

I combine the household and bank's value functions by making the banks part of the households. When the bank dies it gives its net worth back to the household, so $(1 - \sigma)N_t$ is returned to the households each period. The banks will maximize their net worth discounted by the household's marginal utility each period since in the event that they die, they will be giving their net worth back to the households as an estate tax. The banks are ultimately owned by the households so their optimization problem is to maximize household utility.

Households will give banks full insurance every period so that even if a bank run occurs, the households promise to give the banks their share of consumption - every member of the family will consume the same thing. So the household tells the banks to maximize N_t. But there are different states of the world for the households, so the households have different marginal utilities for each state in each period. Therefore I maximize banker's value function discounted by household's value of consumption in each state of the world. Since bankers have linear utility and have N_t equal to zero in the bank run state, they care only about the non-bank run state, so I only need to track the households' marginal utility in the non-bank run state.

5 Adding the Households' Marginal Utility into the Bankers' Value Function

The bankers' value function without considering the Household Marginal Utility is:

$$\Psi_t = \max_{\phi_t}(1-p_t)\beta\left((1-\sigma)+\sigma\Psi_{t+1}\right)\left[\phi_t\frac{(Z_{t+1}+Q_{t+1})}{Q_t}-(\phi_t-1)R_t\right]$$
$$\text{s.t. } \theta\phi_t \leq \Psi_t$$

Adding in the Household Marginal Utility:

$$\Psi_t = \max_{\phi_t}(1-p_t)\beta\left((1-\sigma)\Pi_{t+1}+\sigma\Psi_{t+1}\right)\left[\phi_t\frac{(Z_{t+1}+Q_{t+1})}{Q_t}-(\phi_t-1)R_t\right]$$

5.1 Updating Banker's Utility in Autarky

In order to be consistent, I discount the utility that the banker receives if he absconds with a fraction θ of deposits, and goes into autarky, by the marginal utility of the households in that state. One difference in this article relative to Gertler et al. (2020) is that I assume that banks can divert fraction θ of their leverage, however when they consume it, they still consume it with the household's marginal utility, since they go home and consume and cease their banking activity in the future.

The implementability constraint can still be set to equality. Because the bankers take the households' marginal utility as given and maximize their value function by taking on the maximum amount of leverage. Thus the bankers participation constraint becomes:

$$\theta\Pi_t\phi_t \leq \Psi_t$$

where $\Pi_t = \frac{1}{C_t^h}$.

Their participation constraint indicates that the value of operating honestly and continuing on must be greater than or equal to the value of absconding with the value of leverage that they have received, or depositors will not deposit with the bank. Banks will set their amount of leverage to the maximum of the participation constraint This is because high leverage maximizes their net worth and banks have linear utility, so they only care about the non-run states. The periods in which the household marginal utility of consumption will be the highest is in the period directly following the bank run. So the bankers will care the most about maximizing N in this period. However in order to maximize net worth in the period directly following the bank run, the banker must take on a lot of deposits. In order for the banker to maximize net worth in the periods closer to steady state, the banker must maximize net worth in previous periods since the growth in net worth compounds on each other. Therefore, in order to maximize net worth each period to increase compounding, the bankers must take on as much leverage as possible each period.

For the case with no leverage cap in place:

$$\Pi_t\theta\frac{Q_tK_t^b}{N_t} \quad \leq \quad \frac{V_t}{N_t}$$

Where bankers will take leverage to the maximum, so that:

$$
\begin{aligned}
\Pi_t \theta \phi_t &= \Psi_t \\
\Pi_t \theta \phi_t &= (1-p_t)\beta\left((1-\sigma)\Pi_{t+1} + \sigma\theta\Pi_{t+1}\phi_{t+1}\right)\left[\phi_t \frac{(Z_{t+1}+Q_{t+1})}{Q_t} - (\phi_t - 1)R_t\right] \\
\phi_t &= (1-p_t)\frac{\beta\Pi_{t+1}}{\theta\Pi_t}\left((1-\sigma) + \sigma\theta\phi_{t+1}\right)\left[\phi_t \frac{(Z_{t+1}+Q_{t+1})}{Q_t} - (\phi_t - 1)R_t\right]
\end{aligned}
$$

For the case with a leverage cap in place, where $\bar{\phi}_t$ is the restricted value of leverage and ϕ_t is the market value of leverage that arises out of banks' participation constraint:

$$
\psi_t = \max_{\phi_t} \mathbb{E}_t \left\{ \left[\beta(1-\sigma)\Pi_{t+1} + \beta\sigma\psi_{t+1}\right] \frac{N_{t+1}}{N_t} \right\} \quad \text{s.t.}
$$

$$
\begin{aligned}
\Pi_t \theta \phi_t &\leq \psi_t \\
\phi_t &\leq \bar{\phi}_t, \quad \text{for each } t
\end{aligned}
$$

This can be rewritten:

$$
\psi_t = (1-p_t)\left\{ \left[\beta(1-\sigma)\Pi_{t+1} + \beta\sigma\psi_{t+1}\right] \left[\phi_t \frac{(Z_{t+1}+Q_{t+1})}{Q_t} - (\phi_t - 1)R_t\right] \right\}
$$

Which will be maximized when:

$$
\phi_t = \min\left\{ \frac{\Psi_t}{\Pi_t \theta}, \bar{\phi}_t \right\}
$$

To calculate the system with the leverage cap in place, I calculate the banker's normalized value function Ψ_t incorporating the household's marginal utility each period.

$$
V_t = \mathbb{E}_t \left\{ \sum_{i=1}^{\infty} \beta^i (1-\sigma)\sigma^{i-1} \frac{1}{C^h_{t+i}} N_{t+i} \right\}
$$

Where

$$
\Pi_t = \frac{1}{C^h_t}
$$

In steady state, I calculate the terminal value of banker net worth. To do this, I use the geometric sum of the steady state value of net worth – using the steady state analog of the recursion for N_t. All variables have reached their steady state value and will not change, so the t subscript is removed. In this way, I incorporate the households' marginal utility into the terminal value of the bankers' value function by calculating the infinite sum of:

$$\begin{aligned}
\Psi &= \frac{V}{N} \\
&= (1-\sigma)\Bigg\{\beta(1-p)\Pi\frac{1}{\sigma}\sigma\left[\Phi\frac{(Z+Q)}{Q}-(\Phi-1)R\right] \\
&+ \beta^2\Pi\frac{1}{\sigma}\sigma^2(1-p)^2\left[\Phi\frac{(Z+Q)}{Q}-(\Phi-1)R\right]^2 \\
&+ \beta^3\Pi\frac{1}{\sigma}\sigma^3(1-p)^3\left[\Phi\frac{(Z+Q)}{Q}-(\Phi-1)R\right]^3 \\
&+ \ldots \\
&+ \ldots \\
&+ \beta^\infty\Pi\frac{1}{\sigma}\sigma^\infty(1-p)^\infty\left[\Phi\frac{(Z+Q)}{Q}-(\Phi-1)R\right]^\infty\Bigg\} \\
&= \frac{(1-\sigma)\Pi}{\sigma}\sum_{i=1}^{\infty}\left\{\beta(1-p)\sigma\left[\Phi\frac{(Z+Q)}{Q}-(\Phi-1)R\right]\right\}^i
\end{aligned}$$

From the limit of an infinite series, we know that:

$$\begin{aligned}
\sum_{i=0}^{\infty} r^i &= 1+r+r^2+r^3+...+r^n \\
&= \frac{1}{1-r}
\end{aligned}$$

for $r \leq |1|$. By multiplying both sides by r, we can move the geometric progression forward:

$$\begin{aligned}
\sum_{i=1}^{\infty} r^i &= r+r^2+r^3+r^4+...+r^n \\
&= \frac{r}{1-r}.
\end{aligned}$$

Let:

$$
\begin{aligned}
r &\equiv \beta(1-p)\sigma\left[\Phi\frac{(Z+Q)}{Q}-(\Phi-1)R\right] \\
&\equiv \beta(1-p)\sigma\left[\frac{(Z+Q)K^b-RD}{N}\right]
\end{aligned}
$$

then we can rewrite Ψ as:

$$\Psi=\frac{V}{N}=\frac{1-\sigma}{\sigma}\left(\frac{r}{1-r}\right)$$

So that, following the calculations in Section 3, I have:

$$\Psi=\frac{V}{N}=\frac{(1-\sigma)\Pi}{\sigma}\left(\frac{r}{1-r}\right)$$

Once I have the terminal value of net worth, I calculate the terminal value of the value function. Each period, I calculate the previous period's value function back by adding one period earlier on to the value function. Thus, each period, I roll the normalized value function back one period by calculating:

$$\Psi_t=(1-p_t)\frac{N_{t+1}}{N_t}\beta\big[(1-\sigma)\Pi_{t+1}+\sigma\Psi_{t+1}\big]$$

In the system without a leverage cap in place, I do not need to calculate the bankers' value function in this way. Instead, I simply set $\Psi_t=\Pi_t\theta\phi_t$. Once I make this substitution, I no longer need to solve for Ψ_t, instead, I can solve for ϕ_t and then multiply it by $\Pi_t\theta$ each period to get Ψ_t. It is because I am working in terms of ϕ_t that I directly use the ratio of $\frac{\Pi_{t+1}}{\Pi_t}$ and the future growth rates in Ψ_t are rolled into ϕ_t.

Whereas in the system with a leverage cap in place, I implicitly incorporate this difference when I say that:

$$\phi_t=\min\left\{\frac{\Psi_t}{\Pi_t\theta},\bar{\phi}_t\right\}$$

Since the Π_{t+1} as well as all future values of Π_{t+i} are already incorporated in Ψ_t, I implicitly have the ratio $\frac{\Pi_{t+i}}{\Pi_t}$ and then all future growth rates in Π_{t+i} are rolled into ϕ_t.

6 Equation for the Household's Lifetime Expected Utility

After a bank run, any bank that entered during the bank run period is liquidated. So once a bank run hits, banks lose all net worth and are never able to acquire net worth again. Since there is a friction where banks can only operate with retained earnings and debt, once they have zero retained earnings, they will never have non-zero net worth and be able to consume again. This shuts down all paths in the bankers' binomial tree except the top path where, with probability $\Pi_{i=0}^{\infty} p_{t+i}$ the bank does not go bankrupt for each period i. Households however are different. In the bank run period, they consume an amount of consumption, which is lower in the crisis period, and then begin working their way out of the crisis. However, each period there is a chance of another bank run.

6.1 Household's Lifetime Expected Utility

$$U_t = \mathbb{E}_t \left(\sum_{i=0}^{\infty} \beta^i \ln C_{t+i}^h \right)$$

Writing this recursively, where C_h^* is the value of household consumption in the bank run period.

$$\begin{aligned}
V(1) &= \log C_h^* + \beta V(2) \quad \text{for } t = 1 \\
V(t) &= \log C_t^h + \beta \mathbb{E}_t V(t+1) \quad \text{for } t > 1 \\
&= \log C_t^h + \beta \left[p_t V(1) + (1 - p_t) V(t+1) \right] \quad \text{for } t > 1
\end{aligned}$$

The system will be in steady state after approximately 122 periods. Therefore, when $t = 122$ or steady state:

$$\begin{aligned}
V_{ss} &= \log C^h + \beta \left[pV(1) + (1-p)V(ss) \right] \\
&= \log C^h + \beta \Big[pV(1) + (1-p) \Big(\log C^h + \beta \left[pV(1) + (1-p)V_{ss} \right] \Big) \Big] \\
&= \log C^h + \beta \Big[pV(1) + (1-p) \Big(\log C^h + \beta \Big[pV(1) + (1-p) \Big(\log C^h + \beta \big[pV(1) + (1-p)V_{ss} \big] \Big) \Big] \Big) \Big] \\
&= \log C^h + \beta pV(1) + \beta(1-p) \log C^h + \beta^2 (1-p) pV(1) + \beta^2 (1-p)^2 V_{ss} \\
&= \log C^h + \beta pV(1) + \beta(1-p) \log C^h + \beta^2 (1-p) pV(1) + \beta^2 (1-p)^2 \log C^h + \beta^3 (1-p)^2 pV(1) \\
&+ \beta^3 (1-p)^3 V_{ss}
\end{aligned}$$

In this way, I can keep iterating on the following period's $V(ss)$ and each new $V(ss)$ will be discounted by an additional β. To solve this numerically, I guess $V(1)$ and calculate:

$$
\begin{aligned}
V_{ss} &= \sum_{i=0}^{\infty} \beta^i \left[(1-p)^i \log C^h + \beta(1-p)^i p V(1)\right] + \beta^{\infty}(1-p)^{\infty} V_{ss} \\
&= \sum_{i=0}^{\infty} \beta^i (1-p)^i \log C^h + \beta^{i+1}(1-p)^i p V(1) \\
&= \sum_{i=0}^{\infty} \beta^i (1-p)^i \left[\log C^h + \beta p V(1)\right] \\
&= \frac{1}{[1-\beta(1-p)]} \left[\log C^h + \beta p V(1)\right]
\end{aligned}
$$

7 Households' Utility Function

$$
U_t = \max_{C_t^h, D_t, K_t^h} \mathbb{E}_t \left(\sum_{t=0}^{\infty} \beta^i \ln C_{t+i^h} \right)
$$

s.t.

$$
C_t^h + D_t + Q_t K_t^h + \frac{\alpha (K_t^h)^2}{2} = Z W^h + R_t D_{t-1} + (Z + Q_t) K_{t-1}^h + (1-\sigma) N_t
$$

Adding household marginal utility into the banker's problem enables the researcher to study total household consumption units in the economy. The net worth of the exiting bankers' can be added into the household's constraint. This enables the calculation of the economy wide utility which gives rise to the welfare in the economy. With this welfare value in hand, the researcher can compare the welfare in the economy where leverage is not constrained to the welfare in the economy where leverage is constrained by macroprudential policy.

7.1 Welfare Implications of Leverage Restrictions

In this section, I present different types of leverage restrictions and their resulting affects on household utility from the model above solved numerically, for illustrative purposes. Two factors drive changes in the household's lifetime expected continuation utility. The first is increased consumption which is increasing in economic productivity so that all else equal, the inefficient households will consume more when the productive bankers operate more of the capital. The second factor driving household utility is the probability of a costly bank run, since if a bank run occurs, the household will be plunged into periods of low consumption.

In Figure 1, I present leverage restrictions in the second period (t=2) only. In the trio of figures, the first two plots illustrate a lenient leverage restriction of maximum 99.99% of the value of leverage that bankers in the unrestricted system, the laissez-faire system, would choose in the second period only. Under this lenient leverage restriction, the household's lifetime expected continuation utility is higher under the leverage cap regime than it is under the no leverage cap regime at every period. Period 2 is the first period after a bank run occurs, since every time a bank run occurs, the economy restarts along its recovery path in period 1 (t=1). In a crisis period, all banks are cleared out of deposits and have a net worth equal to zero so that they can never borrow again. Therefore, in period 2 the net worth in the banking sector is very small, since the only banks in the economy with non-zero net worth are the entering banks. Concurrently at this time, the price of capital is severely depressed at its fire sale value. This implies that the return on capital will be at its largest at this time. The small net worth in the banking sector coupled with the large returns on capital allow banks to take on extreme leverage in the periods following a crisis period. This high leverage is necessary because it allows the banks to buy back capital from the inefficient households faster and improve production in the economy.

If I make the leverage cap in period 2 even slightly more restrictive and do not restrict leverage in any future period, the household lifetime expected continuation utility falls below the value in the laissez-faire system at every period. This implies that high bank leverage following a crisis is necessary in order to eliminate the largest amount of deadweight losses which are incurred when households are operating all of the capital stock. Further, the initial increase in economic productivity between periods one and two is necessary to set the economy on a higher growth path. Restricting bank leverage too much following a financial crisis can keep capital prices depressed too low for too long and lead to persistently lower household utility.

Conversely, Figure 4 presents the household's lifetime expected continuation utility under leverage restrictions in the long run states only, relative to the household's utility in the world with no exogenous leverage restriction in place. If I restrict bank leverage only in the long run states, the household utility increases above its Laissez-Faire value. The plot shows a leverage restriction of 99.99% of the steady state leverage value in the Laissez-Faire model in five states prior to the steady state. This implies that the benefit of decreasing the probability of a costly bank run in the long run states more than compensates for constraining the productive bank's ability to buy capital. In sum, the framework described in the previous section enables the assessment of leverage caps at different points of the banks' recovery path. Consistent with Gertler et al. (2020) these results provide evidence in favor of more lenient bank leverage ratio restrictions in periods during an economic downturn when household's marginal utility of consumption is highest and stricter during periods where household's marginal utility of consumption is relatively lower.

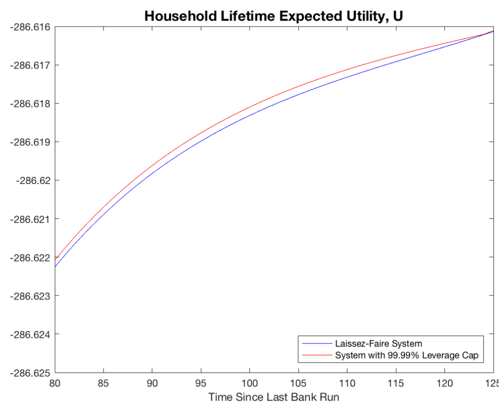

Figure 1: Household Utility with 99.99% Leverage Cap in Period 2 Only

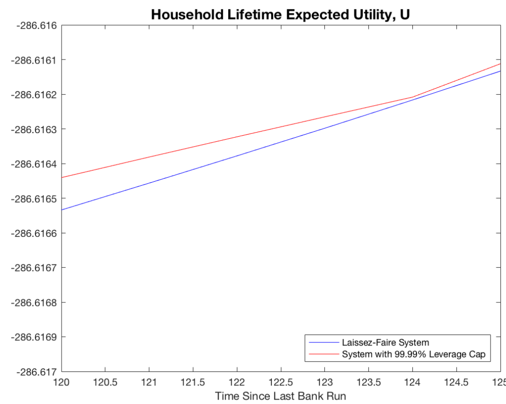

Figure 2: Household Utility with 99.99% Leverage Cap in Period 2 Only (Zoomed In)

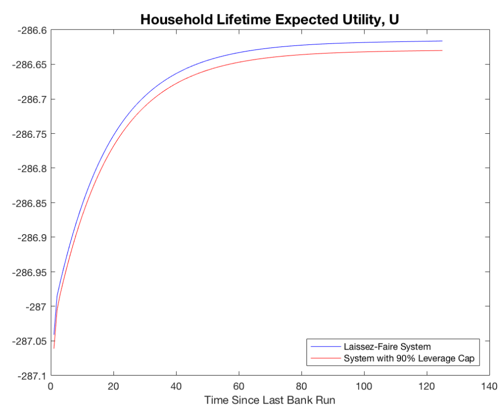

Figure 3: Household Utility with 90% Leverage Cap in Period 2 Only

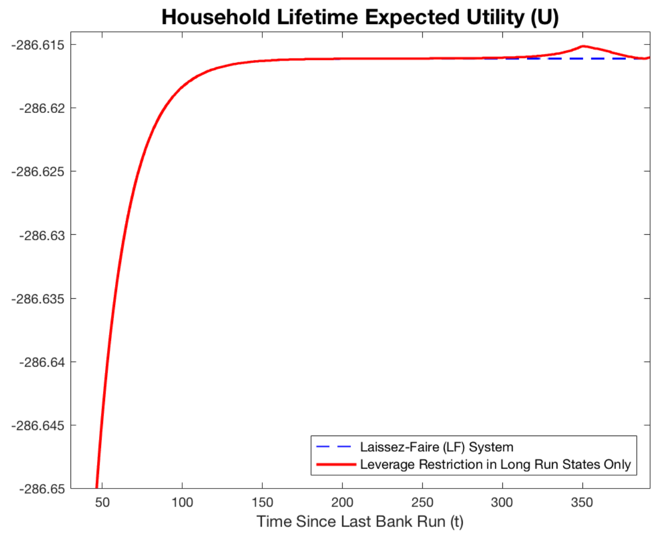

Figure 4: Household Utility with 99.99% Leverage Cap in the Long Run States Only

7.2 System Dynamics Under Multi-period Leverage Restriction

7.2.1 Results for Multiple Period Leverage Restriction

In this section, I study the effects of an exogenous leverage requirement that restricts the maximum amount of leverage that bankers can choose for all periods where banker total assets are greater than 15 times net worth in the uncapped laissez-faire regime. In other words, the leverage cap binds when leverage is 15 times equity or larger – these are the periods following the bank run, when leverage is highest – and then the cap is removed. I chose to cap periods with bank leverage above15 times net worth because, taking the reciprocal, this corresponds to net worth equaling 6.67% of total assets. This number is slightly more conservative than the U.S. baseline requirement, proposed in July 2013 by the Federal Reserve Board, FDIC, and Office of the Comptroller of the Currency, that the country's systemically important banks maintain equity capital worth 6% of total assets to be considered well capitalized. Understanding the dynamics at play under the leverage cap and when it is removed is instructive to understand how economic variables respond and the implications for the size of banks under each system.

The leverage restriction requires that the amount of deposits banks take on be the minimum of either 90% of the optimal leverage chosen in the laissez-faire regime, or the maximum amount of leverage allowed by their incentive constraint. For this calibration of the model, the leverage cap ceases to bind after period 67 or 16.75 years and the economy reaches the steady state in 120 periods or 30 years.

Figure 5 illustrates the recovery path that bank leverage in the economy, ϕ_t, follows

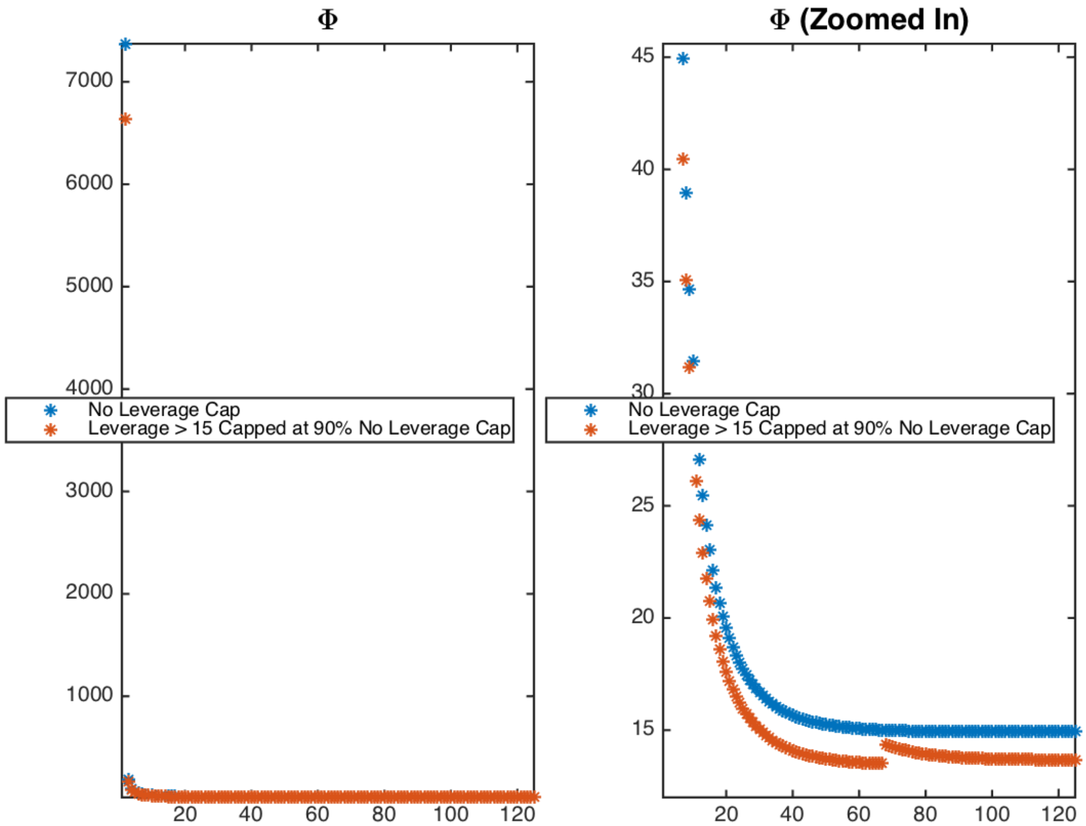

Figure 5: Leverage Φ_t with Multiple Period Leverage Cap

both with and without the leverage restriction in place. The figure on the right hand side is a zoomed in version that omits the first periods directly following a bank run since these periods have enormous leverage. For all plots in this section, the x-axis denotes t or the number of periods since the last bank run. The first period, $t = 1$ is the period in which the bank run occurs and the plots illustrate the recovery path that the variable follows from the bank run period to the steady state value ($t = 125$). In the plots, I compare the path that the variables follow in the unrestricted model versus the path that they follow in the model with the same parameter values but with the leverage restriction in place. Each period, there is a probability p_t that a bank run occurs however the plots reflect the variable's trajectory in the case that no subsequent bank run occurs before the economy reaches steady state.

Figure 6 plots the path that the probability of a bank run, p_t, follows from the time of

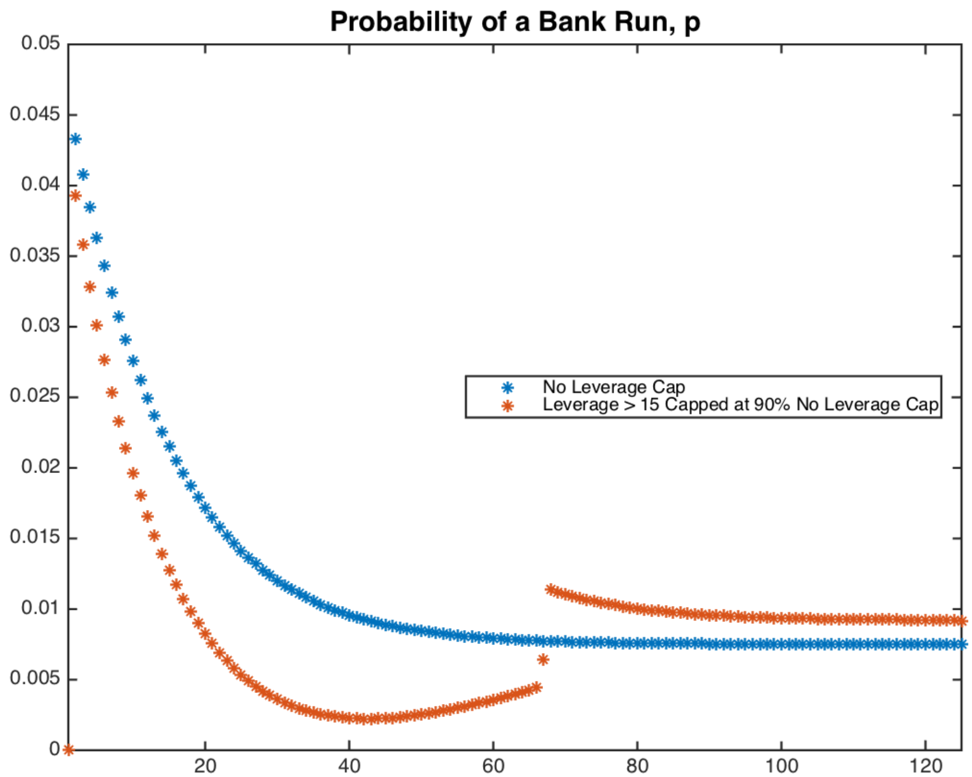

Figure 6: Probability of Bank Run p_t with Multiple Period Leverage Cap

a bank run in period one to steady state. The probability of a bank run p_t decreases in the capped model relative to the uncapped model for the first 67 periods. As seen in the formula for p_t the drop in p_t relative to the unrestricted system in the first 66 periods is driven by the decrease in leverage ϕ_t.

$$p_t = 1 - \min\left\{ \overbrace{\frac{(Z_{t+1} + Q^*_{t+1})(1 - K^h_t)}{(\phi_t - 1) N_t R_{t+1}}}^{\text{Recovery Rate, } x_{t+1}}, 1 \right\}$$

There is a feedback loop at work via the recovery rate. The probability of a bank run is inversely related to the recovery rate x_{t+1}. The recovery rate depends not only on leverage but also on the price that capital takes on in the bank run period Q^*_{t+1}. During periods of extremely high leverage following the bank run, the changes in the leverage ratio dominate the effect on p_t. However for periods where the leverage ratio is close to its steady state value, changes in the price of capital in a bank run dominate changes in p_t. However, it is by taking on more leverage that banks can purchase more capital and drive up the price of capital. Therefore forcing banks to take on lower leverage initially after a bank run can inadvertently depress the price of capital to the point that the probability of a bank run increases in the steady state.

The first term in the minimum operator is the recovery rate or the total value of bank assets in the bank run state divided by the total cost of deposits that a bank would owe in the bank run state. The recovery value is driven up as the leverage cap forces banks to take less deposits than households are willing to give them. This decreases the probability of a bank run initially, bringing it to a minimum of of 0.0022% in period 41 or about 10 years after the bank run, if the economy reaches that period without falling into another bank run. However, the probability of a bank run increases after the leverage cap stops binding because the cap causes irreparably low bank capital holdings while the cap was in place, which drive down the price of capital in the bank run state, Q_t^*, as well as the bank's current capital holdings relative to their steady state values in the lassez-faire system. Once the leverage cap ceases to bind, banks are able to increase their net worth due to relatively higher returns on capital when prices are depressed. However the depressed price of capital in the bank run state drives down the steady state probability of a bank run, p_t, tightening banker incentive constraints so that higher values of net worth do not translate into a higher franchise value. Bankers cannot restore their capital holdings to steady state levels because depositors are not willing to lend them enough in the form of deposits. The denominator of the recovery value decreases as the steady state value of leverage decreases, however banks in steady state are slightly larger which offsets the decrease in leverage. The numerator of the recovery value falls by more due to the decrease in Q_t^* and bank capital holdings than the denominator does with a simultaneous fall in ϕ_t and rise in N_t.

Return on deposits stays relatively similar between the uncapped and capped systems. For the first 6 periods after the bank run, the return on deposits in the capped system is lower than in the uncapped system by maximum of 0.0013 or 0.129% in the period directly following the bank run and then by about 0.000037 or 0.004% for the next 5 periods. After that, it fluctuates between very slight increases and decreases that seem to offset each other, other than a jump in the period where the leverage cap stops binding. This jump is caused by high capital returns as well as a relative increase in the probability of a bank run as banks take on a discontinuous amount of leverage.

The banker net worth N_t is mechanically equal to zero in the period after the bank run and equal to the banker's endowment in the second period in both systems since all existing banks are liquidated in the period that the bank run occurs and banks in the period following the bank run enter with net worth equal only to their endowment. From periods 2 to 67, banks have a smaller net worth in the capped system by on average 0.0023 over this time period. Once the leverage cap stops binding, banks in the capped system

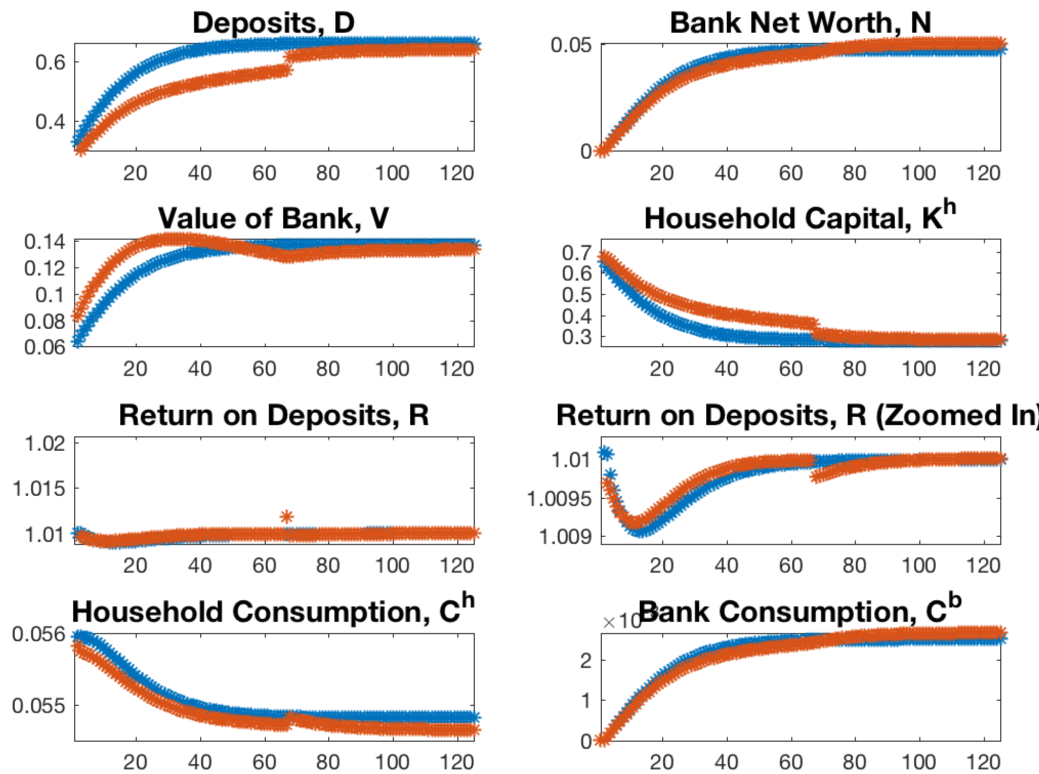

Figure 7: Recovery Path after Bank Run with and without Multiple Period Leverage Cap (Blue: No Leverage Cap, Red: Leverage > 15 Capped at 90% No Leverage Cap)

begin increasing their net worth relative to the uncapped system and have a net worth that is greater than the capped system by 0.0029 in the steady state.

When the leverage is capped, bankers are not allowed to take on as many deposits as the households are willing to give to them based on their participation constraint. Since bankers are financially constrained, the households must directly hold the capital themselves. This leads to households holding more capital in the capped model than they do in the uncapped model over the entire recovery path of the economy. From periods 1 to 67 households hold on average 0.0835 units, or 22% of the average in the Laissez-faire system, more capital in the capped system than they do in the uncapped system. From periods 68 to 120, they hold on average 0.0107 more units, or 3.8% of the average, of capital and in steady state, they hold 0.0042 units or 1.5% more capital.

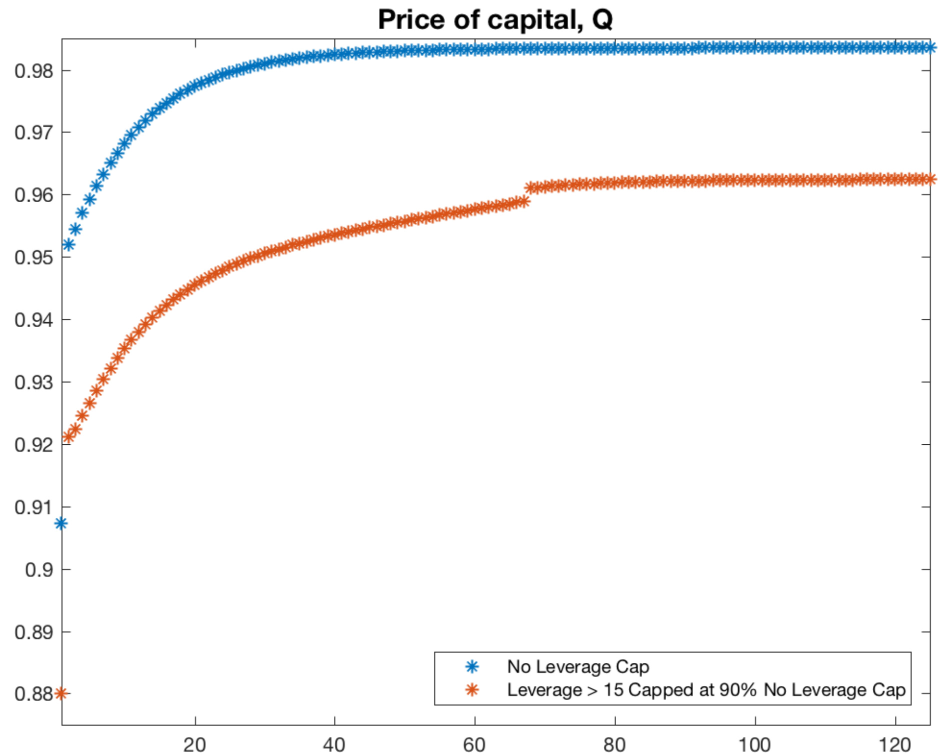

Figure 8: Price of Capital Q_t with and without Multiple Period Leverage Cap

The household's first order condition in part helps determine the price of capital, Q_t, when the household is holding any units of capital.

$$1 = E_t \left[(1-p_t)\beta\Lambda \frac{Z_{t+1} + Q_{t+1}}{Q_t + f'(K_t^h)} + p_t\beta\Lambda^* \frac{Z_{t+1} + Q_{t+1}^*}{Q_t + f'(K_t^h)} \right]$$

Where $\Lambda^* = \frac{C_t^h}{C_{+1}^{h*}}$ is the household's intertemporal marginal rate of substitution conditional on a bank run occurring at time $t+1$ and $f'(K_t^h) = \alpha K_t^h$. The market price of capital tends to be decreasing in household capital, K_t^h holdings since the household's management cost for operating capital is increasing in household capital holdings.

As seen in Figure 8, relative to the system with no leverage cap, the price of capital Q_t is depressed along the economy's entire recovery path after a bank run and remains depressed in steady state. In the system with capped leverage, the price of capital during a bank run is depressed to 0.8800, a decrease of 3% from its laissez fair value of 0.9072. While the leverage cap is in place, from periods 2 through 67, the price of capital is depressed by 3% on average. After the cap is no longer binding, the price of capital remains depressed by 2% on average and stays depressed by 2% in the steady state.

This is because in the bank run period, all banks are liquidated so that their net worth drops to zero. Once a bank has zero net worth, the assumed financial friction that banks can only increase net worth through retained earnings implies that it will never have non-zero net worth at any time in the future. Therefore, in the period following the bank run, only the bankers that enter in that period will have non-zero net worth. These banks are lucky to be born at this time. They enter the economy at a time when the households hold all of the capital in the economy, since households have a convex and increasing management cost associated with operating the capital, this means that the households' management costs are at their maximum. In the laissez-faire regime, the entering bankers are therefore able to extract the total surplus from their advantage in operational efficiency in the form of the maximum returns on bank capital possible. These high returns increase the bankers' value functions, loosening their participation constraints because the households are willing to lend them a lot of money to take advantage of these high returns. These very highly levered periods following a bank run are crucial to allow bankers to purchase as much capital as possible.

Capping leverage in one period decreases the amount of capital that banks are able to purchase from the households. Therefore the households hold relatively more capital and have higher management costs than in the uncapped system. Since households demand similar returns to the uncapped system, the current price of capital decreases as $f'(K_t^h)$ in their first order condition rises. This mechanism causes returns to fall for the first two periods after a bank run relative to the uncapped model since the price at which entering bankers purchase the capital in the period following the bank run is the same in both models. However beginning in the fourth period, the period returns in the capped system begin to surpass those in the uncapped system. This is because of the convex management costs that households shoulder as they operate more capital. As the leverage cap regime bears on, each period the banks are able to purchase less capital from the households, leaving households to operate incrementally more capital each period than they would in the lassez-faire system. The difference in returns between the capped and uncapped system in Figure 9 reflect the convexity of the management cost. Since the households hold more capital in the capped model than they would in the uncapped model, prices are depressed and the bankers in the capped system are able to purchase the capital at a lower price and extract rents from their advantage in operating efficiency for longer than they would be able to in the uncapped model.

Once the leverage cap ceases to bind, the banks take on the maximum amount of leverage that their participation constraint allows, or that the depositors are willing to give them. This causes bank returns to jump discontinuously as the banks buy capital for relatively cheap and drive the price of Q_t up discontinuously in this period. This coupled with returns, elevated from uncapped levels, drives up bank net worth. However

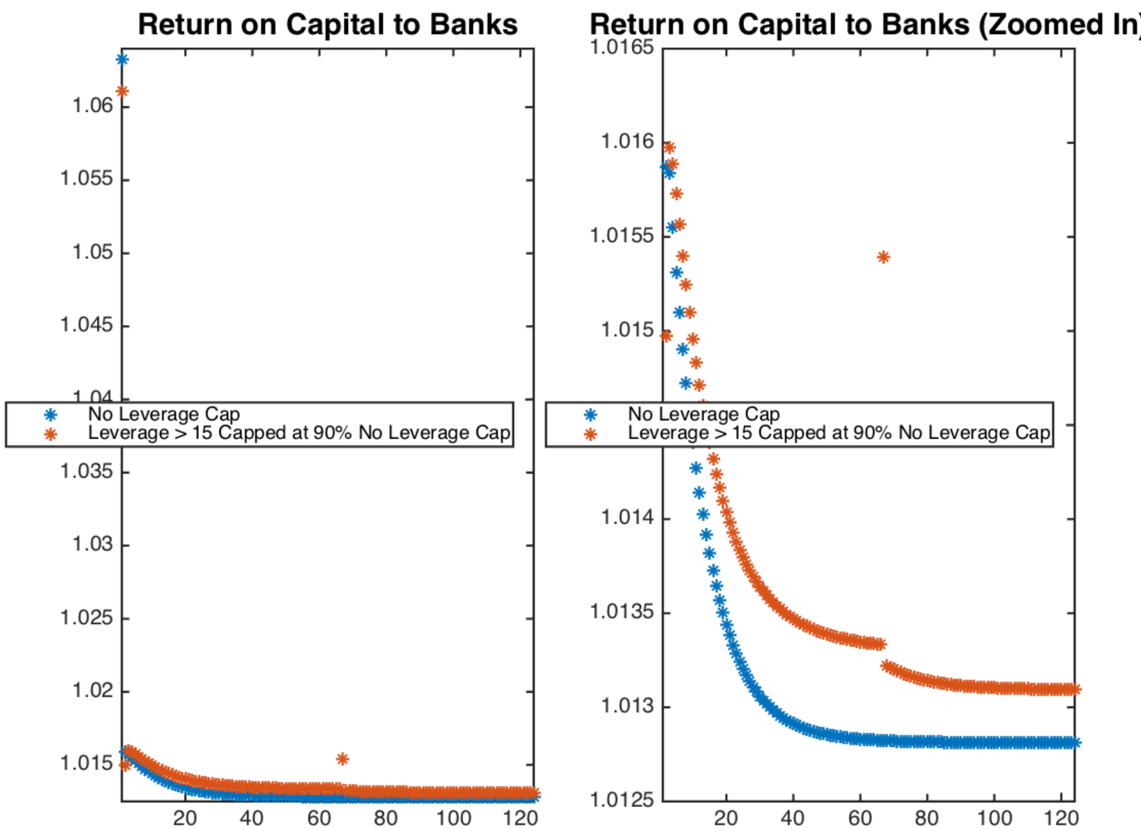

Figure 9: Return on Bank Assets after a Bank Run with and without Multiple Period Leverage Cap

the increase in net worth relative to the uncapped level does not translate into higher capital holdings by the banks because the higher net worth and depressed price of capital in a bank run state decrease the recovery value, increasing p_t. This increase in p_t decreases the banker value value function and tightened banker participation constraints relative to the uncapped model. Therefore, even though the banks are slightly bigger, they cannot take on enough leverage to buy as much capital from the inefficient households as in model with no leverage caps.

The banker's value function is the sum of all future consumption discounted by the banker's discount rate as well as the probability that the banker reaches a given period. Wrapped into this probability that a banker reaches a given period is the probability that there is no bank run in that given period. In steady state, the banker net worth under the leverage cap increases slightly (which increases banker consumption which is equal to the net worth of the fraction of bankers that exit the economy each period). However the probability of a bank run is increasing as the depressed Q_t^* and bank capital holdings begin to dominate the effect of decreased leverage and higher net worth in the recovery rate x_{t+1}. The increase in p_t increases the discount rate on future values of banker consumption, lowering the bankers value function, tightening the participation constraint and decreasing the amount of deposits that households will lend them. Therefore, even though banker net worth is increasing, the simultaneous decrease in leverage relative to the uncapped system makes the banks unable to purchase the same fraction of capital as they can in the uncapped system. This results in households operating elevated levels of capital which directly leads to decreased capital prices throughout the entire recovery path that the economy follows after a bank run. These results seem to imply the leverage cap introduces a wedge in the economy that allow steady state banks to be bigger and generate higher returns. However because banks in the capped system never acquire as much leverage as in the uncapped system, they cannot purchase the lassez-faire fraction of capital. The wedge therefore forces the inefficient households to operate elevated levels of capital and allows the efficient banks to extract higher operating rents from them each period.

7.2.2 Implications of Leverage Restriction for Simulated Economy

I simulate the economy under both the leverage cap regime in Section 7.2.1 and the uncapped regime for 10,000 periods. The economies both begin in the period following a bank run and are allowed to evolve according to their recovery paths solved for above. Each period, they are subject to a potential run on the banking sector which occurs with probability p_t. Regardless of what period the economy had reached before the bank run, if the economy falls into another bank run, it will need to start at the beginning of its recovery path and begin working sequentially toward its steady state again. Each period, I draw a random number distributed on the unit interval. The stochastic simulation begins at period 1, the period when the bank run occurs. Before the system may evolve to period 2, I first draw a random number. If the number drawn is less than p_2, then the economy is thrown back into a bank run. If not, the economy is allowed to progress to period 2 and I repeat the process, this time checking whether the random number drawn is less than p_3 before allowing the economy to advance to period 3, and so on. In the model, given the banks lose all of their net worth if a bank run occurs, their net worth in period one equals zero. Their participation constraint implies that banks in period one will not be able to take on any deposits given that their net worth is equal to zero, so the probability of a bank run occurring in period two is equal to zero. However, in period three and every future period, there is a positive probability of a bank run occurring. Intuitively, each period with probability $1 - p_t$, the economy evolves along the recovery path plotted in the figures above and with probability p_t a bank run occurs and throws the economy back into period one. p_t is decreasing as the economy moves further away from the bank run in period one. Therefore the economy is the most fragile during the periods immediately following a bank run and may suffer several bank runs that happen in rapid succession and prolong its recovery process after an initial crisis.

After simulating both the economy with a leverage cap in place and the economy with no exogenous leverage cap, I calculate the average number of periods between bank runs, the average number of periods that the economy stays in steady state once it has reached steady state, and the average number of periods that the economy takes to return to steady state after suffering a bank run. I find that on average, a bank run occurs nearly every 81.3 periods or 20.3 years in the uncapped model and nearly every 109.9 periods or 27.5 years in the capped model. The system reaches the steady state about 44.2 periods or 11 years faster when the leverage cap is in place. The longer amount of time between bank runs and the ability for the economy to reach the steady state faster under a leverage cap regime are due to the decreased probability of a bank run, p_t, while the leverage cap binds in the economy with a leverage cap in place. Conditional on reaching steady state however, the system with the leverage cap regime falls out of the steady state into a bank run on average 1.3 years or 5.2 periods earlier than it would without a leverage cap in place. This is due to the fact that p_t is driven up by the decreased price of capital in the bank run state caused by allocational efficiency losses that result from capping bank leverage in the periods directly following a bank run. These results provide evidence supporting a leverage cap's ability to stabilize the economy. However the results also imply that there could be allocational efficiency losses that increase the risk in the economy if bank leverage is too harshly restricted directly following a bank run as this can permanently slow the economy's ability to recover from a crisis.

Table 1: Recovery Times Economy Simulated 10,000 Periods (Multi-Period Leverage Cap)

Average Number of Periods	No Leverage Cap	Leverage Cap
Between Bank Runs	81.3	109.9
To Reach SS	318.1	273.9
In SS (Conditional on Reaching)	87.1	82.0

8 Conclusion

I provide a methodological contribution for analyzing welfare effects of bank leverage ratio restrictions over the business cycle in a general equilibrium framework. The contribution of this article is to provide a technical note that writes the steady state values of both the household lifetime expected utility and the bankers' value function so that each value function can be iterated backward using the specified laws of motion. This innovation facilitates solving the banks' value function directly by induction backward from the steady state and thus facilitates the solution to the model with restricted leverage. Using the method described, I study different types of leverage restriction regimes to understand the dynamics at play under the leverage cap and when it is removed, and how economic variables respond and the implications for the size of banks under each system.

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