Problem 1 - 1D Elliptic PDE

$u : (0,1) \to \mathbb{R}$

$$-u'' = f \quad \text{for } x \in (0,1)$$

$$\left.\begin{aligned} u(0) &= \alpha \\ u(1) &= \beta \end{aligned}\right\} \text{Dirichlet Boundaries}$$

$f : (0,1) \to \mathbb{R}$ $\qquad \alpha \in \mathbb{R}$ $\qquad \beta \in \mathbb{R}$

1a) Discretisation

Mesh : $x_i = ih$ for $0 \le i \le (m+1)$

where we use n nodes

so $\underbrace{h = \frac{1}{(n-1)}}_{\text{mesh width}}$ and $\underbrace{m = n-2}_{\text{no. of interior mesh nodes}}$

Second-Order Difference Scheme :

Denote U_j as the approximation $u(x_i)$

$$-\left(\frac{U_{(j-1)} - 2U_j + U_{(j+1)}}{h^2}\right) = f(x_j) \quad \text{for } j = 1, \ldots, m$$

with boundaries : $U_0 = \alpha$ & $U_{m+1} = \beta$

@ $j=1$

$$\frac{-\alpha + 2U_1 - U_2}{h^2} = f(x_1)$$

$$\Rightarrow \frac{2U_1 - U_2}{h^2} = f(x_1) + \frac{\alpha}{h^2}$$

@ $j=m$:

$$\frac{-U_{m-1} + 2U_m - \beta}{h^2} = f(x_m)$$

$$\Rightarrow \frac{-U_{m-1} + 2U_m}{h^2} = f(x_m) + \frac{\beta}{h^2}$$