Problem 2 - Poisson Equation (2D domains)

$\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$

$u : \Omega \to \mathbb{R}$

$$-(u_{xx} + u_{yy}) = f(x,y) \quad \text{for } (x,y) \in \Omega$$

$$u(x,y) = g(x,y) \quad \text{for } (x,y) \in \partial\Omega$$

$f : \Omega \to \mathbb{R}$ $\qquad g : \partial\Omega \to \mathbb{R}$

2a) Discretisation

Mesh: (x_i, y_j) for $x_i = ih$ & $y_j = jh$ $\quad 0 \leq i,j \leq (m+1)$

m is number of interior mesh nodes.

$h = \frac{1}{(m+1)}$ is the mesh width.

Difference Scheme:

$$\frac{-U_{i-1,j} - U_{i,j-1} + 4U_{i,j} - U_{i,j+1} - U_{i+1,j}}{h^2} = f(x_i, y_j)$$

for $1 \leq i \leq m$ & $1 \leq j \leq m$.

with boundary nodes at: $i=0$, $i=(m+1)$, $j=0$, $j=(m+1)$

so,
$$U_{0,j} = g(0, hj) \qquad U_{(m+1),j} = g((m+1)h, hj)$$
$$U_{i,0} = g(hi, 0) \qquad U_{i,(m+1)} = g(hi, (m+1)h)$$

for $0 \leq i \leq (m+1)$ & $0 \leq j \leq (m+1)$