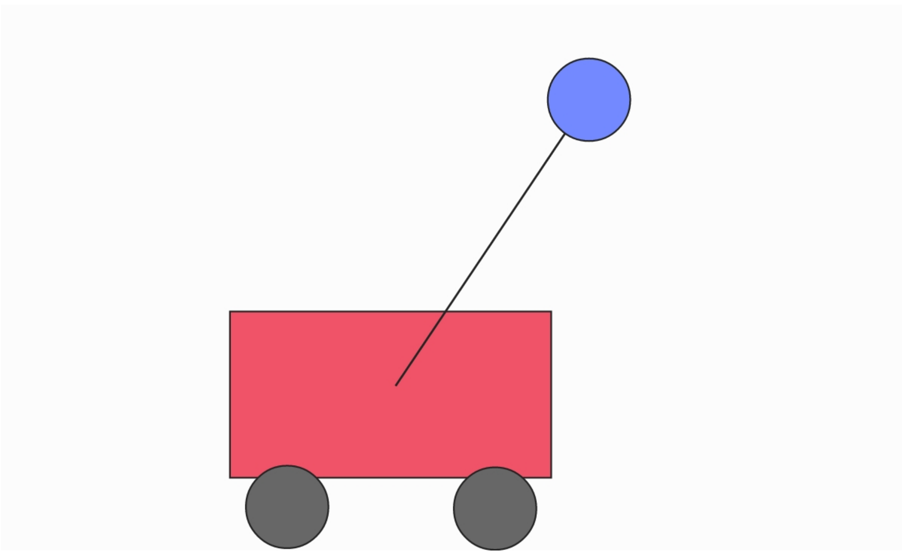
UNIVERSITÀ DELLA CALABRIA

Dynamical System Theory Project

Modeling, analysis and control of an inverted pendulum on a cart

Robotics and Automation Engineering
2023-2024



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1 Introduction

A common problem in the field of control engineering is the position control of an inverted pendulum. In particular, the aim of this project is to model, analyse and control a mechanical system made up of an inverted pendulum mounted on a cart.

Inverted pendulum on a cart is a 2-DoF system; each Degree of Freedom is associated to a motion which can be performed by this system:

- One degree is exploited by the cart subsystem, which is allowed to move back and forward along horizontal direction
- The second degree derives from the pendulum subsystem, consisting of a mass attached through a rod to a pivot mounted on the cart; the pendulum is able to swing around the pivot fixed on the cart.

The desired goal for this project is to synthetize a controller which allows the cart to move while keeping the pendulum in upright position.

The roadmap to achieve the target result is divided in four phases:

1. **Modeling phase**, focused on translating description and components of the system in a set of analytical laws of evolution;
2. **Analysis phase**, during which the behaviour of the system is observed and discussed. Also a linearized version of the system is proposed, in order to simplify successive phases;
3. **Structural description phase**, where structural properties of the system are investigated. This is a crucial part, since the possibility to synthetize the desired controller depends on results of this phase;
4. **Synthesis**, the last phase of the project. Here feedback control logic are implemented to achieve the declared goal.

2 Modeling

2.1 Physical description

A cart of mass M can move along its longitudinal direction, which coincides with $\hat{\imath}$ direction of the coordinate reference system. In particular, suppose this cart is motorized and able to produce a force $F\hat{\imath}$; moreover, there's no friction between cart and surface. This first subsystem is fully described by position x_c and speed v_c of the cart.

A pendulum is attached to the cart: a pivot is mounted on the cart and, through a non-deformable rod of length L, a mass m is able to rotate without friction around the pivot. Pendulum subsystem is described if an angle θ and an angular speed ω are known. Angle θ, which is null along $-\hat{\jmath}$ direction, defines relative position of mass m with respect to cart; angular speed ω is the rate of change of θ.

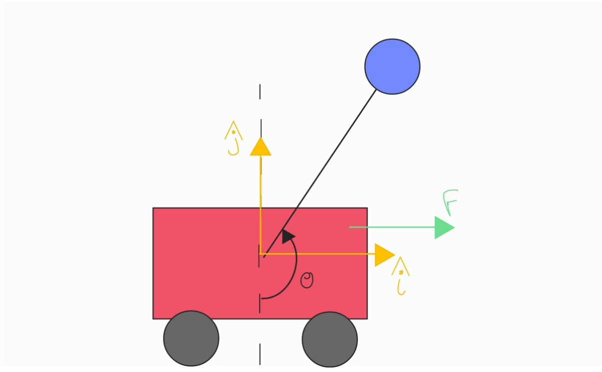


Figure 1: Picture of the system under analysis.

Each subsystem can be furtherly analyzed highlighting which are the forces in action and where them act. Once this physical characterization is completed, balance equations between forces will be used to derive a mathematical model for cart-pendulum system. Free body diagrams for both subsystems are showed in figures 2a and 2b.

Applying Newton's second law to cart subsystem, the following equations are obtained

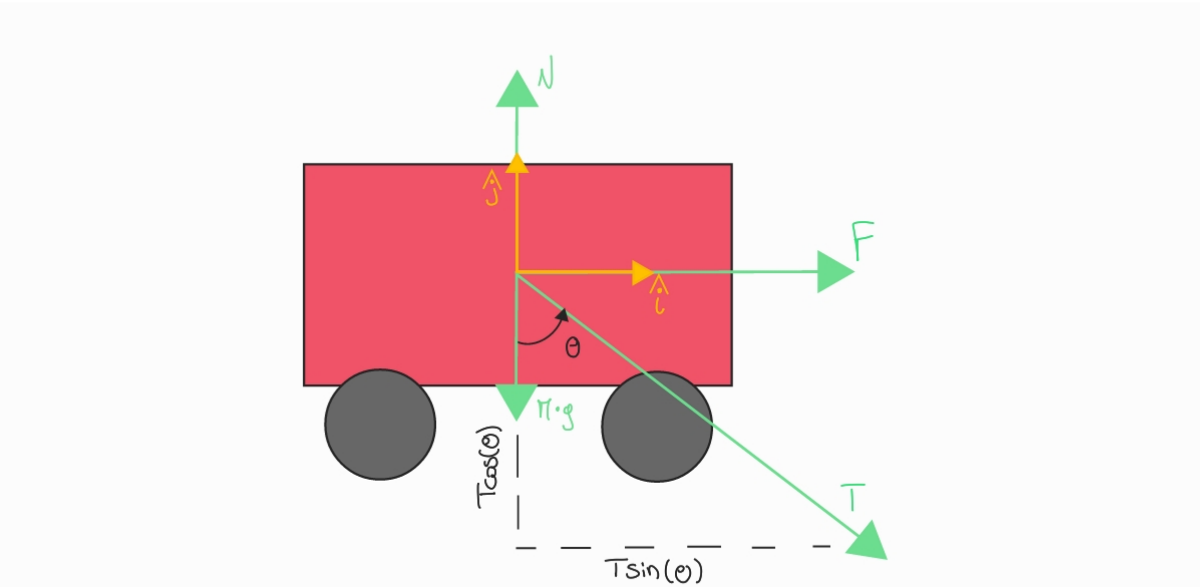
$$F + T\sin\theta = M\ddot{x}_c \tag{1}$$
$$N - Mg = M\ddot{y}_c \tag{2}$$

where (1) refers to $\hat{\imath}$ direction, while (2) refers to $\hat{\jmath}$ direction.

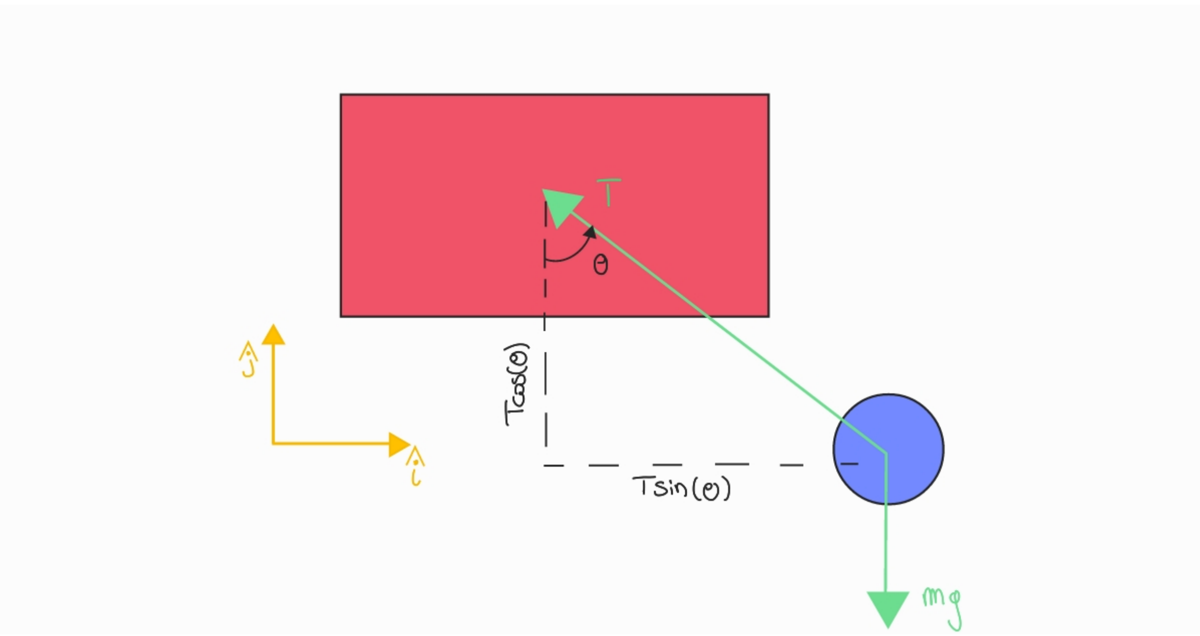
Similarly, pendulum subsystem is described by equations

$$-T\sin\theta = m\ddot{x}_p \tag{3}$$
$$T\cos\theta - mg = m\ddot{y}_p \tag{4}$$

which refer respectively to $\hat{\imath}$ direction and $\hat{\jmath}$ direction.



(a) Forces acting on cart subsystem.



(b) Forces acting on pendulum subsystem.

Accelerations $\ddot{x}_p$ and $\ddot{y}_p$ depend on two different motions which are:

1. The longitudinal motion of the cart,
2. A relative rotation of pendulum around the cart.

Hence, a complete expression for $\ddot{x}_p$ and $\ddot{y}_p$ is given by the following equations

$$\ddot{x}_p = \ddot{x}_c + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta \tag{5}$$

$$\ddot{y}_p = \ddot{y}_c + L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta \tag{6}$$

Finally, table 1 collects all parameters characterizing the system. It specifies, for each parameter, a description and the value assigned during simulation processes.

M	Mass of the cart	$5[Kg]$
m	Mass of the pendulum	$1[Kg]$
L	Length of the rod in pendulum subsystem	$2[m]$
g	Gravitational acceleration	$9.81[\frac{m}{s^2}]$

Table 1: Parameters of the system.

2.2 Model derivation

A mathematical model for cart-pendulum system can be retrieved starting from equations (1), (3), (4) and the full description of accelerations $\ddot{x}_p$ and $\ddot{y}_p$.

To obtain a first equation, use (3) to substitute $T\sin\theta$ in (1); also, replace $\ddot{x}_p$ with its full expression. These operations result in

$$F - m\cdot\left(\ddot{x}_c + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta\right) = M\ddot{x}_c \tag{7}$$

A second relation is found combining equations (3) and (4). In particular, the outcome of summing

$$\cos\theta\cdot(3) + \sin\theta\cdot(4)$$

is an equation with no dependence on rod's tension T.

$$-g\sin\theta = \ddot{x}_c\cos\theta + L\ddot{\theta} \tag{8}$$

Finally, we came up with a pair of coupled equations describing the behavior of the whole system. Table 2 resumes which are the variables of cart and pendulum subsystems and their symbols; also, it defines a new set of symbols we'll use from now on to describe the system. Symbols v and ω refers, respectively, to horizontal cart's speed and pendulum's angular speed.

Variable	Description	New Symbol
x_p	Cart's position	p
$\ddot{x}_p$	Cart's acceleration	$\dot{v}$
θ	Pendulum's angle wrt down position	θ
$\ddot{\theta}$	Pendulum's acceleration	$\dot{\omega}$

Table 2: Variables characterizing cart-pendulum system.

$$
\begin{cases} F + mL\omega^2 \sin\theta = (M+m)\cdot\dot{v} + mL\dot{\omega}\cos\theta \\ -g\sin\theta = \dot{v}\cos\theta + L\dot{\omega} \end{cases} \tag{9}
$$

Thanks to the introduction of variables v and ω, we are able to move from system (9) to a state space model representation. Vector $[p, v, \theta, \omega]^T$ is the state of the whole system, while cart's position p and pendulum's angle θ are outputs.

$$
\begin{cases} \dot{p} = v \\ \dot{v} = \frac{F + mL\omega^2 \sin\theta + mg\sin\theta\cos\theta}{M + m(\sin\theta)^2} \\ \dot{\theta} = \omega \\ \dot{\omega} = -\frac{F\cos\theta + mL\omega^2\sin\theta\cos\theta + (m+M)g\sin\theta}{M + m(\sin\theta)^2} \\ y_1 = p \\ y_2 = \theta \end{cases} \tag{10}
$$

2.3 Simulink implementation

A Simulink scheme of non-linear model (10) has been built to simulate the system in different scenarios. Subsystems showed in the scheme implement exactly, using Simulink's blocks, non-linear equations defining $\dot{v}$ and $\dot{\omega}$.

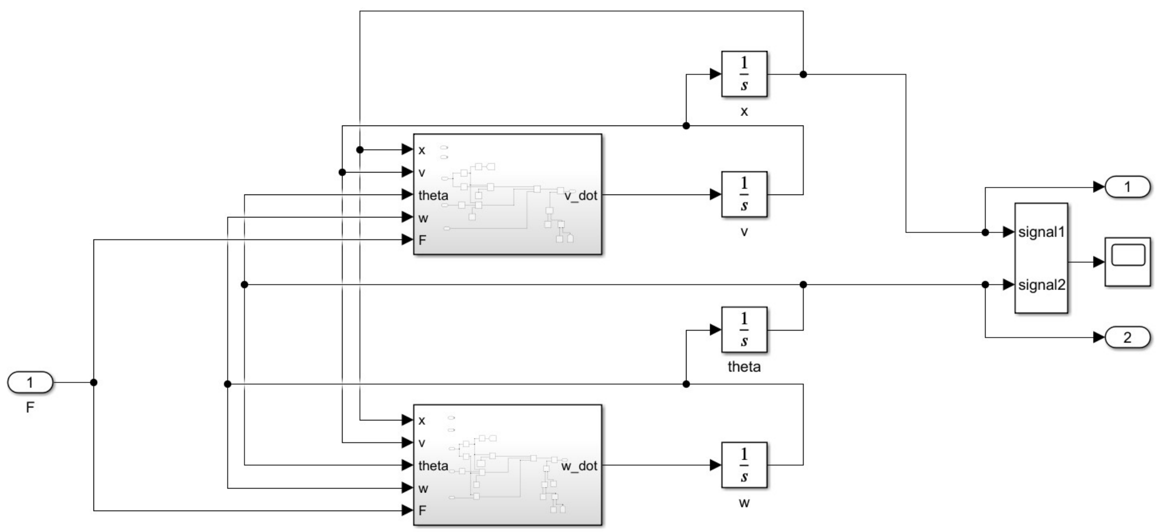
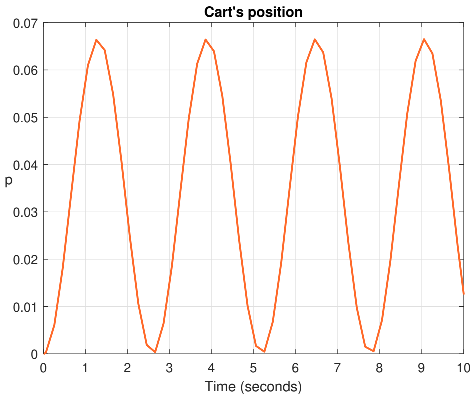
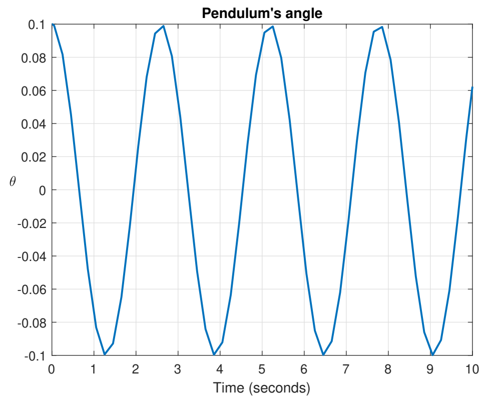


Figure 3: Simulink Scheme of non-linear model.

For example, figures 4a and 4b show how system evolves for still cart condition and θ initially set to $0.1[rad]$.



(a) Cart's position evolution with zero input and $\theta = 0.1[rad]$.



(b) Pendulum's angle evolution with zero input and $\theta = 0.1[rad]$.

Figure 4

3 Analysis

In this section we'll study cart-pendulum system behavior; in particular, we focus on upright pendulum position and which mathematical terms characterize the evolution near this state.

3.1 Equilibrium points

Our analysis starts checking equilibrium points, or states, for the considered system; assuming a zero input signal $u(t) = F = 0$, equilibrium states are those which "stop" the evolution of the system: once an equilibrium x_e is reached by the system, x_e is mantained in time. Mathematically, x_e is a point in state space which, for TC systems, is solution of equation

$$\dot{x}(t) = f(x, u)\Big|_{x=x_e, u=0} = 0$$

To find equilibria for cart-pendulum system, we need to solve the homogeneus system

$$\begin{cases} \dot{p} = v = 0 \\ \dot{v} = \frac{mL\omega^2 \sin\theta + mg \sin\theta \cos\theta}{M + m(\sin\theta)^2} = 0 \\ \dot{\theta} = \omega = 0 \\ \dot{\omega} = -\frac{mL\omega^2 \sin\theta \cos\theta + (m+M)g \sin\theta}{M + m(\sin\theta)^2} = 0 \end{cases} \tag{11}$$

where F does not show up in equations because, as already said, the considered input is zero.

First and third equations from system (11) constraint two components of solution vector

$$v = 0, \quad \omega = 0$$

and, substituting $\omega = 0$ in the other two equations, we obtain

$$\begin{cases} mg \sin\theta \cos\theta = 0 \\ (m + M)g \sin\theta = 0 \end{cases}$$

Both equations are verified for

$$\sin\theta = 0 \implies \theta = n \cdot \pi, \quad n \in \mathbb{Z}$$

so cart-pendulum system has infinite equilibrium states which, in reality, are just two: pendulum down and pendulum up. Notice also that:

- Despite pendulum down or up position, equation $v = 0$ implies a still condition on cart subsystem;
- Equilibrium states are indipendent from cart's position; in other words, position component p in equilibrium states is a free variable.

3.2 Linearize around upright pendulum equilibrium state

Since the aim of this project is to stabilize the pendulum in upright position, equilibrium state

$$x_e = [p, 0, \pi, 0]^T$$

is the one we are interested in; in particular, we would like to study cart-pendulum system when its states approches x_e.

Linearization is a powerful tool we use to apply linear system theory to non-linear ones. The key idea is that a non-linear system, if considered when its state is in a neighboord centered in a nominal state $\bar{x}$, can be analysed as if it would be a linear system; anyway, performed analysis are valid only for states near the nominal one.

The result of linearization process is a linear model of the system, so $\dot{x} = Ax + Bu$, where A and B are, respectively, the Jacobians of vector field $f(x, u)$ with respect to state x and input u

$$A = \left(\frac{\partial f}{\partial x}\right)\Big|_{x=\bar{x},\, u=\bar{u}} \qquad B = \left(\frac{\partial f}{\partial u}\right)\Big|_{x=\bar{x},\, u=\bar{u}}$$

evaluated for nominal state $\bar{x}$ and nominal input $\bar{u}$.

Instead, outputs equation of linearized cart-pendulum system remain unchanged because they are already linear. Matlab function *linmod* can be used to obtain a linearized model starting from Simulink scheme showed in figure 3. The model provided by *linmod* function is

$$\begin{bmatrix} \dot{\tilde{p}} \\ \dot{\tilde{v}} \\ \dot{\tilde{\theta}} \\ \dot{\tilde{\omega}} \end{bmatrix} = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.9620 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 5.8860 & 0 \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \tilde{v} \\ \tilde{\theta} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2000 \\ 0 \\ 0.1000 \end{bmatrix} F \tag{12}$$

Linear state equation (12) can be used to study cart-pendulum system for small variations centered in equilibrium state $x_e = [0, 0, \pi, 0]^T$. Note that linearized and non-linear states are different: the state of the former captures just the small variations of the latter.

3.2.1 Differences between linearized and non-linear model

Despite linearization provides a linear approximation of the system near a nominal state, the observed evolution for the two models is not always so ”similar”.

In our case, linearizing around equilibrium point $x_e = [0, 0, \pi, 0]^T$, it is possible to observe that linearized and non-linear model evolve differently. Consider, for the non-linear system, an initial condition $x_0 = [0, 0, \pi - \delta, 0]^T$ where $\delta = 0.15$ is a perturbation; the corresponding initial condition for linearized model is $\tilde{x_0} = x_0 - x_e = [0, 0, -\delta, 0]^T$. Figure 5 highlight that:

1. Linearized model outputs diverge to infinity,
2. In non-linear model, the pendulum keeps oscillating in the range $[\pi - \delta, -(\pi - \delta)]\,[rad]$.

To provide a more complete picture of which are limits of linearization, figure 6 show differences between linearized and non-linear models considering pendulum down equilibrium.

3.3 Stability of upright pendulum equilibrium state

Thanks to linearized model (12), now we can use Linear System Theory to analyse in deeper details the equilibrium of interest.

Stability of equilibrium state is the first property we discuss. An equilibrium state may be stable or unstable. The difference is how the system reacts to perturbations of internal state:

- If equilibrium is **stable** then the system replies to state perturbation mantaining a bounded difference with respect to equilibrium state; moreover, if equilibrium is attractive the system we’ll reach equilibrium state as time passes;

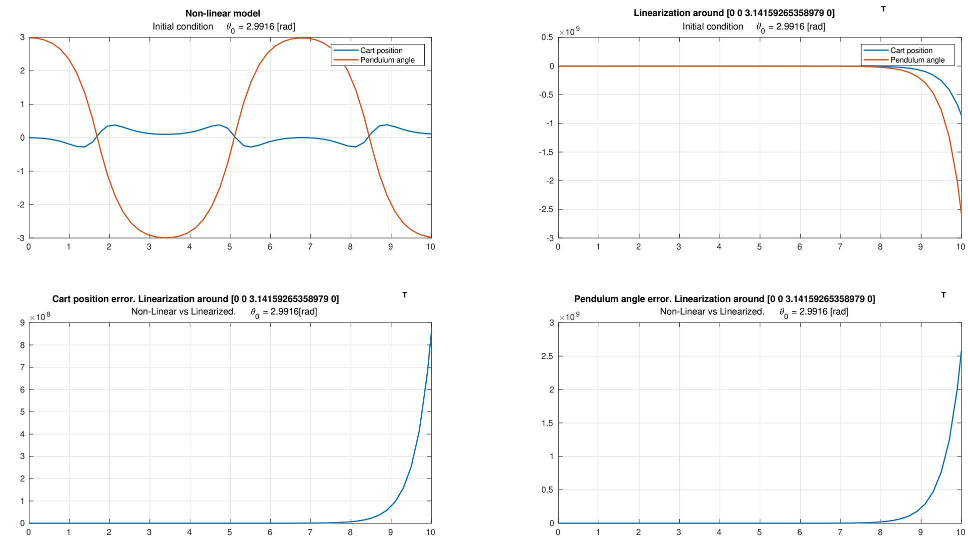


Figure 5: Above: evolution of non-linear and linearized models starting from perturbed pendulum up state. Below: error between non-linear and linearized model.

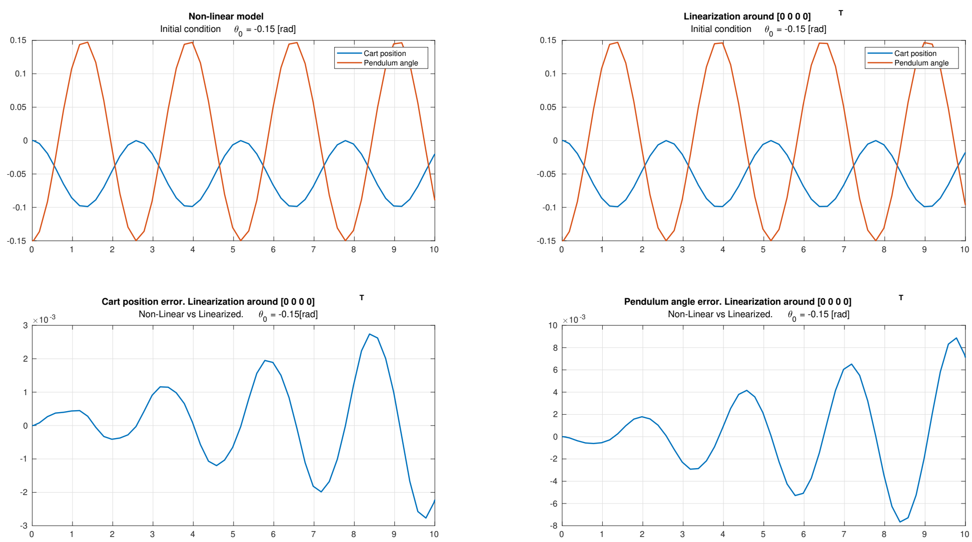


Figure 6: Above: evolution of non-linear and linearized models starting from perturbed pendulum down state. Below: error between non-linear and linearized model.

- If equilibrium is **unstable** the system replies to state perturbation deviating, not boundedly, from equilibrium state.

Since we already know the linearized model, we can use *Reduced Lyapunov Criterion* to study the stability of $x_e = [0, 0, \pi, 0]^T$. This criterion is based on the sign of real part of eigenvalues of A, where $A = \frac{\partial f}{\partial x}\Big|_{x=x_e, u=0}$. Matlab helps us to compute the eigenvalues: in fact, A is available thanks to *linmod* method; Matlab function *eig(A)* returns eigenvalues of matrix A, which in our case are

$$\lambda_1 = \lambda_2 = 0 \quad \lambda_3 = 2.4261 \quad \lambda_4 = -2.4261$$

Lyapunov's criterion states that *if matrix A has **at least** an unstable eigenvalue, i.e. with positive real part for TC systems, then equilibrium state studied is **unstable***. Applying this criterion, we conclude equilibrium state $x_e = [0, 0, \pi, 0]^T$ is unstable for cart-pendulum system.

3.4 Free response analysis

Once we found a linear approximation for our system, it is worth to understand of which terms its state and output signals are made up of.

Free response analysis concerns on how the system behaves when it is let free to evolve starting from an initial state x_0, without being driven by any input (i.e. $u(t) = 0$). Starting point for this analysis is the *Global Input-State-Output model* of the system; thanks to Lagrange equation, we move from local to global I/S/O model of the linearized system

$$\begin{cases} \dot{x}(t) = A \cdot x(t) \\ y(t) = CA \cdot x(t) \end{cases} \rightarrow \begin{cases} x(t) = e^{At} \cdot x_0 \\ y(t) = Ce^{At} \cdot x_0 \end{cases} \tag{13}$$

finally finding a closed solution to know state and output of the system at a given time.

Despite the advantage to have a closed solution for state and output equations, this solution has not a clear geometrical interpretation. *Modal Decomposition* highlights how free response is achieved.

3.4.1 Modal decomposition

Core concept of modal decomposition is the change of base to represent state space. In particular, state $x(t)$ expressed by the canonical base is equivalent to a state $r(t)$ expressed in a different base; between $x(t)$ and $r(t)$ exists a linear relationship

$$x(t) = T \cdot r(t) \qquad T^{-1} \cdot x(t) = r(t)$$

where $T \in \mathbb{R}^{n \times n}$ is an invertible linear transformation; column vectors of T build up a base for state space, and $r(t)$ expresses system state with respect to this base.

Applying this transformation to global model (13) we obtain

$$\begin{cases} T \cdot r(t) = e^{At} T \cdot r_0 \\ y(t) = Ce^{At} T \cdot r_0 \end{cases} \rightarrow \begin{cases} r(t) = T^{-1} e^{At} T \cdot r_0 = e^{T^{-1} A T t} \cdot r_0 \\ y(t) = Ce^{At} T \cdot r_0 \end{cases} \tag{14}$$

which is a linear system *similar* to the starting one.

Choosing the right T matrix we can shift the system to modal coordinates, putting in evidence the *modes* of the system. Usually, if A is non-defective, matrix T is obtained collecting right eigenvectors of A. Matrix A found for cart-pendulum system linearized around pendulum upright position is defective, so we use Jordan decomposition to obtain a quasi-diagonal similar matrix.

It is possible to find a Jordan decomposition $AT = TJ$ calling Matlab's method *jordan(A)*; matrices T and J returned by the method are

$$T = \begin{bmatrix} -0.3333 & 0 & -0.0687 & 0.0687 \\ 0 & -0.3333 & 0.1667 & 0.1667 \\ 0 & 0 & -0.2061 & 0.2061 \\ 0 & 0 & 0.5000 & 0.5000 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2.4261 & 0 \\ 0 & 0 & 0 & 2.4261 \end{bmatrix} \tag{15}$$

and so, state equation in model (14) becomes

$$r(t) = e^{Jt} \cdot r_0, \quad r_0 = T^{-1} \cdot x_0$$

Solving exponential matrix e^{Jt}, starting from an initial condition x_0, we obtain the evolution for the system expressed in modal coordinates

$$r(t) = e^{Jt} \cdot r_0 = \begin{bmatrix} r_{01} + r_{02}t \\ r_{02} \\ r_{03}e^{-2.4261 \cdot t} \\ r_{04}e^{2.4261 \cdot t} \end{bmatrix} \tag{16}$$

and, at the end, moving back from $r(t)$ to $x(t)$ modal decomposition of the system is discovered.

$$x(t) = T \cdot r(t) = \begin{bmatrix} -0.3333 & 0 & -0.0687 & 0.0687 \\ 0 & -0.3333 & 0.1667 & 0.1667 \\ 0 & 0 & -0.2061 & 0.2061 \\ 0 & 0 & 0.5000 & 0.5000 \end{bmatrix} \cdot \begin{bmatrix} r_{01} + r_{02}t \\ r_{02} \\ r_{03}e^{-2.4261 \cdot t} \\ r_{04}e^{2.4261 \cdot t} \end{bmatrix} \tag{17}$$

$$= (r_{01} + r_{02}t) \cdot \begin{bmatrix} -0.3333 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r_{02} \cdot \begin{bmatrix} 0 \\ -0.3333 \\ 0 \\ 0 \end{bmatrix} + r_{03}e^{-2.4261 \cdot t} \cdot \begin{bmatrix} -0.0687 \\ 0.1667 \\ -0.2061 \\ 0.5000 \end{bmatrix} + r_{04}e^{2.4261 \cdot t} \cdot \begin{bmatrix} 0.0687 \\ 0.1667 \\ 0.2061 \\ 0.5000 \end{bmatrix}. \tag{18}$$

Each element $\alpha_i e^{\beta_i \cdot t} \cdot v_i$ of sum (18) is a *mode* of the system. A *mode* is defined by two parts: a vector v_i, which specifies a direction in state space; a weight $\alpha_i e^{\beta_i \cdot t}$, which determines how increases or decreases in time the contribute of vector v_i.

3.4.2 Modes of linearized model

Linearized cart-pendulum system is characterized by:

- One *convergent* mode, which corresponds with $e^{-2.4261 \cdot t}$ term;
- Two *bounded* modes, which act on the first two component of the state through weights r_{01} and r_{02};
- One *polynomial divergent* mode, since there's a term proportional to time t influencing the first component of the state;

- One *exponential divergent* mode, due to $e^{2.4261 \cdot t}$ term.

Moreover, divergent mode affects all state components of the linearized system since the associated vector has all non-zero component. This is consistent with instability analysis provided by Reduced Lyapunov's Criterion for upright pendulum state.

4 Structural Properties

Structural Properties characterize a dynamical system and, in details, highlight how we can interact with the physical process through input and output signals.

These properties can be classified in two groups:

- One which investigates how input signal can affect the state of the process;
- The other analyses what can be learned about the state of the process knowing input and output signals.

4.1 Controllability and Reachability

Controllability and Reachability are two structural properties strictly related; they are *input-state properties*, which means they state some relationships between input and state signals.

Reachability answers to the question: *for which final states $x(t_f)$ there exists an input signal $u(t)$ such that the system reaches $x(t_f)$ starting from an initial condition $x(0)$?* For an LTI system, such as the linearized cart-pendulum model, it is equivalent to ask whether or not a certain state is reachable starting from the zero state.

For continuous time systems, the answer is provided by the *reachability matrix*. Assuming initial condion $x(0) = 0_x$, state evolution for an LTI-TC system can be decomposed as follows

$$x(t) = R \cdot \alpha(t) = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \cdot \begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_{mn}(t) \end{bmatrix} \tag{19}$$

where R is known as *reachability matrix*. Linear transformation (19) emphasises that state $x(t)$ belongs to the subspace generated by R, which is the image set $Im(R)$ also defined as *set of reachable states*.

Controllability is slightly different; it replies to the question: *which initial contitions $x(0)$ can be driven, through a certain input signal $u(t)$, to a final state $x(t_f)$?* Usually, for LTI systems, final state $x(t_f)$ is supposed to be 0_x.

Controllability and reachability properties coincide for LTI systems: the sets of reachable and controllable states are equivalent, and both are equal to $Im(R)$; in this case, full controllability, or full reachability, are checked looking ad the rank of matrix R. If R is a full-rank matrix, then its column space coincides with the whole state-space.

Cart-pendulum system reachability has been verified using Matlab's funcion *rank* and *crtb*. The result is that cart-pendulum system has full reachability (controllability).

4.2 Observability and Reconstructability

Observability and Reconstructability are dual properties with respect to the previous two; they provide information about *how many components of the state can be estimated given input signals $u(t)$ provided to a system and the corrisponding outputs $y(t)$*.

In details, **observability** concerns about estimating initial condition of a system knowing output history; on the other hand, **reconstructability** analyses the possibility to compute a state $x(t)$ starting from input signals $u(t)$, outputs $y(t)$ and initial condition provided by observability.

For LTI systems these two properties coincide; they can be checked computing the rank, or null-space dimension, of the *observability matrix* which is

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

and can be computed in Matlab calling function *obsv*. Cart-pendulum model has a full-rank observability matrix, which means the system is full observable and reconstructable.

5 Synthesis

Last and crucial phase of this project is the *synthesis* of a compensator capable of mantaining pendulum subsystem in upright position while the cart is moving.

Synthesis process is divived in three steps:

1. First, a *state feedback controller* is developed in order to stabilize upright-pendulum equilibrium. The linearized model (12) is the plant considered to design the controller; here we assume the whole state vector is always available to update control signal.

2. Then, always based on model (12), an *observer* is designed to estimate system's internal state.

3. Finally, controller and observer are grouped to obtain a *dynamic compensator* and a reference signal is introduced to specify the desired final position of the cart.

For each steps, simulations are performed for both linearized and non-linear system.

5.1 State feedback controller

Our first objective is to stabilize upright-pendulum equilibrium. Assume to know, for every instant t, the whole internal state x of the system; we can use this information to generate a proportional control force u able to drive the system back to the desired equilibrium state.

Let $u = Kx$ be our proportial control action, where $K \in \mathbb{R}^{m \times n}$ is a static matrix of gains. For the moment, consider this feedback action as the only contribute to control signal. Defined u, the closed-loop model of linearized system (12) becomes

$$\begin{cases} \dot{x} = Ax + Bu = (A + BK) \cdot x \\ y = Cx + Du = (C + DK) \cdot x \end{cases}$$

Matrix $(A + BK)$ models the dynamic of closed-loop system: if we would be able to find a gain-matrix K such that $(A + BK)$ has only stable eigenvalues, then we will stabilize the linearized model.

Structural properties analysis proved full controllability for cart-pendulum system, or equivalently that the couple (A, B) is reachable. Because of this property, *poles allocation theorem* ensures that the spectrum of $(A + BK)$ can be arbitrarily assigned. In particular, we computed through Matlab's function *acker* a gain matrix K such that feedback system eigenvalues are $-1, -3.5, -2, -2.4261$; here's K

$$K = \begin{bmatrix} -17.3116 & -38.0492 & 376.1797 & 165.3594 \end{bmatrix}$$

and the produced outputs are showed in figure 7.

Moreover, we could use the same control law for non-linear model of the system. Few considerations should be done: the linearized model is able to reproduce the behavior of non-linear model for small perturbation around pendulum-up equilibrium, so states as $[0, 0, \pi+\delta, 0]^T$; anyway, linearized dynamic referes just to perturbation term $[0, 0, \delta, 0]^T$.

This means that, to use the same controller consistently with the non-linear plant we should feed in the difference $x - x_e$, where x is the internal state of non-linear process and x_e is upright-pendulum equilibrium state. See figure 8 to appreciate performances of state feedback for non-linear model.

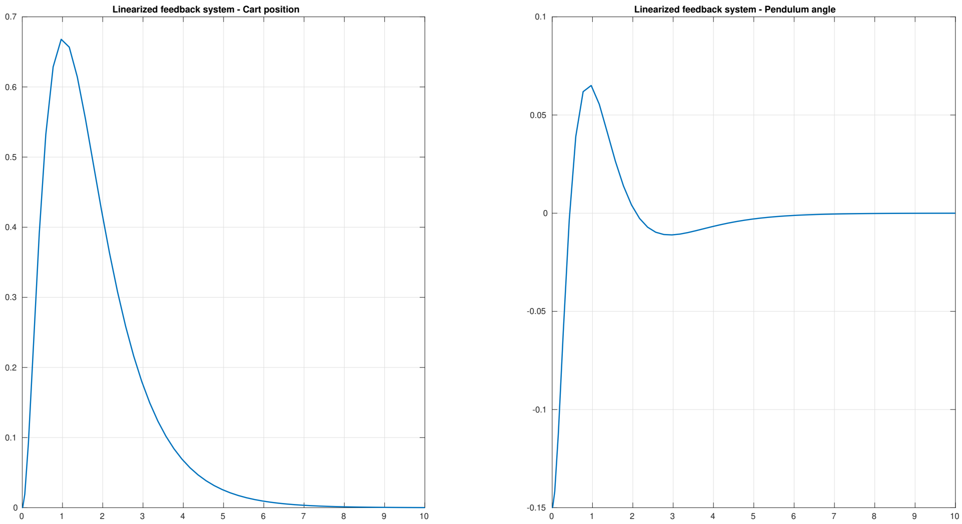


Figure 7: Linearized system performance including state feedback

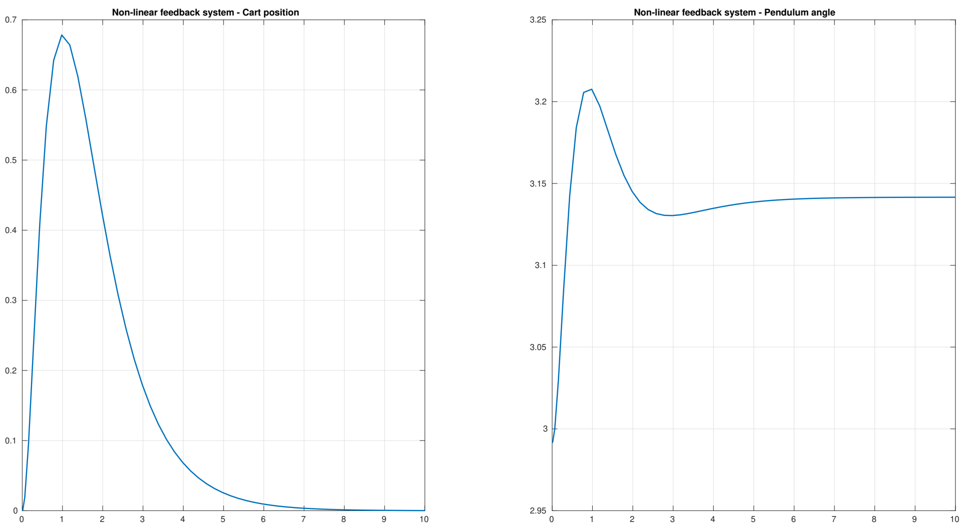


Figure 8: Non-linear system performance including state feedback

5.2 Asymptotical state observer

State feedback tecnique is based on the assumption that process internal state x is always known. Anyway, this does not happen so frequently. An *observer* system can be introduced to solve the issue.

An observer is a dynamical system which estimates the state of a process P knowing input and output signals characterizing P. Let's $\hat{x}$ be the estimated state. A Luenberger observer mimics the dynamic of process P, but it includes also a correction term; this quantity is proportional to the error between real output y and output computed through state estimation $\hat{y} = C\hat{x} + Du$.

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L \cdot (\hat{y} - y) \\ \hat{y} = C\hat{x} + Du \end{cases} \tag{20}$$

Remembering that cart-pendulum system is strictly causal, so $D = 0$, evolution of the error between real and estimate state is defined by the following law

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = \\ &= Ax + Bu - A\hat{x} - Bu - L(\hat{y} - y) = \\ &= A(x - \hat{x}) - LC(\hat{x} - x) = \\ &= (A + LC) \cdot (x - \hat{x}) = \\ &= (A + LC) \cdot e \end{aligned}$$

and so, it is sufficient to find a gain matrix L such that $(A + LC)$ has stable eigenvalues in order to let the error go asymptotically to zero.

From linear algebra we know that $spec(A + LC) = spec(A + LC)^T$, and the same is true also for the ranks. This means that, as already done for feedback controller, we can check if the couple (A^T, C^T) is reachable and solve pole allocation for $(A + LC)^T = (A^T + C^T L^T)$.

Structural properties analysis proved that observability matrix is full rank, so by dualty (A^T, C^T) is a full-reachable couple. Matlab's routine *place* is used to compute matrix gain L in order to place $(A + LC)$ poles in $-20, -24, -26, -28$. The obtained L is

$$L = \begin{bmatrix} 49.3632 & -3.0306 \\ 605.1886 & -74.8992 \\ -2.9269 & 48.6368 \\ -74.0654 & 592.6993 \end{bmatrix}$$

and performances of the observer for both linearized and non-linear model are highlighted in figures 9 and 10.

Note as, since the observer is based on a linearized plant of cart-pendulum system, it captures correctly the state of linearized model. On the other hand, we can not say the same thing for non-linear case: the dynamic of the observer does not fit exactly the non-linear system; it is just able to, at least, estimate the components of the state which coincide with the output of the system.

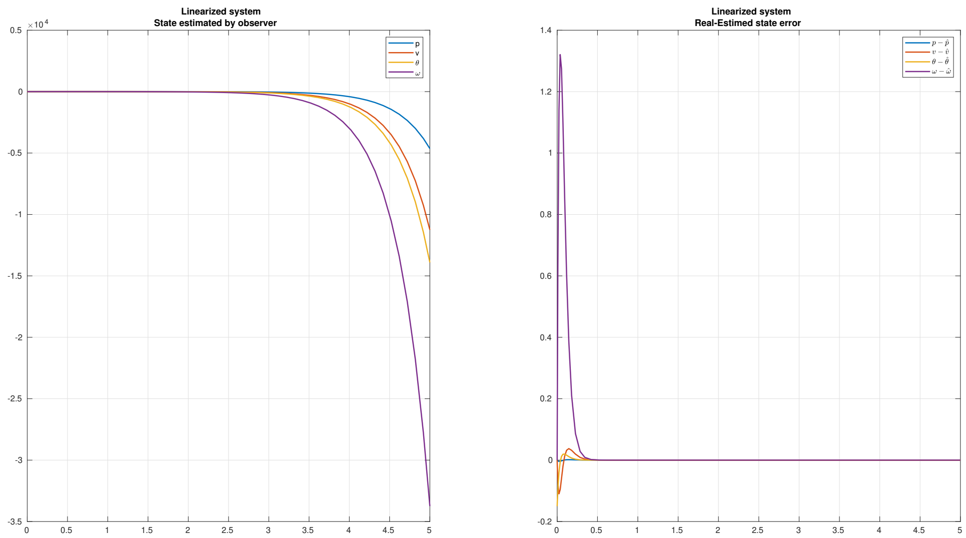


Figure 9: Observer performance estimating linearized model state

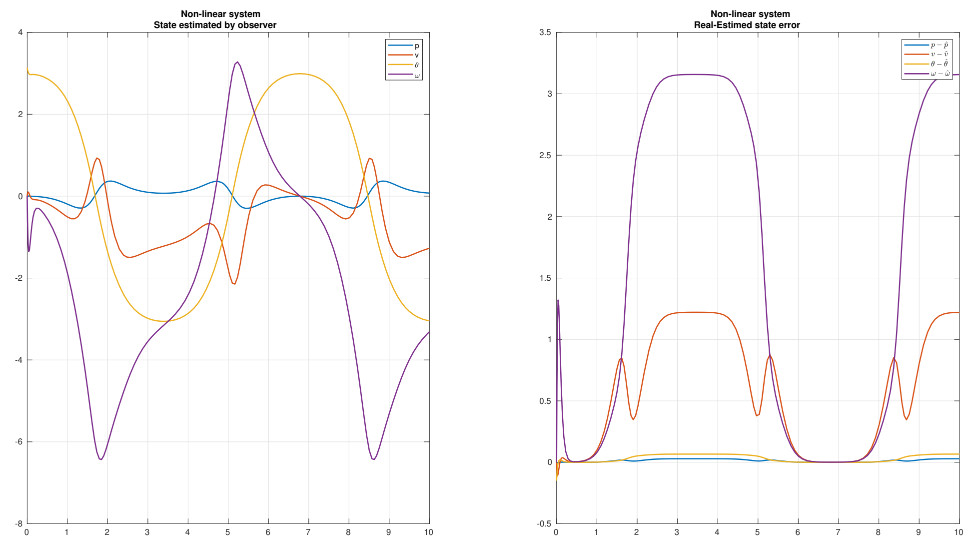


Figure 10: Observer performance estimating non-linear model state

5.3 Dynamic compensator

Finally, state feedback controller and observer can be combined in a system which acts as a compensator: it produces control signals to keep the system at the desired equilibrium point; those signals are computed through an estimated state.

Controller and observer already designed in previous sections are grouped in a compensator; performances achieved with linearized and non-linear system are quite similar as shown in figure 11.

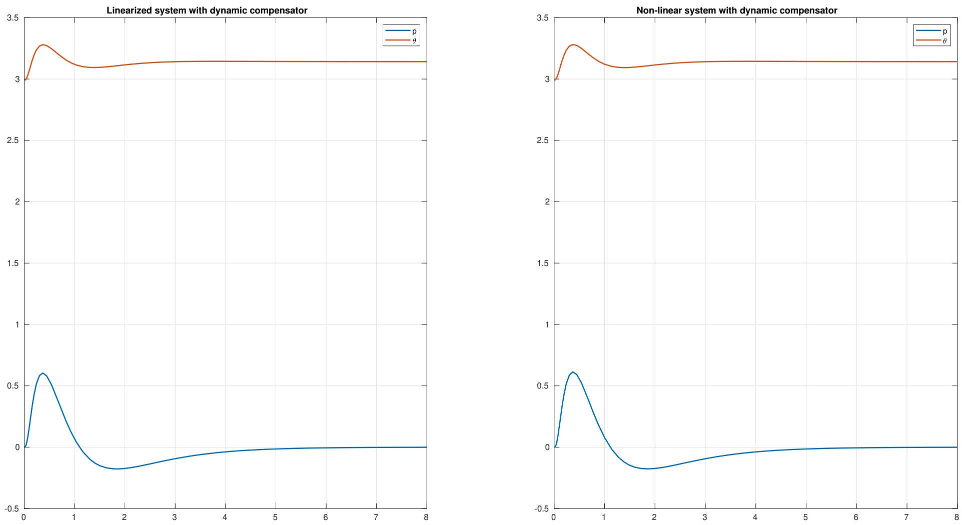


Figure 11: System performance with dynamic compensator

5.4 Regulation problem for cart position

Main goal of this project is to be able to change cart position keeping the pendulum upright. Now that pendulum stability has been solved, last thing to do is to model force F in the system showed in figure 2a as a reference signal for the output y_1.

This means that control signal is now expressed as sum of two terms $u = Kx + K_r r$, where:

- Kx is a feedback component, and it produces commands to mantain pendulum position;
- r is a reference signal, which represents the desired final position of the cart.

Regulation problem can be solved tuning gain K_r; the amplification produced by K_r has to normalize input-output gain characterizing the transfer function between $u(t)$ and $y_1(t)$.

5.4.1 Step response of linearized-closed-loop model

A brief forced response analysis of the system is useful to tune parameter K_r. Starting from linearized-open-loop model (12) we developed a state-feedback control law which we applied to cart-pendulum system; the new model, obtained introducing a loop in the system, is

$$\begin{cases} \dot{x} = (A + BK)x \\ y = (C + DK)x \end{cases}$$

which, in the end, changing also control signal in $u = Kx + K_r r$ becomes

$$\begin{cases} \dot{x} = (A + BK)x + BK_r r \\ y = (C + DK)x + DK_r r \end{cases} \tag{21}$$

In order to study the forced response, from state space model (21) it is possible to derive the *transfer function* of the system $W_{yu}(s)$; it is an $p \times m$ matrix of functions representing direct links between inputs and outputs of the system. Indipendent variable of transfer function is the generalized frequency $s \in \mathbb{C}$.

Analytically, transfer function can be derived from state space model (21) as

$$W_{yu}(s) = (C + DK)[sI - (A + BK)]^{-1}B + D$$

and thanks to Matlab's function *ss2tf* we know its expression, reported in equation (22).

$$W_{yu}(s) = \begin{bmatrix} \frac{0.2(s-2.215)(s+2.215)}{(s+3.5)(s+2.426)(s+2)(s+1)} \\ \\ \frac{0.1(s-2.506e-08)(s+2.506e-08)}{(s+3.5)(s+2.426)(s+2)(s+1)} \end{bmatrix} \tag{22}$$

An LTI system receiving in input a signal $ue^{\lambda t}$, assuming $e^{\lambda t}$ is not a mode characterizing the system, reproduces in output the same exponential but scaled of $W_{yu}(\lambda)$.
Reference r we want to plug into the system is a step, which can be modelled as $1 \cdot e^{0t}$. So, substituting $s = 0$ in $W_{yu}(s)$ we discover the scale factor of the system with respect to a step input signal.

$$W_{yu}(s)\Big|_{s=0} = \begin{bmatrix} -0.0578 \\ -3.6981 \cdot 10^{-18} \end{bmatrix}$$

5.4.2 Tuning K_r gain

For LTI systems, provided an input signal $ue^{\lambda t}$ which is not a mode of the system, the permanent component of output signal is

$$y(t)\Big|_{t\to\infty} = W_{yu}(\lambda)ue^{\lambda t}$$

and so, applying the reference $K_r r$, cart-pendulum system permanent output is

$$\begin{bmatrix} p(t) \\ \theta(t) \end{bmatrix}\Bigg|_{t\to\infty} = \begin{bmatrix} -0.0578 \\ -3.6981 \cdot 10^{-18} \end{bmatrix} K_r r$$

Choosing K_r such that $p(t)\Big|_{t\to\infty} = r$ we would solve the regulation problem. In particular, in our case

$$K_r = -\frac{1}{0.0578} \tag{23}$$

5.4.3 Performance

Parameter K_r found in previous section has been implemented as a gain for a step reference input signal to feed the non-linear model. Figure 12 shows results of a simulation starting from initial condition $x(0) = [0, 0, \pi - 0.15, 0]^T$ and characterized by $r(t) = step(t)$; $r(t)$ expresses position 1 as the target that cart has to achive.

As desired, the cart moved from position 0 to position 1 keeping the pendulum up.

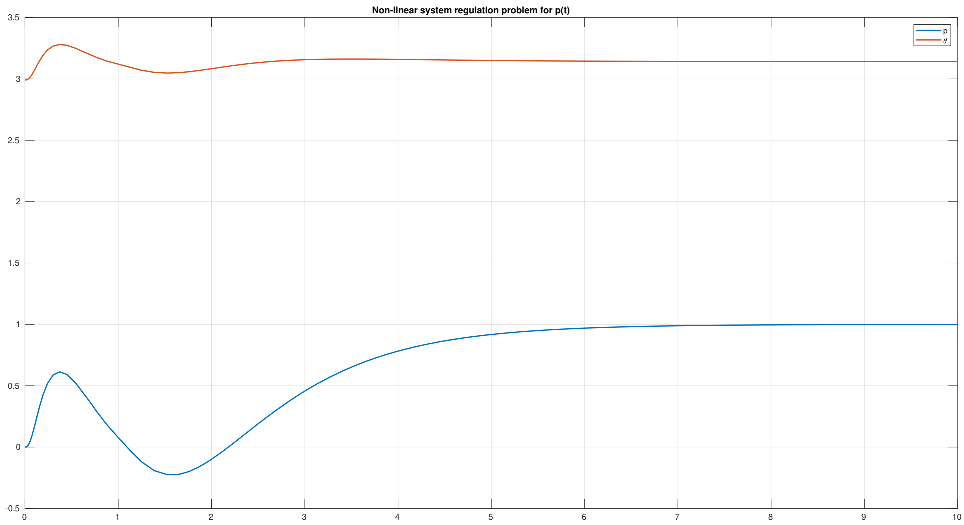


Figure 12: Simulation of regulation problem for cart-position output.