Robust Pricing Optimization Under Uncertain Demand Elasticity

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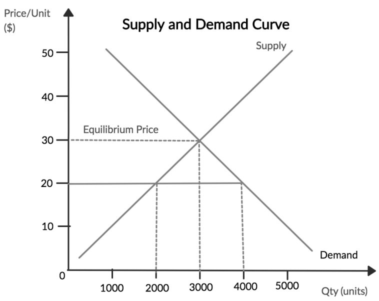
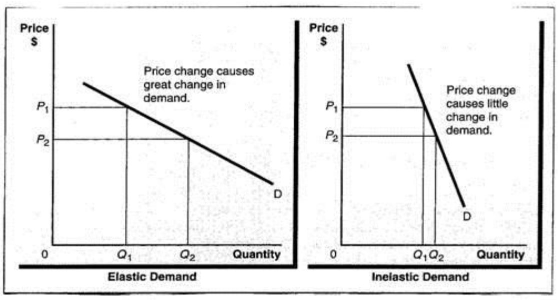
1 PROBLEM STATEMENT

In classical economics, the customer demand is modeled as a downward sloping line and the supply as an upward sloping line (Figure 1.1 a). While this is a simplistic model, it is highly telling of how consumers and firms behave and react to each other. For example, the slope of the demand curve is a representation of the consumer's price sensitivity. A very steep line means that a small change in price has drastic changes in quantity demanded by consumers (e.g. in perfect competition markets) and meanwhile a flat line with zero slope means that regardless of the price charged, the quantity demanded is fixed (e.g. in monopolistic markets) (Figure 1.1 b). Therefore, a critical part of profit maximization for firms is understanding their customers through the elasticity of demand for their products and setting prices accordingly.



(a) Supply and Demand Curves



(b) Elasticity of Demand

Figure 1.1: Economic Models

In this project, we study robust profit maximization in a highly competitive market - the taxi and ride share industry, where customer demand can rapidly change throughout the day. For example, if we take the perspective of Uber in New York, clearly people trying to take a ride to work in the morning absolutely need a ride and are more willing to pay high amounts. Meanwhile, people taking a leisurely trip to the mall in the afternoon or weekend can afford to consider other options (bus/train). On top of this, consumers can easily check prices of competitor services, such as Lyft and see if they are offering reduced prices or coupons. This represents an adversarial force on the demand for Uber's products and creates a setting where robust optimization can help us handle the worst-case uncertainty in demand sensitivity.

Beyond the ride-share industry, this notion of profit optimization under demand elasticity uncertainty can be applied to many different markets, especially ones that are competitive and face unstable consumer behavior in demand. These include retail products, the food delivery industry, and many other places.

2 Data

2.1 Parameter Generation

Our online search of ride share data unsurprisingly did not yield anything useful. We therefore decided to generate our own data. For this, we simulated the following parameters to create nominal supply and demand sets, where $Supply_t = s_t p_t$ and $Demand_t = \alpha_t - \beta_t * p_t$ where $t \in 1, ...24$ represents the hour of the day. When generating these figures, we based our numbers on our intuitions of busyness throughout the day. For instance, it takes a lot of motivation to get drivers out in the middle of the night, so supply elasticity s_t is low and customers who need a ride at 2 AM likely really need it, so they similarly have a low demand elasticity. Using this heuristic, we defined a "busyness" rating scale from [0,100] for each hour of the day, and randomly generated the supply and demand slopes accordingly for example $\beta_t \sim Norm(\frac{busyness_t}{100}, \frac{busyness_t}{1000})$.

Parameter	Definition	Dimension	Range
α_t	Inelastic demand	24	[7, 100]
β_t	Demand Slope	24	[0.10, 0.95]
s_t	Supply Slope	24	[0.05, 1.00]
c_t	Wage share for drivers[1]	24	[0.6,0.7]
δ_t	Driver sunk costs[2]	24	[$2,$4]

2.2 Uncertainty Sets

In our formulation, the uncertainty factor z is integrated in the demand such that $D_t = \alpha_t - (\beta_t + \sigma_t z) p_t$. For example, if our competitors offer coupons, we can expect our customers

[01] This is the proportion of the ride price that the drivers receive

[02] This is the amount we pay to each driver for being active on the ride share platform (i.e. part of the supply) but not actively giving rides

to become more price-sensitive (steeper demand). We use a few uncertainty sets here that work based on our intuition.

- Box (ρ): the box uncertainty set says that our uncertainty z lies uniformly in a box such that the worst case demand is handled.
- Budget (ρ, Γ): the budget uncertainty set is designed so that throughout the day, there are Γ hours where the demand can fluctuate away from its nominal value by ρ, representing for example uncertainty in the day hours but stability in the night.
- Polyhedron (D, d): the polyhedral uncertainty set is designed so that consecutive uncertainties from t to $t+1$ are chained together. This is intuitive because if we see that one hour is particularly busy, we expect the next one to be as well.

3 Main Results

In this section, we first provide the nominal and robust formulations and then show the results of various experiments. For this project, we focus exclusively on the case where supply is greater than demand (and hence, the quantity of rides is equal to the demand). This is an extremely realistic scenario - imagine if as a rider, you logged onto Uber at a dire time and needed a ride, but literally no drivers were present regardless of the price that you would have paid. This type of situation would undermine the reliability and reputation of the service and is highly undesirable.

Moving forward with the supply is greater than demand assumption, the nominal problem is defined for $t \in [0, 24]$ (each hour of the day):

$$\begin{aligned}
\underset{p_t}{\text{Max}} \quad & \sum_{1}^{T} D_t p_t - D_t c_t - \delta_t v_t \\
\text{subject to} \quad & v_t = S_t - D_t \\
& v_t \geq 0 \\
& D_t = \alpha_t - \beta_t p_t \\
& S_t = s_t p_t \\
& |p_{t+1} - p_t| \leq 10
\end{aligned}$$

In the objective, we pick price p_t to maximize profit as given by the quantity times price, minus quantity times cost (paid to drivers), minus the cost of idle drivers (we have to reimburse them for being ready to go when a request comes in). We also add a price constraint that we can't change the dollar charge by more than $10 between each hour of the day for realism. In the robust formulation of the problem, the tweak is changing the demand to:

$$D_t = \alpha_t - (\beta_t + \sigma_t z_t) p_t \quad \forall z \in Z$$

Under these formulations, the nominal problem is a quadratic maximization and the robust problem has an objective that is concave in p, z. To solve this, we rewrite our nominal optimization problem using the epigraph form, removing all uncertain variables from the objective function and allowing us to take the robust counterpart of the epigraph constraint. The derivation can be found in the appendix section 5.1.

Ultimately, our robust formulation is:

$$
\begin{aligned}
\underset{p_t,\tau,V_t}{\text{Max}} \quad & \tau \\
\text{subject to} \quad & \tau + \sum_1^T r\beta_t p_t^2 + \sum_1^T \delta_t \beta_t p_t - \sum_1^T p_t(r\alpha_t - \delta_t s_t) - \sum_1^T \delta_t \alpha_t + \delta^*(V|Z) \leq 0 \\
& r_t\theta_t p_t^2 + \delta_t \theta_t p_t \leq V_t \quad \forall t \in [T] \\
& (-s_t - \beta_t)p_t + \delta^*(-\theta_t p_t|Z) + \alpha_t \leq 0 \quad \forall t \in [T] \\
& |p_{t+1} - p_t| \leq 10 \quad \forall t \in [2:T]
\end{aligned}
$$

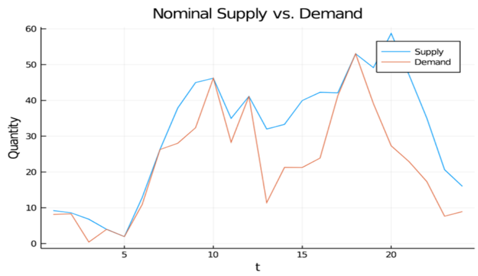
For each of the different uncertainty sets mentioned in Section 2.2, we find the terms $\delta^*(V|Z)$ and $\delta^*(-\theta_t p_t|Z)\forall t \in [T]$ and replace them in the robust formulation. The following table contains for each of the constraints their respective $\delta^*(P^T x|Z)$:

Uncertainty set	Z	$\delta^*(V\|Z)$	$\delta^*(-\theta_t p_t\|Z)$
Box	$\|\|z\|\|_\infty$	$\rho\|\|V\|\|_1$	$\rho p_t \theta_t$
Budget	$\|\|z\|\|_\infty \leq \rho$ $\|\|z\|\|_1 \leq \Gamma$	$\rho\|\|y\|\|_1 + \Gamma\|\|V-y\|\|_\infty$	$\rho\|y_t\| + \Gamma\|\theta_t p_t + y_t\|$
Polyhedral	$Dz <= d$	$\hat{a}^T x + d^T y \leq 0$ [3] $D^T y = V$ $y \geq 0$	$\hat{a}^T x + d^T y^{(t)} \leq 0$ $D^T y^{(t)} = p^T P^{(t)}$ [4] $y^{(t)} \geq 0$

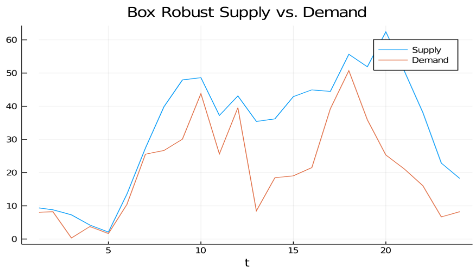
3.1 Nominal Versus Robust Problem and Solutions

Here, we visualize the difference between the nominal and robust problem. In the graphs of figure 3.1, the most noticeable difference is that for the nominal solution we have some periods where the supply term exactly meets the demand curve. Meanwhile, in the robust solution the supply "floats" above the demand at all periods.

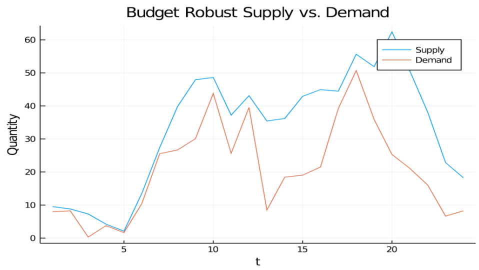
[03] The term $\bar{a}^T x$ just represents the nominal part of the constraint we are transforming.
[04] Each of the different $P^{(t)}$ is just a square matrix of zeros of size T, where the element (t,t) is $-\theta_t$.



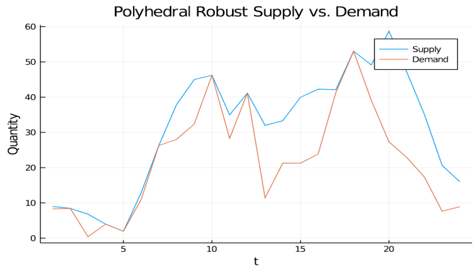
(a) Nominal



(b) Box Robust ($\rho = 1$)



(c) Budget Robust ($\rho = 1, \Gamma = 15$)



(d) Polyhedral Robust

Figure 3.1: Supply vs Demand based on different price policies

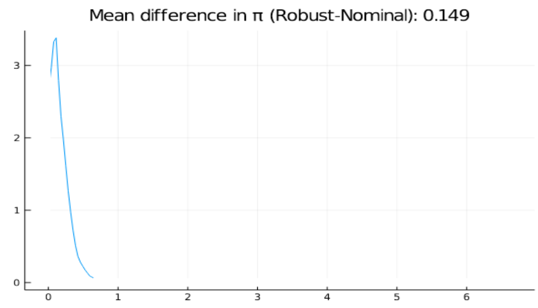
3.2 Profit Under Simulations

To set up the simulations simulations, we use simulate different realizations of demand elasticity throughout the day under

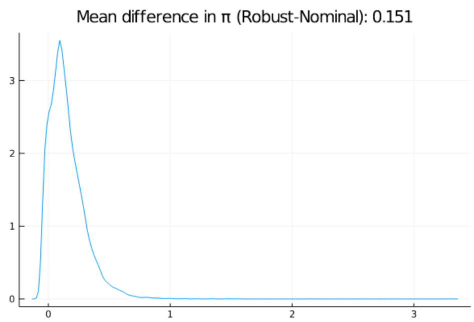
$$\beta_t \sim \hat{\beta}_t + \sigma_t z_t$$

To simulate Z, we try simulating $Z \sim N(0,1)$ which is the distribution we obtained β_t from. Importantly, we also put a penalty fee in the profit of \$250 for each violation of $Supply_t \geq Demand_t$. This is roughly equivalent to saying whenever there are more customers than drivers, we suffer a reputation damage worth about 5 rides of revenue because the dissatisfied customers will look for alternative transportation. Using the pricing policy from the nominal optimization violates this constraint quite often, and placing this penalty helps us compare the profit under the nominal and robust policies.

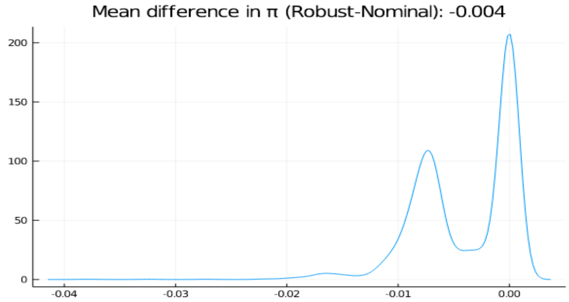
As we see in Figure 3.2, all the robust price policies on average generates more profit than the nominal policy. In the case of polyhedral uncertainty set, we do not observe an improvement over the nominal solution. This might be due to the D and d we used. However, the price policies based on the box and the budget uncertainty sets are seen to improve significantly the profit, when comparing with the nominal solution. For one side, the box based price policy, on average generates 14.9% more than the nominal. On the other side, the budget based price policy, on average generates a 15.1% more than the nominal.



(a) Box vs Nominal Profit



(b) Budget vs Nominal Profit ($\rho = 1$)



(c) Polyhedral vs Nominal Profit

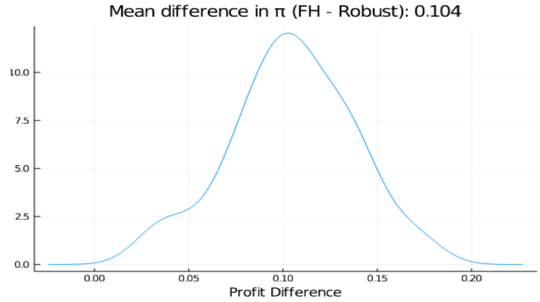
Figure 3.2: Simulated Profit: Nominal vs Robust

3.3 Folding Horizon Approach

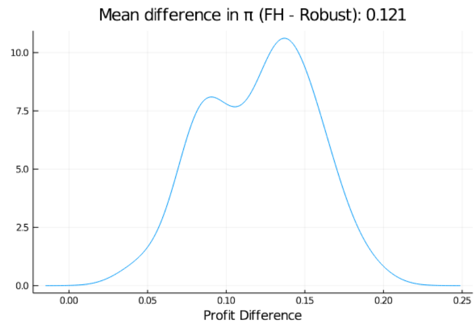
In the folding horizon approach, we first imagine a scenario where at current timestep t, we can perfectly predict the demand at $t+1$ and can optimize p_{t+1} as the here-and-now variable. At the same time, all previous timestep prices $(p_1, ..., p_t)$ are fixed and the remaining prices $(p_{t+2}, ..., p_{24})$ are still optimized under the robust setting. However, this is a non-trivial problem because we still have price continuity constraints where $|p_{t+1} - p_t| \leq 10$ across all timesteps, so we can't simply go for the fully greedy optimal solution to solve for p_{t+1}. Instead, we have to pick a price now while accounting for what prices we may want to set in the future given demand uncertainty.

Similar to the approach in the previous section, we can simulate the advantage of using the folding horizon method, where we have perfect foresight on next-step demand elasticity, versus static robust policy. We conduct these simulations with the Box and Budget uncertainty set. In Figure 3.3, we can see that the folding horizon approach in general overcomes the robust price policies in terms of profit. For the box uncertainty set, we observe an average of 10.4% increase in profit against the a static robust price policy. For the budget uncertainty set, we observe an average of 12.1% increase in profit when comparing to a static robust price policy. These 2 results demonstrate that the folding horizon approach can be useful to generate price policies that are more robust and yield a more stable profit.

From a practical view, predicting demands for the next 24 hours is extremely difficult, but it is feasible that with enough historical data and a high quality model, the demand in the



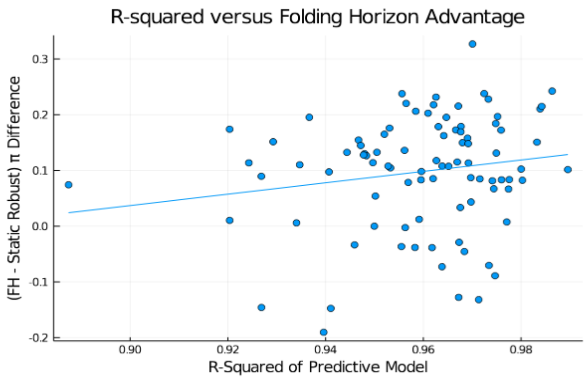
(a) Box Profit: Folding vs Robust



(b) Budget Profit: Folding vs Robust ($\rho = 1, \Gamma = 15$)

Figure 3.3: Simulation results of profit: Folding vs Robust

immediate next period could be predicted with high accuracy (e.g. feed the time, traffic conditions, weather conditions, event occurrences, competition activeness into a model trained to predict the next timestep's demand). Another experiment we modeled is to propose that our model for the next-step demand is imperfect and instead has Gaussian errors in its predictions of $\hat{\beta_{t+1}}$. In this case, we simulate having a model with a predictive power quantified by some $R^2 \in [0.9, 1.0]$ and then apply folding horizon on the predicted demand elasticity, instead of the ground truth [5]. The results are in figure 3.4. This figure shows that as one would expect, higher quality models (higher R^2) provide more benefit in using a folding horizon approach. This also has a sensitivity interpretation, as it seems that the performance of the folding horizon is quite sensitive as well - if we cannot produce a model that has very high R^2, perhaps we are better off not using folding horizon because it will produce negative benefits compared to the static robust policy.



(a) Box Profit vs Rsquared

Figure 3.4: Simulation results of profit: Folding vs Robust

[05] In this experiment, we changed the robustness parameters of the box set from $\rho = 1$ to $\rho = 4$ to better handle the simulated predictive errors

4 Future Work

First, when can relax the constraint that $S \geq D$. For one side, we need to change the way the objective function to include the number of rides as $Min(D_t, S_t)$. In this scenario, we will possibly be facing with non-convex constraints, quadratic terms and uncertainty combined, which will make the problem more challenging to solve as it could involve integer programming to separate the cases.

Second, we can also consider the supply elasticity s_t to be also impacted by uncertainty. In this case, we include to the formulation $s_t = \bar{s}_t + \theta_t^{(s)} z_t^{(s)}$. Another topic to explore is how the uncertainty in the estimations of s_t and of β_t are related to each other, as potentially drivers have some knowledge of how riders behave at certain times of day.

Third, we can also consider more than n products instead of only 1. In this context, we will have to work with correlated demands D_{it} for each product i at each time t. To do it, we need to include the following expression $D_{st} = \alpha_{st} - \beta_{st} p_{st} + \sum_{s' \neq s} \beta_{ss't} p_t$. Here, the term $\beta_{ss't}$ represents how the demand of the product s varies when the prices of the product s' varies.

5 Appendix

5.1 Derivation of Robust Counterpart

Our reformulated epigraph nominal problem is:

$$
\begin{aligned}
\underset{p_t, \tau}{\text{Max}} \quad & \tau \\
\text{subject to} \quad & \tau + \sum_1^T r\beta_t p_t^2 + \delta_t \beta_t p_t - p_t(r\alpha_t - \delta_t s_t) - \delta_t \alpha_t \leq 0 \\
& (-s_t - \beta_t) p_t + \alpha_t \leq 0 \quad \forall t \in [T] \\
& |p_{t+1} - p_t| \leq 10 \quad \forall t \in [2:T]
\end{aligned}
$$

Here we have a quadratic constraint on the decision variable p_t, so we applied the next transformation. Let matrix $A(z)$ and vector $b(z)$ be linear in the uncertain variables such that

$$
A = \bar{A} + \sum_{t=1}^T z_t A_t, \quad b = \bar{b} + \sum_{t=1}^T z_t b_t
$$

Since we do not assume any correlation structure for the uncertainties of prices, our A_t matrix is a diagonal entry where $A_{t,t} = \sigma_t$ the estimated standard deviation of the demand elasticity and A is zero everywhere else.

$$
x^T A x + b^T x_c \leq 0
$$

Equivalent to:

$$
\begin{aligned}
& x^T \bar{A}^T x + \bar{b}^T x + \delta^*(V|Z) + c \leq 0 \quad \forall z \in Z \\
& x^T A_t x + b_t^T x \leq V_t \quad \forall t
\end{aligned}
$$

With this robust formulation, we can pick an uncertainty set Z, find the appropriate support function $\delta^*(V|Z)$ and solve the robust problem as a quadratically constrained program.

6 Contributions

Armando formulated the robust optimization problem and found the robust counterparts of the different uncertainty sets, because we couldn't just use JuMPeR. He coded the functions to solve the different robust optimization problems.

Brian coded the data generation process, and formulated and coded the nominal problem. He coded the folding horizon approach and the pipeline to run the simulations to evaluate the different price policies.