Statistical Learning II

Lecture 11 - LASSO

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BSS: orthogonal covariates

Putting together, the solution of the BSS problem:

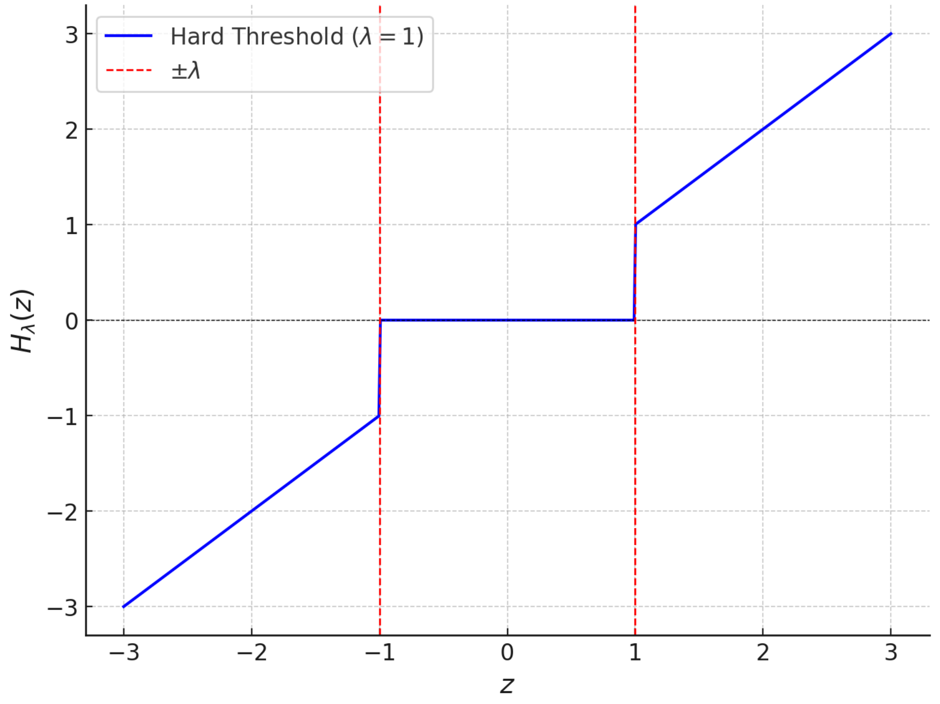
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^d} \frac{1}{2n} \sum_{i=1}^{n} \left(y_i - \langle \boldsymbol{\theta}, \boldsymbol{x}_i \rangle \right)^2 + \lambda ||\boldsymbol{\theta}||_0$$

Under the assumption of $\boldsymbol{X}^\top \boldsymbol{X} = \boldsymbol{I}_d$ is given by:

$$\hat{\boldsymbol{\theta}}_\lambda = H_{\sqrt{2n\lambda}}(\boldsymbol{X}^\top \boldsymbol{y})$$

Where:

$$H_\lambda(z) = \begin{cases} 0 & \text{if } |z| < \lambda \\ z & \text{otherwise} \end{cases}$$



BSS: orthogonal covariates

To understand better this solution, consider a linear model for the data:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\theta}_\star + \boldsymbol{\varepsilon}$$

With $\mathbb{E}[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\top] = \sigma\boldsymbol{I}_n$ and $\boldsymbol{\theta}_\star$ a k-sparse vector

$\mathbb{E}[\boldsymbol{\varepsilon}] = 0$

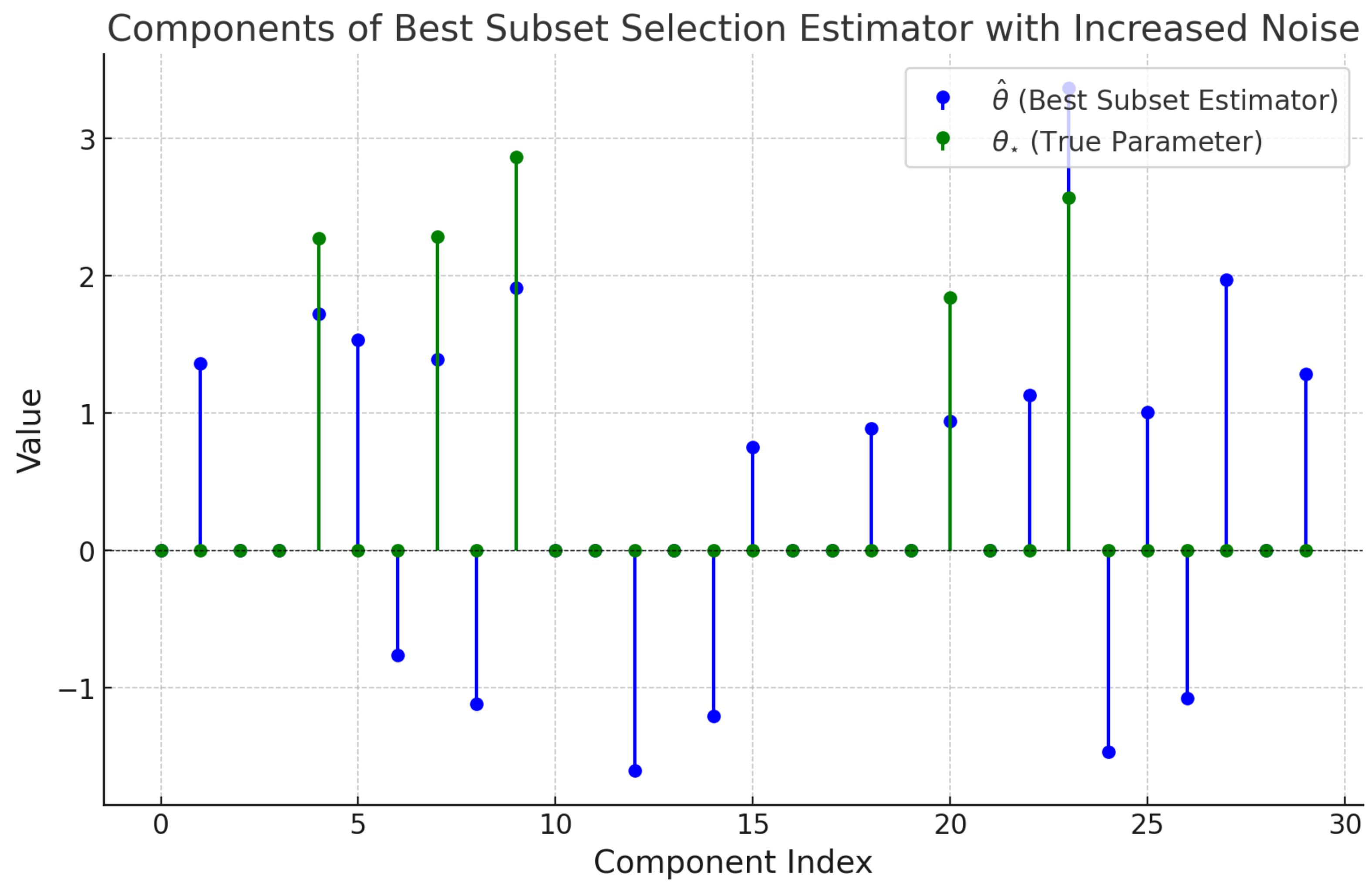
The, the solution is given by:

$$\hat{\boldsymbol{\theta}}_\lambda = H_{\sqrt{2n\lambda}}(\boldsymbol{\theta}_\star + \boldsymbol{X}^\top\boldsymbol{\varepsilon})$$

BSS: orthogonal covariates

Example: $n = 40$ $\lambda = 0.5$ $\boldsymbol{\theta}_\star$ 5-sparse

$d = 30$ $\sigma^2 = 1$ $||\boldsymbol{\theta}_\star||_2^2 = 5.35$



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Denoting:

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We can write:

$$\hat{\boldsymbol{\theta}}_S = \boldsymbol{X}_S^+ \boldsymbol{y}$$

In other words, BSS = OLS in the support!

The hard part is to find S as a function of $\boldsymbol{X}, \boldsymbol{y}, \lambda$...

Pitfalls of BSS

More generally, BSS is that it is a non-convex problem

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^d} \frac{1}{2n} \sum_{i=1}^{n} \left(y_i - \langle \boldsymbol{\theta}, \boldsymbol{x}_i \rangle \right)^2 + \lambda ||\boldsymbol{\theta}||_0$$

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💡 That's the key idea of the LASSO.

LASSO

The Least Absolute Shrinkage and Selection Operator (LASSO) is defined as the solution of the following problem:

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^d} \frac{1}{2n} \sum_{i=1}^{n} \left(y_i - \langle \boldsymbol{\theta}, \boldsymbol{x}_i \rangle \right)^2 + \lambda \, || \boldsymbol{\theta} ||_1$$

where $|| \cdot ||_1 : \mathbb{R}^d \to \mathbb{R}_+$ is the ℓ_1-norm:

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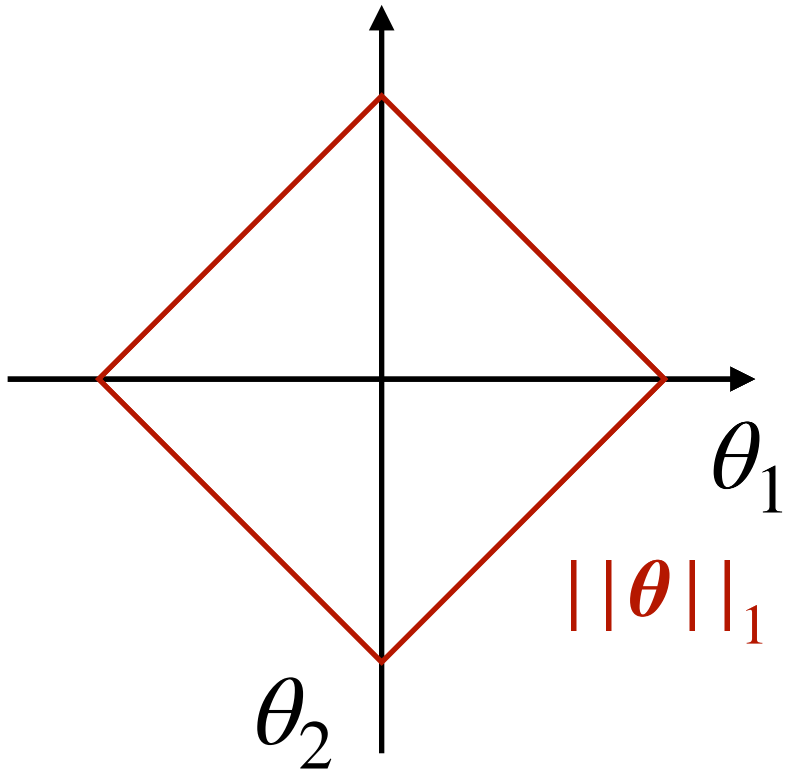
Note that both $||\cdot||_1$ and $||\cdot||_2$ are small for sparse vectors... why this is different?

LASSO: visualisation



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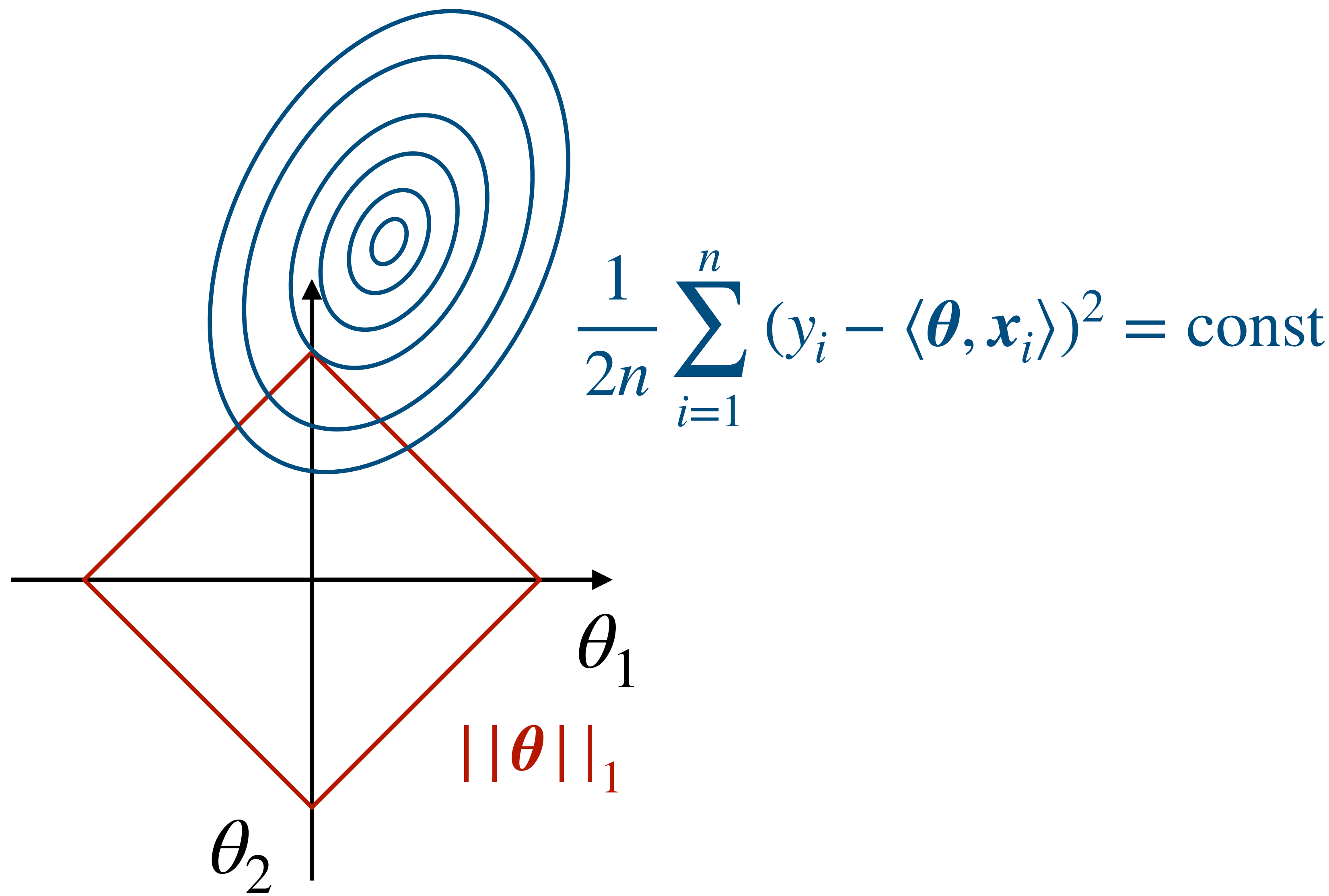

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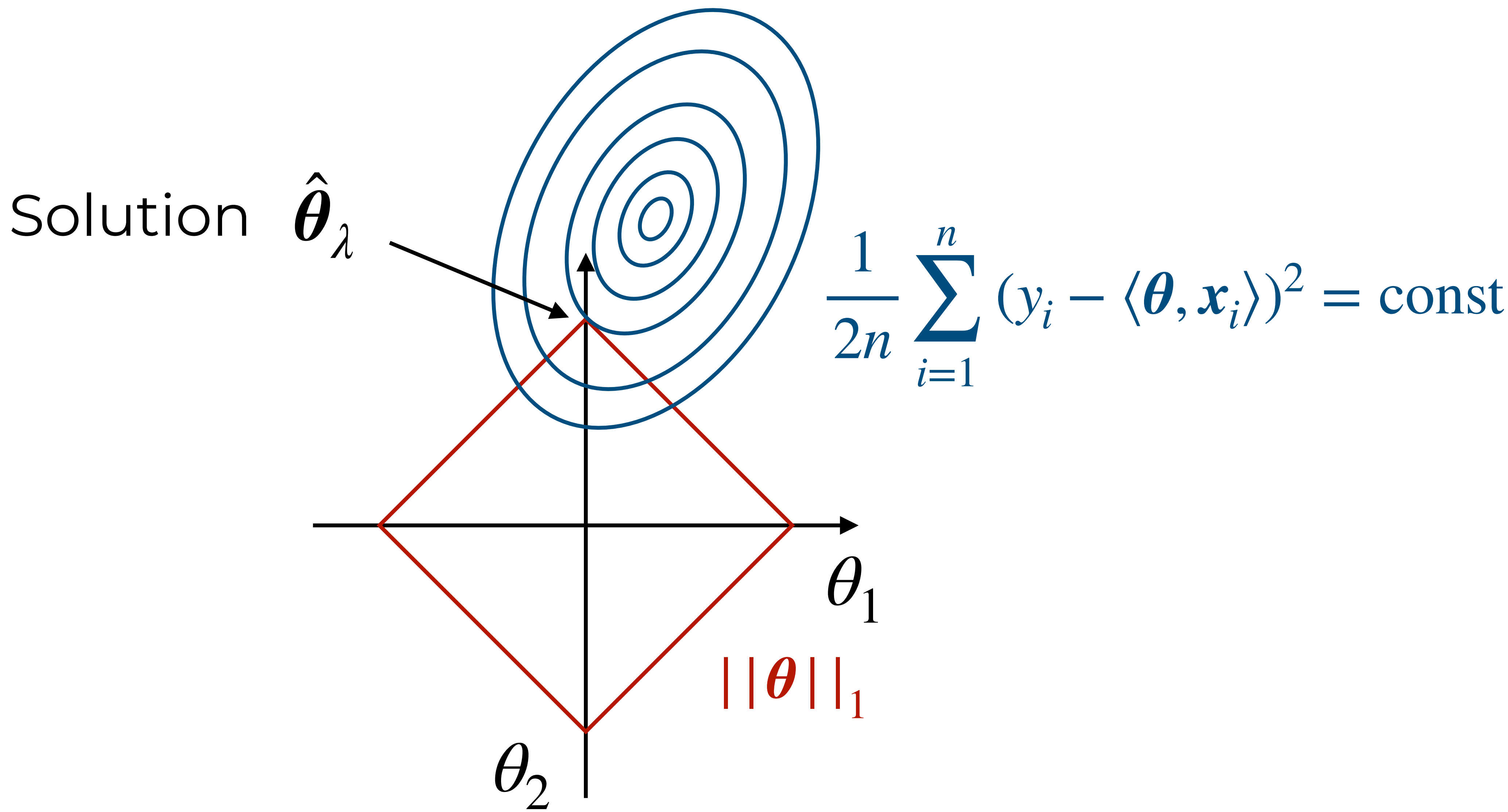

Sharper corners favours sparser solutions!

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Following exactly the same steps from before, in this case we need to solve the following coordinate wise problem:

$$\min_{\theta_j \in \mathbb{R}} L(\theta_j) := \left\{ \frac{1}{2n}(z_j - \theta_j)^2 + \lambda \, | \, \theta_j \, | \right\}$$

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As before, we note that: $$L(\theta_j) = \begin{cases} \frac{1}{2n}(z_j - \theta_j)^2 + \lambda\theta_j & \text{for } \theta_\mathrm{j} > 0 \quad \text{(a)} \\ \frac{z_j^2}{2n} & \text{for } \theta_\mathrm{j} = 0 \quad \text{(b)} \\ \frac{1}{2n}(z_j - \theta_j)^2 - \lambda\theta_j & \text{for } \theta_\mathrm{j} < 0 \quad \text{(c)} \end{cases}$$

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Putting together:
$$\theta_j = \begin{cases} z_j - \mathrm{sign}(z_j)n\lambda & \text{for } |z_j| > \lambda \\ 0 & \text{for } |z_j| \in [-\lambda, \lambda] \end{cases}$$

Soft-thresholding function

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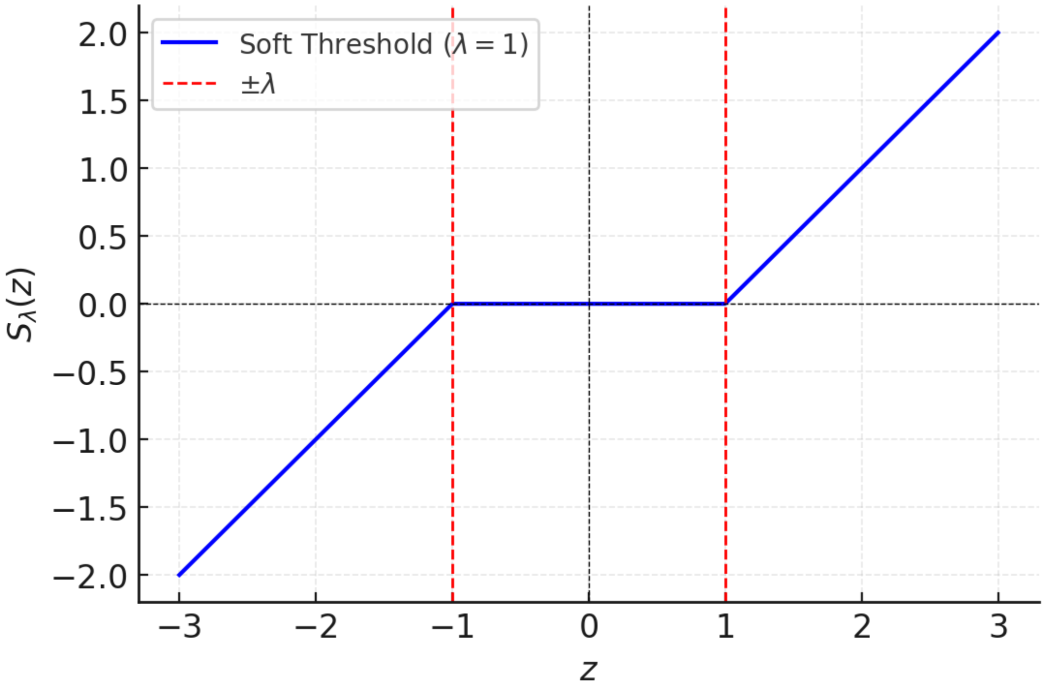
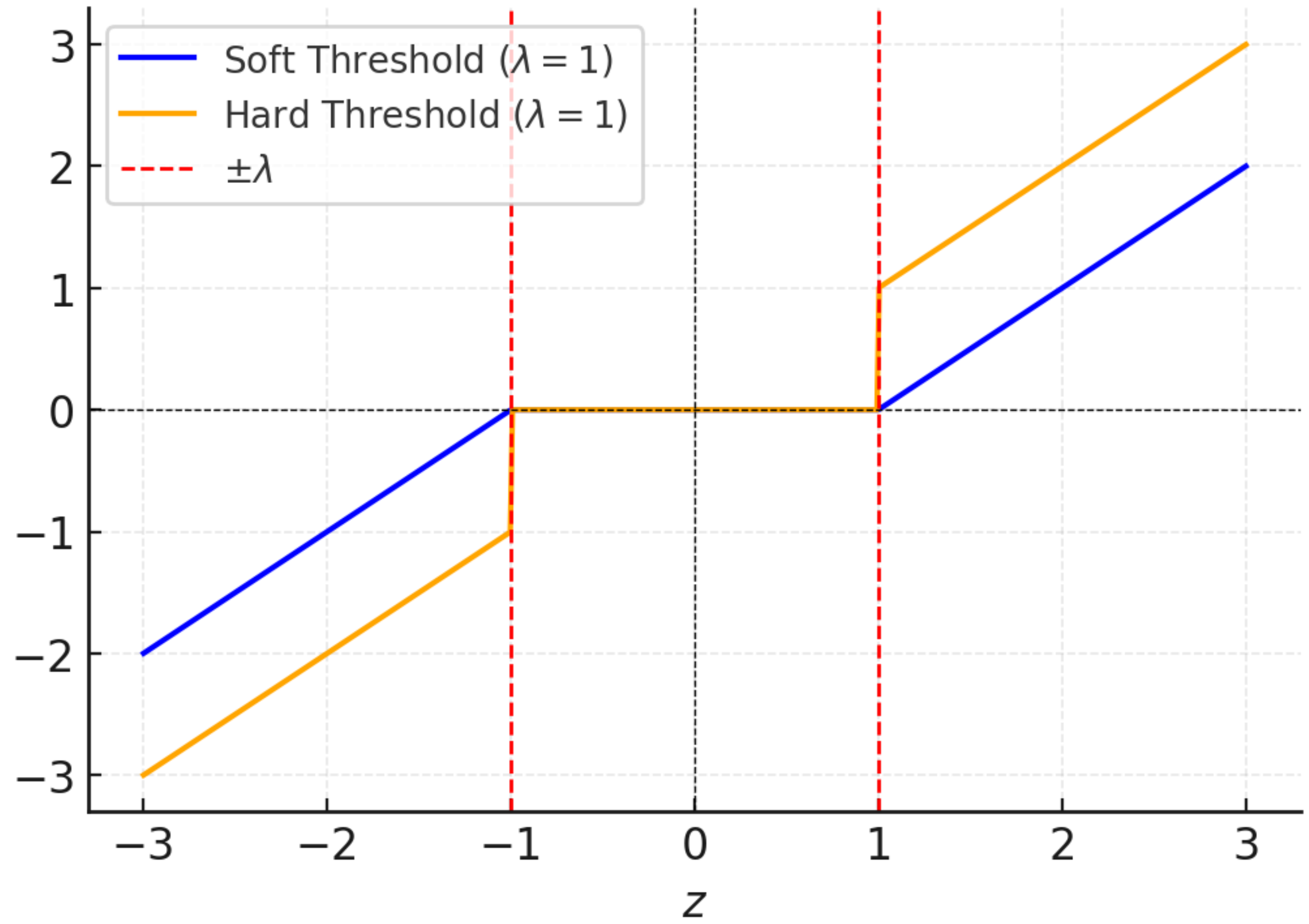


BSS vs. LASSO

It is instructive to compare the BSS and LASSO solutions in the orthogonal covariate case



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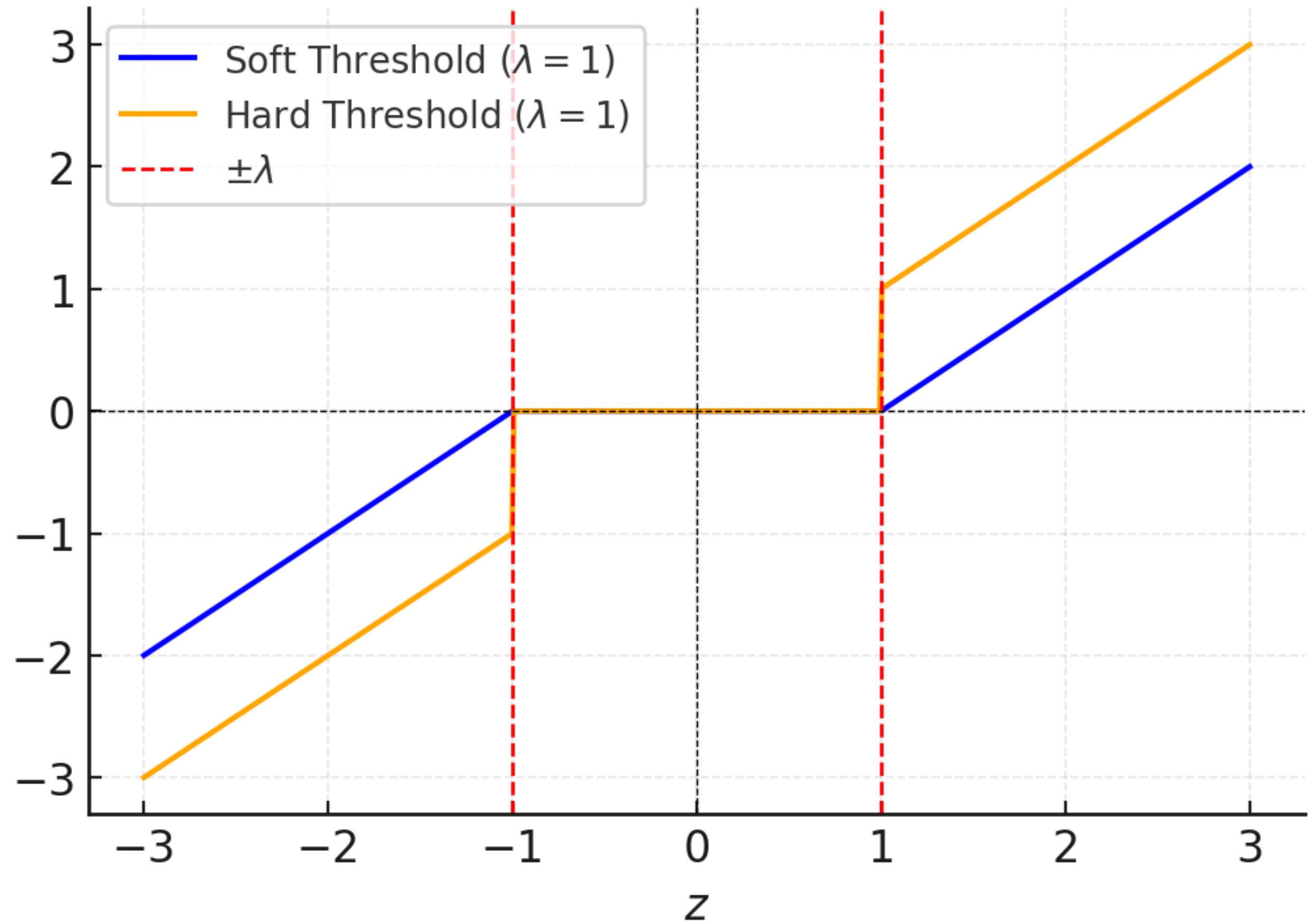
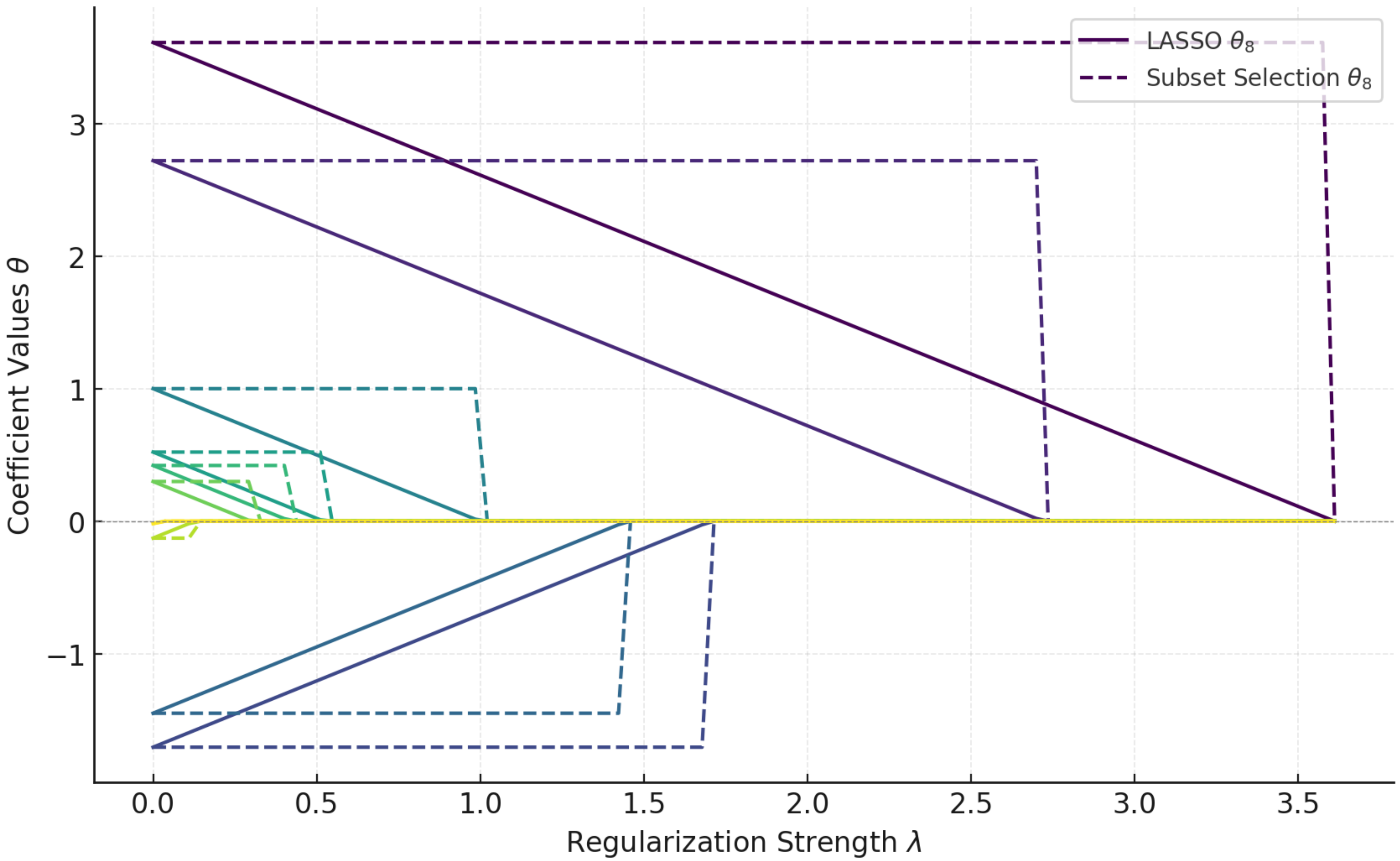


- <u>Key similarity</u>: both solutions induce sparsity
- <u>Key differences:</u> LASSO is convex and induce shrinkage (e.g. $z - \lambda$ for $z > \lambda$)

BSS vs. LASSO

$n = 20$ $\quad d = 10$ $\quad y_i = \langle \boldsymbol{\theta}_\star, \boldsymbol{x}_i \rangle + \varepsilon_i$ $\quad \varepsilon_i \sim \mathcal{N}(0,1)$ $\quad \boldsymbol{X}^\top \boldsymbol{X} = \boldsymbol{I}_{10},$ $\quad \boldsymbol{\theta}_\star$ is 5-sparse



- BSS is discontinuous
- LASSO is piece-wise continuous

⚠ For general design, non-zero path not simply a line

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OLS Shrinkage

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$||\hat{\boldsymbol{\theta}}_{\text{LASSO}}||_1 \leq ||\hat{\boldsymbol{\theta}}_{\text{OLS}}||_1$!!!

LASSO in practice

Beyond the orthogonal case, the LASSO problem:

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^d} \frac{1}{2n} \sum_{i=1}^{n} \left(y_i - \langle \boldsymbol{\theta}, \boldsymbol{x}_i \rangle \right)^2 + \lambda || \boldsymbol{\theta} ||_1$$

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<u>Idea</u>: alternate between these two.

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Iterative Shrinkage-Thresholding Algorithm (ISTA)

$$\boldsymbol{\theta}^{k+1} = S_{\eta\lambda}\left(\boldsymbol{\theta}^{k} + \frac{\eta}{n}\boldsymbol{X}^{\top}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\theta}^{k})\right)$$

LASSO in practice

$n = 10$

$d = 2$

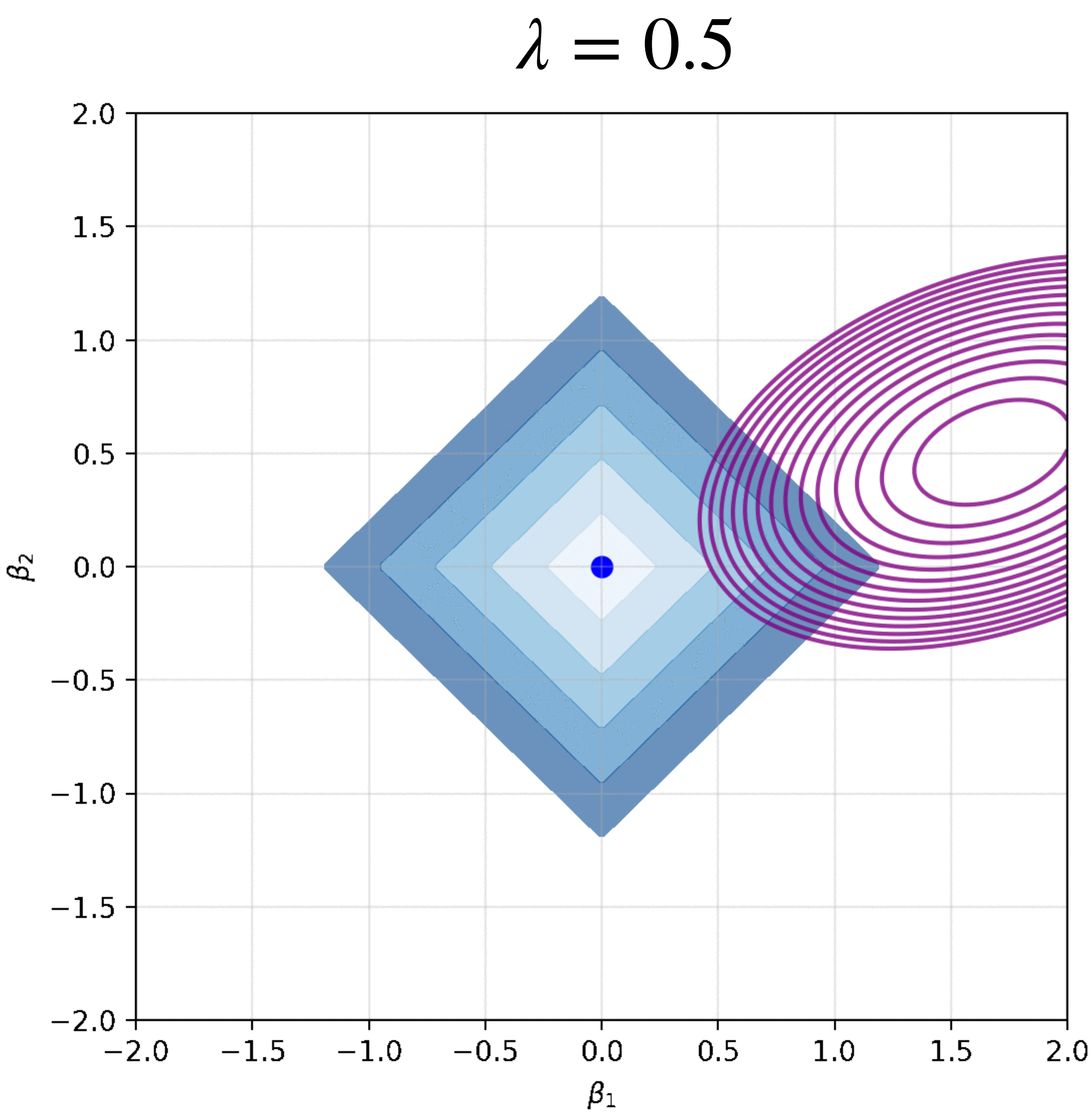
$y_i = \langle \boldsymbol{\theta}_\star, \boldsymbol{x}_i \rangle + \varepsilon_i$

$\boldsymbol{x}_i \sim \mathcal{N}(0, \boldsymbol{I}_2)$

$\varepsilon_i \sim \mathcal{N}(0,1)$

$$\boldsymbol{\theta}_\star = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

$\eta = 0.1$



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$$\lambda = 0.1$$

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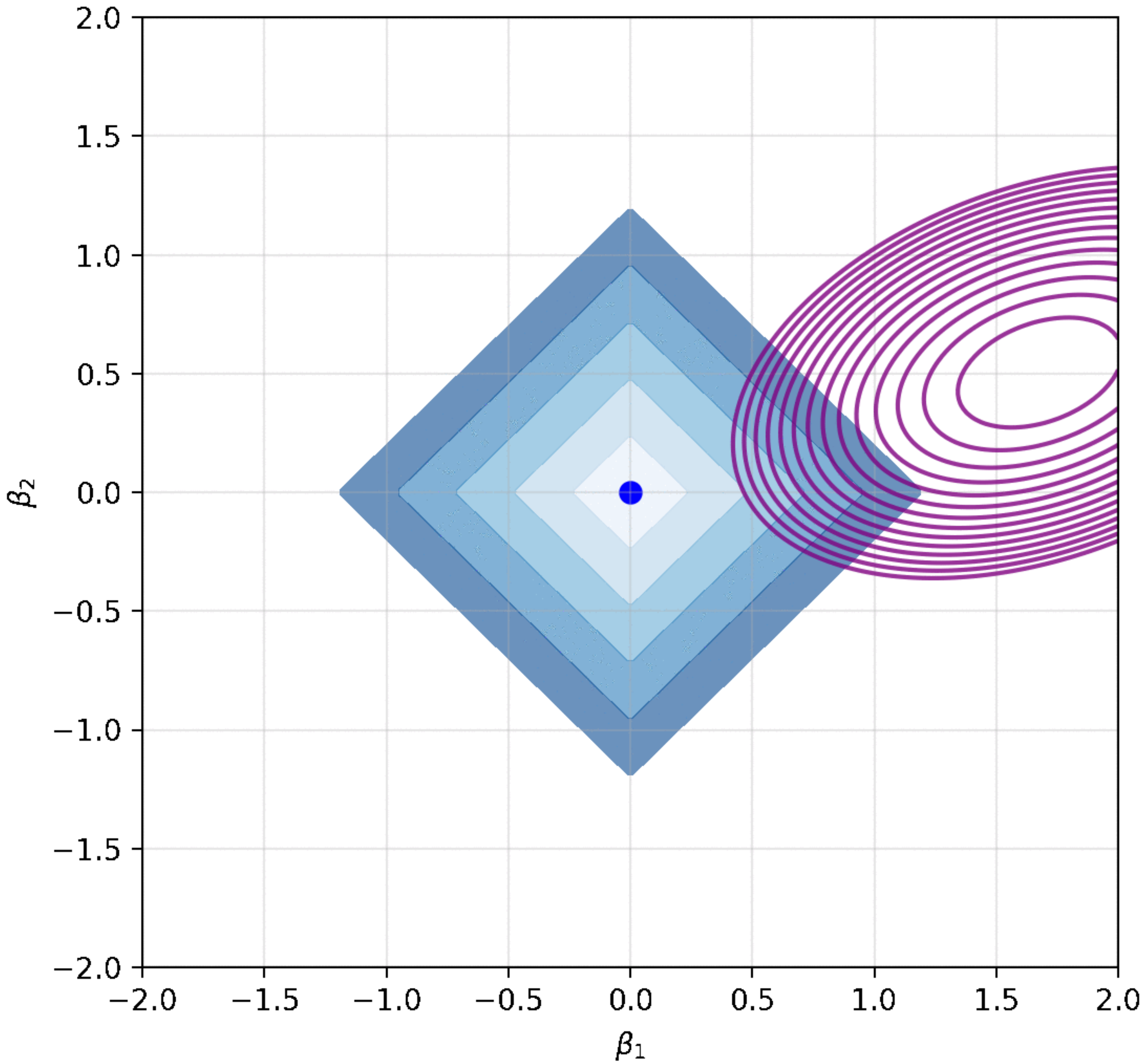

Elastic Net

The elastic net algorithm combines ridge with LASSO:

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^d} \frac{1}{2n}\sum_{i=1}^{n}\left(y_i - \langle \boldsymbol{\theta}, \boldsymbol{x}_i\rangle\right)^2 + \lambda_1 ||\boldsymbol{\theta}||_1 + \frac{\lambda_2}{2}||\boldsymbol{\theta}||_2^2$$

And is particularly suited to the case where the covariate $\boldsymbol{X}$ is badly conditioned.