

Homework Week 2

MATHEMATICS OF DEEP LEARNING
MASH & IASD 2025

Lecturer: Bruno Loureiro, bruno.loureiro@di.ens.fr

Instructions: This homework is **due on Monday 27/01/2025**. Please send your solutions in a PDF file named HW2_NOM_PRENOM.PDF to the above address with the subject “[MATHSDL2025] Homework 2”. Formats accepted: LaTeX or a **readable** scan of handwritten solutions.

1 Exercises

Exercise 1.

Show that the $L^p(\mu)$ norm:

$$\|f\|_p := \left(\int \mu(d\mathbf{x}) |f(\mathbf{x})|^p \right)^{1/p} \quad (1)$$

is an increasing function of $p \in [1, \infty]$:

$$\|f\|_{L^\infty(\mu)} \geq \dots \geq \|f\|_{L^2(\mu)} \geq \|f\|_{L^1(\mu)} \quad (2)$$

where we recall $\|f\|_{L^\infty(\mu)} = \sup_{x \in \text{supp}(\mu)} |f(x)|$. Conclude that we have the inclusion:

$$L^\infty(\mu) \subset \dots \subset L^2(\mu) \subset L^1(\mu) \quad (3)$$

Exercise 2.

Give an example of a probability measure on μ on \mathbb{R}^d and compact set $K \subset \mathbb{R}^d$ such that:

$$\|f\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(K)} \quad (4)$$

Exercise 3.

Consider the following continuous function on $[0, 1]$:

$$g(x) = \begin{cases} 0 & x < 1/2 \\ 2x & x \in [1/2, 1] \end{cases} \quad (5)$$

How many neurons p are needed to approximate g within a precision $\epsilon > 0$ using the two-layer neural network with step-size activation from Proposition 2 in the lectures? Show that we could do as well by using less neurons if we adapt the partition to the function.

Exercise 4.

Show that:

$$\inf_{f_\theta \in \mathcal{F}_{\text{relu},1}} \sup_{x \in \mathbb{R}} |f_\theta(x) - \sin(x)| \geq 1 \quad (6)$$

where $\mathcal{F}_{\text{relu},1}$, the class of two-layer neural networks over \mathbb{R} with relu activation and unbounded width. Conclude that the compactness assumption in the definition of a universal approximator is crucial to define meaningful approximation results.