



#### Statistical Learning II

Lecture 7 - Ridge regression

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#### Marvels and pitfalls of OLS

Recall that:

$$\hat{\boldsymbol{\theta}}_{OLS}(X, \mathbf{y}) = \boldsymbol{\theta}_{\star} + \frac{1}{n} \hat{\boldsymbol{\Sigma}}_{n}^{-1} X^{\mathsf{T}} \boldsymbol{\varepsilon}$$

$$= \boldsymbol{\theta}_{\star} + \sum_{j=1}^{d} \frac{1}{\sigma_{j}} \langle \boldsymbol{u}_{j}, \boldsymbol{\varepsilon} \rangle \boldsymbol{v}_{j}$$

Hence: • signal is stronger in directions with larger s.v.

noise dominates directions with smaller s.v.

OLS has larger variance for data with small "effective dimension".

#### What to do?

Classical strategies to mitigate variance:

- Dimensionality reduction: PCA, random projections (sketching), etc.
- Variable subset selection: Stepwise selection, best Subset Selection, etc.

Regularisation: ridge, LASSO, etc.

Note the averaged norm of the OLS is given by:

$$\mathbb{E}_{\boldsymbol{\varepsilon}} \left[ ||\hat{\boldsymbol{\theta}}_{OLS}||_2^2 \right] = ||\boldsymbol{\theta}_{\star}||_2^2 + \sigma^2 \sum_{j=1}^d \frac{1}{\sigma_j^2}$$

Therefore, small s.v.s lead to larger expected norm.

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Key idea: penalise the norm.

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^d} \frac{1}{2n} ||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\theta}||_2^2 + \frac{\lambda}{2} ||\boldsymbol{\theta}||_2^2$$

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Least squares empirical risk

Regularisation or "ridge" penalty

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$$\min_{\boldsymbol{\theta} \in \mathbb{R}^d} \hat{\mathcal{R}}_n^{\lambda}(\boldsymbol{\theta}) := \frac{1}{2n} ||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\theta}||_2^2 + \frac{\lambda}{2} ||\boldsymbol{\theta}||_2^2$$

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#### Remarks:

• The regularised empirical risk is a strongly convex function of  $\theta \in \mathbb{R}^d$ 

$$\nabla_{\boldsymbol{\theta}} \hat{\mathcal{R}}_{n}^{\lambda}(\boldsymbol{\theta}) = -\frac{1}{n} \boldsymbol{X}^{\top} \left( \boldsymbol{y} - \boldsymbol{X} \boldsymbol{\theta} \right) + \lambda \boldsymbol{\theta}$$

$$\nabla_{\boldsymbol{\theta}}^{2} \hat{\mathcal{R}}_{n}^{\lambda}(\boldsymbol{\theta}) = \frac{1}{n} \boldsymbol{X}^{\top} \boldsymbol{X} + \lambda \boldsymbol{I}_{d} > 0$$

$$(= \hat{\boldsymbol{\Sigma}}_{n} + \lambda \boldsymbol{I}_{n})$$

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In other words, minimiser always exist and is unique.

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For 
$$\lambda \to 0^+$$
,  $\hat{\boldsymbol{\theta}}_{\lambda} \to \hat{\boldsymbol{\theta}}_{\mathrm{OLS}}$ 

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rank(X)

Remarks: • As before, consider s.v.d. of  $X = \sum_{j=1}^{\infty} \sigma_j u_j v_j^{\top}$ 

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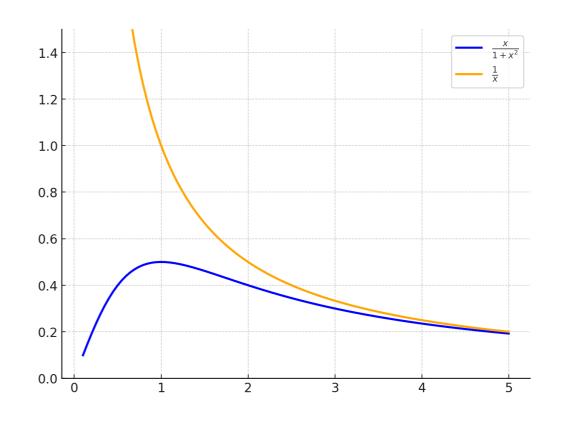
$$\hat{\boldsymbol{\theta}}_{\lambda}(\boldsymbol{X}, \boldsymbol{y}) = \sum_{j=1}^{\operatorname{rank}(\boldsymbol{X})} \frac{\sigma_{j}}{\sigma_{j}^{2} + n\lambda} \langle \boldsymbol{u}_{j}, \boldsymbol{y} \rangle \boldsymbol{v}_{j}$$

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Ridge performs shrinkage: small s.v.s are suppressed!



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