1\. (a) $f;g$ is a function.

reason

① all $s \in S$ there is exactly one $t \in T$ such that $(s,t) \in f$ — Function definition and $f: S \to T$ is function

② all $t \in T$ there is exactly one $u \in U$ such that $(t,u) \in g$ — Function definition and $g: T \to U$ is function

from the definition of ; operation, we can know

$f;g := \{(a,c) : \text{There exists } b \in T, \text{ such that } (a,b) \in f \text{ and } (b,c) \in g\}$

NEXT, according to statement ① and ②, and co-domain of f is equal to the domain of g

so we can know that. all $s \in S$ there is exactly one $u \in U$.

According to function definition. we get the conclusion that $f;g$ is a function.

(b) prove:

According to ① ; operation, $R_1;R_2 = \{(a,c) : \text{there exist } b \in T \text{ such that } (a,b) \in R_1 \text{ and } (b,c) \in R_2\}$

② $R \subseteq S \times S$

We can know that $R;R = \{(a,c) : \text{there exist } b \in S \text{ such that } (a,b) \in R \text{ and } (b,c) \in R\}$

and ~~R have sa~~ the domain of R is equal to the co-domain of R., and R is transitive

$\Rightarrow$ So we can ~~conclusion~~ get that $R;R$ have the relationship : $(m,n) \in R;R$ — m can be any element in S, n can be any one in S, too

It is same as the definition of $R \Rightarrow R = (R;R)$

In conclusion, $R \cup (R;R) = R \cup R$ (Idempotence)

$= R$

(c) if $R^i = R^{i+1}$ for some i

[Induction base]: $R^i = R^{i+1}$ for some i

[Induction step]: Assume that $j \geq i$, satisfy $R^j = R^i$, then

$R^{j+1} = R^j \cup (R^j ; R)$ (definition of R^{i+1}, $R^{i+1} := R^i \cup (R^i ; R)$ for $i \geq 0$)

$= R^i \cup (R^i ; R)$ ($R^j = R^i$)

$= R^{i+1}$ ~~($R^i = R^{i+1}$)~~ (definition of R^{i+1})

$= R^i$ ($R^i = R^{i+1}$)

[Conclusion] From induction above, we can get that if $R^i = R^{i+1}$ for some i, then $R^i = R^j$ for all $j \geq i$

(d) <1> $0 \leq k \leq i$, According to $R^{i+1} = R^i \cup (R^i ; R)$, we can know $R^i \subseteq R^{i+1}$

So we have $R^{k+1} \supseteq R^k$ in this area

Then, according to transitive, $R^{k+1} \supseteq R^k$, $R^k \supseteq R^{k-1}$ ..... $R^1 \supseteq R^0$

$R^k \subseteq R^i$ ~~in~~ when $0 \leq k \leq i$

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<ii> $k > i+1$

Because $R^i = R^{i+1}$ for some i. from conclusion in (c), we can know that $R^i = R^j$ for all $j \geq i$

so $R^j \subseteq R^i$ for all $j \geq i$ ($R^i = R^j \Leftrightarrow R^i \subseteq R^j$ and $R^j \supseteq R^i$)

In conclusion, if $R^i = R^{i+1}$ for some i. then $R^k \subseteq R^i$ for all $k \geq 0$

(e) ① First, we can know that $R^n \subseteq R^{n+1}$ because the definition $R^{i+1} = R^i \cup (R^i ; R)$ and the formula ($R^i = R^j \Leftrightarrow R^i \subseteq R^j$ and $R^j \subseteq R^i$)

② ~~THE~~ Then,

because $|S| = n$, so now assume $S = \{c_1, c_2, c_3 \dots c_n\}$

if $(a,b) \in R^{n+1}$, this ~~is~~ mean that $(a, c_1) \in R$; $(c_1, c_2) \in R$, ... $(c_n, b) \in R$, $a, b \in c_i$ $i \in [1, n]$

In (d), $0 \leq k \leq i$, then is $R^k \subseteq R^i$ $\Rightarrow R \subseteq R^i \Rightarrow (a, c_1) \in R^{i-1}$; $(c_1, c_2) \in R^{i-1}$; .... $(c_n, b) \in R^{i-1}$

We can get $(a,b) \in R^i$ because $R^i = R^{i-1} \cup (R^{i-1}; R)$ and $(a,b) \in (R^{i-1}; R)$

In this case, if $(a,b) \in R^{n+1}$ ~~for some i<n~~ then $(a,b) \in R^i$ for some $i < (n+1)$

$\Rightarrow R^{n+1} \subseteq R^n$

In Conclusion $R^n = R^{n+1}$

(f)

$R^0 = R \subseteq R^n$ (the result from (d))

if $(a,b) \in R^n$ and $(b,c) \in R \subseteq R^n$, then $(a,c) \in (R^n; R)$ ~~$\subseteq R^n$~~ and $(R^n; R) \subseteq R^n$

~~because~~ due to the definition of relation composition.

Hence, ~~According~~ according the definition of transitive, R^* is transitive.

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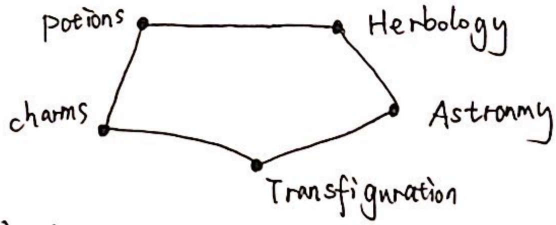
2. In the graph,

(a) vertices would be several subjects

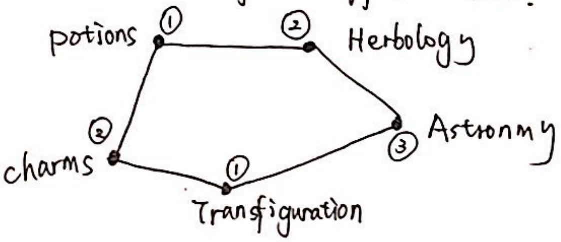
edges would be a student have different subjects at the same time

relate: consider subject as vertices in graph, if two ~~e~~ vertices have one edge, means that at least one student enroll these two classes. So these two classes cannot exam at the same time. the graph is:



(b) the minimum number of timeslots required is 3

In the graph above, two vertices connected by some edge cannot exam at the same time. so we can relate this problem to graph coloring. Just like graph below. ①②③ stand for different color.



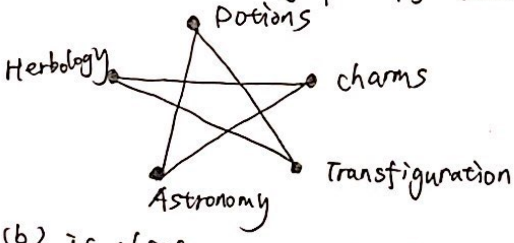
(c)

(a) is changed to

vertices would be different subjects

edges would be two different course can exam without conflict

relate: The graph is below



(b) is change to the largest number of subjects that can be examined at the same time without conflict is 2 because the maximum degree of edges is 2

3.

(a) n edges

In order to have 1 face, there is a loop in this graph, so n vertices would have n edges.

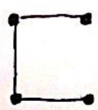
(b) For instance

K3

vertices = 3
edges = 3
faces = 1

K4

vertices = 4
edges = 6
faces = 3



vertices = 4
edges = 3
faces = 0

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In conclusion, $n+f-m=1$

(C) prove: (Induction by f)

[B] Induction Basis: If face $=1$ ~~then~~ and the graph consists of n vertices then graph have n edges (because graph have 1 loop)
now, $n+f-m = n+1-n = 1$

[I] Induction step: when face $=k \geq 2$,
assume that a graph satisfies the folowing conditions: vertices$(n)=u$ and edges$(m)=v$ $k+u-v=1$
Next, because face $=k$, there are k loops in this graph. We delete a edge in one of k loops in order to change face from k to $k-1$. Now, vertices and edges is u and $v-1$, respectively. It is also satisfy this rule.
Conclusion: $n+f-m=1$ is proved.

4. (a)

p	q	$p \circ q$	$(p\circ q)\circ(p\circ q)$
0	0	1	0
0	1	1	0
1	0	1	0
1	1	0	1

$$
\begin{aligned}
(p\circ q)\circ(p\circ q) &= \neg(p\wedge q)\circ(\neg(p\wedge q)) && \text{(definition of } \circ \text{ operation)}\\
&= \neg(\neg(p\wedge q)\circ(\neg(p\wedge q))) && \text{(definition of } \circ \text{ operation)}\\
&= \neg(\neg(p\wedge q)) && \text{(Idempotence)}\\
&= p\wedge q && \text{(Double Negation)}
\end{aligned}
$$

(b) <i>
$$
\begin{aligned}
\neg p &= \neg(p\wedge p) && \text{(Idempotence)}\\
&= p\circ p && \text{(definition of } \circ \text{ operation)}
\end{aligned}
$$

<ii>
$$
\begin{aligned}
p\vee q &= \neg(\neg p\wedge\neg q) && \text{(De Morgan's Laws)}\\
&= \neg p\circ\neg q && \text{(definition of } \circ \text{ operation)}\\
&= \neg(p\wedge p)\circ\neg(q\wedge q) && \text{(Idempotence)}\\
&= (p\circ p)\circ(q\circ q) && \text{(definition of } \circ \text{ operation)}
\end{aligned}
$$

<iii>
$$
\begin{aligned}
p\to q &= \neg p\vee q && \text{(Implication)}\\
&= \neg(\neg(\neg p\vee q)) && \text{(Double Negation)}\\
&= \neg(p\wedge\neg q) && \text{(De Morgan's Laws)}\\
&= p\circ\neg q && \text{(definition of } \circ \text{ operation)}\\
&= p\circ\neg(q\wedge q) && \text{(Idempotence)}\\
&= p\circ(q\circ q) && \text{(definition of } \circ \text{ operation)}
\end{aligned}
$$

<iv>
$$
p\leftrightarrow q = (p\to q)\wedge(q\to p) \quad \text{(Implication)}
$$

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$= p \circ (q \circ q) \wedge (q \circ (p \circ p))$ (resule of <ii>)

assume that $p \circ (q \circ q) = X$ $q \circ (p \circ p) = Y$

$p \leftrightarrow q = X \wedge Y$

$= \neg(\neg(X \wedge Y))$ (Double Negation)

$= \neg(X \circ Y)$ (Definition of $\circ$ operation)

$= \neg((X \circ Y) \wedge (X \circ Y))$ (Idempotence)

$= (X \circ Y) \circ (X \circ Y)$ (Definition of $\circ$ operation)

$= ((p \circ (q \circ q)) \circ (q \circ (p \circ p))) \circ ((p \circ (q \circ q)) \circ (q \circ (p \circ p)))$ (X. Y definition)

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