1. (a) $f;g$ is a function.

reason

① all $s \in S$ there is exactly one $t \in T$ such that $(s,t) \in f$ — Function definition and $f: S \to T$ is function

② all $t \in T$ there is exactly one $u \in U$ such that $(t,u) \in g$ — Function definition and $g: T \to U$ is function

from the definition of ; operation, we can know

$f;g := \{(a,c): \text{There exists } b \in T, \text{ such that } (a,b) \in f \text{ and } (b,c) \in g\}$

NEXT, according to statement ① and ②, and co-domain of f is equal to the domain of g

so we can know that. all $s \in S$ there is exactly one $u \in U$.

According to function definition. we get the conclusion that $f;g$ is a function.

(b) prove:

According to ① ; operation, $R_1;R_2 = \{(a,c): \text{there exist } b \in T \text{ such that } (a,b) \in R_1 \text{ and } (b,c) \in R_2\}$

② $R \subseteq S \times S$

We can know that $R;R = \{(a,c): \text{there exist } b \in S \text{ such that } (a,b) \in R \text{ and } (b,c) \in R\}$

and the domain of R ~~have sa~~ is equal to the co-domain of R., and R is transitive

$\Rightarrow$ so we can ~~conclusion~~ get that $R;R$ have the relationship: $(m,n) \in R;R$ — m can be any element in S, n can be any one in S, too

It is same as the definition of $R \Rightarrow R = (R;R)$

In conclusion, $R \cup (R;R) = R \cup R$ (Idempotence)

$= R$

(c) if $R^i = R^{i+1}$ for some i

[Induction base]: $R^i = R^{i+1}$ for some i

[Induction step]: Assume that $j \geq i$, satisfy $R^j = R^i$, then

$R^{j+1} = R^j \cup (R^j;R)$ (definition of R^{i+1}, $R^{i+1} := R^i \cup (R^i;R)$ for $i \geq 0$)

$= R^i \cup (R^i;R)$ ($R^j = R^i$)

$= R^{i+1}$ ~~($R^i = R^{i+1}$)~~ (definition of R^{i+1})

$= R^i$ ($R^i = R^{i+1}$)

[Conclusion] From induction above, we can get that if $R^i = R^{i+1}$ for some i, then $R^i = R^j$ for all $j \geq i$

(d) <i> $0 \leq k \leq i$, According to $R^{i+1} = R^i \cup (R^i;R)$, we can know $R^i \subseteq R^{i+1}$

So we have $R^{k+1} \supseteq R^k$ in this area

Then, according to transitive, $R^{k+1} \supseteq R^k$, $R^k \supseteq R^{k-1}$ ..... $R^1 \supseteq R^0$

$R^k \subseteq R^i$ ~~to~~ when $0 \leq k \leq i$

<ii> $k > i+1$

Because $R^i = R^{i+1}$ for some i. from conclusion in (c), we can know that $R^i = R^j$ for all $j \geq i$

so $R^j \subseteq R^i$ for all $j \geq i$ ($R^i = R^j \iff R^i \subseteq R^j$ and $R^j \supseteq R^i$)

In conclusion, if $R^i = R^{i+1}$ for some i. then $R^k \subseteq R^i$ for all $k \geq 0$

(e) ① First, we can know that $R^n \subseteq R^{n+1}$ because the definition $R^{i+1} = R^i \cup (R^i ; R)$ and the formula ($R^i = R^j \iff R^i \subseteq R^j$ and $R^j \subseteq R^i$)

② ~~This~~ Then,

because $|S| = n$, so now assume $S = \{c_1, c_2, c_3 \dots c_n\}$

if $(a,b) \in R^{n+1}$, this ~~is~~ mean that $(a, c_1) \in R$; $(c_1, c_2) \in R$; .... $(c_n, b) \in R$, $a, b \in c_i$, $i \in [1,n]$

In (d), $0 \leq k \leq i$, ~~then~~ is $R^k \subseteq R^i$ $\Rightarrow$ $R \subseteq R^i \Rightarrow (a, c_1) \in R^{i-1}$; $(c_1, c_2) \in R^{i-1}$; .... $(c_n, b) \in R^{i-1}$

We can get $(a,b) \in R^i$ because $R^i = R^{i-1} \cup (R^{i-1}; R)$ and $(a,b) \in (R^{i-1}; R)$

In this case, if $(a,b) \in R^{n+1}$ ~~for some i<n~~ then $(a,b) \in R^i$ for some $i < (n+1)$

$\Rightarrow R^{n+1} \subseteq R^n$

In Conclusion $R^n = R^{n+1}$

(f)

$R^0 = R \subseteq R^n$ (the result from (d))

if $(a,b) \in R^n$ and $(b,c) \in R \subseteq R^n$, then $(a,c) \in (R^n; R)$ ~~$\in R^n$~~ and $(R^n; R) \subseteq R^n$

~~because~~ due to the definition of relation composition.

Hence, ~~According~~ according the definition of transitive, R^* is transitive.

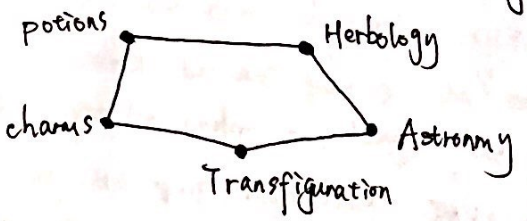
2. (a)

In the graph,

Vertices would be several subjects

edges would be a ~~the same~~ student have different subjects at the same time

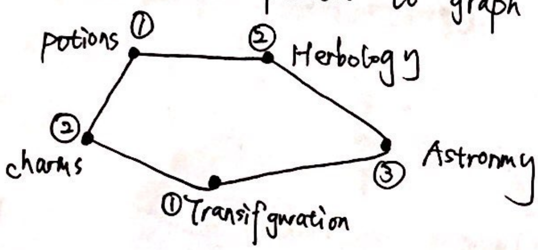
relate : consider subjece as vertices in graph, if two ~~edge~~ vertices have one edge, means that at lease one student enroll ~~this~~ these two classes. So these two classes cannot exam at the same time. The graph is :



(b)

the minimum number of timeslots required is 3.

In the graph above, two vertices connected by same edge cannot ~~pass~~ exam at the same time. So we can relate this problem to graph coloring. Just like graph below. ①②③ stand for different color.



(c)

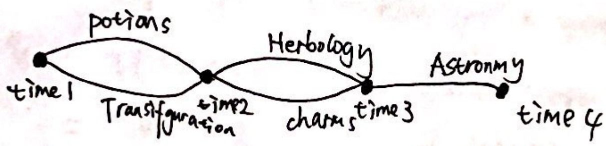
change my goal to determine the largest number of subjects that can be examined at the same time without conflicts.

(a) is changed to

vertices would be different ~~timeslots~~ times, timeslots is between two times

edges would be subjects can be scheduled for this timeslot.

relate:



(b) the minimum timeslots is 3, and from the graph above the largest number of subjects that can be examined at the same time without conflict is 2.

3. (a) n edges

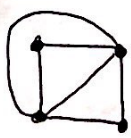
In order to have 1 face, there is a cycle in this graph. So n vertices would have n edges.

(b) For instance

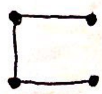
K3

vertices : 3
edges : 3
faces : 1

K4



vertices : 4
edges : 6
faces : 3



vertices : 4
edges : ~~4~~ 3
faces : 0

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In conclusion, $n+f-m=1$

(C) prove: (Induction by f)

[B] Induction Basis: If face $=1$ ~~then~~ and the graph consists of n vertices then graph have n edges (because graph have 1 loop)

now, $n+f-m = n+1-n = 1$

[I] Induction step: when face $=k \geq 2$,

assume that a graph satisfies the following conditions: vertices$(n) = u$ and edges$(m) = v$, $k+u-v=1$

Next, because face $=k$, there are k loops in this graph. We delete a edge in one of k loops in order to change face from k to $k-1$. Now, vertices and edges is u and $v-1$, respectively. It is also satisfy this rule.

Conclusion: $n+f-m=1$ is proved.

4. (a)

p	q	$p \circ q$	$(p\circ q)\circ(p\circ q)$
0	0	1	0
0	1	1	0
1	0	1	0
1	1	0	1

$(p\circ q)\circ(p\circ q) = \neg(p\wedge q)\circ(\neg(p\wedge q))$ (definition of $\circ$ operation)

$= \neg(\neg(p\wedge q)\circ(\neg(p\wedge q)))$ (definition of $\circ$ operation)

$= \neg(\neg(p\wedge q))$ (Idempotence)

$= p\wedge q$ (Double Negation)

(b) <i> $\neg p = \neg(p\wedge p)$ (Idempotence)

$= p\circ p$ (definition of $\circ$ operation)

<ii> $p\vee q = \neg(\neg p\wedge\neg q)$ (De Morgan's Laws)

$= \neg p\circ\neg q$ (definition of $\circ$ operation)

$= \neg(p\wedge p)\circ\neg(q\wedge q)$ (Idempotence)

$= (p\circ p)\circ(q\circ q)$ (definition of $\circ$ operation)

<iii> $p\to q = \neg p\vee q$ (Implication)

$= \neg(\neg(\neg p\vee q))$ (Double Negation)

$= \neg(p\wedge\neg q)$ (De Morgan's Laws)

$= p\circ\neg q$ (definition of $\circ$ operation)

$= p\circ\neg(q\wedge q)$ (Idempotence)

$= p\circ(q\circ q)$ (definition of $\circ$ operation)

<iv> $p\leftrightarrow q = (p\to q)\wedge(q\to p)$ (Implication)

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$= p \circ (q \circ q) \wedge (q \circ (p \circ p))$ (result of <ii>)

assume that $p \circ (q \circ q) = X$ $q \circ (p \circ p) = Y$

$p \leftrightarrow q = X \wedge Y$

$= \neg(\neg(X \wedge Y))$ (Double Negation)

$= \neg(X \circ Y)$ (Definition of $\circ$ operation)

$= \neg((X \circ Y) \wedge (X \circ Y))$ (Idempotence)

$= (X \circ Y) \circ (X \circ Y)$ (Definition of $\circ$ operation)

$= ((p \circ (q \circ q)) \circ (q \circ (p \circ p))) \circ ((p \circ (q \circ q)) \circ (q \circ (p \circ p)))$ (X. Y definition)