2020 Math Camp — Lecture 5

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1 Introduction

Recall the parametric constrained-maximization problem:

$$\text{Maximize (in } z\text{)} \quad F(z,\theta) \quad \text{subject to} \quad z \in A$$

Suppose the solution to the above problem exists. Let $Z(\theta)$ be the set of solutions of this problem for the parameter θ. We are interested in how $Z(\theta)$ vary with θ.

A traditional approach is to apply the Implicit Function Theorem to the first-order maximization conditions. This approach requires some assumptions about the objective function F and the constraint set A: F is twice differentiable and strictly concave (ensures that $Z(\theta)$ contains at most one element), A is convex, $Z(\theta)$ is not empty for each θ and all its elements lie in the interior of A. In addition, θ is a continuous policy variable. Under this conditions the unique $z(\theta) \in Z(\theta)$ solves:

$$F_z(z(\theta), \theta) = 0$$

We can apply the Implicit Function Theorem to show that:

$$z'(\theta) = -\frac{F_{z\theta}(z(\theta), \theta)}{F_{zz}(z(\theta), \theta)}$$

The above approach allows to calculate $z'(\theta)$ exactly if F is known and satisfies the above conditions. However, very often we might not know F exactly, only some of its properties, F does not have to be smooth and strictly concave, A might not be convex (indivisible goods) and sometimes it is not necessary for us to know $z'(\theta)$ exactly, just knowing the sign of $z'(\theta)$ is enough.

This note is devoted to a set of tools that allow us to answer how $z'(\theta)$ depends on θ while relaxing all the assumptions mentioned above.

2 Binary relations

Definition 1. Let X and Y be two nonempty sets. A subset R of $X \times Y$ is called a (binary) relation from X to Y. In case $X = Y$ (relation from X to X) we say that it is a relation on X. If $(x, y) \in R$ then we write xRy.

Example 2. a. "equal to" on $\mathbb{R}$: $R = \{(x, y) \in \mathbb{R}^2 : x = y\}$;

b. "greater than" on $\mathbb{R}$: $R = \{(x, y) \in \mathbb{R}^2 : x > y\}$;

c. "multiple of" on $\mathbb{N}$: $R = \{(x, y) \in \mathbb{N}^2 : x = ky \text{ for some } k \in \mathbb{N}\}$;

d. "a subset of" on $2^{\mathbb{R}}$: $R = \{(A, B) : A \subseteq B\}$.

Definition 3. Let X be a nonempty set and R be a relation on it. R is said to be:

- *refexive* if xRx for all $x \in X$;
- *complete* if either xRy or yRx for each $x, y \in X$;
- *antisymmetric* if xRy and yRx implies $x = y$;
- *transitive* if xRy, yRz implies that xRz.

Definition 4. A relation $\succsim$ on a nonempty set X is called a *partial order* on X if it is reflexive, antisymmetric, transitive.

Definition 5. A partially ordered set, or a poset $(X, \succsim)$ consists of a set X and a partial ordering over that set, $\succsim$.

Remark 6. A partially ordered set $(X, \succsim)$ need not be complete. That is, there can be elements $x, y \in X$ such that neither xRy nor yRx.

Definition 7. A totally ordered set (X, R) is a partially ordered set that is complete, i.e. for every $x, y \in X$, either xRy, or yRx (or both).

Example 8. $(\mathbb{R}, \succsim)$ is a totally ordered set, while $(\mathbb{R}^2, \succsim)$ under the usual (componentwise) order is not.

3 Increasing differences

Definition 9. If X is a partially ordered set, $f : X \to \mathbb{R}$ is increasing if $\bar{x} \geq \underline{x}$ implies $f(\bar{x}) \geq f(\underline{x})$.

Definition 10. Let X_1, X_2 be two partially ordered sets. Then, $f : X_1 \times X_2 \to \mathbb{R}$ has increasing differences if, for all $\bar{x}_1 \geq \underline{x}_1$ and $\bar{x}_2 \geq \underline{x}_2$:

$$f(\bar{x}_1, \bar{x}_2) - f(\underline{x}_1, \bar{x}_2) \geq f(\bar{x}_1, \underline{x}_2) - f(\underline{x}_1, \underline{x}_2)$$

We say that f has *strictly* increasing differences if the above inequalities are strict.

Remark 11. Informally, f is "more increasing" in x_1 for higher values of x_2.

Exercise 1. Let $X_1 = X_2 = \mathbb{R}$. Prove that $f = x_1 x_2$ has increasing differences.

Exercise 2. Prove or disprove: for any partially ordered X_1, X_2 and any increasing $f : X_1 \to \mathbb{R}$ and $g : X_2 \to \mathbb{R}$, $f(x_1)g(x_2)$ has increasing differences.

Proposition 12. *If $X_1 = X_2 = \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously twice differentiable, then increasing differences is equivalent to* $\frac{\partial^2 f}{\partial x_1 \partial x_2} \geq 0$.

Theorem 13 (Topkis's Monotone Selection Theorem)**.** *If $f : X \times \Theta \to \mathbb{R}$ with $X, \Theta \subseteq \mathbb{R}$ has strictly increasing differences and $X^*(\theta) = \arg\max_{x \in X} f(x, \theta)$, then for all $\theta, \theta' \in \Theta$ such that $\theta' > \theta$, $x \in X^*(\theta)$, and $x' \in X^*(\theta')$, we have $x \leq x'$.*

4 Lattices

Definition 14. We say that z is a lower bound for two elements $x, y \in X$ if $z \leq x$ and $z \leq y$. It's the greatest lower bound (GLB) if $z' \leq z$ for every lower bound z'. Define upper bound and the least upper bound (LUB) similarly.

Definition 15. Then $(X, \succsim)$ is a lattice if every pair of elements has the GLB and LUB. In this case, we write $x \wedge y$ ("meet") for GLB and $x \vee y$ ("join") for LUB.

Exercise 3. Prove that GLB and LUB are unique.

Example 16. Lattices:

a. $\mathbb{R}$ with usual ordering;

b. $\mathbb{R}^2$ with componentwise ordering;

c. a set of subsets of a given set X, ordered by inclusion;

d. $A = \{(0,0), (0,1), (1,0), (2,2)\}$ under the usual order. Note that LUB operation is not the same as in $\mathbb{R}^2$.

Not lattices:

a. $A = \{(0,0), (0,1), (1,0), (1,2), (2,1), (2,2)\}$;

b. $A = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 \leq K\}$.

5 Supermodular functions and Topkis' Theorem

Remark 17. We want to generalize Theorem 13 to $X \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^m$. What if an increase in θ pushes x_i and x_j up, but the increase in x_i pushes x_j down? We need complementarity within the components of x in addition to complementarity between x and θ.

Definition 18 (Supermodularity)**.** Let X be a lattice. We say that $f : X \times \Theta \to \mathbb{R}$ is supermodular if, for all $x, y \in X$ and $\theta \in \Theta$,

$$f(x \wedge y, \theta) + f(x \vee y, \theta) \geq f(x, \theta) + f(y, \theta)$$

Exercise 4. Prove or disprove: any function $f : \mathbb{R} \to \mathbb{R}$ is supermodular.

Remark 19. For $f : X_1 \times X_2 \to \mathbb{R}$, where X_1 and X_2 are totally ordered sets, supermodularity is equivalent to increasing differences.

Definition 20. Define a strong set order on subsets of a lattice X. We say $S \leq T$ if, for all $s \in S$ and $t \in T$, we have $s \wedge t \in S$ and $s \vee t \in T$.

Theorem 21 (Topkis' Monotonicity Theorem)**.** *Let $X \subseteq \mathbb{R}^n$ be a lattice and $\Theta \subseteq \mathbb{R}^m$, and $f : X \times \Theta \to \mathbb{R}$ has increasing differences in (x, θ) and is supermodular in x, then $X^*(\theta)$ is increasing in the strong set order.*

6 Other useful results

Proposition 22 (Chain rule)**.** *Let $F(x) = f(g(x))$. Then $F'(x) = f'(g(x))g'(x)$.*

Proposition 23 (Leibniz's rule)**.** *If $-\infty < a(x), b(x) < +\infty$,*

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x, b(x)) \cdot \frac{d}{dx} b(x) - f(x, a(x)) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt$$

Proposition 24 (Jensen's inequality)**.** *If X is a random variable and f is a convex function, then $f(E[X]) \leq E[f(X)]$.*

Proposition 25 (Law of iterated expectations)**.** *If X and Y are random variables, then $E[X] = E[E[X|Y]]$.*

Theorem 26 (Fubini's theorem)**.** *If $\int_{X \times Y} |f(x,y)| \, d(x,y) < \infty$, then*

$$\int_X \left(\int_Y f(x,y)\, dy\right) dx = \int_Y \left(\int_X f(x,y)\, dx\right) dy = \int_{X \times Y} f(x,y)\, d(x,y)$$

Theorem 27 (Envelope theorem)**.** *Consider the following maximization problem:*

$$\max_{x,y} f(x,y,\theta) \quad s.t. \;\; g(x,y,\theta) \leq c$$

Define the Lagrangian:

$$\mathcal{L}(x, y, \lambda, \theta) = f(x, y, \theta) + \lambda[c - g(x, y, \theta)]$$

Let $x^*(\theta)$, $y^*(\theta)$ and λ^* denote solutions to this problem. Define the value function:

$$F(\theta) = f(x^*(\theta), y^*(\theta), \theta)$$

Then, the derivative of the value function with respect to θ is:

$$\frac{dF(\theta)}{d\theta} = \frac{\partial f}{\partial \theta}\Big|_{(x^*(\theta), y^*(\theta), \theta)} - \lambda^* \frac{\partial g}{\partial \theta}\Big|_{(x^*(\theta), y^*(\theta), \theta)}$$

Remark 28. In the unconstrained case or when the constraint is not binding ($\lambda^* = 0$),

$$\frac{dF(\theta)}{d\theta} = \frac{\partial f}{\partial \theta}\Big|_{(x^*(\theta), y^*(\theta), \theta)}$$

Example 29. First price auction with private values. Bidder N has value $v \in [0, +\infty)$, submits bid b. N wins with probability $p(b)$. N solves the following problem:

$$\max_b \ u(v, b) = p(b)v - b$$

Let $b^*(v)$ denote the solution to this problem. N's value function:

$$U(v) = p(b^*(v))v - b^*(v)$$

We want to know how $U(v)$ changes with v. Apply the Envelope theorem:

$$\frac{dU(v)}{dv} = \frac{\partial u(v, b)}{\partial v}(v, b^*(v)) = p(b^*(v))$$

If in equilibrium the highest value bidder wins, $p(b^*(v))$ is equal to the probability that N has the highest value and we can write $p(b^*(v)) = p(v)$. Integrating:

$$U(v) = U(0) + \int_0^v p(\bar{v})d\bar{v}$$

Hence, N's expected utility *only* depends on her probability of winning as a function of her valuation v and her expected utility when she has the lowest possible valuation ($U(0)$).