Fun with Finite Difference

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Outline

- Continuous time and state: expectation and backward equation.
- Fokker-Planck and Dupire forward equations.
- Discrete time and state: Operators as matrices.
- Backward finite difference equation: the Theta scheme.
- Discrete duality: Fokker-Planck and Dupire forward finite difference equations.
- Accuracy.

- Stability.

- The finite difference cheat sheet.

- Coding the Theta solver.

References

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Continuous Time

- Assume that we have some Markov state variable that follows an SDE of the form

$$dx = \mu(t,x)dt + \sigma(t,x)dW \tag{1}$$

- … and we wish to compute

$$f(t,x(t)) = E_t[e^{-\int_t^T r(u,x(u))du} f(T,x(T))] \tag{2}$$

- Then by Ito's lemmas f solves the backward PDE

$$0 = \partial_t f + Af \quad ,A = -r + \mu\partial_x + \frac{1}{2}\sigma^2\partial_{xx} \tag{3}$$

- Of particular interest is the pricing of derivatives in which case you can think of f as the value of some derivative and this case the expectation is taken under the risk-neutral measure.

- In this course we are considering how to numerically solve the expectation problem (2) by crunching the PDE (3) on a computer by use of finite difference methods.

- Before we hit the coding we'll spend a bit of time on the equations.

- Specifically, where the finite difference methods come from.

Forward Equations

- Let

$$p(t,x) = E[e^{-\int_0^t r(u)du}\delta(x(t)-x)]$$

$$c(t,x) = E[e^{-\int_0^t r(u)}(x(t)-x)^+]$$

- … be the initial Arrow-Debreu and European option prices respectively so that

$$f(0,x(0))=\int_{-\infty}^{\infty} f(t,x)\,p(t,x)\,dx$$

$$p(t,x)=\partial_{xx}c(t,x)$$

- The Arrow-Debreu and European option prices satisfy the forward PDEs

$$0=-\partial_t p+A^* p \quad ,A^*=-r-\partial_x\mu+\frac{1}{2}\partial_{xx}\sigma^2$$

$$0=-\partial_t c+Ac \quad ,\mu=0$$

- The forward equations run forward in expiry and strike, rather than backwards in time and spot. Draw.

- We will show that these forward equations also exist in finite difference land.

- In fact, they are much easier to derive there.

Finite Difference Solution

- Consider the backward PDE

$$0 = f_t + Af \qquad ,A = -r + \mu\partial_x + \frac{1}{2}\sigma^2\partial_{xx} \tag{1}$$

- Suppose we work on a discrete grid of states $x_0 < \ldots < x_{n-1}$.

- Introduce the discrete difference operators

$$\begin{aligned}
\delta_x^+ f(x_i) &= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \quad ,0 \le i < n-1 \\
\delta_x^- f(x_i) &= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad ,0 < i \le n-1
\end{aligned} \tag{2}$$

- … and their weighted average and differences

$$\delta_x g(x_i) = (1-\lambda)\delta_x^- g(x_i) + \lambda \delta_x^+ g(x_i) \quad , \lambda = \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}$$

$$\delta_{xx} g(x_i) = 2\frac{(\delta_x^+ - \delta_x^-) g(x_i)}{x_{i+1} - x_{i-1}} \quad , 0 < i < n-1 \tag{3}$$

- We can now discretize the PDE (1) by central differences for the drift

$$\bar{A} = -r + \mu \delta_x + \frac{1}{2}\sigma^2 \delta_{xx} \tag{4}$$

- Or we can use dynamic up- and down-winding for the drift:

$$\bar{A} = -r + \mu^+ \delta_x^+ - \mu^- \delta_x^- + \frac{1}{2}\sigma^2 \delta_{xx} \tag{5}$$

- In this case, dependent on the sign of the drift we'll go either upwards or downwards.

- If we set absorption on the boundaries of the grid, i.e. $\sigma = \mu = 0$ for $i = 0, n-1$, then the PDE can be approximated as a *tridiagonal* matrix ODE

$$0 = f_t + \bar{A} f \tag{6}$$

- Tridiagonal means that the matrix $\bar{A}$ only has non-zero values on the diagonal and just around the diagonal

$$\bar{A}_{ij} = 0 \quad ; i = 0, n-1 \quad ; j < i-1, i+1 < j \tag{7}$$

>>> Draw: tridag matrix <<<

- Examples of tridiagonal matrices on compact form:

$$(\delta_x^+)_{i\cdot} = \left[0 \quad \frac{-1}{x_{i+1}-x_i} \quad \frac{1}{x_{i+1}-x_i} \right] \quad ,0 \le i < n-1$$

$$(\delta_x^-)_{i\cdot} = \left[\frac{-1}{x_{i+1}-x_i} \quad \frac{1}{x_{i+1}-x_i} \quad 0 \right] \quad ,0 < i \le n-1$$

$$(\delta_x)_{i\cdot} = \frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}(\delta_x^-)_{i\cdot} + \frac{x_i - x_{i-1}}{x_{i+1}-x_{i-1}}(\delta_x^+)_{i\cdot} \quad ,0 < i < n-1$$

$$(\delta_{xx})_{i\cdot} = \frac{2}{x_{i+1}-x_{i-1}}((\delta_x^+)_{i\cdot} - (\delta_x^-)_{i\cdot}) \quad ,0 < i < n-1$$

Theta Scheme

- The solution to the matrix ODE (6) is

$$f(t_h)=e^{\Delta t\bar{A}}f(t_{h+1})=\left(\sum_{k=0}^{\infty}\frac{(\Delta t\bar{A})^k}{k!}\right)f(t_{h+1}) \quad ,\Delta t=t_{h+1}-t_h \tag{8}$$

- We can discretize (8) in time using the Theta scheme

$$\begin{aligned} f(t_{h+1/2})&=[I+(1-\theta)\Delta t\bar{A}]f(t_{h+1}) \\ [I-\theta\Delta t\bar{A}]f(t_h)&=f(t_{h+1/2}) \end{aligned} \tag{9}$$

>>> Drawing: theta scheme <<<

- Different value of theta

  - $\theta=0$: Explicit

  - $\theta=1$: Implicit

  - $\theta=1/2$: Crank-Nicolson

- We have now turned the PDE problem into a recursive matrix-vector system.

Arrow Debreu Prices

- The matrix

$$H = [I - \theta \Delta t \bar{A}]^{-1}[I + (1-\theta)\Delta t \bar{A}]$$

- … is a matrix of "discounted transition probabilities" or Arrow-Debreu prices in the sense that H_{ij} is the (t_h, x_i) value of ***one*** paid if we hit $x(t_{h+1}) = x_j$ in the next time step:

$$H_{ij} = E[e^{-\int_{t_h}^{t_{h+1}} r(u)du} 1_{x(t_{h+1})=x_j} | x(t_h) = x_i] \underset{r=0}{=} \Pr[x(t_{h+1}) = x_j | x(t_h) = x_i]$$

- These need not be non-negative.

- But if they are, the discrete finite difference solution will produce arbitrage free option process.

Forward Finite Difference

- Equation (9) is solved backwards in discrete time steps:

$$t_h \leftarrow t_{h+1/2} \leftarrow t_{h+1}$$

- Specifically, we have the sequence of backwards matrix-vector multiplications

$$f(t_0, x(t_0)) = p(t_0)'\{H(t_0)\cdot\ldots\cdot H(t_m)\} f(t_m)$$

$$bwd:\ \Leftarrow\Leftarrow\ldots\Leftarrow\Leftarrow \tag{10}$$

$$fwd:\ \Rightarrow\Rightarrow\ldots\Rightarrow\Rightarrow$$

- … where $p(t_0,x_i)=1_{x_i=x(t_0)}$ is a vector of zero's except in one place – at the initial spot.

- We can run these operations in reverse, i.e. from left to right and thereby forward in time.

- Specifically,

$$\begin{aligned}[I-\theta\Delta t\bar{A}]'\,p(t_{h+1/2}) &= p(t_h) \\ p(t_{h+1}) &= [I+(1-\theta)\Delta t\bar{A}]'\,p(t_{h+1/2})\end{aligned} \tag{11}$$

- We then have

$$f(t_0, x(t_0)) = \sum_{i=0}^{n-1} p(t_m, x_i) f(t_m, x_i) = p(t_m)' f(t_m) \qquad (12)$$

- We thus have that (11) is the *discrete Fokker-Planck* equation for the *backward* system (9).

Dupire Equation

- For European call options we have

$$c(t_h,k)=\sum_{i=0}^{n-1} p(t_h,x_i)(x_i-k)^+ \tag{13}$$

- For the case of $\mu=0$, it can be shown that the European option prices solve

$$\begin{aligned}[I-\theta\Delta t\bar{A}]c(t_{h+1/2})&=c(t_h)\\ c(t_{h+1})&=[I+(1-\theta)\Delta t\bar{A}]c(t_{h+1/2})\end{aligned} \tag{14}$$

- … with initial boundary condition $c(t_0,k)=(x(t_0)-k)^+$.

- This is the discrete Dupire equation.

Finite Difference Duality

- Backward scheme

$$\begin{aligned} f(t_{h+1/2}) &= [I+(1-\theta)\Delta t\bar{A}]\,f(t_{h+1}) \\ [I-\theta\Delta t\bar{A}]\,f(t_h) &= f(t_{h+1/2}) \end{aligned} \tag{15}$$

- Forward scheme

$$\begin{aligned} [I-\theta\Delta t\bar{A}]'\,p(t_{h+1/2}) &= p(t_h) \\ p(t_{h+1}) &= [I+(1-\theta)\Delta t\bar{A}]'\,p(t_{h+1/2}) \end{aligned} \tag{16}$$

- Dupire equation ($\mu=0$)

$$
\begin{aligned}
[I-\theta\Delta t\bar{A}]c(t_{h+1/2}) &= c(t_h) \\
c(t_{h+1}) &= [I+(1-\theta)\Delta t\bar{A}]c(t_{h+1/2})
\end{aligned}
\tag{17}
$$

- … are all mutually ***discretely*** consistent.

- In the sense that the prices they produce will be the same no matter which method you use.

- You can for example calibrate to European options using the forward equation and reprice perfectly using the backward equation.

Accuracy

- By Taylor expansion of $f(t_h) = e^{\Delta t A} f(t_{h+1})$ we get the accuracies:
- Explicit $\theta = 0$: $O(\Delta t + \Delta x^2)$
- Implicit $\theta = 1$: $O(\Delta t + \Delta x^2)$
- Crank-Nicolson $\theta = 1/2$: $O(\Delta t^2 + \Delta x^2)$
- Dropping to $O(\Delta x)$ if up- and down-winding is used or if the grid spacing is strongly varying.

Stability: von Neumann

- Previous accuracy considerations are only local.
- For convergence we additionally need stability in the sense that $\| f \|$ is bounded as $\Delta t, \Delta x \to 0$.
- Von Neumann analysis: for constant grid and parameters look for eigen solutions of the form

$$e(t_h, x_i) = g^{-h} e^{\imath k x_i} \qquad , \imath = \sqrt{-1}, k \in \mathbb{R}, g \in \mathbb{C}$$

- If $|g| \leq 1 + O(\Delta t)$ for all k then we say that the scheme is von Neumann stable.

- For the constant parameter case von Neumann stability is sufficient to conclude that the scheme converges to the right result as $\Delta t, \Delta x \to 0$.

- For the general case, with general parameters and spacing it isn't proven to be sufficient for convergence.

- There are, however, to my knowledge, no counter examples.

- For the Theta scheme we have for constant parameters and grid spacing

$$g = \frac{[I + (1-\theta)\Delta t \bar{D}](e^{\imath kx})}{[I - \theta \Delta t \bar{D}](e^{\imath kx})} = \frac{1-(1-\theta)\Delta t[r - \imath(\mu / \Delta x)\sin(k\Delta x) + 2(\sigma / \Delta x)^2 \sin(k\Delta x / 2)^2]}{1+\theta \Delta t[r - \imath(\mu / \Delta x)\sin(k\Delta x) + 2(\sigma / \Delta x)^2 \sin(k\Delta x / 2)^2]}$$

Stability: Positivity

- Another related property is whether the transition probabilities, i.e. the elements of H are non-negative.

- I.e. whether all discrete option prices are consistent with absence of arbitrage.

- Explicit scheme, $\theta=0$, is vN stable iff $H \geq 0$, i.e. $\Delta t \leq O(\Delta x^2)$.

- Implicit scheme, $\theta=1$, is always vN stable but $H \geq 0$ only if up- and down-winding is used.

- CN scheme, $\theta=1/2$, is always vN stable but $H \geq 0$ only if up-and-down winding is used and the explicit condition $\Delta t \leq O(\Delta x^2)$ is met.

- CN scheme exhibits a (vaguely) oscillatory convergence pattern.
- However, CN converges way quicker $O(\Delta t^2)$ than the other methods and this is the method of choice if model parameters are given and we need to compute option prices as accurate as possible.
- However, for local volatility this is not really the task.
- Here, matching observed prices or producing arbitrage free option prices is the objective.
- In this case, the implicit scheme is the winner because it ensures $H \geq 0$.

>>> Live Example <<<

The Finite Difference Cheat Sheet

Schemes	Explicit $\theta = 0$	Implicit $\theta = 1$	Crank-Nicolson $\theta = 1/2$
PDE	$0 = f_t + Af \quad , A = -r + \mu\partial_x + \frac{1}{2}\sigma^2\partial_{xx}$		
FD central	$\bar{A} = -r + \mu\delta_x + \frac{1}{2}\sigma^2\delta_{xx}$		
FD winding	$\bar{A} = -r + \mu^+\delta_x^+ + \mu^-\delta_x^- + \frac{1}{2}\sigma^2\delta_{xx}$		
Boundaries	Absorption: $\mu = \sigma = 0$ or reflection: $\mu_0 > 0, \mu_{n-1} < 0, \sigma = 0$		
Backward	$f(t_h) = [I - \theta\Delta t\bar{A}]^{-1}[I + (1-\theta)\Delta t\bar{A}]f(t_{h+1})$		
Forward	$p(t_{h+1}) = [I + (1-\theta)\Delta t\bar{A}'][I - \theta\Delta t\bar{A}']^{-1}p(t_h)$		
Dupire	$c(t_{h+1}) = [I + (1-\theta)\Delta t\bar{A}][I - \theta\Delta t\bar{A}]^{-1}c(t_h) \quad , \mu = 0$		
Grid width	$\pm 5 \cdot (\int_0^T \sigma(u, x(0))^2 du)^{1/2}$		
Transform	PDE or grid spacing: $y = \int_{x_0}^{x} \sigma(a)^{-1} da$		
Vanilla strikes	Mid between grid points or $O(\Delta x^2)$ error		
Digitals	Mid between grid points or $O(\Delta x)$ error		
Cont barriers	On grid and absorption or $O(\Delta t^{1/2})$ error		
Von Neumann	$\Delta t \le O(\Delta x^2)$	Always	Always
$p \ge 0, \mu = 0$	$\Delta t \le O(\Delta x^2)$	Always	$\Delta t \le O(\Delta x^2)$
$p \ge 0, \mu \ne 0$	$\Delta t \le O(\Delta x^2)$	With winding	With winding and $\Delta t \le O(\Delta x^2)$

Accuracy central	$O(\Delta t+\Delta x^2)$	$O(\Delta t+\Delta x^2)$	$O(\Delta t^2+\Delta x^2)$
Accuracy winding	$O(\Delta t+\Delta x)$	$O(\Delta t+\Delta x)$	$O(\Delta t^2+\Delta x)$
Time/spatial steps	4	0.5	0.5
Models	Brownian motion	Non-parametric	Parametric
CPU/time step (n=200)	3e-6s	4e-6s	7e-6s

Coding the Theta Solver

- The theta solver consists of the components:

  - kFiniteDifference::dx() and ::dxx() construct the difference operators in compact matrix form.

  - kFd1d::calcAx() constructs the matrix $I + \Delta t\bar{A}$ in compact matrix form.

  - kMatrixAlgebra::banmul() and ::tridag() do band diagonal matrix multiplication and tridiagonal matrix-vector solution.

  - kFd1d::rollBwd() implements the backward roll by calling the above routines.

  - kMatrixAlgebra::transpose() transposes a band-diagonal matrix.

- kFd1d::rollFwd() implements the forward roll.

- All in all in the hood of a few hundred lines of code.
- This is what we are going to implement in this course.