Jesper Andreasen

Holes in Black-Scholes?

In this article, we provide a model-free test for whether the dynamic hedging argument actually works in practice and a quantification of how much jumps influence the delta hedge.

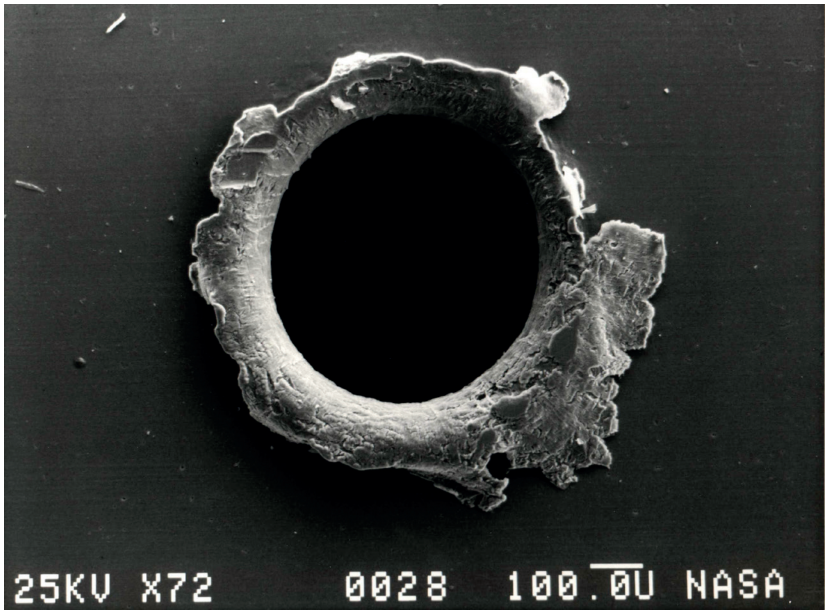
Merton's dynamic hedging argument is central to the derivation of the Black-Scholes partial differential equation and thereby the celebrated Black-Scholes option pricing formula. The key question here is whether the underlying stock can be assumed to evolve continuously or whether jumps matter. In this article, we provide a model-free test for whether the dynamic hedging argument actually works in practice and a quantification of how much jumps influence the delta hedge. For a diverse sets of assets, S&P 500, USD/EUR and Bitcoin, we find that jumps matter surprisingly little, even when options are only re-hedged on a daily basis. It is stochastic volatility that the option hedger needs to worry about.

Introduction

Stock prices generally move in ticks and the stock exchange is closed at night, so as such, stock prices can't really be described (or observed) as a Brownian motion. For the option hedger, however, the question is rather whether the Brownian motion approximation is close enough. This is what we consider in this article. Specifically, we describe a test procedure that can be used to assess whether a stock can be approximated as a continuous process. The test is based on Taylor expanding the logarithm of stock price increments and using these to quantify to what extent higher moments than the second are needed to describe how the stock moves. The test doesn't depend on any option data or interest rate or dividend information. As such, it is model-free.

For stock index, foreign exchange and Bitcoin, we find that jumps impact hedging very little. However, volatility is random and should be of major concern to any option hedger.

The rest of this article is organized as follows: In the next section, we go through Merton's delta hedging argument and derive Black-Scholes' partial differential equation for option prices. We then describe our time series test and our empirical results. Inspired by the empirical results, we return to the issue of volatility risk and how it affects the option hedger and round off with a section on different models with non-constant volatility.



NASA image, Public domain, via Wikimedia Commons

Black & Scholes — rock & roll

Consider a non-dividend-paying stock that follows a geometric Brownian motion:

$$\frac{dS(t)}{S(t)} = \mu\, dt + dW^P \tag{1}$$

where μ and σ are constants and W^P is a Brownian motion under the real physical measure P. Suppose the interest rate is constant and given by r.

Assume option prices can be written as a function of time and spot only, $C = C(t, S)$. Then Ito's lemma applied to C yields

$$dC = (C_t + \mu S C_S + \frac{1}{2}\sigma^2 S^2 C_{SS})dt + C_S \sigma\, S dW^P \tag{2}$$

Let Y be the value of a continuously re-balanced self-financing trading strategy that holds a number of stocks and the rest in the bank. This strategy evolves according to

$$dY = a dS + (Y - aS)\, r dt = a\mu\, S dt + a\sigma\, S dW^P + (Y - aS)\, r dt \tag{3}$$

If we, at a given point in time, assume that $Y = C$ and $a = C_S$ then the diffusion of the portfolio (3) and the option (2) are the same. Hence, it follows by arbitrage that the drift of the portfolio (3) and the option (2) must be the same too and we arrive at the PDE:

$$0 = C_t - rC + rSC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} \tag{4}$$

If the PDE is solved with respect to the boundary condition for a European call option $C(T, S) = (S - K)^+$ we get the celebrated Black-Scholes formula as the result.

On the other hand, if C is the solution to the PDE (4), then (3) defines a self-financing trading strategy that replicates the option. We conclude that the option price can only be given as the solution to (4).

The magic is that the drift μ drops out of the pricing PDE. In fact, the solution to (4) is also the expectation of the discounted terminal payoff under an equivalent probability measure Q, denoted the risk-neutral probability measure, where the expected return of the stock is the risk-free interest rate r:

$$\begin{aligned} C(t, S(t)) &= E_t^Q[e^{-r(T-t)}C(T, S(T))] \\ \frac{dS(t)}{S(t)} &= rdt + \sigma dW^Q \end{aligned} \tag{5}$$

So, investors may have different opinions on the expected return but still agree on the option prices. Hence, the measure P should be denoted the subjective rather than the objective probability measure.

As much as we like the delta hedging argument, there is actually a quicker
[AQ1] way to the target. — if the target is just the risk-neutral pricing equation (5), and no additional information about where it comes from. Suppose we knew that options could be priced according to

$$\begin{aligned} C(t, S(t)) &= E_t^R[e^{-a(T-t)}C(T, S(T))] \\ \frac{dS(t)}{S(t)} &= bdt + \sigma dW^R \end{aligned} \tag{6}$$

Then $a = r$ emerges by setting $C(T, S) = 1$, and $b = r$ follows from pricing $C(T, S) = S$.

Back on track: A crucial assumption for the hedging argument is that the stock evolves continuously. If this is not the case, we would not be able to neutralize the risk of the underlying by only delta hedging. In the next section, we provide a test for whether the stock price process can be assumed to be driven by a Brownian motion.

The log contract

Consider a stock that evolves continuously:

$$\frac{dS(t)}{S(t)} = \mu(t)\,dt + \sigma(t)\,dW^P \tag{7}$$

where μ and σ are general stochastic processes.

Using Ito's lemma, we obtain

$$\ln\frac{S(t)}{S(0)} - \underbrace{\int_0^t \frac{dS(u)}{S(u)}}_{\substack{\text{delta}\\ \text{strategy}}} = -\frac{1}{2}\underbrace{\int_0^t \left(\frac{dS(u)}{S(u)}\right)^2}_{\substack{\text{variance}\\ \text{contract}}} \tag{8}$$

This says that we can replicate a contract that pays the logarithm of the stock with a delta strategy in the underlying stock and a contract that pays the realized variance. Normally, this relation is used the other way around to synthesize the variance contract from a log-profile on the underlying stock. The log-profile can in turn be replicated by a static position in European options. In fact, this relation is used to define the VIX contract.

Equation (8) is a continuous equation whose validity only really holds if the underlying evolves continuously, and the delta strategy is continuously rebalanced. For a sequence of discrete time steps $\{t_h\}$, and discrete re-hedging, we have

$$\ln\frac{S(t_n)}{S(t_0)} - \underbrace{\sum_{h=0}^{n-1}\frac{\Delta S(t_h)}{S(t_h)}}_{\substack{\text{delta}\\ \text{strategy}}} = -\frac{1}{2}\underbrace{\sum_{h=0}^{n-1}\left(\frac{\Delta S(t_h)}{S(t_h)}\right)^2}_{\substack{\text{variance}\\ \text{contract}}} + \frac{1}{3}\underbrace{\sum_{h=0}^{n-1}\left(\frac{\Delta S(t_h)}{S(t_h)}\right)^3}_{\substack{\text{skewness}\\ \text{contract}}} - \frac{1}{4}\underbrace{\sum_{h=0}^{n-1}\left(\frac{\Delta S(t_h)}{S(t_h)}\right)^3}_{\substack{\text{kurtosis}\\ \text{contract}}} + \ldots \tag{9}$$

So, in the discrete time case we do in principle need the whole sequence of delta strategy, variance, skewness and kurtosis contracts and higher moments … to replicate the log-contract.

Our test is essentially for a given re-hedge frequency when you can cut off the sequence in (9). That is, in replicating the log contract, how much do each of the contracts, variance, skewness, and kurtosis, contribute. It seems most convenient to express these contributions in log-normal volatility terms:

$$\begin{aligned} \hat{\sigma}_2 &= \left[\frac{1}{T}\sum\left(\frac{\Delta S}{S}\right)^2\right]^{1/2} \\ \hat{\sigma}_3 &= \left[\frac{1}{T}\left(\sum\left(\frac{\Delta S}{S}\right)^2 - \frac{2}{3}\sum\left(\frac{\Delta S}{S}\right)^3\right)\right]^{1/2} \\ \hat{\sigma}_4 &= \left[\frac{1}{T}\left(\sum\left(\frac{\Delta S}{S}\right)^2 - \frac{2}{3}\sum\left(\frac{\Delta S}{S}\right)^3 + \frac{1}{2}\sum\left(\frac{\Delta S}{S}\right)^4\right)\right]^{1/2} \\ &\vdots \\ \hat{\sigma}_\infty &= \left[\frac{-2}{T}\left(\ln\frac{S(T)}{S(0)} - \sum\left(\frac{\Delta S}{S}\right)\right)\right]^{1/2} \end{aligned} \tag{10}$$

Or

$$\hat{\sigma}_{p,q}(t_m) = \left\{\frac{2}{t_m - t_{m-q}}\left[\sum_{k=2}^{p}\frac{(-1)^k}{k}\sum_{h=m-q}^{m-1}\left(\frac{S(t_{h+1}) - S(t_h)}{S(t_h)}\right)^k\right]\right\}^{1/2} \quad, p \geq 2 \quad, q \geq 1 \tag{11}$$

where p is the maximal moment included in calculating the volatility estimate and q is the number of time periods used.

The test can be performed for different length log-contracts, q, and from time series data for the stock price only. We do not need option prices or interest rate or dividend information.

Figure 1 shows the evolution of a rolling 20 trading-day log contract for S&P 500 from 1928–2021 converted into log-normal volatility. We see that the log contract, in terms of log-normal volatility, $\hat{\sigma}_{\infty,20}$, fluctuates between roughly 8% up to levels over 100% in crisis times. The four big spikes in stock market volatility are 1929, 1987, 2008, and 2020.

Figure 2 shows the contributions from the rolling 20-day skewness and kurtosis contracts to the log contract again in log-normal volatility terms. So $\hat{\sigma}_{3,20}, -\hat{\sigma}_{2,20}$ and $\hat{\sigma}_{4,20}, -\hat{\sigma}_{3,20}$, respectively. The highest contributions from the skewness contract are just over 2% in 1929 and 5% in 1987. Contributions from the kurtosis contract were under 1% over the full period.

Contributions for higher moments than $p = 4$ are really small and not shown here.

Figures 3 and 4 repeat the exercise for EUR/USD over the period 2004–2022. Here, the results are even more in favor of the Brownian motion approximation. Highest contributions for the skewness and kurtosis contracts are 3% and 0.5%, respectively, and outside 2008, there are almost no contributions.

Figures 5 and 6 consider Bitcoin for the period 2014 to 2022. Here, we do see higher contributions from skewness and kurtosis. Skewness contract peeks at 25% and kurtosis at 5%. However, this only happens once over the period. Over the rest of the period, skewness and kurtosis are below 9% and 1.5%, respectively. This should be compared with volatility levels that in many cases exceed 100%, and in a few cases hit 200%.

The conclusion from applying the log-contract test is that the effect of jumps is small and that it's the volatility that the option hedger should be worried about. In the next section, we consider the volatility risk that the option hedger is exposed to.

Fundamental theorem of option trading

Suppose the option trader uses a different volatility $\bar{\sigma}$ for pricing and hedging than the volatility that is realized σ. Also, for simplicity let us assume that the interest rate is zero, $r = 0$. That is, we consider an option book with a value V which satisfies the PDE:

$$0 = V_t + \frac{1}{2}\bar{\sigma}^2 S^2 V_{SS} \tag{12}$$

Using Ito's lemma and the PDE (10), we get

$$\begin{aligned} dV &= (V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS})\, dt + \sigma S V_S dW^P \\ &= \frac{1}{2}(\sigma^2 - \bar{\sigma}^2) S^2 V_{SS}\, dt + V_S(\mu S dt + \sigma S dW^P) \end{aligned} \tag{13}$$

If the book runner keeps his book delta flat, $V_S = 0$, by continuously trading the stock, his P&L will only be flat if either $V_{SS} = 0$, or $\bar{\sigma} = \sigma$.

Alternatively, he needs to be long gamma ($V_{SS} = 0$) when realized volatility is higher than implied volatility and short gamma ($V_{SS} < 0$) when realized volatility is lower than implied volatility. Looking at Figures 1, 3 and 5, they indicate that this may be more difficult than it sounds.

What is happening here is that when pricing volatility is different from realized volatility, $\bar{\sigma} \neq \sigma$, the strategy defined by PDE (12) is no longer self-financing and the book will make or lose money at a rate $O(dt)$.

One important lesson here is that volatility risk is a $O(dt)$ risk. Contrary to popular lingo, option books don't blow up — they bleed. Which may be more painful.

Local stochastic rough tough ARCH GARCH

We have seen that the underlying can be assumed to be moving almost continuously. However, volatility is far from constant and the option hedger with open gamma exposure will get hit, or benefit, at a rate of $O(dt)$ from not guessing the volatility correctly.

So, in this chapter, we will assume that the underlying stock follows:

$$\frac{dS(t)}{S(t)} = \sigma(t) dW^Q \tag{14}$$

where σ is a general process.

A popular way of incorporating the market expectation of volatility into pricing is to use the local volatility model:

$$\frac{dS(t)}{S(t)} = \vartheta(t, S(t)) dW^Q \tag{15}$$

where ϑ is a deterministic function.

If the local volatility model (15) is fitted to option prices, then we know that the local variance $\sigma(t, K)^2$ is the expected instantaneous variance conditional on being at the level $S(t) = K$, i.e.,

$$\vartheta(t, K)^2 = E^Q[\sigma(t)^2 \,|\, S(t) = K] \tag{16}$$

Stochastic local volatility usually means extending the local volatility with a single Markov stochastic volatility factor:

$$\begin{aligned} \frac{dS(t)}{S(t)} &= z(t)\vartheta(t, S(t)) dW^Q \\ dz(t) &= \beta(z(t))dt + \varepsilon(z(t))\, dZ^Q \quad, dW^Q \cdot dZ^Q = \rho dt \end{aligned} \tag{17}$$

Again, perfect fit to market prices of options means that the model shares conditional expectation to local variance with the market:

$$\vartheta(t, K)^2 E^Q[z(t) \,|\, S(t) = K] = E^Q[\sigma(t)^2 \,|\, S(t) = K] \tag{18}$$

Relations (16) and (18) are due to Gyöngy (1986) and Dupire (1994). The results can be derived bys setting the pricing volatility in (13) to zero.

The local volatility and stochastic local volatility models are the industry standard for pricing exotics. The local volatility model is also used as a tool for marking vanilla options.

These models can easily be criticized. They have several drawbacks and miss many empirical observations. First, the local volatility and stochastic local volatility models are by nature typically non-stationary, and forward smiles and skews tend to be very different from today's smiles. This means that one needs to be careful about using them for pricing forward starting options such as cliquets. Also, they tend to be out of whack with the pricing of volatility derivatives. They typically tend to *overvalue* options on VIX, for example.

Whereas there is broad agreement that these models aren't perfect, there's less agreement on what is actually better. Particularly when *better* also weighs in the difficulty of implementing significantly more computationally troublesome models in a trading environment. That is, including things like calibration, risk report generation and general production tasks.

Within the past decade, rough volatility models have received a lot of attention in academic circles. See, for example, Bayer, Friz and Gatheral (2016). The idea is that volatility is driven by a fractional integrated Brownian motion:

$$Z^H(t) = \int_0^t (t-u)^{H-1/2}\, dZ(u) \tag{19}$$

where $H > 0$ is the so-called Hurst exponent. $H = 1/2$ corresponds to a standard Brownian motion and $H \to 0$ to white noise.

Proponents of this model argue that sample paths of implied and realized volatility, like the ones in Figures 1, 3 and 5, look hairy and a lot more as if they were driven fractionally integrated noise with small H, than a standard Brownian motion. Also, implied volatility skews tend to decay at a rate of approximately $O(T^{1/2})$, which is consistent with $H \approx 0$. Further, it seems that rough volatility does a better job at reconciling the pricing of S&P 500 and VIX options than conventional stochastic volatility models. But it is still a struggle.

Recently, Guyon and Lekeufack (2022) argue that volatility can and should be modeled purely as a function of the path of the stock. That is, no extra Brownian motion is necessary. Pure stock price path dependence is sufficient. This is based on empirical time series investigation of VIX and other volatility indexes, as well as the joint pricing of S&P and VIX options. It is also based on economic and mathematical arguments.

The economic argument goes like this: Markets are efficient and all the information about the stock, including its volatility, is included in the path of the stock.

The mathematical argument is the following intriguing result by Brunick and Shreve (2013). When S follows (14), the distribution of the path $\{S(u)\}_{0\le u\le t}$ is the same as the distribution of the path $\{\tilde{S}(u)\}_{0\le u\le t}$ where $\tilde{S}$ follows:

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \vartheta(t, \{\tilde{S}(u)\}_{0\le u\le t})\, dW^Q \tag{20}$$

$$\vartheta(t, \{\tilde{S}(u)\}_{0\le u\le t})^2 = E^Q[\sigma(t)^2 \mid \{S(u)\}_{0\le u\le t}]$$

This is a path-wise extension of the Gyöngy-Dupire relations (16) and (18).

In other words, path dependent models span the space of all processes if we're only concerned with the path of the stock. And, in fact, if we only observe the stock price path, we will never be able to tell the difference anyways. Let that linger for a while.

Does that mean stock and volatility are perfectly correlated? No. One of the fascinating things about the Brownian motion and path dependence is that it can generate decorrelation very quickly. Instantaneously, actually. For example:

$$\mathrm{corr}[Z^H(t), Z(t)] = \frac{\sqrt{2H}}{H + 1/2} \tag{21}$$

Does it mean that we can hedge options or options on options by trading the stock only? If we consider a forward quantity such as

$$F(t,T) = E_t[Z^H(T)] = \int_0^t (T-u)^{H-1/2}\, dZ(u) \tag{22}$$

we see that

$$dF(t,T) = (T-t)^{H-1/2}\, dZ(t) \tag{23}$$

So, if we try to replicate F by trading Z, then the necessary holding will go to infinity as we approach T, i.e., $(T-t)^{H-1/2} \to \infty$ as $t \to T$. This will conflict with reality and practicality. The model risk will be proportional to $(T-t)^{H-1/2}$. So yes, we can hedge F with Z, except when we get close to expiration, then we can't. This is similar to the situation that we have if we try to delta hedge a digital option in Black-Scholes model. This of course compounds on the risk that we may not know exactly what H and other parameters are.

Path-dependent volatility is far from a new invention. Robert Engle wrote about the ARCH model in 1982 and was awarded the Nobel Prize for it in 2003. So history is repeating.

As already touched upon, the challenge with path-dependent models is two-fold. In their native form, they can only be implemented using high-dimensional Monte-Carlo with very frequent, say daily, time stepping. The other, and probably more important problem, is that there is no clear consensus on exactly what path-dependent model to choose. There are zoos of them: rough-tough-this-and-that, and ARCH-GARCH-that-and-this. But I'm not in doubt that at some stage technological development will bring these models into practical use. I'm looking forward to it.

Conclusion

No, no major holes in Black-Scholes. Stock index, foreign exchange and Bitcoins are well approximated as continuous processes. Volatility is non-constant but probably more path-dependent than local or traditional Markovian. Lots of challenges in getting path-dependency to work in practice.

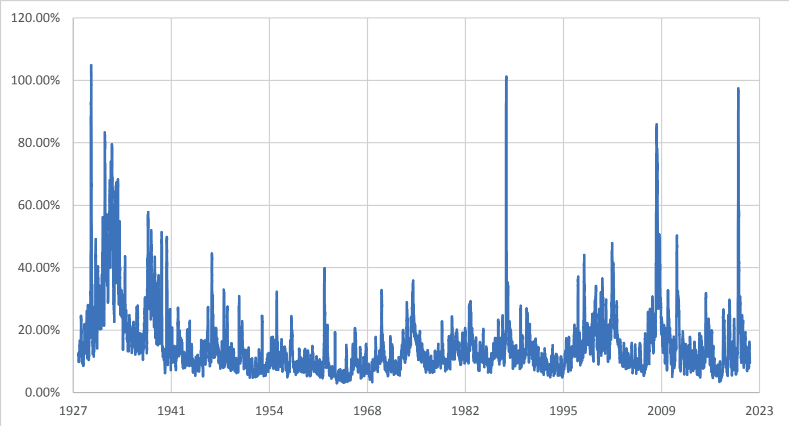
About the Author

Jesper Andreasen
Saxo Bank, Copenhagen
kwant.daddy@saxobank.com

REFERENCES

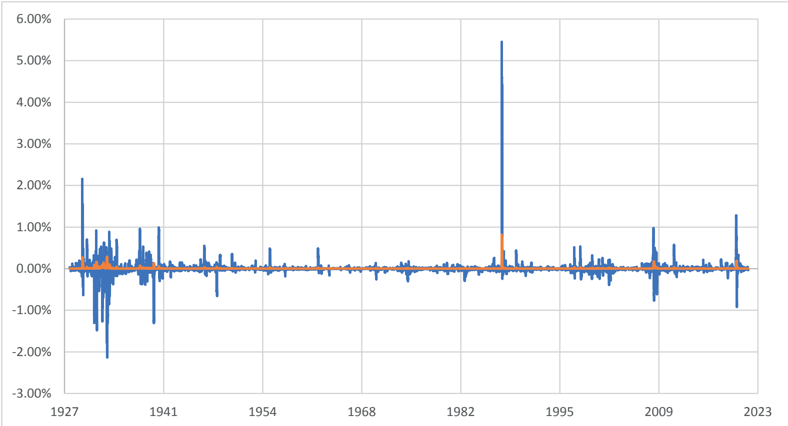
Andreasen, J. 2023. Catch Up. *Wilmott* January.
Bayer, C., Friz, P. and Gatheral, J. 2016. Pricing under Rough Volatility. *Quantitative Finance* 16.
Black, F. and Scholes, M. 1973. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81.
Brunick, G. and Shreve, S. 2013. Mimicking an Ito Process by a Solution of a Stochastic Differential Equation. *Annals of Applied Probability* 23.
Dupire, B. 1994. Pricing with a Smile. *Risk* January.
Engle, R. 1982. Autoregressive Conditional Heteroscedacity with Estimates of Variance of United Kingdom Inflation. *Econometrica* 50.
Guyon, J. and Lekeufack, J. 2022. Volatility is (Mostly) Path-Dependent. *SSRN*.
Gyöngy, I. 1986. Mimicking the One-Dimensional Marginal Distribution of Processes Having an Ito Differential. *Probability Theory and Related Fields* 71.
Merton, R. 1973. Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science* 4

Figure 1: S&P 500 log-contracts.



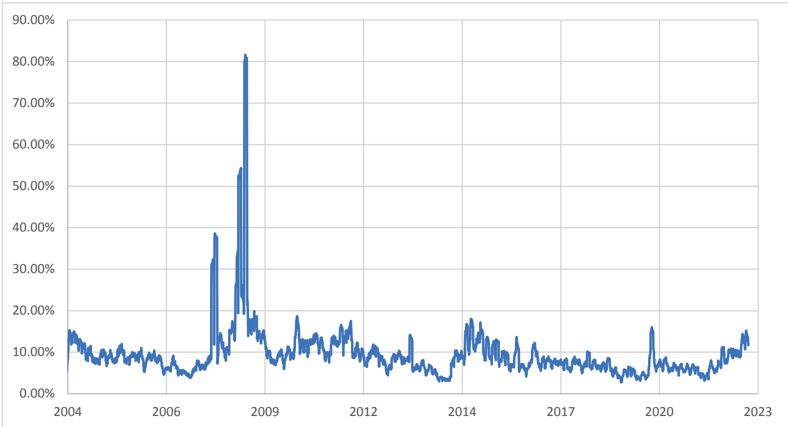
Rolling 20 business day log-contract for S&P 500 in log-normal volatility terms $\hat{\sigma}_{\infty,20}$.

Figure 2: S&P 500 skewness and kurtosis contracts.



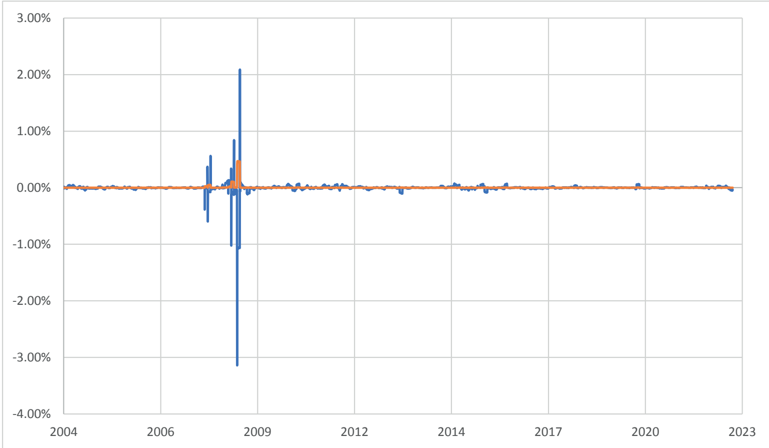
Rolling 20 business day skewness $\hat{\sigma}_{3,20} - \hat{\sigma}_{2,20}$ (blue) and kurtosis $\hat{\sigma}_{4,20} - \hat{\sigma}_{3,20}$ (orange) contributions to the 20 business day rolling log-contracts for the S&P 500 stock index.

Figure 3: EUR/USD log-contracts.



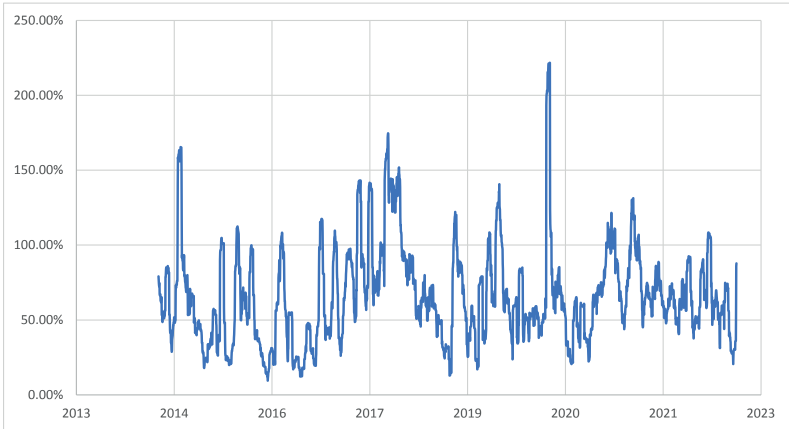
Rolling 20 business day log-contract for EUR/USD in log-normal volatility terms $\hat{\sigma}_{\infty,20}$.

Figure 4: EUR/USD skewness and kurtosis contracts.



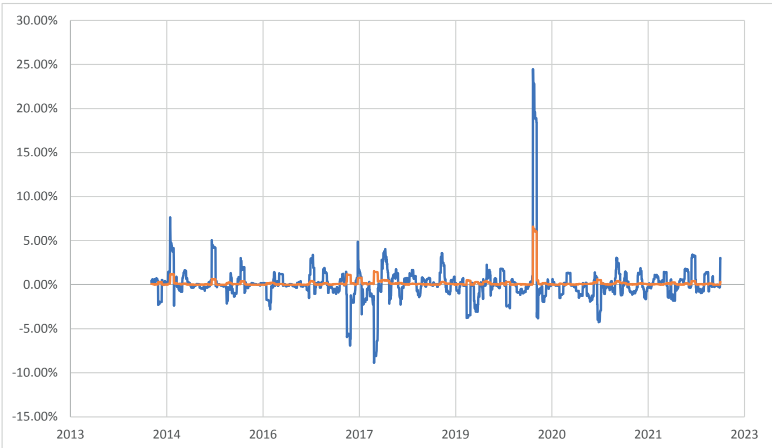
Rolling 20 business day skewness $\hat{\sigma}_{3,20} - \hat{\sigma}_{2,20}$ (blue) and kurtosis $\hat{\sigma}_{4,20} - \hat{\sigma}_{3,20}$ (orange) contributions to the 20 business day rolling log-contracts for EUR/USD.

Figure 5: Bitcoin log-contracts.



Rolling 20 business day log-contract for Bitcoin in log-normal volatility terms $\hat{\sigma}_{\infty,20}$.

Figure 6: Bitcoin skewness and kurtosis contracts.



Rolling 20 business day skewness $\hat{\sigma}_{3,20} - \hat{\sigma}_{2,20}$ (blue) and kurtosis $\hat{\sigma}_{4,20} - \hat{\sigma}_{3,20}$ (orange) contributions to the 20 business day rolling log-contracts for Bitcoin.

AQ1]If the target is just (5) and not why. Could you please recheck this sentence? It seems incomplete.