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St: **Planck-Fokker Boundary Conditions**
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We derive boundary conditions for finite difference solution of the Fokker-Planck equation and consider the continuous limit.

The Backward Equation

Consider the backward PDE equation

$$
\begin{aligned}
&0 = \frac{\partial F}{\partial t} + AF \\
&A(t,x) = -r(t,x) + \mu(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(t,x)^2 \frac{\partial^2}{\partial x^2}
\end{aligned}
\tag{1}
$$

In numerical solution of the system (1) we typically assume $\partial^2 F / \partial x^2 = 0$ or, equivalently, $\sigma = 0$, at the boundaries and accordingly approximate the operator A on the discrete domain $(x_0, \ldots, x_n)$, by

$$
\bar{A} = \begin{bmatrix}
-r_0 - \frac{\mu_0}{\Delta x} & \frac{\mu_0}{\Delta x} & & & & & \\
-\frac{\mu_1}{2\Delta x} + \frac{\sigma_1^2}{2\Delta x^2} & -r_1 - \frac{\sigma_1^2}{\Delta x^2} & \frac{\mu_1}{2\Delta x} + \frac{\sigma_1^2}{2\Delta x^2} & & & & \\
& -\frac{\mu_2}{2\Delta x} + \frac{\sigma_2^2}{2\Delta x^2} & -r_2 - \frac{\sigma_2^2}{\Delta x^2} & \frac{\mu_2}{2\Delta x} + \frac{\sigma_2^2}{2\Delta x^2} & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & -\frac{\mu_{n-2}}{2\Delta x} + \frac{\sigma_{n-2}^2}{2\Delta x^2} & -r_{n-2} - \frac{\sigma_{n-2}^2}{\Delta x^2} & \frac{\mu_{n-2}}{2\Delta x} + \frac{\sigma_{n-2}^2}{2\Delta x^2} & \\
& & & & -\frac{\mu_{n-1}}{2\Delta x} + \frac{\sigma_{n-1}^2}{2\Delta x^2} & -r_{n-1} - \frac{\sigma_{n-1}^2}{\Delta x^2} & \frac{\mu_{n-1}}{2\Delta x} + \frac{\sigma_{n-1}^2}{2\Delta x^2} \\
& & & & & -\frac{\mu_n}{\Delta x} & -r_n + \frac{\mu_n}{\Delta x}
\end{bmatrix}
$$

with $r_i = r(t,x_i), \mu_i = \mu(t,x_i), \sigma_i = \sigma(t,x_i)$ for $i = 0, \ldots, n$. Absorption at the boundaries can be incorporated by setting $\mu_0 = \mu_n = 0$, reflection by setting $\mu_0 > 0, \mu_n < 0$.

Let $f(t) = (f_0(t), \ldots, f_n(t))'$ be the solution to

$$
0 = \frac{\partial f}{\partial t} + \bar{A}f \tag{2}
$$

The Forward Equation

Multiplying (2) by a vector function $p(t) = (p_0(t), \ldots, p_n(t))'$ and integrating over time yields

$$
\begin{aligned}
0 &= \int_0^T p(t)' \frac{\partial f(t)}{\partial t} dt + \int_0^T p(t)' \bar{A}(t) f(t) dt \\
&= p(T)' f(T) - p(0)' f(0) - \int_0^T \frac{\partial p(t)'}{\partial t} f(t) dt + \int_0^T p(t)' \bar{A}(t) f(t) dt \\
&= p(T)' f(T) - p(0)' f(0) + \int_0^T (-\frac{\partial p(t)}{\partial t} + \bar{A}(t)' p(t))' f(t) dt
\end{aligned}
\tag{3}
$$

So if p is the solution to

$$
\begin{aligned}
&0 = -\frac{\partial p}{\partial t} + \bar{A}' p \\
&p_i(0) = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}
\end{aligned}
\tag{4}
$$

Then p has the property that

$$
f_j(0) = \sum_{i=0}^{n} f_i(T) p_i(T) \tag{5}
$$

Hence, p is the Green's function of the discrete backward problem (2) which relates to the density q by $p = q\Delta x$.

We have

$$
\bar{A}' = \begin{bmatrix}
-r_0 - \frac{\mu_0}{\Delta x} & -\frac{\mu_1}{2\Delta x} + \frac{\sigma_1^2}{2\Delta x^2} & & & & & \\
\frac{\mu_0}{\Delta x} & -r_1 - \frac{\sigma_1^2}{\Delta x^2} & -\frac{\mu_2}{2\Delta x} + \frac{\sigma_2^2}{2\Delta x^2} & & & & \\
& \frac{\mu_1}{2\Delta x} + \frac{\sigma_1^2}{2\Delta x^2} & -r - \frac{\sigma_2^2}{\Delta x^2} & -\frac{\mu_3}{2\Delta x} + \frac{\sigma_3^2}{2\Delta x^2} & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \frac{\mu_{n-3}}{2\Delta x} + \frac{\sigma_{n-3}^2}{2\Delta x^2} & -r_{n-2} - \frac{\sigma_{n-2}^2}{\Delta x^2} & -\frac{\mu_{n-1}}{2\Delta x} + \frac{\sigma_{n-1}^2}{2\Delta x^2} & \\
& & & & \frac{\mu_{n-2}}{2\Delta x} + \frac{\sigma_{n-2}^2}{2\Delta x^2} & -r_{n-1} - \frac{\sigma_{n-1}^2}{\Delta x^2} & -\frac{\mu_n}{\Delta x} \\
& & & & & \frac{\mu_{n-1}}{2\Delta x} + \frac{\sigma_{n-1}^2}{2\Delta x^2} & -r_n + \frac{\mu_n}{\Delta x}
\end{bmatrix}
$$

where the discrete boundary conditions are given as the top and bottom two rows of the matrix above. We note that for $x_1 < x < x_{n-1}$ the continuous limit is the usual Fokker-Planck equation

$$0 = -\frac{\partial q}{\partial t} - \frac{\partial}{\partial x}[\mu q] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2 q] \tag{6}$$

Inserting the first row in the second in (4) and substituting $p = q\Delta x$ yields

$$0 = -\frac{\partial q_0 \Delta x}{\partial t} - \frac{\partial q_1 \Delta x}{\partial t} - r_0 q_0 \Delta x - r_1 q_1 \Delta x - \frac{\mu_1}{2\Delta x} q_1 \Delta x - \frac{\mu_2}{2\Delta x} q_2 \Delta x + \frac{1}{2}\frac{1}{\Delta x^2}(\sigma_2^2 q_2 \Delta x - \sigma_1^2 q_1 \Delta x)$$

If we assume q is continuous, then

$$0 = O(\Delta x) - \frac{1}{2}(\mu_1 q_1 + \mu_2 q_2) + \frac{1}{2}\frac{1}{\Delta x}(\sigma_2^2 q_2 - \sigma_1^2 q_1)$$

Taking the limit $\Delta x \to 0$, thus yields the boundary condition

$$0 = -\mu q + \frac{1}{2}\frac{\partial}{\partial x}[\sigma^2 q] \quad , x = x_0 \tag{7}$$

Incidentally, this result coincides with the result found in Lucic (2008).

For numerical solution, however, (6) and (7) are not needed. We can base the numerical solution for q directly on (4).

The analysis extends to the multidimensional case by simply replacing the spatial operator A with the sum of the spatial operators for the different dimensions. For the two dimensional case, for example, we set $A = A_x + A_y + A_{xy}$.