Volatility interpolation

Developing an arbitrage-free, consistent volatility surface in both expiry and strike from a discrete set of option quotes is a difficult and computationally intense problem. In this article, ***Jesper Andreasen*** *and* ***Brian Huge*** *use a non-standard variant of the fully implicit finite difference method to reduce the computational cost by orders of magnitude. An example shows how the model can fit the Eurostoxx 50 option market in approximately 0.05 seconds of CPU time*

Local volatility models such as those of Dupire (1994), Andersen & Andreasen (1999), JP Morgan (1999) and Andreasen & Huge (2010) ideally require a full continuum in expiry and strike of arbitrage-consistent European-style option prices as input. In practice, of course, we only observe a discrete set of option prices.

It is well known that interpolation and extrapolation of a two-dimensional implied volatility surface is a non-trivial problem, particularly if one wishes to preserve characteristics that guarantee arbitrage-free option prices. Previous attempts to solve the problem include: interpolation in the strike dimension via fitting of an implied density; best-fit approaches where parametric option pricing models such as the Heston and SABR models are fitted to observed option prices and subsequently used for interpolation; and full-scale non-parametric optimisation approaches where local volatility models are fitted directly to observed option prices (see, for example, Jackwerth & Rubinstein, 1996, Sepp, 2007, Coleman, Li & Verma, 1999, and Avellaneda *et al*, 1997).

All these approaches, however, suffer from drawbacks: the implied-density route does not directly lend itself to interpolation in the maturity dimension; the parametric model approach will not necessarily exactly match all the observed option prices; and the full-scale optimisation technique is computationally costly.

Our modelling approach is based on the finite difference solution of the Dupire (1994) forward equation for option prices, and, as such, is related to the work by Carr (2008), where it is shown that at one step the implicit finite difference method can be viewed as option prices coming from a local variance gamma model. The methodology is related to the implied-density approach and can be specified to give an exact fit to the observed option prices. But, contrary to the implied-density approach, it directly allows for arbitrage-consistent interpolation in the maturity dimension.

For each maturity, a non-linear optimisation problem has to be solved. The number of free parameters will typically be equal to the number of targets, that is, strikes. An update in the optimisation problem is quick as it only involves one time step in the implicit finite difference method, that is, the solution of one tri-diagonal matrix system, a reduction by an order of magnitude or more on traditional approaches. The model calibration can be bootstrapped in the maturity direction but global optimisation is also an option.

After the model is calibrated, the full continuous surface of option prices is, again, generated by a single time step finite difference solution of Dupire's forward equation. Typical interpolation problems for equity options can be solved in a few hundredths of a second of CPU time.

Discrete expiries

Given a time grid of expiries $0 = t_0 < t_1 < \ldots$ and a set of volatility functions $\{\vartheta(k)\}_{i=0,1,\ldots}$, we construct European-style option prices for all the discrete expiries, by recursively solving the forward system:

$$\left[1-\tfrac{1}{2}\Delta t_i\vartheta_i(k)^2\frac{\partial^2}{\partial k^2}\right]c(t_{i+1},k)=c(t_i,k),\qquad c(0,k)=(s(0)-k)^+,\ \ i=0,1,\ldots \tag{1}$$

where $\Delta t_i = t_{i+1} - t_i$.

If we discretise the strike space $k_j = k_0 + j\Delta k$, $j = 0,1, \ldots , n$ and replace the differential operator by the difference operator, we get the following finite difference scheme:

$$\begin{aligned}&\left[1-\tfrac{1}{2}\Delta t_i\vartheta_i(k)^2\delta_{kk}\right]c(t_{i+1},k)=c(t_i,k),\\ &c(0,k)=(s(0)-k)^+,\ \ i=0,1,\ldots\\ &\delta_{kk}f(k)=\frac{1}{\Delta k^2}\big(f(k-\Delta k)-2f(k)+f(k+\Delta k)\big)\end{aligned} \tag{2}$$

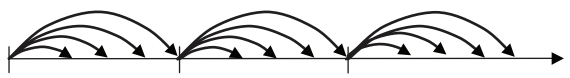
The system (2) can be solved by recursively solving tri-diagonal matrix systems. One can thus view the system (1) as a one-step per expiry implicit finite-difference discretisation of the Dupire (1994) forward equation:

$$0=\frac{\partial c}{\partial t}+\tfrac{1}{2}\sigma(t,k)^2\frac{\partial^2 c}{\partial k^2} \tag{3}$$

For a set of discrete option quotes $\{\hat{c}(t_i, k_{ij})\}$, the system (1) can be bootstrapped forward, expiry by expiry, to find piecewise constant functions:

$$\vartheta_i(k)=a_{ij},\quad b_{i,j-1}<k\le b_{ij} \tag{4}$$

1 Model timeline



that minimise the pricing error in (1). In other words, we solve the optimisation problems:

$$\inf_{\vartheta_i(\cdot)} \sum_j \left(\left(c\left(t_i, k_{ij}\right) - \hat{c}\left(t_i, k_{ij}\right) \right) / w_{ij} \right)^2, w_{ij} = \partial \hat{c}\left(t_i, k_{ij}\right) / \partial \hat{\sigma}\left(t_i, k_{ij}\right) \quad (5)$$

sequentially for $i = 1, 2, \ldots$. Here $\hat{\sigma}$ denotes implied Black volatility. The point here is that for each iteration in (5) only one tri-diagonal matrix system (2) needs to be solved.

Filling the gaps

The system (1) translates the local volatility functions into arbitrage-consistent prices for a discrete set of expiries but it does not directly specify the option prices between the expiries. We fill the gaps by constructing the option prices between two expiries according to:

$$\left[1 - \tfrac{1}{2}\left(t - t_i\right) \vartheta_i\left(k\right)^2 \frac{\partial^2}{\partial k^2} \right] c\left(t, k\right) = c\left(t_i, k\right), \quad t \in \left]t_i, t_{i+1}\right[\quad (6)$$

Note that for expiries that lie between the quoted expiries, the time stepping is non-standard. Instead of multiple small time steps that connect all the intermediate time points, we step directly from t_i to all times $t \in \left]t_i, t_{i+1}\right[$. The time-stepping scheme is illustrated in figure 1. This methodology is essentially what distinguishes our modelling approach from previously presented finite difference-based algorithms, for example, Coleman , Li & Verma (1999) and Avellaneda *et al* (1997).

Absence of arbitrage and stability

Carr (2008) shows that the option prices generated by (1) are consistent with the underlying being a local variance gamma process. From this or from straight calculation we have that (6) can be written as:

$$c\left(t, k\right) = \int_0^\infty \frac{1}{t - t_i} e^{-u/\left(t - t_i\right)} g\left(u, k\right) du, \quad t > t_i \quad (7)$$

where $g(u, k)$ is the solution to:

$$\begin{aligned} 0 &= -\frac{\partial g}{\partial u} + \tfrac{1}{2} \vartheta\left(k\right)^2 \frac{\partial^2 g}{\partial k^2}, \quad u > 0 \\ g\left(0, k\right) &= c\left(t_i, k\right) \end{aligned} \quad (8)$$

In the appendix, we use this to show that the option prices generated by (1) and (6) are consistent with absence of arbitrage, that is, that $c_t(t, k) \geq 0$, $c_{kk}(t, k) \geq 0$ for all (t, k).

For the discrete space case, we note that with the additional

A. SX5E implied volatility quotes (%)

k\t	0.025	0.101	0.197	0.274	0.523	0.772	1.769	2.267	2.784	3.781	4.778	5.774
51.31									33.66	32.91		
58.64									31.78	31.29	30.08	
65.97									30.19	29.76	29.75	
73.30									28.63	28.48	28.48	
76.97				32.62	30.79	30.01	28.43					
80.63				30.58	29.36	28.76	27.53	27.13	27.11	27.11	27.22	28.09
84.30				28.87	27.98	27.50	26.66					
86.13	33.65											
87.96	32.16	29.06	27.64	27.17	26.63	26.37	25.75	25.55	25.80	25.85	26.11	26.93
89.79	30.43	27.97	26.72									
91.63	28.80	26.90	25.78	25.57	25.31	25.19	24.97					
93.46	27.24	25.90	24.89									
95.29	25.86	24.88	24.05	24.07	24.04	24.11	24.18	24.10	24.48	24.69	25.01	25.84
97.12	24.66	23.90	23.29									
98.96	23.58	23.00	22.53	22.69	22.84	22.99	23.47					
100.79	22.47	22.13	21.84									
102.62	21.59	21.40	21.23	21.42	21.73	21.98	22.83	22.75	23.22	23.84	23.92	24.86
104.45	20.91	20.76	20.69									
106.29	20.56	20.24	20.25	20.39	20.74	21.04	22.13					
108.12	20.45	19.82	19.84									
109.95	20.25	19.59	19.44	19.62	19.88	20.22	21.51	21.61	22.19	22.69	23.05	23.99
111.78	19.33	19.29	19.20									
113.62				19.02	19.14	19.50	20.91					
117.28				18.85	18.54	18.88	20.39	20.58	21.22	21.86	22.23	23.21
120.95				18.67	18.11	18.39	19.90					
124.61				18.71	17.85	17.93	19.45		20.54	21.03	21.64	22.51
131.94									19.88	20.54	21.05	21.90
139.27									19.30	20.02	20.54	21.35
146.60									18.49	19.64	20.12	

Note: the table shows implied Black volatilities for European-style options on the SX5E index. Expiries range from two weeks to a little under six years and strikes range from 50–146% of current spot of 2,772.70. Data is as of March 1, 2010

(absorbing) boundary conditions $c_{kk}(t, k_0) = c_{kk}(t, k_n) = 0$, (2) can be written as:

$$Ac\left(t_{i+1}\right) = c\left(t_i\right) \quad (9)$$

where A is the tri-diagonal matrix:

$$A = \begin{bmatrix} 1 & 0 & & & & \\ -z_1 & 1 + 2z_1 & -z_1 & & & \\ & -z_2 & 1 + 2z_2 & -z_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -z_{n-1} & 1 + 2z_{n-1} & -z_{n-1} \\ & & & & 0 & 1 \end{bmatrix}$$

$$z_j = \tfrac{1}{2} \frac{\Delta t}{\Delta k^2} \vartheta_i\left(k_j\right)^2 \quad (10)$$

The tri-diagonal matrix A is diagonally dominant with positive diagonal and negative off-diagonals. Nabben (1999) shows that

B. SX5E calibration accuracy

k\t	0.025	0.101	0.197	0.274	0.523	0.772	1.769	2.267	2.784	3.781	4.778	5.774
51.31									0.00	0.00		
58.64									0.00	−0.02	0.08	
65.97									0.00	0.02	−0.23	
73.30									0.00	−0.02	0.05	
76.97				−0.02	−0.01	0.00	0.00					
80.63				−0.02	−0.01	0.00	0.01	0.00	0.00	0.01	0.06	0.00
84.30				0.00	0.00	0.00	−0.02					
86.13	0.01											
87.96	−0.07	−0.05	0.01	0.02	0.01	−0.01	0.01	0.00	0.00	−0.01	−0.02	0.00
89.79	0.02	0.01	0.00									
91.63	0.01	0.01	0.00	0.02	0.01	0.00	−0.01					
93.46	−0.02	−0.02	0.00									
95.29	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.01	−0.01	0.00
97.12	0.02	0.01	−0.01									
98.96	−0.01	−0.01	0.00	0.00	0.00	0.00	0.00					
100.79	0.01	0.00	0.00									
102.62	0.01	−0.01	0.00	0.00	0.00	−0.01	−0.01	0.00	0.00	−0.03	0.00	0.00
104.45	0.01	0.00	0.02									
106.29	−0.06	−0.01	0.00	0.01	0.00	0.03	0.01					
108.12	0.00	0.00	−0.02									
109.95	−0.10	−0.09	0.00	−0.02	0.00	0.02	−0.01	0.00	−0.01	0.02	−0.02	0.00
111.78	−0.02	0.03	−0.04									
113.62				0.03	0.00	−0.01	0.00					
117.28				−0.03	0.00	0.01	0.00	0.00	0.02	−0.02	0.00	0.00
120.95				0.01	0.00	−0.02	0.00					
124.61				0.00	0.02	0.07	0.02		−0.03	0.02	−0.02	0.00
131.94									0.00	−0.05	0.01	0.00
139.27									0.00	0.01	−0.01	−0.01
146.60									0.02	−0.01	0.00	

Note: the table shows the difference between the model and the target in implied Black volatilities for European-style options on the SX5E index. Data is as of March 1, 2010

for this type of matrix:

$$A^{-1} \geq 0 \tag{11}$$

This implies that the discrete system (2) is stable. As we also have:

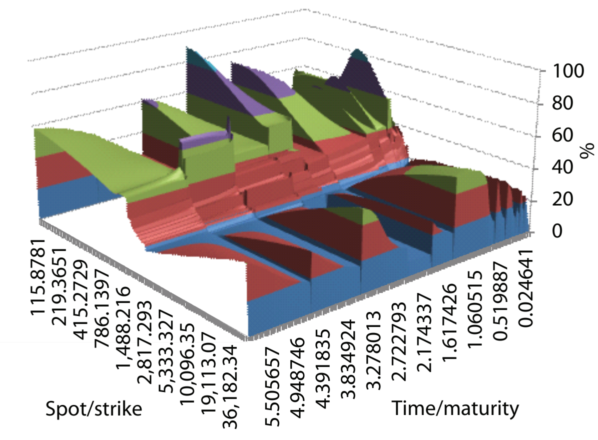
$$A^{-1}\iota = \iota, \quad \iota = (1,\ldots,1)' \tag{12}$$

we can further conclude that the discrete system (2) is arbitrage-free. Because of the tri-diagonal form of the matrix and the discretisation, this also holds if the spacing is non-equidistant.

If the problem is formulated in logarithmic space, $x = \ln k$, as would often be the case, then the discrete system (2) becomes:

$$\begin{aligned}
&\left[1 - \tfrac{1}{2}\Delta t_i \vartheta_i(x)^2 (\delta_{xx} - \delta_x)\right] c(t_{i+1}, x) = c(t_i, x), \\
&c(0,x) = \left(s(0) - e^x\right)^+, \quad i = 0,1,\ldots \\
&\delta_x f(x) = \frac{1}{2\Delta x}\left(f(x+\Delta x) - f(x-\Delta x)\right) \\
&\delta_{xx} f(x) = \frac{1}{\Delta x^2}\left(f(x-\Delta x) - 2f(x) + f(x+\Delta x)\right)
\end{aligned} \tag{13}$$

2 Local volatility derived from model option prices



Note: the graph shows the local volatility surface in the model after it has been fitted to the SX5E market. Data is as of March 1, 2010

It follows that the system is stable if $\Delta x = \ln(k_{j+1}/k_j) \leq 2$, not a constraint that will be breached in any practical application.

As shown in the appendix, (1) and (6) can be slightly generalised by introducing a deterministic time-change $T(t)$:

$$\left[1 - \tfrac{1}{2}\left(T(t) - t_i\right)\vartheta_i(k)^2 \frac{\partial^2}{\partial k^2}\right] c(t,k) = c(t_i,k), \quad t \in \left]t_i, t_{i+1}\right] \tag{14}$$

where $T(t_i) = t_i$ and $T'(t) > 0$. In this case, the local volatility function (3) consistent with the model is given by:

$$\begin{aligned}
\sigma(t,k)^2 &= 2\frac{c_t(t,k)}{c_{kk}(t,k)} \\
&= \vartheta_i(k)^2 \left[T'(t) + \left(T(t) - t_i\right)\frac{\partial \ln c_{kk}(t,k)}{\partial t}\right]
\end{aligned} \tag{15}$$

The introduction of the time-change facilitates the interpolation in the expiry direction. For example, a choice of piecewise cubic functions $T(t)$ can be used to ensure that implied volatility is roughly linear in expiry.

Algorithm

In summary, a discrete set of European-style option quotes is interpolated into a full continuously parameterised surface of arbitrage-consistent option quotes by:

■ **Step 1.** For each expiry, solve an optimisation problem (6) for a piecewise constant volatility function with as many levels as target strikes at the particular expiry. Each iteration involves one update of (1) and is equivalent to one time step in a fully implicit finite difference solver.

■ **Step 2.** For expiries between the original expiries, the volatility functions from step 1 are used in conjunction with (7) to generate option prices for all strikes.

Note that step 2 does not involve any iteration. The process of the time stepping is shown in figure 1.

Numerical example

Here, we consider fitting the model to the Eurostoxx 50 (SX5E) equity option market. The number of expiries is 12, with up to 15 strikes per expiry. The target data is given in table A. We choose to fit a lognormal version of the model based on a finite difference solution with 200 grid points. The local volatility function is set up to be piecewise linear with as many levels as calibration strikes per expiry. The model fits to the option prices in approximately 0.05 seconds of CPU time on a standard PC. The average number of iterations is 86 per expiry. Table B shows the calibration accuracy for the target options. The standard deviation of the error is 0.03% in implied Black volatility.

After the model has been calibrated, we use (6) to calculate option prices for all expiries and strikes, and deduce the local volatility from the option prices using (3). Figure 2 shows the resulting local volatility surface. We note that the local volatility surface has no singularities. So, as expected, the model produces arbitrage-consistent European-style option prices for all expiries and strikes.

Conclusion

We have shown how a non-standard application of the fully implicit finite difference method can be used for arbitrage-free interpolation of implied volatility quotes. The method is quick and robust, and can be used both as a pre-pricing step for local volatility models as well as for market-making in option markets. ■

References

Andersen L and J Andreasen, 1999
Jumping smiles
Risk November, pages 65–68

Andreasen J, 1996
Implied modeling
Working paper, Aarhus University

Andreasen J and B Huge, 2010
Expanded smiles
Risk May, pages 78–81

Avellaneda M, C Friedman, R Holmes and D Samperi, 1997
Calibrating volatility surfaces via relative entropy minimization
Applied Mathematical Finance 4, pages 37–64

Carr P, 2008
Local variance gamma
Working paper, Bloomberg, New York

Coleman T, Y Li and A Verma, 1999
Reconstructing the unknown local volatility function
Journal of Computational Finance 2(3), pages 77–100

Dupire B, 1994
Pricing with a smile
Risk January, pages 18–20

Jackwerth J and M Rubinstein, 1996
Recovering probability distributions from options prices
Journal of Finance 51(5), pages 1,611–1,631

JP Morgan, 1999
Pricing exotics under the smile
Risk November, pages 72–75

Nabben R, 1999
On decay rates of tridiagonal and band matrices
SIAM Journal on Matrix Analysis and Applications 20, pages 820–837

Sepp A, 2007
Using SABR model to produce smooth local volatility surfaces
Working paper, Merrill Lynch

Jesper Andreasen is head of and Brian Huge is chief analyst in the quantitative research department at Danske Markets, Danske Bank, Copenhagen. Email: kwant.daddy@danskebank.com, brno@danskebank.com

Appendix: technical results

■ Proposition 1: absence of arbitrage. The surface of option prices constructed by the recursive schemes (1) and (6) is consistent with absence of arbitrage, that is:

$$c_t(t,k) \geq 0 \qquad c_{kk}(t,k) \geq 0 \tag{16}$$

for all (t, k).

■ Proof of proposition 1. Consider option prices generated by the forward equation:

$$0 = -\frac{\partial g}{\partial t} + \tfrac{1}{2}\vartheta(k)^2 \frac{\partial^2 g}{\partial k^2} \tag{17}$$

which is solved forward in time t given the initial boundary condition $g(0, k)$.

As also noted in Andreasen (1996), (17) can also be seen as the backward equation for:

$$g(t,k) = E^k\left[g(0,k(0)) \middle| k(t) = k\right] \tag{18}$$

where k follows the process:

$$dk(t) = \vartheta(k(t))\,dZ(t) \tag{19}$$

and Z is a Brownian motion running backwards in time. The mapping $g(0, \cdot) \mapsto g(t, \cdot)$ given by (17) thus defines a positive linear functional, in the sense that:

$$g(0,\cdot) \geq 0 \Rightarrow g(t,\cdot) \geq 0 \tag{20}$$

Further, differentiating (17) twice with respect to k yields the forward equation for $p = g_{kk}$:

$$0 = \frac{\partial p}{\partial t} + \tfrac{1}{2}\frac{\partial^2}{\partial k}\left[\vartheta(k)^2 p\right]$$
$$p(0,k) = g_{kk}(0,k) = \int g_{kk}(0,l)\,\delta(k-l)\,dl \tag{21}$$

Equation (18) is equivalent to the Fokker-Planck equation for the process:

$$dx(t) = \vartheta(x(t))\,dW(t) \tag{22}$$

where W is a standard Brownian motion. From this, we conclude that (17) preserves convexity:

$$g_{kk}(0,\cdot) \geq 0 \Rightarrow g_{kk}(t,\cdot) \geq 0 \tag{23}$$

Let $T(u)$ be a strictly increasing function. Define the (Laplace) transform of the option prices by:

$$h(u,k) = \int_0^\infty \frac{1}{T(u)} e^{-t/T(u)} g(t,k)\,dt \tag{24}$$

Multiplying (17) by $e^{-t/T(u)}$ and integrating in t yields:

$$\left[1 - \tfrac{1}{2}T(u)\vartheta(k)^2 \frac{\partial^2}{\partial k^2}\right]h(u,k) = g(0,k) \tag{25}$$

From (20) and (23), we conclude that (25) defines a positive linear functional that preserves convexity.

Differentiating (25) with respect to u yields:

$$\left[1 - \tfrac{1}{2}T(u)\vartheta(k)^2 \frac{\partial^2}{\partial k^2} h_u(u,k) = \tfrac{1}{2}T'(u)\vartheta(k)^2 h_{kk}(u,k)\right] \tag{26}$$

Using (25) as a positive linear functional that preserves convexity, we have that if $g(0, \cdot)$ is convex then:

$$h_u(u,k) \geq 0 \tag{27}$$

for all (u, k).

We conclude that the option prices constructed by (1) and (6) are consistent with absence of arbitrage.