Complex Dynamics

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1 Julia and Mandelbrot Set Definitions

Let $P_c(z) = z^2 + c$

Julia set for P_c, fixes c

$$J_c = \{z \mid \left|P_c^n(z)\right|_n \text{ bounded}\}$$

Mandelbrot set instead fixes z

$$M = \{c \mid \left|P_c^n(0)\right|_n \text{ bounded}\}$$

To be more precise

- Def above is the "filled-in" Julia set
- "True" Julia set is the boundary
- Above is a Quadratic Julia set, can use other rational functions (ratio of complex polynomials)

2 Julia Set Fractal Nature

Sequence $z_n = P_c^n(z_0)$, iterating P_c for a given z_0 in the Julia set.

Easier to visualize the pre-image (single iteration) of a point $z_n = P_c^{-1}(z_{n+1})$

$$\begin{aligned} z_{n+1} &= \left(|z_n|\, e^{i\text{Arg}z}\right)^2 + c \\ z_{n+1} - c &= |z_n|^2\, e^{i2\text{Arg}z} \end{aligned}$$

Solving for z_n in terms of $z_{n+1} - c$

$$\begin{aligned} 2\text{Arg}z_n &= \text{Arg}(z_{n+1} - c) + 2\pi k \\ \text{Arg}z_n &= .5\text{Arg}(z_{n+1} - c) + \pi k,\ k \in \{0, 1\} \\ |z_n| &= \sqrt{|z_{n+1} - c|} \end{aligned}$$

Karl Sims gives a visualization for a disk and how fractal arises in Fig 1. For one iteration,

- Translate the disk by $-c$, halve angle for each point, shift radius towards 1 for each point
- Mirror across real axis (pre-image maps one point to two with π arg offset)

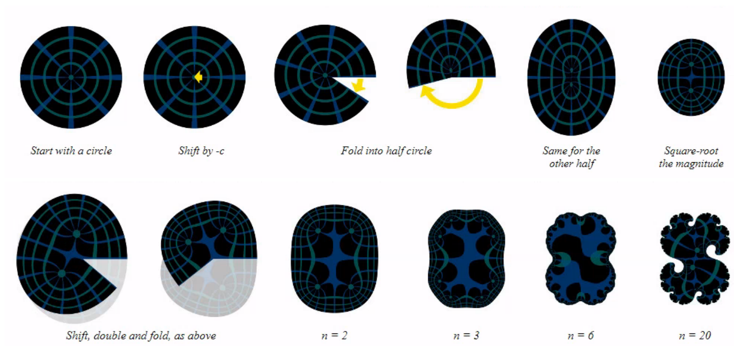


Figure 1: Approaching Julia set by pre-images of disk [11]

We can iterate taking the pre-image of a set containing J_c to approach its shape,

- Take nbhd of infinity S s.t. $J_c \subset \mathbb{C} - S$
- $z_0 \notin J_c \iff \exists N \ni \forall n > N,\ z_n \in S$
- $\mathbb{C} - J_c = \bigcup_n P_c^{-n}(S)$
- DeMorgan's, pre-image of complement: $J_c = \bigcap_n P_c^{-n}(\mathbb{C} - S)$

Figure 2: Julia set, bands around set indicate "escape" iterations [5]

3 Julia Set Escape Radius

How big does the disk need to be such that $J_c \subset D[0, R]$?

- Called "Escape Radius"
- Find R where $z_0 \notin D[0, R] \implies |z_{n+1}| > |z_n|$

For $P_c(z) = z^2 + 1$, take $|z_n| > |R| > 1$,

$$\begin{aligned} |z_{n+1}| &\geq \left|z_n^2\right| - |c| && \text{Reverse Triangle Inequality} \\ &\geq \left|z_n^2\right| - |z_n| \frac{|z_n|}{R} |c| \\ &= |z_n| \frac{|z_n|}{R} (R - |c|) \end{aligned}$$

Thus $R > 1 + |c| \implies |z_{n+1}| > |z_n|$ [9].

Plotting algorithm: take points in disk, iterate P_c, remove points that "escape" the disk.

4 Julia Set Connectedness

- Julia Set is either connected (Fatou Set)...
  - Start from $D[0, R]$, each pre-image iteration preserves connectedness
- Or something that's like the Cantor set (Fatou Dust)
  - Start from $D[0, R]$, each pre-image iteration cuts the previous pieces in two
- If $0 \in J_c$, then it's connected, otherwise Fatou Dust
- J_c connected $\iff 0 \in J_c \iff c \in M$ (M is Mandelbrot set)

Figure 3: Cantor set interval visualization [1]

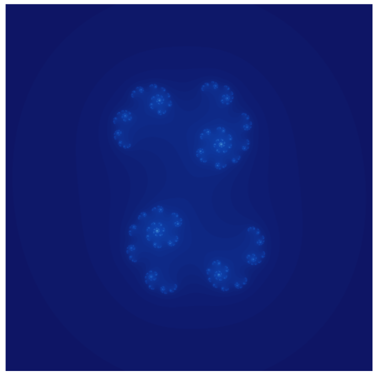


Figure 4: Fatou dust, $c = .45 + .1428i$

5 Mandelbrot Set Properties

- Mandelbrot set is quasi-fractal, whereas Julia is fractal
  - "Copies" are deformed (see Figure 5)
  - "Index" of (connected) Julia sets ($c \notin M$ for disconnected J_c)
  - Mandelbrot at c resembles J_c near 0
- Mandelbrot set is connected
  - Mandelbrot conjectured it was not from initial images!
  - Conformal (angle-preserving) isomorphism to complement of a closed disk
- Open question: Is Mandelbrot locally connected (MLC)?
  - $\forall x \in M,\ x \in \text{open } U \implies \exists \text{open, connected } W,\ x \in W \subset U$
  - Example: Topologist's sine curve connected, not locally connected

6 Mandelbrot Escape Radius

Let $z_n = P_c^n(0)$,

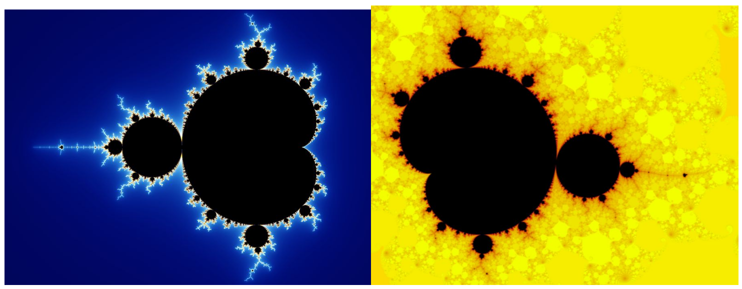
$$c \in M \iff \exists n \ni |z_n| > 2 \text{ [12]}$$

Given $|z_n| > 2$, for $|c| \leq 2$

$$\begin{aligned} |z_{n+1}| &\geq |z_n|^2 - |c| && \text{Reverse Triangle Inequality} \\ &\geq 2|z_n| - 2 = |z_n| + (|z_n| - 2) > |z_n| \end{aligned}$$

For $|c| > 2$, inductively $|z_n| > |c|$, so

$$|z_{n+1}| \geq |z_n|^2 - |c| > |c||z_n| - |c| > 2|z_n| - |c| > |z_n|$$



(a) (b)

Figure 5: Mandelbrot [6] and a deformed "Mini-Mandelbrot" [10]

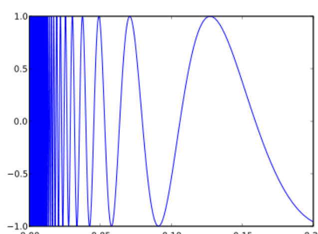


Figure 6: Topologist's Sine Curve [7]

7 Mandelbrot Fixed Points and Cycles

- "Main" Cardioid contains points whose sequence (orbit) approach attractive fixed point
- For attractive fixed point x^*, $x^* = f(x^*)$ and $\left|f'(x)\right| < 1$
  - $x_{n+1} = f(x_n) \approx f(x^*) + f'(x^*)(x_n - x^*)$
  - $|x_{n+1} - x^*| = \left|x_{n+1} - f(x^*)\right| \approx \left|f'(x^*)\right| |x_n - x^*| < |x_n - x^*|$
- Other cardioids have cycles, periods labeled in Fig 7

For c with an attractive fixed point z^* of M, we must have

$$z^* = (z^*)^2 + c$$

and

$$\left| \frac{d}{dz}\bigg|_{z^*} \left[z^2 + c\right] \right| < 1,\ |z^*| < \frac{1}{2}$$

For c on the boundary of those with fixed points, we have $|z^*| = \frac{1}{2}$. Combining with the first condition, the solutions are the main cardioid curve.

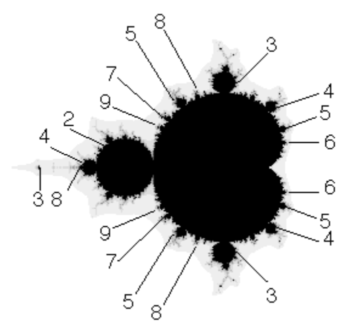


Figure 7: Cardiods labeled with period [8]

8 Visualizations

- Common: color based on iterations takes to "escape" (see Fig 2, 8)
- Orbit trap (see Fig 9)
  - Color point based on how close orbit (sequence of iterations) reaches a "trap" (fixed point or line)
  - Color point based on where it lands on the trap
- Pickover Stalks use a cross for the trap, some think they look "biological" (see Fig 9)

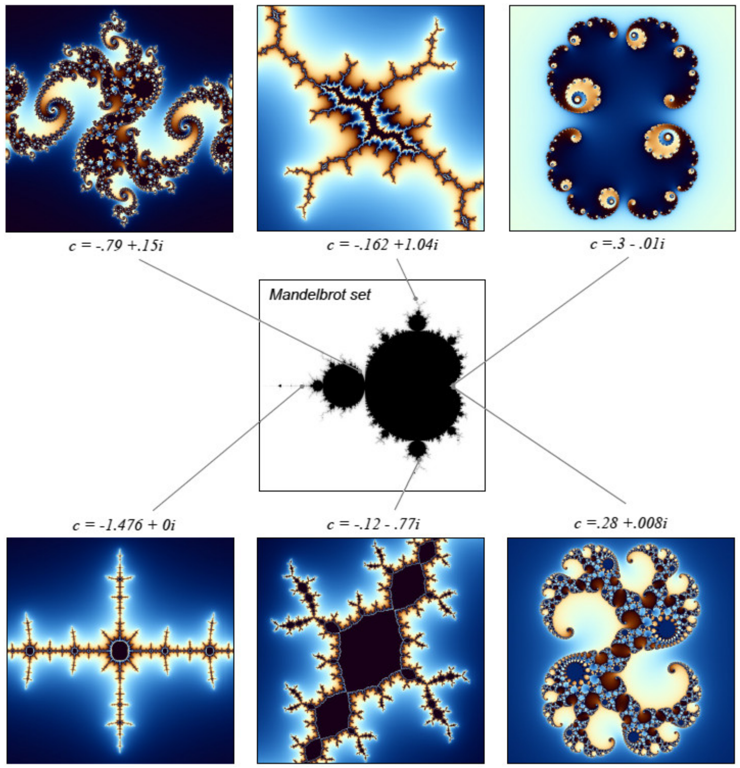
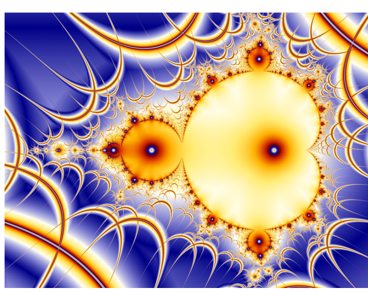
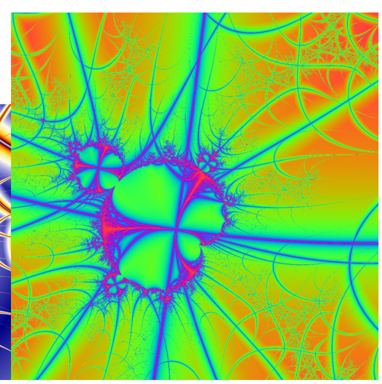


Figure 8: Julia sets colored by "escape" iterations [11]



(a) Line and point Orbit Trap [3]



(b) Pickover Stalk [4]

Figure 9: Orbit trap

9 General Julia Sets

Definitions

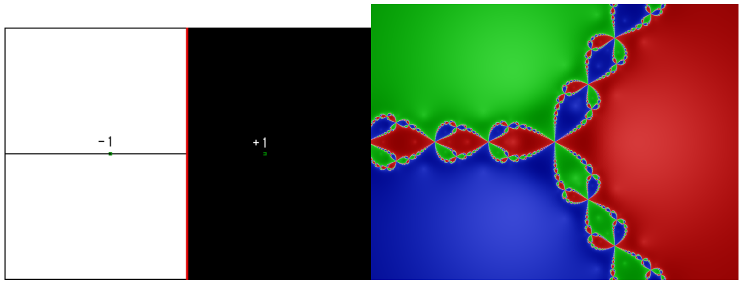
- In general, iterating function $f(x) = p(x)/q(x)$ for complex polynomials p, q, no shared roots
  - f maps Riemann sphere onto itself
  - f is holomorphic
- f is invariant on open sets, Fatou Domains
- Fatou Set is union of Fatou Domains, it's dense!
- "True" Julia Set is the complement of the Fatou Set (boundary), also invariant

In the context of the Quadratic Julia Set described earlier...

- "True" Julia Set is boundary of filled-in Julia Set
- Fatou Domains are...
  - interior of filled-in Julia Set
  - other side of Julia Set boundary, with unbounded orbit

10 Newton Fractals

- Newton's method
  - Iteratively approximate zeroes using tangent lines
  - $x_{n+1} = x_n - p(x_n)/p'(x_n)$
  - Works with complex functions
- $f(x) = x - p(x)/p'(x)$, for p polynomial, can be used as the iteration function for a Julia Set
- One Fatou Domain for each zero of p (points that converge to a zero make up a Domain)
- True Julia Set is the boundary where the method fails!



(a) $z^2 - 1$, two roots [8]   (b) $z^3 - 1$, three roots [2]

Figure 10: Newton's Method

References

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