

# Complex Dynamics

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## 1 Julia and Mandelbrot Set Definitions

Let  $P_c(z) = z^2 + c$

Julia set for  $P_c$ , fixes  $c$

$$J_c = \{z \mid |P_c^n(z)|_n \text{ bounded}\}$$

Mandelbrot set instead fixes  $z$

$$M = \{c \mid |P_c^n(0)|_n \text{ bounded}\}$$

To be more precise

- Def above is the "filled-in" Julia set
- "True" Julia set is the boundary
- Above is a Quadratic Julia set, can use other rational functions (ratio of complex polynomials)

## 2 Julia Set Fractal Nature

Sequence  $z_n = P_c^n(z_0)$ , iterating  $P_c$  for a given  $z_0$  in the Julia set.

Easier to visualize the pre-image (single iteration) of a point  $z_n = P_c^{-1}(z_{n+1})$

$$z_{n+1} = \left(|z_n| e^{i\text{Arg}z}\right)^2 + c$$
$$z_{n+1} - c = |z_n|^2 e^{i2\text{Arg}z}$$

Solving for  $z_n$  in terms of  $z_{n+1} - c$

$$2\text{Arg}z_n = \text{Arg}(z_{n+1} - c) + 2\pi k$$

$$\text{Arg}z_n = .5\text{Arg}(z_{n+1} - c) + \pi k, \quad k \in \{0, 1\}$$

$$|z_n| = \sqrt{|z_{n+1} - c|}$$

Karl Sims gives a visualization for a disk in Fig 1. For one iteration,

- 
- Translate the disk by  $-c$ , halve angle for each point, shift radius towards 1 for each point
  - Mirror across real axis (pre-image maps one point to two with  $\pi$  arg offset)

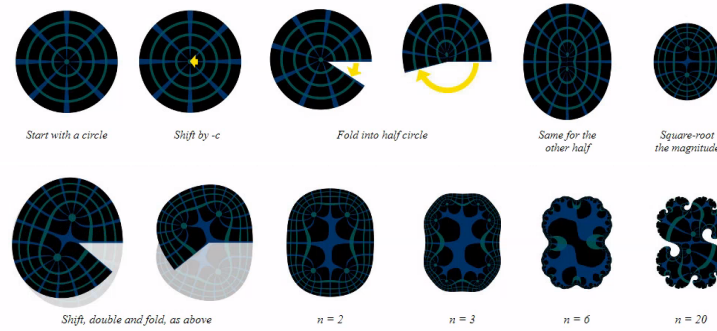


Figure 1: Approaching Julia set by pre-images of disk [11]

We can iterate taking the pre-image of a set containing  $J_c$  to approach its shape,

- Take nbhd of infinity  $S$  s.t.  $J_c \subset \mathbb{C} - S$
- $z_0 \notin J_c \iff \exists N \ni \forall n > N, z_n \in S$
- $\mathbb{C} - J_c = \bigcup_n P_c^{-n}(S)$
- DeMorgan's, pre-image of complement:  $J_c = \bigcap_n P_c^{-n}(\mathbb{C} - S)$

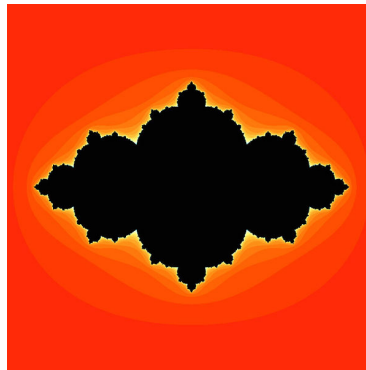


Figure 2: Julia set, bands around set indicate "escape" iterations [5]

### 3 Julia Set Escape Radius

How big does the disk need to be such that  $J_c \subset D[0, R]$ ?

- Called "Escape Radius"
- Find  $R$  where  $z_0 \notin D[0, R] \implies |z_{n+1}| > |z_n|$

For  $P_c(z) = z^2 + 1$ , take  $|z_n| > |R| > 1$ ,

$$\begin{aligned} |z_{n+1}| &\geq \left| z_n^2 \right| - |c| && \text{Reverse Triangle Inequality} \\ &\geq \left| z_n^2 \right| - |z_n| \frac{|z_n|}{R} |c| \\ &= |z_n| \frac{|z_n|}{R} (R - |c|) \end{aligned}$$

Thus  $R > 1 + |c| \implies |z_{n+1}| > |z_n|$  [9].

Plotting algorithm: take points in disk, iterate  $P_c$ , remove points that "escape" the disk.

### 4 Julia Set Connectedness

- Julia Set is either connected (Fatou Set)...
  - Start from  $D[0, R]$ , each pre-image iteration preserves connectedness
- Or something that's like the Cantor set (Fatou Dust)
  - Start from  $D[0, R]$ , each pre-image iteration cuts the previous pieces in two
- If  $0 \in J_c$ , then it's connected, otherwise Fatou Dust
- $J_c$  connected  $\iff 0 \in J_c \iff c \in M$  (M is Mandelbrot set)



Figure 3: Cantor set interval visualization [1]

TODO something about Julia set fixed points, 0 is circle, -2 is chaotic, Mandelbrot explore the transition

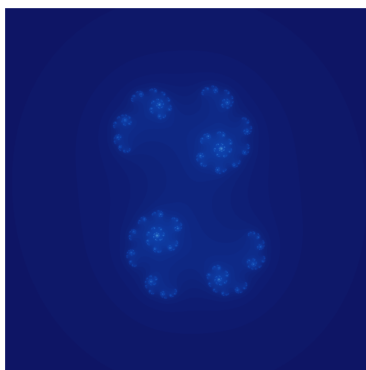


Figure 4: Fatou dust,  $c = .45 + .1428i$

## 5 Mandelbrot Set Properties

- Mandelbrot set is quasi-fractal, whereas Julia is fractal
  - "Copies" are deformed (see Figure 5)
  - "Index" of (connected) Julia sets ( $c \notin M$  for disconnected  $J_c$ )
  - Mandelbrot at  $c$  resembles  $J_c$  near 0
- Mandelbrot set is connected
  - Mandelbrot conjectured it was not from initial images!
  - Conformal (angle-preserving) isomorphism to complement of a closed disk
- Open question: Is Mandelbrot locally connected (MLC)?
  - $\forall x \in M, x \in \text{open } U \implies \exists \text{ open, connected } W, x \in W \subset U$
  - Example: Topologist's sine curve connected, not locally connected

## 6 Mandelbrot Bound / Escape Radius

Let  $z_n = P_c^n(0)$ ,

$$c \in M \iff \exists n \ni |z_n| > 2 \text{ [12]}$$

Given  $|z_n| > 2$ , for  $|c| \leq 2$

$$\begin{aligned} |z_{n+1}| &\geq |z_n|^2 - |c| && \text{Reverse Triangle Inequality} \\ &\geq 2|z_n| - 2 = |z_n| + (|z_n| - 2) > |z_n| \end{aligned}$$

For  $|c| > 2$ , inductively  $|z_n| > |c|$ , so

$$|z_{n+1}| \geq |z_n|^2 - |c| > |c||z_n| - |c| > 2|z_n| - |c| > |z_n|$$

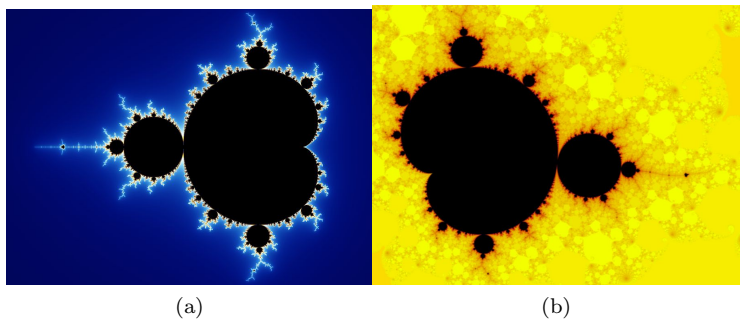


Figure 5: Mandelbrot [6] and a deformed "Mini-Mandelbrot" [10]

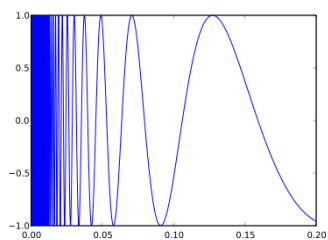


Figure 6: Topologist's Sine Curve [7]

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## 7 Mandelbrot Fixed Points and Cycles

- "Main" Cardioid contains points whose sequence (orbit) approach attractive fixed point
- For attractive fixed point  $x^*$ ,  $x^* = f(x^*)$  and  $|f'(x)| < 1$ 
  - $x_{n+1} = f(x_n) \approx f(x^*) + f'(x^*)(x_n - x^*)$
  - $|x_{n+1} - x^*| = |x_{n+1} - f(x^*)| \approx |f'(x^*)||x_n - x^*| < |x_n - x^*|$
- Other cardioids have cycles, periods labeled in Fig 7

For  $c$  with an attractive fixed point  $z^*$  of  $M$ , we must have

$$z^* = (z^*)^2 + c$$

and

$$\left| \frac{d}{dz} \Big|_{z^*} [z^2 + c] \right| < 1, |z^*| < \frac{1}{2}$$

For  $c$  on the boundary of those with fixed points, we have  $|z^*| = \frac{1}{2}$ . Combining with the first condition, the solutions are the main cardioid curve.

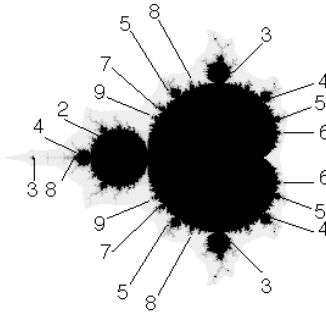


Figure 7: Cardioids labeled with period [8]

## 8 Visualizations

- Common: color based on iterations takes to "escape" (see Fig 2, 8)
- Orbit trap (see Fig 9)
  - Color point based on how close orbit (sequence of iterations) reaches a "trap" (fixed point or line)
  - Color point based on where it lands on the trap
- Pickover Stalks use a cross for the trap, some think they look "biological" (see Fig 9)

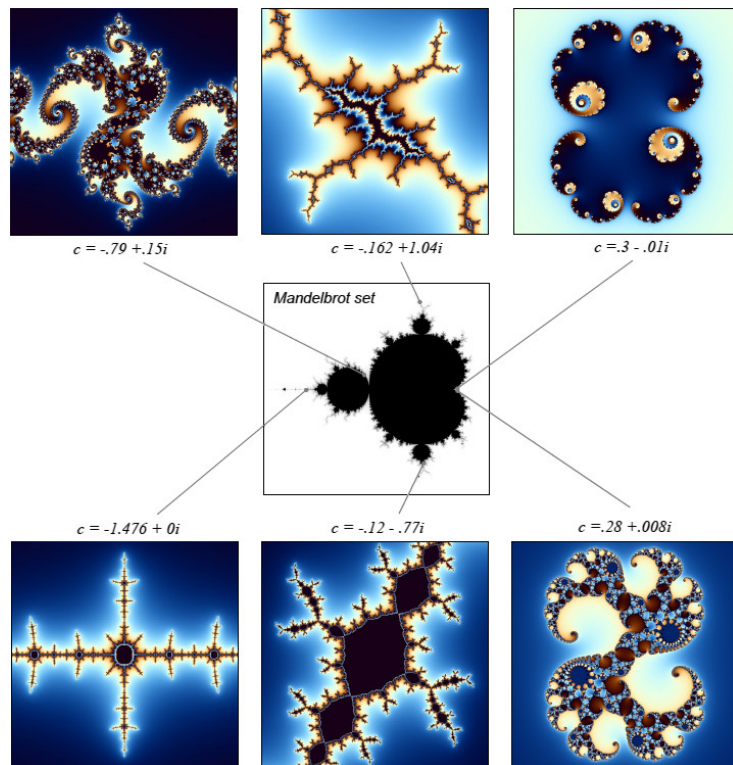


Figure 8: Julia sets colored by "escape" iterations [11]

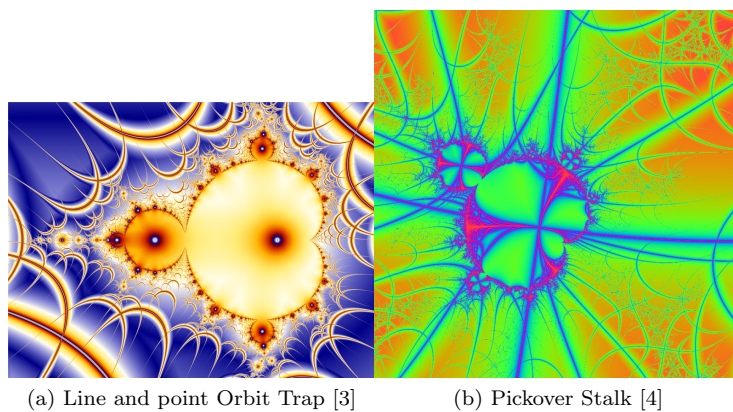


Figure 9: Orbit trap

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## 9 General Julia Sets

### Definitions

- In general, iterating function  $f(x) = p(x)/q(x)$  for complex polynomials  $p, q$ , no shared roots
  - $f$  maps Riemann sphere onto itself
  - $f$  is holomorphic
- $f$  is invariant on open sets, Fatou Domains
- Fatou Set is union of Fatou Domains, it's dense!
- "True" Julia Set is the complement of the Fatou Set (boundary), also invariant

In the context of the Quadratic Julia Set described earlier...

- "True" Julia Set is boundary of filled-in Julia Set
- Fatou Domains are...
  - interior of filled-in Julia Set
  - other side of Julia Set boundary, with unbounded orbit

## 10 Newton Fractals

- Newton's method
  - Iteratively approximate zeroes using tangent lines
  - $x_{n+1} = x_n - p(x_n)/p'(x_n)$
  - Works with complex functions
- $f(x) = x - p(x)/p'(x)$ , for  $p$  polynomial, can be used as the iteration function for a Julia Set
- One Fatou Domain for each zero of  $f$
- True Julia Set is the boundary where the method fails!



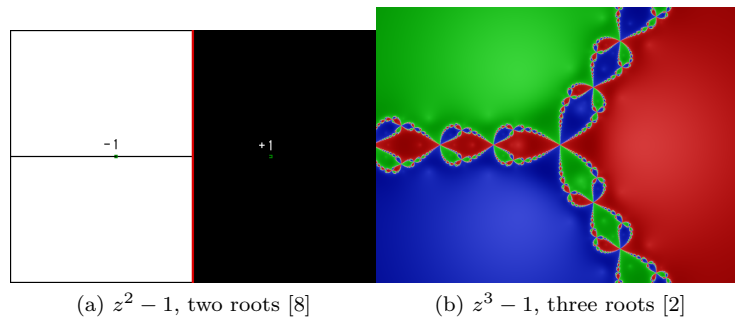


Figure 10: Newton's Method

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