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Homework 0

Problem 1

(a) Use proof by induction:

Inductive Hypothesis: Assume that $digsum(x \bullet y) = digsum(x) + digsum(y)$ is true for all arbitrary strings x such that 0 ≤ |x| < |w| and any arbitrary string y, and that w is some arbitrary string

Base Case: prove for x = ϵ :

$digsum(x \bullet y) = digsum(x) + digsum(y)$	
$digsum(\epsilon \bullet y) = digsum(\epsilon) + digsum(y)$	Replace x with ϵ
$digsum(y) = 0 + digsum(y)$	Definition of digsum
$digsum(y) = digsum(y)$	Additive Identity

Inductive Step: Let $w = a \bullet x$:

$digsum(w \bullet y) = digsum(w) + digsum(y)$	
$digsum\big((a \bullet x) \bullet y\big) = digsum(a \bullet x) + digsum(y)$	Replace w with ax
$digsum\big(a \bullet (x \bullet y)\big) = digsum(a \bullet x) + digsum(y)$	Associative property of concatenation
$a + digsum(x \bullet y) = a + digsum(x) + digsum(y)$	Definition of digsum
$a + digsum(x) + digsum(y) = a + digsum(x) + digsum(y)$	Induction Hypothesis

∴ It has now been shown that $digsum(xy) = digsum(x) + digsum(y)$ for all strings x and y.

(b) Use proof by induction:

Inductive Hypothesis: Assume that $digsum(x^R) = digsum(x)$ is true for all arbitrary strings x such that 0 ≤ |x| < |w|, with w being some arbitrary string.

Base Case: prove for x = ϵ

$digsum(x^R) = digsum(x)$	
$digsum(\epsilon^R) = digsum(\epsilon)$	Replace x with ϵ
$digsum(\epsilon) = digsum(\epsilon)$	Definition of reversal
$0 = 0$	Subtract from both sides

Inductive Step: Let $w = a \bullet x$:

$digsum(w^R) = digsum(w)$	
$digsum((a \bullet x)^R) = digsum(a \bullet x)$	Replace w with ax
$digsum(x^R \bullet a) = digsum(a \bullet x)$	Definition of reversal
$digsum(a) + digsum(x^R) = a + digsum(x)$	$digsum(xy) = digsum(x) + digsum(y)$ from problem 1A.

$a + digsum(x^R) = a + digsum(x)$ Definition of digsum

$a + digsum(x) = a + digsum(x)$ Inductive hypothesis

∴ It has now been shown that $digisum(x) = digisum(x^R)$ for all strings x.

Problem 2

(a) '374' is a 3-digit string, therefore it can only be represented in the form 'ax' or 'axb':

- Let '374' be represented in the form axb with a = 3, b = 4, and x = 7. We know that '374' is not in L_{odd} because, by definition, $a, b \in \{1,3,5,7,9\}$ must be true but $b = 4 \notin \{1,3,5,7,9\}$.
- Let '374' be represented in the form ax then, with a = 3 and x = 74. It is not in L_{odd} in this case either, because $a \notin \{0, 2, 4, 6, 8\}$ as necessary. And $x = 74 \notin L_{odd}$ because if we represent '74' in the same ax form $x \notin \{1,3,5,7,9\} \rightarrow ax = 74 \notin L_{odd}$

(b) Prove by induction:

Inductive Hypothesis: Suppose that for any $x \in L_{odd}$, $digsum(x)$ is odd for any arbitrary string x such that 0 < |x| < |w|, with w being some arbitrary string.

Base Case: Let x = a where $a \in \{1, 3, 5, 7, 9\}$. Since x is only one digit long, that means the result of $digsum(x)$ is simply equal to a. Because a is limited to 1, 3, 5, 7, or 9 (all odd numbers), the result of $digsum(x)$ will also be odd.

Inductive Step: Let w be any arbitrary string in L_{odd}.

W can be composed in the following two ways:

1.) $w = a \bullet x \bullet b$; $\mathrm{a, b} \in \{1,3,5,7,9\}$:

1. $digsum(w) = digsum(a \bullet x \bullet b) = digsum(a) + digsum(x) + digsum(b)$. This is true by the theorem of problem 1A
2. We know that $digsum(a)$ and $digsum(b)$ are both odd, because a and b are both odd-number one-character strings. We know that $digsum(x)$ is odd by our inductive hypothesis.
3. $digsum(a) + dgisum(b)$ is an even number, because an odd number plus an odd number is an even number.
4. $\left(digsum(a) + digsum(b)\right) + digsum(x)$ is an even number plus an odd number, which results in an odd number (see below proof).
5. $digisum(w)$ is then an odd number from (1.)

2.) $w = a \bullet x$; $a \in \{0, 2, 4, 6, 8\}$:

1. $digsum(w) = digsum(a \bullet x) = digsum(a) + digsum(x)$. This is true from the theorem of problem 1A.
2. We know that $digsum(a)$ is even because a is a one-character string of an even number, therefore $digsum(a)$ is even because a is even
3. We know that $digsum(x)$ is an odd number by our inductive hypothesis
4. $digsum(a) + digsum(x)$ is an even number plus an odd number, which results in an odd number.
4. $digisum(w)$ is then an odd number from (1.)

∴ In both cases it is clearly shown then that for any $x \in L_{odd}$, $digisum(x)$ is odd for all strings x.

Problem 3

Define “bar” operation as follows on $\Sigma = \{0,1\}$

$$\overline{1} = 0$$
$$\overline{0} = 1$$

Define L_{bad} as follows:

- $a \in L_{bad}$ for $a \in \{0, 1\}$
- $bx \in L_{bad}$ for ($x \in L_{bad}$) and ($x \neq \mathrm{b} \bullet b \bullet \overline{b} \bullet z$, for any string z) and ($b \in \{0, 1\}$)
- $xc \in L_{bad}$ for ($x \in L_{bad}$) and ($x \neq z \bullet \overline{c} \bullet c \bullet c$, for any string z) and ($c \in \{0, 1\}$)

Justification:

We can define L_{bad} recursively. We start off with our base case, either 0 or 1, to which we can build onto. Assume that Z is any string in L_{bad}. We then can make a new string from Z by either adding a character to the front (second bullet) or to the back (third bullet).

To add a character to the front three conditions must be satisfied: First, the string we are building onto must be a member of L_{bad}. Second, the character must be 0 or 1. Third, the string Z cannot have the character you wish to add twice in a row at the beginning. This means that if you wish to add a 0, Z cannot be of the form 001x, and if you wish to add a 1, Z cannot be of the form 110x, where x is any string. This is because adding the character in this case would result in 3 1’s or 0’s in a row, thereby eliminating it from L_{bad}.

The third bullet point about appending a character to the end uses the same logic as appending a character to the front, but just with the directions reversed.