ASTR 589 – Physics of Astrophysics– Fall 2021
Assignment IV, on Fluid Dynamics
Due Date: Thursday November 4 at 5pm.

1. **The Energy Equation:** In class, we derived the equation for the evolution of the total energy density of a fluid

$$\frac{\partial}{\partial t}\left(\frac{3}{2}P+\frac{1}{2}\rho u^2\right)+\frac{\partial}{\partial x_i}\left[\left(\frac{3}{2}P+\frac{1}{2}\rho u^2\right)u_i\right]=\rho a_i u_i-\frac{\partial}{\partial x_i}\left(\frac{1}{2}H_i+\Psi_{ij}u_j\right),$$

where H_i is the heat flux vector and $\Psi_{ij}=P\delta_{ij}-\sigma'_{ij}$ is the stress tensor of the fluid. Convert it into an equation for the evolution of the entropy of the fluid, i.e., show that

$$\rho T\frac{DS}{dt}=-\frac{\partial H_i}{\partial x_i}+\frac{(\sigma'_{ij})^2}{2\eta},$$

where the entropy S is defined as

$$S\equiv\frac{1}{\gamma-1}k\ln\left(\frac{P}{\rho^\gamma}\right),$$

$\gamma=5/3$, and η is the coefficient of shear viscosity. *Hints:* You will need to subtract the momentum equation multiplied by u_j and use the continuity equation; you will also need to show that

$$\sigma'_{ij}\frac{\partial u_j}{\partial x_i}=\frac{(\sigma'_{ij})^2}{2\eta}.$$

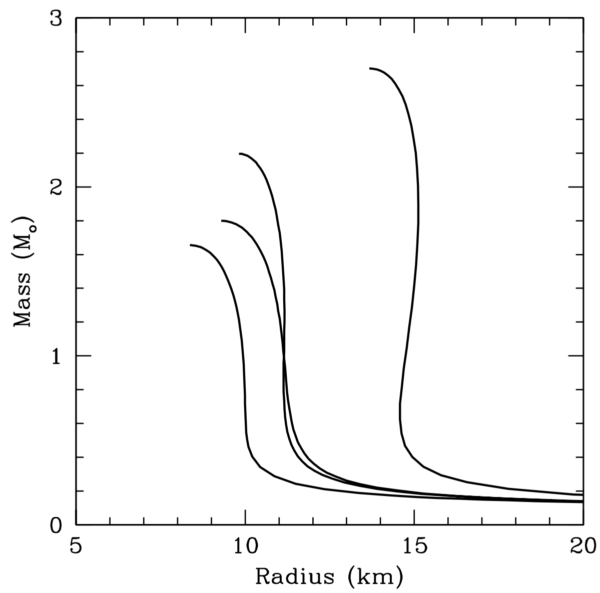
2. **Polytropes:** Use the fact that the solution to the Lane-Emden equation for a polytrope of index n can be expanded to a series of only even powers in ξ, i.e.,

$$\phi(\xi)=1+C_1\xi^2+C_2\xi^4+...$$

to show that

$$\phi(\xi)=1-\frac{\xi^2}{6}+\frac{n\xi^4}{120}+...$$

3. **Neutron Stars:** Several equations of state for neutron-star matter predict that, at masses smaller than their maximum possible mass, the radii of neutron stars depend very weakly on their mass, i.e., $R\simeq M^0$ (see figure in next page). What does this imply for these equations of state of neutron-star matter, if we treat them as polytropes? How do they compare to the equation of state of a non-relativistic degenerate neutron gas? For masses smaller than the maximum neutron-star mass, you can assume that general relativistic corrections are negligible.



Conceptual Questions and Review

Do not turn in these answers. They are for your own use when studying and reviewing class materials.

The Equations of Fluid Mechanics and Polytropes

1. Under what conditions can you use the equations of fluid mechanics to model a physical system? What are the basic assumptions involved in deriving them? When do you have to resort to solving the Boltzmann equation?

2. Describe at least one astrophysical setting, the modeling of which requires solving the Boltzmann equation instead of the fluid equations.

3. What is the physical meaning of the pressure gradient and the viscosity terms in the Euler equation, when the fluid consists of non-interacting particles?

4. What is the closure problem in deriving the equations of fluid mechanics? How is it addressed (in the absence of turbulent motions)?

5. Use dimensional arguments to derive an approximate expression for the coefficient of shear viscosity in a fluid in terms of its density, the sound speed, and a characteristic length scale. What is this characteristic length scale?

6. What is the difference between a polytropic and an adiabatic equation of state? What determines the polytropic index in the first case and the adiabatic index in the second?

7. Describe two astrophysical settings in which a polytropic or an adiabatic equation of state is exact.

8. Use the fact that the radius of a polytropic object with index $n \geq 5$ is infinite to show that the mass in an isothermal sphere in hydrostatic equilibrium is also infinite. The polytropic index is the exponent in the relation $P = K\rho^{1+1/n}$.