Lec 12

FLUID ~~S~~ MECHANICS

What is a fluid?

Rigid body: A collection of particles in which the distances between particles remain fixed in time (neglect deformations and thermal motion)

Fluid: A continuous distribution of matter that can be deformed, and cannot resist shear (tangential) stresses

Criterium:
If l is the collisional mean free path and L the physical size of the system, then if

$$l \ll L$$

we can introduce the concept of a fluid, ~~[illegible]~~ In particular

We can introduce the concept of a fluid element with vol V whose linear size is much smaller than L, i.e.

$$V^{\frac{1}{3}} << L,$$

but $V^{\frac{1}{3}} >> l$.

There are *many* particles in V and we may effectively define a mean (bulk or fluid) velocity u for the collection, with individual particle motions $\vec{v}$ having a random component $\vec{w}$, above the mean i.e.

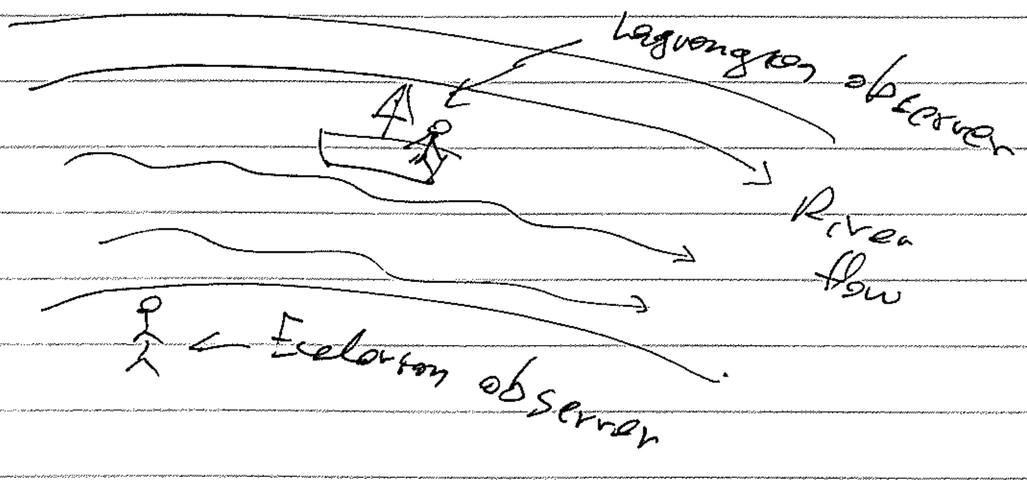
$$\vec{v} = \vec{u} + \vec{w}$$

$\vec{w}$ does not carry a particle away because of collisions, instead causes the particle to undergo random walk about the mean motion $\vec{u}$.

Eulerian & Lagrangian Reference frames

- With a Eulerian frame we study the time evolution of physical quantities at fixed points in space.
- With a Lagrangian frame we study the time evolution of physical quantities (e.g. density, temperature) as they vary along trajectories of individual fluid elements.

Eulerian: $Q = Q(x_1, x_2, x_3, t)$
Lagrangian: $Q = Q(\xi_1, \xi_2, \xi_3, t)$



Eulerian fixed coordinate system

$$Q = Q(x_1, x_2, x_3, t)$$

define Eulerian or local derivative as

$$\frac{\partial Q}{\partial t} \equiv \left.\frac{\partial Q}{\partial t}\right|_{x_i}$$

Lagrangian coord. system

$$Q = Q(\xi_1, \xi_2, \xi_3, t)$$

ξ_i are parameters labeling each ~~vol~~ fluid element uniquely. For example, they may be the Eulerian coordinates of each vol. element at a given time. These coordinates would change in time as the medium flows/deforms. The Lagrangian derivative is defined as

$$\frac{DQ}{dt} = \left.\frac{\partial Q}{\partial t}\right|_{\xi_i}.$$

but each ~~vol~~ fluid element has an eqn. of motion of the form

$$x_i = x_i(\xi_1, \xi_2, \xi_3, t)$$

So, the Lagrangian derivative is

$$\frac{DQ}{\partial t} = \frac{\partial Q}{\partial t}\Big|_{\xi_i} = \frac{\partial Q}{\partial t}\Big|_{x_i} + \sum_i \frac{\partial Q}{\partial x^i}\,\frac{\partial x_i}{\partial t}\Big|_{\xi_i}$$

Eulerian derivative

(What is $\frac{Dx^j}{dt}$? $\frac{\partial x^j}{\partial t}\Big|_{\xi_i} = 0 + \sum_i \frac{\partial x^j}{\partial x^i}\frac{\partial x^i}{\partial t}\Big|_{\xi_i} = \frac{\partial x^j}{\partial t}\Big|_{\xi_i}$)

Also $\frac{\partial x_i}{\partial t} = u_i$

Thus

$$\frac{DQ}{dt} = \frac{\partial Q}{\partial t} + \sum_i u_i \frac{\partial Q}{\partial x_i} =$$

$$= \frac{\partial Q}{\partial t} + \vec{u}\cdot\vec{\nabla} Q$$

directional derivative

relation between Eulerian and Lagrangian derivatives for a scalar quantity Q

What about acceleration?

$$\vec{a} = \frac{D\vec{u}}{dt}$$

or

$$a_i = \frac{Du_i}{dt} = \left.\frac{\partial u_i}{\partial t}\right|_{x_i} + \sum_j \left.\frac{\partial u_i}{\partial x_j}\frac{\partial x_j}{\partial t}\right|_{\xi_i}$$

Thus: $a_i = \frac{\partial u_i}{\partial t} + (\vec{u}\cdot\vec{\nabla})\cdot u_i$

$\boxed{\text{FLUID MECHANICS}}$

Basic Eqn's

The fluid properties can be described using the particle distribution function in phase space

$$\underline{f(\bar{x},\bar{p},t)}\, d^3x\, d\bar{p}$$

\# particles per unit phase space volume

In the absence of sources/sinks for particles, Liouville's theorem

$$\frac{Df}{dt} = 0 \Rightarrow \frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} = 0$$

Vlasov eqn

Since $v_i = \dot{x}_i$, $p_i = m_i v_i$ ← mass

$\dot{p}_i = m_i a_i$ ← acceleration

$$\boxed{\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + a_i \frac{\partial f}{\partial v_i} = 0}$$

Collection of particles → averaged quantities, such as density, mean velocity, etc. based on ~~the~~ f.

Let's define the moments of the dist. function

$$\rho(\bar{x},t) = \int_{d_p} f(\bar{x},p,t)\, dp =$$

$$= \int_{dv} m\, f(x,\bar{v},t)\, d\bar{v}$$

density - zeroth moment

$$u_i(\bar{x},t) = \frac{1}{\rho} \int m v_i f(x,\bar{v},t)\, d\bar{v}$$

mean velocity - 1st moment

Taking moments of the Vlasov eqn. will allow us to derive Eqn's of motion for these average quantities

General concept: Let Q be a physical quantity with a mean defined by

$$\langle Q \rangle(\bar{x},t) = \frac{1}{\rho} \int_{dv} m\, Q\, f(\bar{x},\bar{v},t)\, dv$$

To derive an eqn. for Q multiply the Vlasov eqn by Q and integrate over all velocities

$$\underbrace{\int Q \frac{\partial f}{\partial t} d^3v}_{A} + \underbrace{\int Q v_i \frac{\partial f}{\partial x_i} d^3v}_{B} + \underbrace{\int a_i Q \frac{\partial f}{\partial v_i} d^3v}_{C} = 0$$

$\cancel{A = \int Q \frac{\partial f}{\partial t} d^3v}$

Assume that Q is not an explicit function of x^i, t, e.g. $Q =$

$Q_i = u_i$, $Q = \frac{1}{2} v^2$, then

$$A = \int Q \frac{\partial f}{\partial t} d^3v = \frac{\partial}{\partial t} \int Q f \, d^3v = \frac{\partial}{\partial t} (\rho \langle Q \rangle)$$

$$B = \int Q v_i \frac{\partial f}{\partial x_i} d^3v = \frac{\partial}{\partial x^i} \int Q v_i f \, d^3v =$$

$$= \frac{\partial}{\partial x_i} (\rho \langle v_i Q \rangle)$$

$$C = \int Q a_i \frac{\partial f}{\partial v_i} d^3v = \int \frac{\partial}{\partial v_i} (f Q a_i) d^3v -$$

$$- \int f \frac{\partial}{\partial v_i} (Q a_i) d^3v =$$

$$= \overbrace{\cancel{\int_{ds} f \cdot Q \cdot \vec{a} \, ds}}^{0} - \int f a_i \frac{\partial Q}{\partial v_i} d^3v \; *$$

$$= -\rho \left\langle a_i \frac{\partial Q}{\partial v_i} \right\rangle$$

* for external forces either ind. of v or for which $\frac{\partial a_i}{\partial v_i} = 0$

Is $\frac{\partial a_i}{\partial v_i} = 0$ for B-fields?

Putting it all together

$$\boxed{\frac{\partial}{\partial t}\left(\rho \langle Q \rangle\right) + \frac{\partial}{\partial x_i}\left(\rho \langle v_i Q \rangle\right) - \rho \left\langle a_i \frac{\partial Q}{\partial v_i} \right\rangle = 0}$$

Can also write it as

$$\boxed{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}} + \rho \underbrace{\frac{\partial u_i}{\partial x_i}}_{\nabla \bar{u}} = 0$$

$$\underbrace{\left(\frac{\partial}{\partial t} + u \cdot \nabla\right)\rho}_{\frac{D}{Dt}}$$

Hence $\frac{D\rho}{Dt} + \rho \bar{\nabla} \cdot \vec{u} = 0$

Thus, the density of a fluid element does not change when $\boxed{\bar{\nabla} \cdot \bar{u} = 0}$

Incompressible = divergence-free velocity field