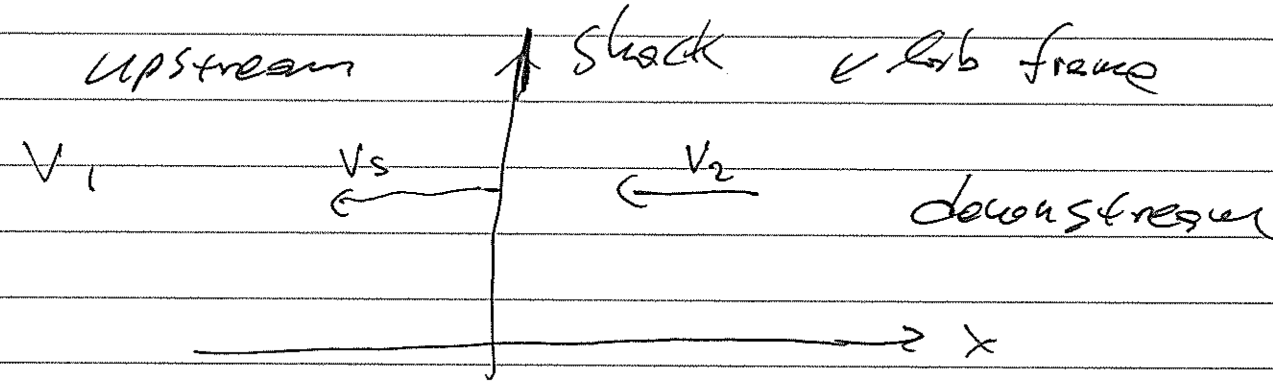
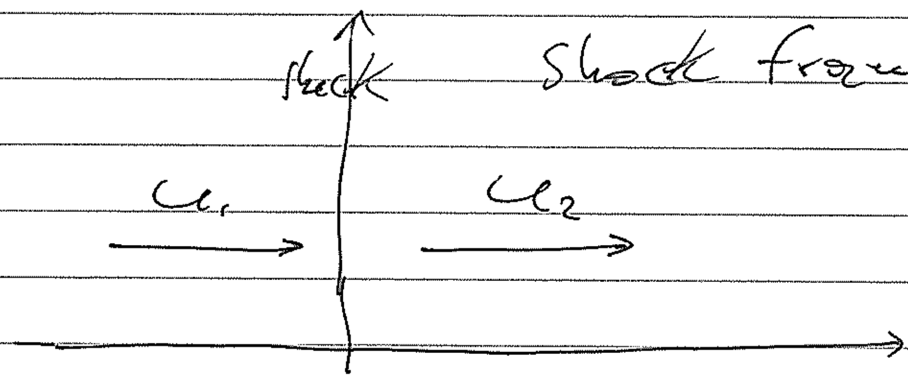
Lec 21

Let's consider a steady shock (time independent)

Let's transform quantities to the shock frame





upstream enters shock at $u_1 = V_1 - V_s$

downstream leaves shock at $u_2 = V_2 - V_s$

Cons. laws

$$\frac{d(\rho u)}{dx} = 0$$

$$\frac{d(\rho u^2 + p)}{dx} = 0$$

$$\frac{d}{dx}\left[\rho u\left(h + \frac{1}{2}u^2\right)\right] =$$

$h \rightarrow \varepsilon + \frac{p}{\rho}$ specific endalpy

internal → ε energy

Integrating left to right

we must have

$\rho_1 u_1 = \rho_2 u_2 \equiv \dot{m}$ → mass flux through shock

$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$ (jump conditions)

$h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2$

upstream variable index "1"
downstream — " — "2"

These can be manipulated ~~to~~

$\cancel{h_2 - h_1 = \frac{1}{2}(V_1 + V_2)(p_2 - p_1)}$

$p = (\gamma - 1)\rho\varepsilon$

for a perfect gas with $p = k\rho^\gamma$ → $h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$

to $$\frac{p_2}{p_1} = \frac{(\gamma+1)V_1 - (\gamma-1)V_2}{(\gamma+1)V_2 - (\gamma-1)V_1}$$

$$\frac{V_2}{V_1} = \frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2} = \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$$

$V_i = \frac{1}{\rho_i}$

~~jump~~

Another useful form is with the Mach number (upstream)

$$M_1 = \frac{u_1}{c_s} \quad , \quad c_s^2 = \gamma \frac{P}{\rho}$$

Shock Mach number

Then for a perfect we find

$$\left.\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{(\gamma+1) M_1^2}{(\gamma+1) + (\gamma-1)(M_1^2-1)} = \frac{u_1}{u_2} \\ \frac{P_2}{P_1} &= \frac{(\gamma+1) + 2\gamma (M_1^2-1)}{\gamma+1} \end{aligned}\right\} \text{Rankine-hugoniot condition}$$

If $M_1 \geq 1$, then notice that

$$P_2 \geq P_1 \quad , \quad \rho_2 \geq \rho_1 \quad , \quad u_1 \leq u_2$$

In the limit of a very strong shock $M_1 \to \infty$, the density jump is bounded $\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1} \overset{\gamma=\frac{5}{3}}{=\!=} 4$

but $\frac{P_2}{P_1}$ is unbounded

While not obvious from R-H $\to$ $M_2 < 1$, $M_1 > 1$

Width of a shock?

If we had included the viscous term, the eqn of momentum would be

$$\rho u^2 + P - \frac{4}{3}\eta \frac{du}{dx} = \text{const}$$

We expect the viscous term to be important in the shock, so

$$\rho u^2 \sim \frac{4}{3}\eta \frac{du}{dx} \sim \rho \nu \frac{du}{dx} \sim \rho\nu \frac{\Delta u}{\Delta x}$$

$$\Rightarrow \quad u^2 \sim \nu \frac{\Delta u}{\Delta x} \Rightarrow \Delta x \sim \nu \frac{\Delta u}{u^2}$$

For a strong shock $\Delta u = u_2 - u_1 \sim u_2$

and $\nu = \underbrace{l}_{m.f.p} \cdot c_s \sim l \cdot v_T$ $\left(\begin{array}{l} c_s^2 = \gamma \frac{P}{\rho} = \gamma \frac{kT}{m} \\ v_T^2 \approx \frac{kT}{m} \end{array} \right)$

$$\Delta x \sim \frac{l\, c_s}{u} \sim l$$

Since u_2 becomes subsonic if u_1 is supersonic

Thus, Δx is of order the m.f.p

Part 3: Stellar Dynamics

So far, we studied properties of gases and now we're moving to systems where the collisional m.f.p. is comparable to the size of the system.

Typical stellar systems: galaxies, glob. cluster, stellar clusters, planetary systems

Consider a typical globular cluster

$$n_* = 0.4 - 10^3 \ \frac{\text{stars}}{pc^3} \quad \left(\begin{array}{l} \text{from outskir} \\ \text{to center} \end{array} \right.$$

$$R = 20 - 50 \ pc$$

Typical stellar radii $R_* \sim 10^{12} cm$ ($\sim 10 R_\odot$

Collision mfp is $\quad \sigma_{coll} \sim \pi R_*^2$

$$\lambda_{coll} = \frac{1}{n_* \sigma_{coll}} \sim 10^{30} cm \sim 10^{11} pc >> R$$

So, how do we model these systems? Definitely not like gas

N-body approach: Describe motion of each star in the smoothed-out gravitational field of the rest of the stars

$$\frac{d^2\bar{x}_n}{dt^2} = -\bar{\nabla}\phi(\bar{x}_n, t) \quad , \quad \nabla^2\phi = 4\pi G \sum_i m_i \delta(x-x_i)$$

$$= 4\pi G \rho$$

So, we can evolve the positions and momenta of every particle as a function of time

Statistical

Boltzmann eqn. describing the evolution of the distribution function

$$\frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

Now, we can no longer assume frequent collisions and get moments of the B-eqn. as we did in gases where $f \to$ M-Boltzmann distribution.

We need a way to describe the collision terms. Two standard approx

i) Binary collisions or two body relaxation which last for a relatively short time but maybe important in evolving systems

ii) Distant encounters, where we make the small-angle scattering approximation; these typically dominate. This approach gives rise to the Fokker-Planck Equation

Relevant timescales

Important for identifying important phenomena and how far from equilibrium a system is.

Mixing time: τ_{mix}

Essentially the crossing time of a star with a "typical" velocity v, across the cluster diameter. It is the timescale over which a cluster can exhibit collective phenomena, such as oscillation

$$\tau_{mix} = \frac{2R}{\langle v^2 \rangle^{1/2}}$$

Relaxation time: τ_{rel}

The timescale at which a star loses memory of its initial conditions, by the effect of close and distant encounters. It also sets the approx. timescale, as a result, at which orbits become randomized, distribution function approaches M-B, and statistical description approaching a fluid may be employed (but N is small & $\lambda_{coll} \gg R$ still!)

We will see how τ_{rel} is estimated later.

Evaporation time: τ_{evap}

A velocity distribution, especially one approaching M-B, implies that some stars have $v \geq v_{escape}$, so the cluster will lose stars and may eventually be depleted.

$$\tau_{evap} = -\frac{N}{dN/dt}$$

For $N \gg 1$ (the number of stars in the system)

As stars escape, the remaining stars relax to M-B, and fill in the tail of the distribution, so relaxation-evaporation continuous.

Note equipartition of energy gives higher velocity to lower-mass stars, so these tend to escape first (the mass distribution of stars in a cluster changes with time).

Observations of clusters/galaxies show that

$$\tau_{mix} < \tau_{rel} < \tau_{evap}.$$

Before studying encounters, deriving timescales etc. let's look at the case of statistical equilib.

Statistical Equilibrium

There are 3 constants of motion for N stars under their mutual grav. attraction (assuming isolated cluster)

(i) linear momentum of the center of mass

$$\vec{p}_{cm} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i$$

(ii) Angular momentum

$$\vec{J} = \sum_{i=1}^{N} m_i \vec{r}_i \times \dot{\vec{r}}_i$$

(iii) Total energy

$$E = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{r}_i^2 + U$$

$$U = -\sum_{i>j} \frac{G m_i m_j}{r_{ij}}$$

sum over $N(N-1)/2$ distinct pairs

In the center-of-mass frame

(i) $\vec{p}_{cm} = 0$

ii) $E = T + U = \text{const}$

iii) $\vec{J}_{cm} = \sum_i m_i \vec{r}_{i,cm} \times \dot{\vec{r}}_{i,cm} = \text{const}$

With these we can prove the Virial Theor