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Physics for Astronomy (ASTR 589)

# Homework 1

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1. **Analytically or numerically integrate the 1D, time independent radiative transfer equation through a slab (or use the formal solution to the transfer equation) to find $I_\nu(\mu)$ for the following source functions and temperature profiles: Here, $\tau$ is the dimensionless depth variable at $\mu = 1$, $B_\nu(T)$ is the blackbody function, and you may take $\tau_{\max} = 10$ through the entire slab for $\mu = 1$ (i.e., at normal incidence to the slab). The boundary conditions are $I_\nu(\mu) = 0$ for $-1 \leq \mu < 0$ specified at $\tau_0 = 0$ and $I_\nu(\mu) = 0$ for $0 \leq \mu \leq 1$ specified at $\tau = \tau_{\max}$.**

   (a) $S_\nu(T) = B_\nu(T)$ **and** $T(z) =$ **const.**

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Starting from the full RTE,

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{n}\nabla I_\nu = \eta_\nu - \chi_\nu I_\nu \tag{1}$$

the time dependent portion can be removed as ${}^{\partial I_\nu}/_{\partial t} = 0$, and the dimensionality can be reduced in slab symmetry as $\hat{n}\nabla = n_z {}^{\partial}/_{\partial z} = \mu {}^{\partial}/_{\partial z}$

$$\mu\frac{\partial I_\nu}{\partial z} = \eta_\nu - \chi_\nu I_\nu. \tag{2}$$

Defining the source function as $S_\nu = {}^{\eta_\nu}/_{\chi_\nu}$ and the optical depth as $d\tau_\nu = -\chi_\nu dz$, this can be rewritten as

$$\mu\frac{\partial I_\nu}{\partial \tau_\nu} = I_\nu - S_\nu. \tag{3}$$

Substituting in the source function, the equation becomes

$$\mu\frac{\partial I_\nu}{\partial \tau_\nu} = I_\nu - B_\nu(T). \tag{4}$$

Manipulating this, we have that

$$\frac{\partial I_\nu}{\partial \tau_\nu} - \frac{1}{\mu}I_\nu = -\frac{1}{\mu}B_\nu(T) \tag{5}$$

$$\frac{\partial}{\partial \tau_\nu}(I_\nu e^{-\tau_\nu/\mu}) = \frac{\partial I_\nu}{\partial \tau_\nu}e^{-\tau_\nu/\mu} + I_\nu\frac{\partial}{\partial \tau_\nu}e^{-\tau_\nu/\mu} = \frac{\partial I_\nu}{\partial \tau_\nu}e^{-\tau_\nu/\mu} - \frac{I_\nu}{\mu}e^{-\tau_\nu/\mu} = e^{-\tau_\nu/\mu}\left(\frac{\partial I_\nu}{\partial \tau_\nu} - \frac{1}{\mu}I_\nu\right) \tag{6}$$

$$\frac{\partial}{\partial \tau_\nu}\left(I_\nu e^{-\tau_\nu/\mu}\right) = e^{-\tau_\nu/\mu}\left(\frac{\partial I_\nu}{\partial \tau_\nu} - \frac{1}{\mu}I_\nu\right) = e^{-\tau_\nu/\mu}\left(-\frac{1}{\mu}B_\nu(T)\right) = -\frac{1}{\mu}e^{-\tau_\nu/\mu}B_\nu(T) \tag{7}$$

Because there is no intensity entering from the top, and the downward radiation does not affect the observed (upward) radiation, this function can simply be integrated from the bottom up. Doing so,

$$\int_{\tau_0}^{\tau_{\max}} \frac{\partial}{\partial \tau'_\nu}\left(I_\nu e^{-\tau'_\nu/\mu}\right)\, d\tau'_\nu = -\frac{1}{\mu}\int_{\tau_0}^{\tau_{\max}} e^{-\tau'_\nu/\mu}B_\nu(T)\, d\tau'_\nu \tag{8}$$

The left hand side is simply

$$\begin{aligned}\int_{\tau_0}^{\tau_{\max}} \frac{\partial}{\partial \tau_\nu'}\left(I_\nu e^{-\tau_\nu'/\mu}\right)\, d\tau_\nu' &= I_\nu(\tau_{\max})e^{-\tau_{\max}/\mu} - I_\nu(\tau_0)e^{-\tau_0/\mu} \\ &= -I_\nu(\tau_0)e^{-\tau_0/\mu}\end{aligned} \tag{9}$$

The right hand side is then

$$\begin{aligned}-\frac{1}{\mu}\int_{\tau_0}^{\tau_{\max}} e^{-\tau_\nu'/\mu} B_\nu(T)\, d\tau_\nu' &= -\frac{1}{\mu} B_\nu(T)\int_{\tau_0}^{\tau_{\max}} e^{-\tau_\nu'/\mu}\, d\tau_\nu' \\ &= B_\nu(T)\left(e^{-\tau_{\max}/\mu} - e^{-\tau_0/\mu}\right) \\ &= B_\nu(T)\left(e^{-\tau_{\max}/\mu} - 1\right)\end{aligned} \tag{10}$$

Combining, the RTE is simply

$$-I_\nu(\tau_0)e^{-\tau_0/\mu} = B_\nu(T)\left(e^{-\tau_{\max}/\mu} - 1\right) \implies \tag{11}$$

$$\begin{aligned}I_\nu(\tau_0) &= -B_\nu(T)\left(e^{-(\tau_{\max}\,-\,\tau_0)/\mu} - e^{\tau_0/\mu}\right) \\ &= B_\nu(T)\left(e^{\tau_0/\mu} - e^{-(\tau_{\max}\,-\,\tau_0)/\mu}\right) \\ &= \boxed{B_\nu(T)\left(1 - e^{-\,10/\mu}\right) = I_\nu(\tau_0 = 0)}\end{aligned} \tag{12}$$

(b) $S_\nu = B_\nu(T)$ **and** $T(z) = T_0\tau^2$**.**

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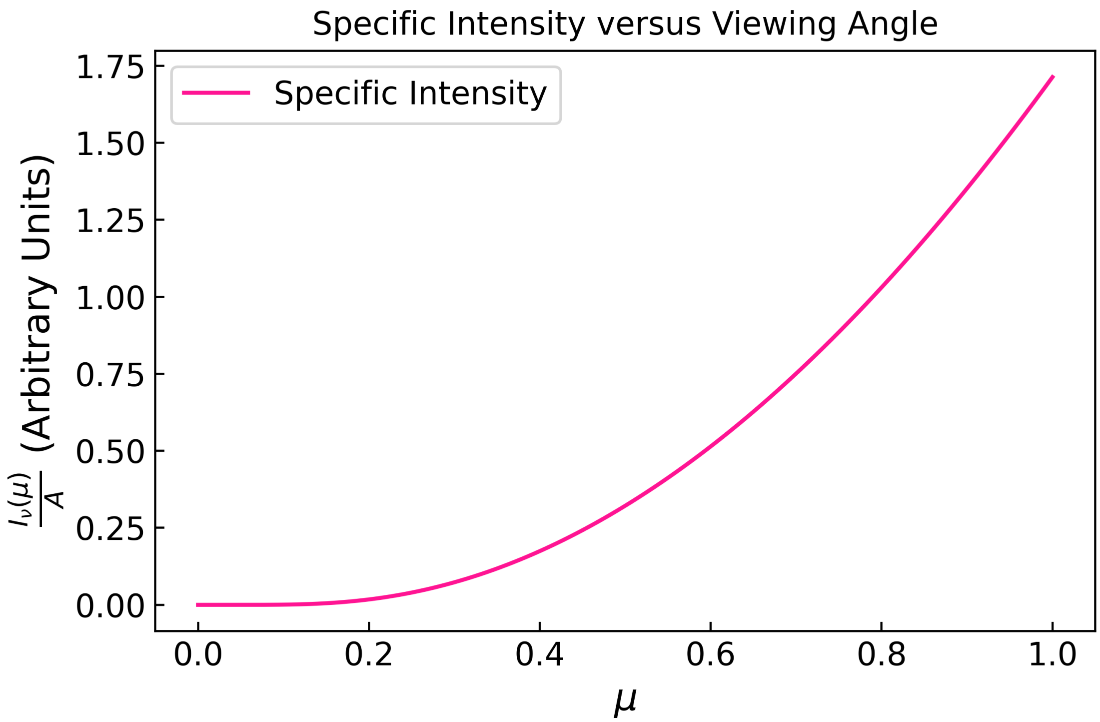
From the solution above, we know that

$$\begin{aligned}I_\nu(\tau_0)e^{-\tau_0/\mu} &= \frac{1}{\mu}\int_{\tau_0}^{\tau_{\max}} B_\nu(T_0\tau_\nu'^2)e^{-\tau_\nu'/\mu}\, d\tau_\nu' \\ &= \frac{1}{\mu}\int_{\tau_0}^{\tau_{\max}} \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT_0\tau_\nu'^2} - 1\right)^{-1} e^{-\tau_\nu'/\mu}\, d\tau_\nu' \\ &= \frac{2h\nu^3}{c^2\mu}\int_{\tau_0}^{\tau_{\max}} \left(e^{h\nu/kT_0\tau_\nu'^2} - 1\right)^{-1} e^{-\tau_\nu'/\mu}\, d\tau_\nu' \implies \\ I_\nu(\tau_0) &= \frac{2h\nu^3}{c^2\mu}\int_{\tau_0}^{\tau_{\max}} \frac{e^{-\tau_\nu'/\mu}}{e^{h\nu/kT_0\tau_\nu'^2} - 1}\, d\tau_\nu'\end{aligned} \tag{13}$$

Working in units where ${}^{h\nu}/_{kT_0} = 1$ and ${}^{2h\nu^3}/_{c^2} = A$, this becomes

$$I_\nu(\tau_0) = \frac{A}{\mu}\int_0^{10} \frac{e^{-\tau_\nu'/\mu}}{e^{-\tau_\nu'^2} - 1}\, d\tau_\nu' \tag{14}$$

This must be numerically integrated for each viewing angle $\mu$. Doing so gives




```python
import numpy as np
import scipy.integrate as int
import matplotlib.pyplot as plt
# BP Necessary imports.

def integrand(t,u):
    top = np.exp(-(t/u))
    bottom = np.exp(1/(t**2)) - 1
    return top / bottom
# BP Defining function to integrate.

I_nu = []
mu = np.linspace(1e-5, 1, 1000)
# BP Creating arrays to loop over.

for x in mu:
    result = int.quad(lambda t: integrand(t,x), 1e-4, 10)[0] / x
    I_nu.append(result)
# BP Integrating the inner function for each value of mu and appending to result
↪ array.

plt.rc('axes', labelsize=14)
plt.rc('figure', titlesize=30)
plt.rc('xtick', labelsize=12)
plt.rc('ytick', labelsize=12)
plt.rc('legend', fontsize=12)
fig, ax = plt.subplots(1, 1, figsize=(6,4), layout='tight')
# BP Stylization parameters.

ax.plot(mu, I_nu, color='deeppink', label='Specific Intensity')
ax.set_xlabel(r'$\mu$')
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```
ax.set_ylabel(r'$\frac{{I_{{\nu}}({{\mu}})}}{{A}}$ (Arbitrary Units)')
ax.set_title('Specific Intensity versus Viewing Angle')
ax.tick_params(axis='both', direction='in', which='both')
ax.legend(loc = 'upper left')
fig.tight_layout()
plt.savefig(r'/home/baparker/GitHub/Coursework/ASTR-589/HW/HW 1/HW1_1.png',dpi = 500)
plt.show()
# BP Plotting and labelling.
```

2. **A distant source emits blackbody radiation at temperature $T_s$. There is an intervening nebula at temperature $T_n < T_s$. Calculate the specific intensity of the radiation you would observe by looking at the source through the nebula as well as along a line of sight to the nebula that does not intersect the source. You may assume that the frequency of observation is smaller than both $kT_s$ and $kT_n$ so that you may work in the Rayleigh-Jeans limit. Can you use these two observations to infer the optical depth through the nebula at that wavelength?**

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Approximating the nebula as a slab at LTE with optical depth $\tau_n$ and source function $B_\nu(T_n)$ with no incident intensity from the observer, the observed specific intensity along a line of sight that does not intersect the source, following the derivation above where $\mu = 1$, is simply

$$I_\nu(\tau_0 = 0) = \boxed{B_\nu(T_n)\left(1 - e^{-\tau_n}\right) = I_n} \tag{15}$$

Similarly, when looking at the distant source through the nebula, $I_\nu(\tau_n) = B_\nu(T_s)$

$$\begin{aligned} I_\nu(\tau_n)e^{-\tau_n} - I_\nu(\tau_0)e^{-\tau_0} = B_\nu(T_s)e^{-\tau_n} - I_\nu(\tau_0)e^{-\tau_0} = B_\nu(T_n)\left(e^{-\tau_n} - 1\right) \implies \\ I_\nu(\tau_0) = \boxed{B_\nu(T_s)e^{-\tau_n} + B_\nu(T_n)\left(1 - e^{-\tau_n}\right) = I_s} \end{aligned} \tag{16}$$

Assuming both temperatures are known, these can be used to solve for the optical depth of the nebula.

$$\begin{aligned} I_s - I_n &= B_\nu(T_s)e^{-\tau_n} + B_\nu(T_n)\left(1 - e^{-\tau_n}\right) - B_\nu(T_n)\left(1 - e^{-\tau_n}\right) = B_\nu(T_s)e^{-\tau_n} \implies \\ e^{-\tau_n} &= \frac{I_s - I_n}{B_\nu(T_s)} \implies \\ -\tau_n &= \ln\frac{I_s - I_n}{B_\nu(T_s)} \implies \boxed{\tau_n = \ln\frac{B_\nu(T_s)}{I_s - I_n}} \end{aligned} \tag{17}$$

In the Rayleigh-Jeans limit,

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT_0\tau_\nu'^2} - 1\right)^{-1} \approx \frac{2\nu^2 k_B T}{c^2} \tag{18}$$

3. **Photoionization is a process in which a photon is absorbed by an atom and an electron is ejected. An energy at least equal to the ionization potential is required. Let this energy be $h\nu_0$ and let $\sigma_\nu$ be the cross section for photoionization. Show that the number of photoionizations per unit volume and per unit time is**

$$4\pi n_a \int_{\nu_0}^{\infty} \frac{\sigma_\nu J_\nu}{h\nu}\, d\nu,$$

**where $n_a$ is the number density of atoms.**

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The overall rate for ionization in a unit volume at a given frequencies is given by the number of scattering particles in that volume, $n_a$, times the number of ionizing photons, $n_\gamma$, times the probability for scattering, the cross section $\sigma_\nu$, times the differential speed, which for nonrelativistic particles is approximately $c$.

$$N_\nu = n_\gamma n_a \sigma_\nu c \tag{19}$$

Integrating this quantity over all frequencies above the ionization potential gives the total photoionization rate, $N$.

$$N = \int_{\nu_0}^{\infty} N_\nu \, d\nu = \int_{\nu_0}^{\infty} n_\gamma n_a \sigma_\nu c \, d\nu \tag{20}$$

The quantity $n_\gamma c$ is the total number of photons of frequency $\nu$ passing through a volume. Multiplying this by $h\nu$ gives the energy per area per time at frequency $\nu$, or the total flux. Thus, we have that

$$n_\gamma c h\nu = \frac{dE}{ds\ dt\ d\nu} = \int \frac{dE}{ds\ dt\ d\nu\ d\Omega}\, d\Omega = \int I_\nu \, d\Omega = 4\pi J_\nu \tag{21}$$

$$n_\gamma c = \frac{4\pi J_\nu}{h\nu} \tag{22}$$

Combining this with the previous equation, we get that the total photoionization rate is

$$N = \int_{\nu_0}^{\infty} n_\gamma n_a \sigma_\nu c = \int_{\nu_0}^{\infty} n_a \sigma_\nu \frac{4\pi J_\nu}{h\nu}\, d\nu = \boxed{4\pi n_a \int_{\nu_0}^{\infty} \frac{J_\nu \sigma_\nu}{h\nu}\, d\nu = N} \tag{23}$$

4. **We discussed in class that there is a close connection but not a one-to-one relationship between the photon mean-free-path $\lambda_p$ and the degree of isotropy of the radiation field. Give an example for each case (where $l$ is the characteristic length scale of the medium):**

(a) $\lambda_p \ll l$ **but** $I_\nu$ **is anisotropic**

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The first case describes a medium where the radiation is anisotropic while still being optically thick over large distances. Such objects include AGN and protoplanetary disks as the angle of observation determines the incident radiation.

For a protoplanetary disk, the characteristic length is simply the radius of the disk, $l \approx 1000$ AU $= 1.5 \times 10^{14}$ m. Assuming the disk is made of mostly hydrogen with a density of $n \approx 10^9$ m$^{-3}$, the scattering cross section is simply the area of the hydrogen atom using the Bohr radius of $\sigma = \pi \cdot (5.3 \times 10^{-11}\ \mathrm{m}^2)^2 = 8.8 \times 10^{-21}$m. This makes the mean free path

$$\lambda_m = \frac{1}{n\sigma} = \frac{1}{10^9\ \mathrm{m}^{-3} \cdot 8.8 \times 10^{-21}\ \mathrm{m}^2} = 1.1 \times 10^{11}\ \mathrm{m} \ll 1.5 \times 10^{14}\ \mathrm{m} \tag{24}$$

Additionally, the optical depth for embedded protoplanetary disks range from $\tau \sim 10$ to $\tau \sim 10^4$. Assuming that $\chi_\nu$ is constant across the characteristic length,

$$\tau_\nu = \int_0^l \chi_\nu\, ds = \frac{l}{\lambda_p} \tag{25}$$

meaning that any $\tau_\nu$ greater than one implies more than one mean free path across the object. As such, the derived optical depth means the mean free path is much less than the characteristic length.

(b) $\lambda_p > l$ **but** $I_\nu$ **is isotropic. Note that the second case is more rare.**

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The second case describes an optically thin medium where the radiation is isotropic over the photon mean free path. Some examples include the cosmic microwave background, which is almost perfectly isotropic, and hot plasma. For the CMB, the characteristic size at reionization was approximately $l \approx 1$ Mpc $= 3.1 \times 10^{22}$ m. Assuming pure Thomson scattering with an electron density of $n \approx 0.22$ m$^3$ and Thomson cross section $\sigma_e = 6.7 \times 10^{-29}$ m$^2$, the mean free path is approximately

$$\lambda_m = \frac{1}{n\sigma} = \frac{1}{0.22\ \mathrm{m}^{-3} \cdot 6.7 \times 10^{-29}\ \mathrm{m}^2} = 6.7 \times 10^{28}\ \mathrm{m} > 3.1 \times 10^{22}\ \mathrm{m} \tag{26}$$

This is further shown by the optical depth at the time of reionization, which is approximately $\tau_\nu \approx 0.06$, implying that the mean free path is larger than the scale length.