Lec 15

Polytropes: Simple stellar models

Last time we derived the eqns. of fluid dynamics

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}(\rho u_i) = 0$$

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial}{\partial x^i} u_j = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} \overbrace{- \frac{\partial \Phi}{\partial x_j}}^{a_j} + \cancelto{0}{\frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_i}}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}P + \frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x^i}\left[\left(\frac{3}{2}P + \frac{1}{2}\rho u^2\right)u_i\right] =$$

$$= \rho a_j u_j + \cancelto{0}{\text{heat conduction, viscous heating}}$$

We will solve for the equilibrium structure of spherically symm. configuration

(1) steady state $\Rightarrow \frac{\partial}{\partial t} = 0$

(2) static $\Rightarrow u_i = 0$

Looking at the energy eqn, there will be a non-trivial solution only if there is an additional source of energy (e.g. nuclear reaction) and a sink (e.g. radiative cooling) all of which we neglected. So, we won't explicitly solve the energy eqn.

Spherical symmetry $\frac{\partial}{\partial\theta} \to 0$, $\frac{\partial}{\partial\phi} \to 0$

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{G M(r)}{r^2}$$ → enclosed mass

$$M(r) = \int_0^r 4\pi r^2 \rho \, dr \text{, i.e. } \frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{r^2}{\rho}\frac{dP}{dr} = -G M(r) \Rightarrow$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM}{dr} = 0$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho} \quad (1)$$

Now we need an EOS. In general

$P = P(\rho, T, X_i)$, $T \to$ from energy equ.

But, a reasonable simplification is a "polytropic equation of state", where T is such that

$$P = K\rho^{\Gamma} \quad \text{or} \quad P = K\rho^{1+\frac{1}{n}}$$

$P = P(\rho) \equiv$ barotropic EOS

n = polytropic index

Note:

1) For degenerate matter inside of white dwarf, such an EOS is (almost) exact.

2) For normal gases, this EOS implies $T = T(\rho)$

3) A polytropic EOS is not the same as an adiabatic EOS.
$P = K\rho^{\gamma}$, with γ determined by the

the constant entropy condition ~~[illegible]~~ and connects to microphysics through degrees of freedom

$$\gamma = \frac{C_p}{C_v}$$

To make the solution ~~tractable~~ we introduce dimensionless variables

$$\xi = \left(\frac{\rho}{\rho_c}\right)^{\frac{1}{n}}, \quad \rho_c = \text{central density}$$

Then

$$\rho = \rho_c \xi^n$$

$$P = K\rho^{1+\frac{1}{n}} = K\rho_c^{\frac{n+1}{n}} \cdot \xi^{n+1}$$

$$\frac{dP}{dr} = K\rho_c^{\frac{n+1}{n}}(n+1)\,\xi^n \frac{d\xi}{dr}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho_c \xi^n} K\rho_c^{\frac{1}{n}}(n+1)\,\xi^n \frac{d\xi}{dr}\right) = -4\pi G \rho_c \xi^n$$

$$\Rightarrow (n+1)K\rho_c^{\frac{1}{n}-1}\,\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\xi}{dr}\right] = -4\pi G \xi^n$$

ξ is dimensionless, so

$\dfrac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}$ must have dimensions of length^2

Call $\alpha \equiv \left[\dfrac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}\right]^{\frac{1}{2}}$

Then letting $\xi = \dfrac{r}{\alpha}$

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n}$$ Lane-Emden eqn.

2nd-order ODE and requires two boundary conditions

1. By definition at $\xi=0$ $\rho=\rho_c$

thus $\theta(\xi=0)=1$

2. Second B.C.

Now Consider near $\xi=0$ $\left\{\begin{array}{l}\xi \to -\xi \Rightarrow \\ \theta(-\xi)=\theta(\xi) \\ \text{thus } \theta'(0)=0\end{array}\right.$

$$\theta(\xi) = 1 + \cancel{\theta'\xi} + \frac{1}{2}\theta''(0)\xi^2$$

$$\frac{d\theta}{d\xi} = \theta'(0) + \theta''(0)\xi \Rightarrow$$

$$\xi^2\frac{d\theta}{d\xi} = \xi^2\theta'(0) + \theta''(0)\xi^3$$

$$\frac{d}{d\xi}\left[\xi^2 \frac{d\theta}{d\xi}\right] = 2\xi\theta'(0) + 3\xi^2\theta''(0)$$

$$\theta^n = (1+\theta'(0)\xi)^n = (1 + n\theta'(0)\xi + \frac{n}{2}\theta''\xi^2)$$

$$\cancel{\frac{2}{\xi}\theta'(0)} + 3\theta''(0) = 1 + \frac{n\theta''(0)\xi^2}{2}$$

higher order

$$\theta''(0) = -\frac{1}{3}$$

then $\boxed{\theta(\xi) = 1 - \frac{1}{3}\xi^2}$

For each value of n, we need to solve the ODE once!

Different stars with different densities can be rescaled once we know $\theta(\xi)$.

- Lane-Emden analytically solvable for $n = 0, 1, 5$
- Solutions are monotonically decreasing
- for other values of n we must solve numerically.

We can use general num. solution to find various properties of the stars

① Radius

For $n < 5$, there exists a radial coordinate ξ_1 at which $\theta(\xi) = 0$, i.e. density $= 0 \rightarrow$ the surface of the star.

n	ξ_1	$-\xi_1^2 \frac{d\theta}{d\xi}\Big\vert_{\xi_1}$
0	2.45	4.9
1	3.14	3.14
2	4.35	2.41
3	6.9	2.01
4	14.97	1.79
5	∞	1.73

Radius of star $R = a\xi_1 \Rightarrow$

$$R = \xi_1 \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \rho_c^{\frac{1-n}{2n}}$$

Note that $0 < n < 1$ $R \uparrow$ as $\rho_c \uparrow$ but for $n > 1$, $R \downarrow$ as $\rho_c \uparrow$

- Note that isothermal sphere has

$$P = \frac{k}{m} \rho T_0 \Rightarrow n = \infty$$

So, isothermal sphere has no surface

(ii) Mass

$$M = \int_0^R 4\pi r^2 \rho \, dr = \int_0^{\xi_1} 4\pi (\xi a)^2 \rho_c \theta^n a \, d\xi$$

$$= 4\pi a^3 \rho_c \int_0^{\xi_1} \theta^n \xi^2 d\xi$$

from Lane-Emden

$$\theta^n = -\frac{1}{\xi^2} \frac{d}{d\xi}\left[\xi^2 \frac{d\theta}{d\xi}\right] \Rightarrow \xi^2 \theta^n = -\frac{d}{d\xi}\left[\xi^2 \frac{d\theta}{d\xi}\right]$$

So, $M = -4\pi a^3 \rho_c \int_0^{\xi_1} d\xi \frac{d}{d\xi}\left[\xi^2 \frac{d\theta}{d\xi}\right] =$

$$= -4\pi a^3 \rho_c \left[\xi^2 \frac{d\theta}{d\xi}\right]_{\xi_1} \quad \text{or}$$

$$\boxed{M = \left[-\xi_1^2 \frac{d\theta}{d\xi}\right]_{\xi_1} 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-n}{2n}}}$$

M - R relation

$$\left.\begin{aligned} R &= A \cdot \rho_c^{\frac{1-n}{2n}} \\ M &= B\, \rho_c^{\frac{3-n}{2n}} \end{aligned}\right\} \xrightarrow{\text{eliminate } \rho_c}$$

$$M = B \left(\frac{R}{A}\right)^{\frac{3-n}{1-n}}$$

M ↑ or ↓ with R↑ depending on n

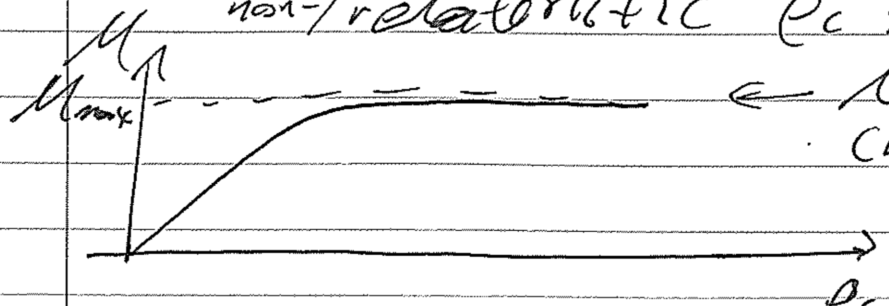
For $n=3$, $M = $ const.!!!

Example. For cold degenerate e^- (and other fermions)

$P \sim \rho^{5/3}$, $n = \frac{3}{2}$ non-relativistic e^-

$P \sim \rho^{4/3}$, $n = 3$ relativistic e^-

non-/relativistic ρ_c small/large



← M_{max} → Chandrasekhar mass

ρ_c Max mass supported by WD