Lecture 19

Let's write the hydro equ's and integrate them over z

Con't equ. (axisymmetry)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{u}) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u^r) + \cancel{\frac{\partial}{\partial z}(\rho u^z)} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int \rho\, dz + \frac{\partial}{\partial r}\int (r\rho u^r)\, dz$$

$$\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial r}(r\Sigma u^r) = 0$$

The mom. equ.

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = -\frac{\partial t_{ij}}{\partial x^i} - \rho a_j$$

$$\cancel{\frac{\partial}{\partial t}(\rho u_\varphi) + \frac{\partial}{\partial x^i}(\rho u_i u_\varphi) = -\partial}$$

Integrate over z, focus only on $u_\varphi = \Omega r$, then

$$\frac{\partial}{\partial t}\left(r^2 \Sigma \Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r u^r \Sigma r^2 \Omega\right) =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\nu \Sigma r^3 \frac{d\Omega}{dr}\right)$$

$\downarrow$

Expand use z-integrate
cont' eqn

$\downarrow$

$$r u^r \Sigma = \left[\frac{\partial}{\partial r}\left(r^2 \Omega\right)\right]^{-1} \left[\frac{\partial}{\partial r}\left(\nu \Sigma r^{-3} \frac{\partial \Omega}{\partial r}\right)\right]$$

$\downarrow$

Plug back into cont' eqn.

and assume Keplerian vel Ω
around a point mass

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}$$

$\downarrow$

$$\boxed{\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(\nu \Sigma r^{1/2}\right)\right]}$$

What kind of eqn. is this?

Diffusion eqn. !!!

What are the characteristic scales in the problem?

$$\frac{\cancel{\Sigma}}{t_0} \sim \frac{3}{r_0}\frac{1}{r_0}\cancel{r_0^{1/2}}\frac{1}{\cancel{r_0}}\cancel{\Sigma}\,\nu\,\cancel{r_0^{1/2}}$$

$$\Rightarrow t_0 \sim \frac{1}{3}\frac{r_0^2}{\nu}$$ ~~Charact~~

Normally, the normalization is chosen s.t.

$$t_0 = \frac{r_0^2}{12\nu}$$ so the time

variable is $\tau = \frac{12\nu}{r_0^2}t$

and the radial variable $x = r/r_0$

The characteristic time for evolution of the density is

$$t_0 \sim \frac{r_0^2}{3\nu} \sim \text{viscous timescale}$$

In a thin disk the inward drift timescale is

$$t \sim \frac{R}{u_r} \sim \frac{R}{\frac{3}{2}\frac{\nu}{R}} \sim \frac{2}{3}\frac{R^2}{\nu} \sim \frac{R^2}{\nu}$$

~~Hous~~ Hence the same as the viscous timescale!

<u>How fast is this?</u>

Consider molecular viscosity:

$\nu = c_s \lambda \longrightarrow$ mean particle distance

c_s: soundspeed

$$\tau = \frac{R^2}{c_s \lambda} \simeq \frac{R^2}{c_s (\rho/m)^{1/3}} \simeq \frac{R^2}{\left(\frac{kT}{m}\right)^{\frac{1}{2}} \left(\frac{m}{\rho}\right)^{\frac{1}{3}}}$$

$$= \underbrace{7 \times 10^{23}\, s}\left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{4000K}\right)^{-\frac{1}{2}} \left(\frac{\rho}{1 g\, cm^{-3}}\right)^{\frac{1}{3}}$$

$\sim 10^{16}$ years

<u>extremely long!</u>

Molecular viscosity fails by many orders of magnitude!

~~What do~~

The only parameter we can control is λ. What does it ~~as~~ take to make $\tau \sim$ week

$$\lambda \simeq 10^{10}\,\text{cm}\left(\frac{R}{R_\odot}\right)^2\left(\frac{T}{4000\,K}\right)^{-\frac{1}{2}}\left(\frac{\tau}{1\,wk}\right)^{-1}$$

To make λ ~~this~~ small it must correspond to some other length scale. ~~The relevant scale may~~

This means we need some "anomalous" source of viscosity.

Current belief is that the relevant scale is the turbulent scale, and the turbulent velocity field such that

$$\nu = v_{turb} \cdot l_{turb}$$

size of largest turbulent cells

relative to mean gas motion

Shock at the turbulent scale dissipate turbulent kinetic energy into heat when the motion is supersonic, so one requires

$$\underline{v_{turb} \le C_s}.$$

The cell sizes are limited by the disk thickness so $\underline{l_{turb} \le h}$

Since the viscous stress is

$$f_{r\varphi} = -\sigma_{\varphi r} = -\frac{3}{2}\eta\Omega = \cancel{\ldots}$$

And from hydrostatic equilib.

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GM}{r^2}\frac{z}{r} \Rightarrow \frac{1}{\rho}\frac{P}{h} \simeq \frac{GMh}{r^3}$$

$$\Rightarrow C_s^2 = \frac{GM}{r^3}h^2 \Rightarrow C_s = \Omega \cdot h$$

$$\cancel{f_{r\varphi} = l\frac{3}{2}\eta l\frac{C_s}{h}}$$

$$f_{r\varphi} = -\frac{3}{2}\eta\Omega = -\frac{3}{2}\nu\rho\Omega = -\frac{3}{2}l_{turb}v_{turb}\rho\Omega$$

$$\le \frac{3}{2}C_s \cdot h\rho\Omega = \frac{3}{2}\rho C_s^2 \simeq P$$

We then write in general

$$f_\phi = \alpha P \quad \text{with } \alpha \lesssim 1$$

α viscosity parameter!

Shakura - Sunyaev 1972

Since $f_\varphi = \frac{3}{2} \nu \rho \Omega = \alpha P \Rightarrow$

$$\nu = \alpha \frac{2}{3} \frac{P}{\rho} \Omega^{-1} \simeq \alpha \frac{2}{3} \frac{c_s^2}{\Omega} \simeq \alpha \frac{2}{3} c_s \cdot h$$

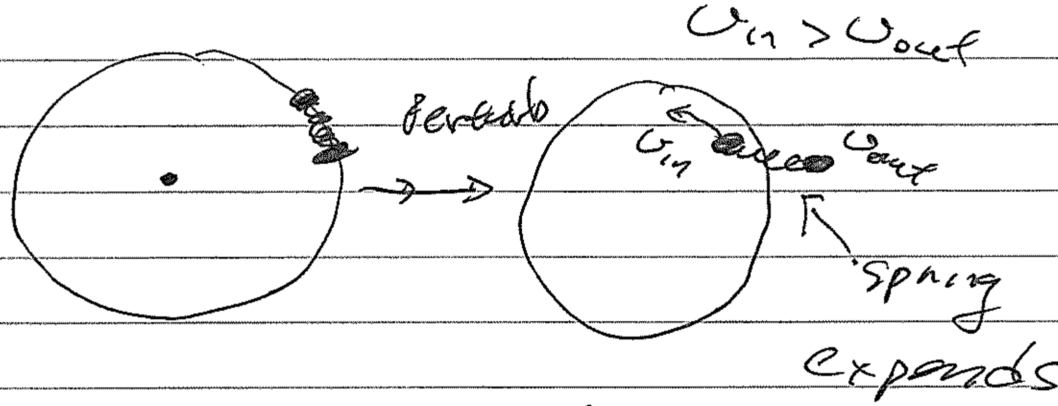
Typical values of $\alpha \simeq [10^{-3}, 10^{-1}]$

MHD simulations suggest

$\alpha \simeq 10^{-2}$

What can lead to such turbulent anomalous viscosity?

Magnetorotational Instabilities
(MRI)



Spring force pulls
"in" particle back → loses ang. momentum

→ "in" particle goes to smaller r

Spings force accelerates "out" particle

→ "out" particle move to larger r

→ now spring expands further

→ cascade → ang. mom. transport!

Result turbulence!
Balbus-Hawley Instabilities