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Physics for Astronomy (ASTR 589)

Homework 4

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1. **In class, we derived the equation for the evolution of the total energy density of a fluid**

$$\frac{\partial}{\partial t}\left(\frac{3}{2}P+\frac{1}{2}\rho u^2\right)+\frac{\partial}{\partial x_i}\left[\left(\frac{3}{2}P+\frac{1}{2}\rho u^2\right)u_i\right]=\rho a_i u_i-\frac{\partial}{\partial x_i}\left(\frac{1}{2}H_i+\Psi_{ij}u_j\right),$$

where H_i is the heat flux vector and $\Psi_{ij}=P\delta_{ij}-\sigma'_{ij}$ is the stress tensor of the fluid. Convert it into an equation for the evolution of the entropy of the fluid, i.e., show that

$$\rho T\frac{DS}{dt}=-\frac{1}{2}\frac{\partial H_i}{\partial x_i}+\frac{(\sigma'_{ij})^2}{2\eta},$$

where the entropy S is defined as

$$S=\frac{1}{\gamma-1}k\ln\left(\frac{P}{\rho^\gamma}\right),$$

$\gamma=5/3$, and η is the coefficient of shear viscosity. *Hints:* You will need to subtract the momentum equation multiplied by u_j and use the continuity equation; you will also need to show that (assuming zero bulk viscosity coefficient)

$$\sigma'_{ij}\frac{\partial u_j}{\partial x_i}=\frac{(\sigma'_{ij})^2}{2\eta}.$$

Expanding the energy equation, we have that

$$\frac{3}{2}\frac{\partial P}{\partial t}+\frac{1}{2}\frac{\partial}{\partial t}(\rho u^2)+\frac{3}{2}\frac{\partial}{\partial x_i}(Pu_i)+\frac{1}{2}\frac{\partial}{\partial x_i}(\rho u^2 u_i)=\rho a_j u_j-\frac{1}{2}\frac{\partial H_i}{\partial x_i}-\frac{\partial}{\partial x_i}(\Psi_{ij}u_j) \tag{1}$$

The momentum equation is

$$\frac{\partial u_j}{\partial t}+u_i\frac{\partial u_j}{\partial x_i}=-\frac{1}{\rho}\frac{\partial \Psi_{ij}}{\partial x_i}+a_j \tag{2}$$

Multiplying this by ρu_j gives

$$\rho u_j\frac{\partial u_j}{\partial t}+\rho u_j u_i\frac{\partial u_j}{\partial x_i}=-u_j\frac{\partial \Psi_{ij}}{\partial x_i}+a_j\rho u_j \tag{3}$$

Subtracting this from the energy equation, we have that

$$\begin{aligned}\frac{3}{2}\frac{\partial P}{\partial t}+\frac{1}{2}\frac{\partial}{\partial t}(\rho u^2)+\frac{3}{2}\frac{\partial}{\partial x_i}(Pu_i)+\frac{1}{2}\frac{\partial}{\partial x_i}(\rho u^2 u_i)-\rho u_j\frac{\partial u_j}{\partial t}-\rho u_j u_i\frac{\partial u_j}{\partial x_i}&\\ =\rho a_j u_j-\frac{1}{2}\frac{\partial H_i}{\partial x_i}-\frac{\partial}{\partial x_i}(\Psi_{ij}u_j)+u_j\frac{\partial \Psi_{ij}}{\partial x_i}-a_j\rho u_j=-\frac{1}{2}\frac{\partial H_i}{\partial x_i}-\frac{\partial}{\partial x_i}(\Psi_{ij}u_j)+u_j\frac{\partial \Psi_{ij}}{\partial x_i}&\\ =-\frac{1}{2}\frac{\partial H_i}{\partial x_i}-\Psi_{ij}\frac{\partial u_j}{\partial x_i}-u_j\frac{\partial \Psi_{ij}}{\partial x_i}+u_j\frac{\partial \Psi_{ij}}{\partial x_i}=-\frac{1}{2}\frac{\partial H_i}{\partial x_i}-\Psi_{ij}\frac{\partial u_j}{\partial x_i}&\\ =-\frac{1}{2}\frac{\partial H_i}{\partial x_i}-P\delta_{ij}\frac{\partial u_j}{\partial x_i}+\sigma_{ij}\frac{\partial u_j}{\partial x_i}=-\frac{1}{2}\frac{\partial H_i}{\partial x_i}-P\frac{\partial u_j}{\partial x_j}+\sigma_{ij}\frac{\partial u_j}{\partial x_i}&\end{aligned} \tag{4}$$

Examining the last term on the right hand side, we have that

$$\sigma_{ij}\frac{\partial u_j}{\partial x_i}=\eta\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}-\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)\frac{\partial u_j}{\partial x_i} \tag{5}$$

Solving for $\frac{\partial u_j}{\partial x_i}$ from the definition of σ_{ij} and substituting back into the above equation,

$$\sigma_{ij}\frac{\partial u_j}{\partial x_i}=\sigma_{ij}\left(\frac{\sigma_{ij}}{\eta}-\frac{\partial u_i}{\partial x_j}+\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)=\frac{\sigma_{ij}\sigma^{ij}}{\eta}-\sigma_{ij}\frac{\partial u_i}{\partial x_j}+\frac{2}{3}\delta_{ij}\sigma_{ij}\frac{\partial u_k}{\partial x_k} \tag{6}$$

Because we are summing over both i and j in the Einstein notation, the indices on the left hand side can be flipped and the equality is still true.

$$\sigma_{ji}\frac{\partial u_i}{\partial x_j}=\frac{\sigma_{ij}\sigma^{ij}}{\eta}-\sigma_{ij}\frac{\partial u_i}{\partial x_j}+\frac{2}{3}\delta_{ij}\sigma_{ij}\frac{\partial u_k}{\partial x_k} \tag{7}$$

Similarly, because σ_{ij} is symmetric, $\sigma_{ij}=\sigma_{ji}$, meaning

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j}=\frac{\sigma_{ij}\sigma^{ij}}{\eta}-\sigma_{ij}\frac{\partial u_i}{\partial x_j}+\frac{2}{3}\delta_{ij}\sigma_{ij}\frac{\partial u_k}{\partial x_k} \tag{8}$$

$$\implies 2\sigma_{ij}\frac{\partial u_i}{\partial x_j}=\frac{\sigma_{ij}\sigma^{ij}}{\eta}+\frac{2}{3}\delta_{ij}\sigma_{ij}\frac{\partial u_k}{\partial x_k} \tag{9}$$

$$\implies \sigma_{ij}\frac{\partial u_i}{\partial x_j}=\frac{\sigma_{ij}\sigma^{ij}}{2\eta}+\frac{1}{3}\delta_{ij}\sigma_{ij}\frac{\partial u_k}{\partial x_k} \tag{10}$$

Because the summation on the right hand side is over i and j independently of k, the term $\frac{\partial u_k}{\partial x_k}$ is unaffected by the summation over i and j. Expanding out this summation, only the diagonal terms are nonzero, giving

$$\delta_{ij}\sigma_{ij}=\delta_{xx}\sigma_{xx}+\delta_{yy}\sigma_{yy}+\delta_{zz}\sigma_{zz}=\sigma_{xx}+\sigma_{yy}+\sigma_{zz} \tag{11}$$

However we know that $tr(\sigma_{ij})=\sigma_{xx}+\sigma_{yy}+\sigma_{zz}=0$, meaning the summation is 0 and the equation above becomes

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j}=\frac{\sigma_{ij}\sigma^{ij}}{2\eta}=\frac{\sigma_{ij}^2}{2\eta} \tag{12}$$

The original equation now becomes

$$\frac{3}{2}\frac{\partial P}{\partial t}+\frac{1}{2}\frac{\partial}{\partial t}(\rho u^2)+\frac{3}{2}\frac{\partial}{\partial x_i}(Pu_i)+\frac{1}{2}\frac{\partial}{\partial x_i}(\rho u^2 u_i)-\rho u_j\frac{\partial u_j}{\partial t}-\rho u_j u_i\frac{\partial u_j}{\partial x_i}=-\frac{1}{2}\frac{\partial H_i}{\partial x_i}-P\frac{\partial u_j}{\partial x_j}+\frac{\sigma_{ij}^2}{2\eta} \tag{13}$$

Moving $P\frac{\partial u_j}{\partial x_j}$ to the left hand side and swapping the indices, the left hand side is

$$\begin{aligned}
-\frac{1}{2}\frac{\partial H_i}{\partial x_i}+\frac{\sigma_{ij}^2}{2\eta} &= \frac{3}{2}\frac{\partial P}{\partial t}+\frac{1}{2}\frac{\partial}{\partial t}(\rho u^2)+\frac{3}{2}\frac{\partial}{\partial x_i}(Pu_i)+\frac{1}{2}\frac{\partial}{\partial x_i}(\rho u^2 u_i)-\rho u_j\frac{\partial u_j}{\partial t}-\rho u_j u_i\frac{\partial u_j}{\partial x_i}+P\frac{\partial u_i}{\partial x_i}\\
&= \frac{3}{2}\frac{\partial P}{\partial t}+\frac{1}{2}u^2\frac{\partial \rho}{\partial t}+\frac{1}{2}\rho\frac{\partial u^2}{\partial t}+\frac{3}{2}u_i\frac{\partial P}{\partial x_i}+\frac{3}{2}P\frac{\partial u_i}{\partial x_i}+\frac{1}{2}\rho u_i\frac{\partial}{\partial x_i}(u^2)+\frac{1}{2}u^2\frac{\partial}{\partial x_i}(\rho u_i)\\
&\qquad -\rho u_j\frac{\partial u_j}{\partial t}-\rho u_j u_i\frac{\partial u_j}{\partial x_i}+P\frac{\partial u_i}{\partial x_i}\\
&= \frac{3}{2}\frac{\partial P}{\partial t}+\frac{1}{2}u^2\frac{\partial \rho}{\partial t}+\frac{1}{2}\rho\frac{\partial u^2}{\partial t}+\frac{3}{2}u_i\frac{\partial P}{\partial x_i}+\frac{5}{2}P\frac{\partial u_i}{\partial x_i}+\frac{1}{2}\rho u_i\frac{\partial}{\partial x_i}(u^2)+\frac{1}{2}u^2\frac{\partial}{\partial x_i}(\rho u_i)\\
&\qquad -\rho u_j\frac{\partial u_j}{\partial t}-\rho u_j u_i\frac{\partial u_j}{\partial x_i}
\end{aligned} \tag{14}$$

The continuity equation states

$$\frac{\partial \rho}{\partial t}=-\frac{\partial}{\partial x_i}(\rho u_i)=-u_i\frac{\partial \rho}{\partial x_i}-\rho\frac{\partial u_i}{\partial x_i} \tag{15}$$

Substituting this in gives

$$\begin{aligned}
-\frac{1}{2}\frac{\partial H_i}{\partial x_i}+\frac{\sigma_{ij}^2}{2\eta} &= \frac{3}{2}\frac{\partial P}{\partial t}+\frac{1}{2}u^2\frac{\partial \rho}{\partial t}+\frac{1}{2}\rho\frac{\partial u^2}{\partial t}+\frac{3}{2}u_i\frac{\partial P}{\partial x_i}+\frac{5}{2}P\frac{\partial u_i}{\partial x_i}+\frac{1}{2}\rho u_i\frac{\partial}{\partial x_i}(u^2)-\frac{1}{2}u^2\frac{\partial \rho}{\partial t}-\rho u_j\frac{\partial u_j}{\partial t}-\rho u_j u_i\frac{\partial u_j}{\partial x_i}\\
&= \frac{3}{2}\frac{\partial P}{\partial t}+\frac{1}{2}\rho\frac{\partial u^2}{\partial t}+\frac{3}{2}u_i\frac{\partial P}{\partial x_i}+\frac{5}{2}P\frac{\partial u_i}{\partial x_i}+\frac{1}{2}\rho u_i\frac{\partial}{\partial x_i}(u^2)-\rho u_j\frac{\partial u_j}{\partial t}-\rho u_j u_i\frac{\partial u_j}{\partial x_i}
\end{aligned} \tag{16}$$

Because of the implicit summations in the Einstein notation, we have that $u^2=u_iu^i+u_ju^j=u_ku^k$. Thus we have that

$$\begin{aligned}
-\frac{1}{2}\frac{\partial H_i}{\partial x_i}+\frac{\sigma_{ij}^2}{2\eta} &= \frac{3}{2}\frac{\partial P}{\partial t}+\frac{1}{2}\rho\frac{\partial u_i^2}{\partial t}+\frac{3}{2}u_i\frac{\partial P}{\partial x_i}+\frac{5}{2}P\frac{\partial u_i}{\partial x_i}+\frac{1}{2}\rho u_i\frac{\partial}{\partial x_i}(u_i^2)-\rho u_j\frac{\partial u_j}{\partial t}-\rho u_j u_i\frac{\partial u_j}{\partial x_i}\\
&= \frac{3}{2}\frac{\partial P}{\partial t}+\rho u_i\frac{\partial u_i}{\partial t}+\frac{3}{2}u_i\frac{\partial P}{\partial x_i}+\frac{5}{2}P\frac{\partial u_i}{\partial x_i}+\rho u_i u_i\frac{\partial u_i}{\partial x_i}-\rho u_j\frac{\partial u_j}{\partial t}-\rho u_j u_i\frac{\partial u_j}{\partial x_i}
\end{aligned} \tag{17}$$

From the above identity, we have that $u_j\frac{\partial u_j}{\partial x_i}=\frac{1}{2}\frac{\partial u_j^2}{\partial x_i}=\frac{1}{2}\frac{\partial u_i^2}{\partial x_i}=u_i\frac{\partial u_i}{\partial x_i}$. This gives us

$$\begin{aligned}
-\frac{1}{2}\frac{\partial H_i}{\partial x_i}+\frac{\sigma_{ij}^2}{2\eta} &= \frac{3}{2}\frac{\partial P}{\partial t}+\rho u_i\frac{\partial u_i}{\partial t}+\frac{3}{2}u_i\frac{\partial P}{\partial x_i}+\frac{5}{2}P\frac{\partial u_i}{\partial x_i}+\rho u_i u_i\frac{\partial u_i}{\partial x_i}-\rho u_i\frac{\partial u_i}{\partial t}-\rho u_i u_i\frac{\partial u_i}{\partial x_i}\\
&= \frac{3}{2}\frac{\partial P}{\partial t}+\frac{3}{2}u_i\frac{\partial P}{\partial x_i}+\frac{5}{2}P\frac{\partial u_i}{\partial x_i}
\end{aligned} \tag{18}$$

For an ideal gas with mass of $m = 1$, the equation of state is simply $P = \rho k T$. Substituting this and the continuity equation in, combined with the definition of Lagrangian derivative, gives

$$\begin{aligned}
&-\frac{1}{2}\frac{\partial H_i}{\partial x_i} + \frac{\sigma_{ij}^2}{2\eta} \\
&= \frac{3}{2}\frac{\partial}{\partial t}(\rho k T) + \frac{3}{2}u_i\frac{\partial}{\partial x_i}(\rho k T) + \frac{5}{2}(\rho k T)\frac{\partial u_i}{\partial x_i} = \frac{3}{2}\rho k\frac{\partial T}{\partial t} + \frac{3}{2}kT\frac{\partial \rho}{\partial t} + \frac{3}{2}u_i\rho k\frac{\partial T}{\partial x_i} + \frac{3}{2}u_i kT\frac{\partial \rho}{\partial x_i} + \frac{5}{2}\rho k T\frac{\partial u_i}{\partial x_i} \\
&= \frac{3}{2}\rho k\frac{\partial T}{\partial t} + \frac{3}{2}u_i\rho k\frac{\partial T}{\partial x_i} + \frac{3}{2}kT\left(\frac{\partial \rho}{\partial t} + u_i\frac{\partial \rho}{\partial x_i} + \rho\frac{\partial u_i}{\partial x_i}\right) + \rho k T\frac{\partial u_i}{\partial x_i} \\
&= \frac{3}{2}\rho k\frac{\partial T}{\partial t} + \frac{3}{2}u_i\rho k\frac{\partial T}{\partial x_i} + \frac{3}{2}kT\left(\frac{\partial \rho}{\partial t} - \frac{\partial \rho}{\partial t}\right) + \rho k T\frac{\partial u_i}{\partial x_i} = \frac{3}{2}\rho k\frac{\partial T}{\partial t} + \frac{3}{2}u_i\rho k\frac{\partial T}{\partial x_i} + \rho k T\frac{\partial u_i}{\partial x_i} \\
&= \frac{3}{2}\rho k\frac{DT}{dt} + kT\left(\rho\frac{\partial u_i}{\partial x_i}\right) = \frac{3}{2}\rho k\frac{DT}{dt} + kT\left(\rho\frac{\partial u_i}{\partial x_i}\right) = \frac{3}{2}\rho k\frac{DT}{dt} - kT\left(\frac{\partial \rho}{\partial t} + u_i\frac{\partial \rho}{\partial x_i}\right) = \frac{3}{2}\rho k\frac{DT}{dt} - kT\frac{D\rho}{dt}
\end{aligned} \tag{19}$$

Using the definition of entropy from above, the Lagrangian derivative of entropy is

$$\begin{aligned}
\frac{DS}{dt} &= \frac{D}{dt}\left(\frac{1}{\gamma-1}k\ln\left(\frac{P}{\rho^\gamma}\right)\right) = \frac{1}{\gamma-1}k\frac{D}{dt}\ln\left(\frac{P}{\rho^\gamma}\right) = \frac{1}{\gamma-1}k\frac{D}{dt}\left(\ln P - \gamma\ln\rho\right) \\
&= \frac{1}{\gamma-1}k\frac{1}{P}\frac{DP}{dt} - \frac{\gamma}{\gamma-1}k\frac{1}{\rho}\frac{D\rho}{dt}
\end{aligned} \tag{20}$$

For an ideal gas, we know that $\gamma = 5/3$ and $P = \rho k T$. Thus

$$\begin{aligned}
\frac{DS}{dt} &= \frac{1}{5/3-1}k\frac{1}{\rho k T}\frac{D}{dt}(\rho k T) - \frac{5/3}{5/3-1}k\frac{1}{\rho}\frac{D\rho}{dt} = \frac{3}{2}k\frac{kT}{\rho k T}\frac{D\rho}{dt} + \frac{3}{2}k\frac{\rho k}{\rho k T}\frac{DT}{dt} - \frac{5}{2}k\frac{1}{\rho}\frac{D\rho}{dt} \\
&= \frac{3}{2}\frac{k}{\rho}\frac{D\rho}{dt} + \frac{3}{2}\frac{k}{T}\frac{DT}{dt} - \frac{5}{2}\frac{k}{\rho}\frac{D\rho}{dt} = \frac{3}{2}\frac{k}{T}\frac{DT}{dt} - \frac{k}{\rho}\frac{D\rho}{dt}
\end{aligned} \tag{21}$$

Multiplying this by ρT,

$$\rho T\frac{DS}{dt} = \rho T\frac{3}{2}\frac{k}{T}\frac{DT}{dt} - \rho T\frac{k}{\rho}\frac{D\rho}{dt} = \frac{3}{2}\rho k\frac{DT}{dt} - kT\frac{D\rho}{dt} \tag{22}$$

The Lagrangian derivative of the defined entropy is thus exactly the left hand side of the above equation. We thus have the relation

$$\rho T\frac{DS}{dt} = \frac{3}{2}\rho k\frac{DT}{dt} - kT\frac{D\rho}{dt} = -\frac{1}{2}\frac{\partial H_i}{\partial x_i} + \frac{\sigma_{ij}^2}{2\eta} \implies \tag{23}$$

$$\boxed{\rho T\frac{DS}{dt} = -\frac{1}{2}\frac{\partial H_i}{\partial x_i} + \frac{\sigma_{ij}^2}{2\eta}} \tag{24}$$

exactly as postulated.

2. **Use the fact that the solution to the Lane-Emden equation for a polytrope of index n can be expanded to a series of only even powers in ζ, i.e.,**

$$\Theta(\xi) = 1 + C_1\xi^2 + C_2\xi^4 + \dots$$

to show that

$$\Theta(\zeta) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + \dots$$

For small values of ξ, the solution to the Lane-Emden equation can be Taylor expanded as

$$\Theta(\xi) = 1 + \Theta'(0)\xi + \frac{1}{2}\Theta''(0)\xi^2 + \frac{1}{6}\Theta'''(0)\xi^3 + \frac{1}{24}\Theta''''(0)\xi^4 + \dots \tag{25}$$

Taking the derivative of this function with respect to ξ,

$$\frac{d\Theta}{d\xi} = \Theta'(0) + \Theta''(0)\xi + \frac{1}{2}\Theta'''(0)\xi^2 + \frac{1}{6}\Theta''''(0)\xi^3 + \dots \tag{26}$$

Multiplying this by ξ^2,

$$\xi^2\frac{d\Theta}{d\xi} = \Theta'(0)\xi^2 + \Theta''(0)\xi^3 + \frac{1}{2}\Theta'''(0)\xi^4 + \frac{1}{6}\Theta''''(0)\xi^5 + \dots \tag{27}$$

Taking the derivative of this again with respect to ξ,

$$\frac{d}{d\xi}\left[\xi^2\frac{d\Theta}{d\xi}\right] = 2\Theta'(0)\xi + 3\Theta''(0)\xi^2 + 2\Theta'''(0)\xi^3 + \frac{5}{6}\Theta''''(0)\xi^4 + \dots \tag{28}$$

Dividing this once more by ξ^2,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\Theta}{d\xi}\right] = \frac{2}{\xi}\Theta'(0) + 3\Theta''(0) + 2\Theta'''(0)\xi + \frac{5}{6}\Theta''''(0)\xi^2 + \dots \tag{29}$$

This expression is now exactly the Taylor series expansion of the left-hand side of the Lane-Emden equation. The same Taylor expansion can be done for the right-hand side.

$$-\Theta^n = -\left(1 + \Theta'(0)\xi + \frac{1}{2}\Theta(0)''\xi^2 + \frac{1}{6}\Theta'''(0)\xi^3 + \frac{1}{24}\Theta''''(0)\xi^4 + \dots\right)^n \tag{30}$$

Because we assumed in the beginning that ξ is small, it is safe to assume that $\Theta'(0)\xi + \frac{1}{2}\Theta(0)''\xi^2 + \frac{1}{6}\Theta'''(0)\xi^3 + \frac{1}{24}\Theta''''(0)\xi^4 + \dots \ll 1$. Thus we can assume that $(1+x)^n = 1 + xn$. Thus we have that

$$-\Theta^n = -1 - n\Theta'(0)\xi - \frac{n}{2}\Theta(0)''\xi^2 - \frac{n}{6}\Theta'''(0)\xi^3 - \frac{n}{24}\Theta''''(0)\xi^4 - \dots \tag{31}$$

Equating both the left-hand side from earlier with the Taylor expansion of the right-hand side above, we have that

$$\frac{2}{\xi}\Theta'(0)+3\Theta''(0)+2\Theta'''(0)\xi+\frac{5}{6}\Theta''''(0)\xi^2+\ldots=-1-n\Theta'(0)\xi-\frac{n}{2}\Theta(0)''\xi^2-\frac{n}{6}\Theta'''(0)\xi^3-\frac{n}{24}\Theta''''(0)\xi^4-\ldots \quad (32)$$

Ignoring the higher order terms, we have that

$$\frac{2}{\xi}\Theta'(0)+3\Theta''(0)+2\Theta'''(0)\xi+\frac{5}{6}\Theta''''(0)\xi^2=-1-n\Theta'(0)\xi-\frac{n}{2}\Theta(0)''\xi^2-\frac{n}{6}\Theta'''(0)\xi^3-\frac{n}{24}\Theta''''(0)\xi^4 \quad (33)$$

However, because $\Theta(\xi)$ can be written as a series of only even powers, all of the odd power ξ coefficients must be zero. $\Theta'(0)=\Theta'''(0)=0$, giving

$$3\Theta''(0)+\frac{5}{6}\Theta''''(0)\xi^2=-1-\frac{n}{2}\Theta''(0)\xi^2-\frac{n}{24}\Theta''''(0)\xi^4 \quad (34)$$

Because each of the ξ coefficients must be equal, we then have that

$$3\Theta''(0)=-1 \implies \Theta''(0)=-\frac{1}{3} \quad (35)$$

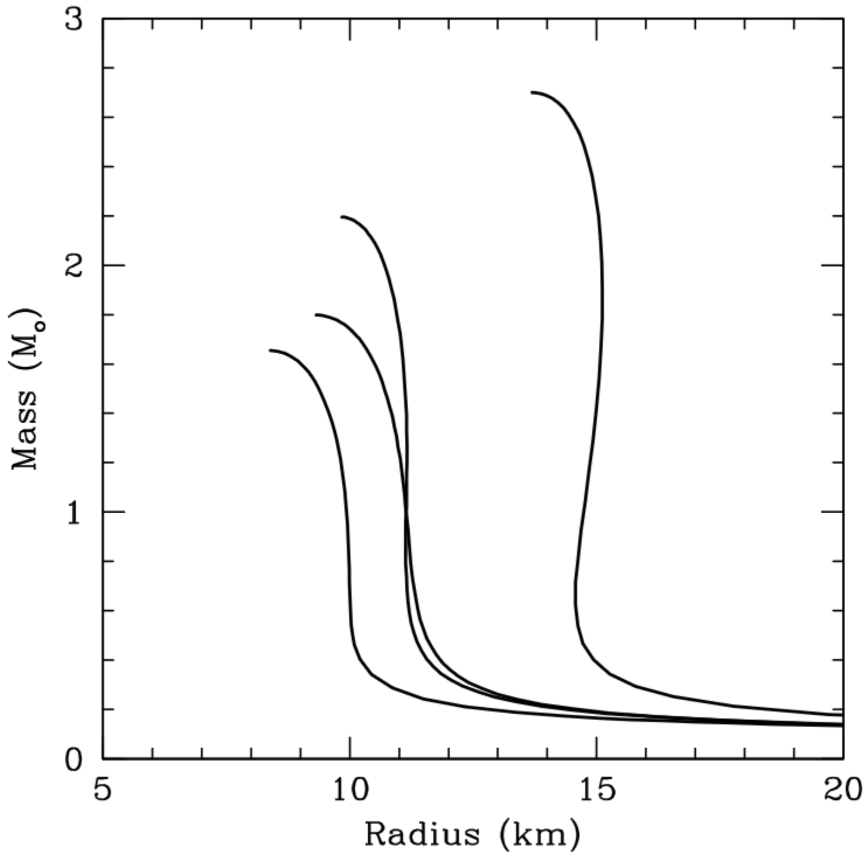
$$\frac{5}{6}\Theta''''(0)=-\frac{n}{2}\Theta''(0)=\frac{1}{6} \implies \Theta''''(0)=\frac{n}{5} \quad (36)$$

Plugging all of these values back into the Taylor expansion, we have that the solution to the Lane-Emden equation can be approximated as

$$\Theta(\xi)=1+(0)\xi-\frac{1}{2}\left(-\frac{1}{3}\right)\xi^2+\frac{1}{6}(0)\xi^3+\frac{1}{24}\frac{n}{5}\xi^4+\ldots \quad (37)$$

$$=\boxed{1-\frac{1}{6}\xi^2+\frac{n}{120}\xi^4+\ldots=\Theta(\xi)} \quad (38)$$

3. **Several equations of state for neutron-star matter predict that, at masses smaller than their maximum possible mass, the radii of stars depend very weakly on their mass, i.e., $R \simeq M^0$. What does this imply for these equations of state of neutron-star matter, if we treat them as polytropes? How do they compare to the equation of state of a non-relativistic degenerate neutron gas? For masses smaller than the maximum neutron-star mass, you can assume that general relativistic corrections are negligible.**



For a polytrope, we know that the mass-radius relationship is approximately

$$M \propto R^{\frac{3-n}{1-n}} \tag{39}$$

or, equivalently,

$$R \propto M^{\frac{1-n}{3-n}}. \tag{40}$$

For a polytrope to exhibit no mass dependence on the radius, this exponent must be zero, making the polytropic index exactly $n = 1$, which makes $\gamma = 1 + 1/n = 2$. Thus the equation of state of these neutron star polytrope models is approximately

$$P \propto \rho^2. \tag{41}$$

For a non-relativistic degenerate neutron gas (what neutron are mostly composed of), the equation of state is the same as that for a non-relativistic degenerate electron gas, $n = 3/2$ and $P \propto \rho^{5/3}$. Neutron star polytropic models would thus be expected to follow this same equation of state. However, this clearly would be a poor model as the radius would always depend on mass, unlike the given mass independence. Even for an ultra-relativistic degenerate Fermi gas, the polytropic index is $n = 3$, far from the predicted $n = 1$. Despite all of this, however, the $n = 1$ polytrope is still a good estimate for neutron stars, despite the non-intuitive interpretation. In reality, a better approximation lies somewhere between $n = 0.5$–1, however none of these values are expected solely from the neutron star's composition approximate coposition of non-relativistic degenerate neutrons.