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Physics for Astronomy (ASTR 589)

Homework 3

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1. **The Coma Cluster is a galaxy cluster with a radius of approximately $R = 3$ Mpc. X-ray observations reveal some amount of hot gas, with a thermal spectrum indicating a temperature of $T = 8.8 \times 10^7$ K. Assume that the gas consists of 75% hydrogen and 25% helium by mass. Wherever necessary, use a Gaunt factor of 1.2.**

   (a) **Do you expect the gas to be fully ionized? Explain your reasoning (or derive the ionization fraction using the Saha Equation).**

This gas is very likely fully ionized. For hydrogen, the large majority of atoms are fully ionized above 10000 K, and for helium the vast majority are fully ionized above 29000 K. Additionally, at temperatures of 8.8×10^7 K, the average kinetic energy of a hydrogen atom is approximately $\langle E \rangle \approx 3kT/2 = 3 \cdot 1.381 \times 10^{-23} \cdot 8.8 \times 10^7/2 = 1.823 \times 10^{-15}$ J $= 11378$ eV, which is significantly larger than the ionization energy of both hydrogen, 13.6 eV, and helium, 24.6 eV. Thus, because the average particle has much more energy than is required to ionize any particle in this gas, it is very likely that all of the particles are ionized.

More concretely, the Saha equation states that the ratio of singly ionized atoms to neutral atoms in a gas is given by

$$\frac{n_1}{n_0} n_e = \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} \frac{2g_1}{g_0} e^{-\chi/kT} \tag{1}$$

Assuming that all of the free electrons come from ionized atoms, for hydrogen $n_e = n_1$ and $g_1 = 1$, $g_0 = 2$, and for helium $n_e = 2n_1$ and $g_0 = 2$, $g_1 = 4$, $g_2 = 2$. We can thus solve for the number density of electrons, and hence the number density of ionized hydrogen. For hydrogen, using the ionization potential of $\chi = 13.54$ eV we then have that

$$\frac{n_1}{n_0} n_e = 1.99 \times 10^{27} \text{ cm}^{-3} \tag{2}$$

Because for hydrogen $n_1 = n_e$, this means that $n_e^2 \gg n_0$. We can then also assume that $n_e \gg n_0$, meaning that the gas is practically fully ionized.

For helium, the above equation must be calculated for each ionization level ratio. These can then be multiplied to give the ionization ratio between the ground state and fully ionized state.

$$\frac{n_1}{n_0} n_e \frac{n_2}{n_0} n_e = \frac{n_2}{n_0} n_e^2 \tag{3}$$

Using the first ionization energy of $\chi_1 = 24.48$ eV and the second ionization energy of $\chi_2 = 54.17$ eV, we obtain

$$\frac{n_1}{n_0} n_e = 7.948 \times 10^{27} \text{ cm}^{-3} \tag{4}$$

$$\frac{n_2}{n_1} n_e = 9.896 \times 10^{26} \text{ cm}^{-3} \implies \tag{5}$$

$$\frac{n_2}{n_0} n_e^2 = 7.865 \times 10^{54} \text{ cm}^{-3} \tag{6}$$

As before, ignoring the extra electron produced from doubly ionized helium, the same conclusions can be drawn, mainly that $n_1 \gg n_0$ and $n_2 \gg n_1$, implying then that $n_2 \gg n_0$. Thus for helium we also have that the vast majority of the atoms are fully ionized.

Using the results from the next section, we can verify this statement and calculate the actual ionization ratios. For hydrogen we have that

$$\boxed{\frac{n_1}{n_0} = 2.132 \times 10^{31}} \tag{7}$$

and for helium we have that

$$\boxed{\frac{n_2}{n_0} = 2.132 \times 10^{31}}. \tag{8}$$

Thus, it is clear that both hydrogen and helium are fully ionized.

(b) **Verify that most of the emission is radiated in the X-ray regime and calculate the average free electron density in the gas, if the X-ray luminosity is $L_X = 5 \times 10^{44}$ erg s^{-1}. Assume here the gas is optically thin.**

In order to emit a photon of energy $h\nu$, the kinetic/thermal energy of the electron must be at least $h\nu$. From the equipartition theorem, we know that on average the kinetic energy of a single electron is given by

$$E = \frac{3}{2}kT = h\nu \tag{9}$$

Thus the average photon frequency produced from an electron gas at temperature T, assuming all of the energy is emitted as a photon via Bremsstrahlung, is simply

$$\bar{\nu} = \frac{3kT}{2h} = 2.75 \times 10^{18} \text{ Hz} = \bar{\nu} \tag{10}$$

which corresponds to X-ray radiation. This value corresponds to the maximum energy photon that can be produced by an electron at the RMS velocity of the Maxwellian distribution. Some particles will produce less energetic photons, while others will produce more energetic photons. However, since X-rays span frequencies down to $\sim 10^{16}$ Hz, the vast majority of the thermal electrons will emit in the X-ray.

Additionally, the derivation of Bremsstrahlung contained a high energy cutoff that arose from photon discreteness. Using this definition of the upper cutoff frequency,

$$\nu_{\text{cutoff}} = \frac{kT}{h} = 1.834 \times 10^{18} \text{ Hz} \tag{11}$$

which is also at X-ray frequencies.

Knowing that the majority of the atoms in the gas are ionized at low optical depth, the main source of emission is from Bremsstrahlung. Assuming that the electrons follow a non-relativistic Maxwellian velocity distribution, the total emissivity from the gas is given as as

$$\eta^{\text{ff}} = \left(\frac{2\pi kT}{3m_e}\right)^{1/2} \frac{2^5 \pi e^6}{3hmc^3} Z^2 n_e n_i \bar{g}_B. \tag{12}$$

Because the gas is optically thin, the total luminosity is simply

$$L = V\eta^{\text{ff}} \tag{13}$$

where V is the volume of the radiating gas, $\frac{4}{3}\pi R^3$.
Additionally, because the gas contains a mixture of gas species, the interacting ion charge, Z, must be expressed as an effective Z, taking into account the charge contributions from both hydrogen and helium. Knowing that hydrogen contributes one third of the mass as hydrogen from the mass fraction, as well as assuming both are homogeneously distributed inside the entire volume, we have that

$$m_H n_H = 3 m_{He} n_{He} \implies \tag{14}$$

$$\frac{n_H}{n_{He}} = \frac{3 m_{He}}{m_H} = \frac{3 \cdot 4}{1} = 12. \tag{15}$$

The effective ion charge is given by the total charge contributed from each species divided by the total number of ions, which is simply $n_H + n_{He}$. Doing so gives

$$Z = \frac{n_e}{n_i} = \frac{n_H + 2n_{He}}{n_H + n_{He}} = \frac{12n_{He} + 2n_{He}}{12n_{He} + n_{He}} = \frac{14n_{He}}{13n_{He}} = \frac{14}{13}. \tag{16}$$

which is reasonable as we expect the average to be slightly above 1.
From the above derivation, we also find that the ion density can be expressed in terms of the electron density as

$$n_i = \frac{13}{14} n_e \tag{17}$$

Putting all of this together, the total Luminosity of the gas from thermal Bremsstrahlung now becomes

$$L = V\eta^{\text{ff}} = V \left(\frac{2\pi kT}{3m_e}\right)^{1/2} \frac{2^5 \pi e^6}{3hm_e c^3} Z^2 n_e n_i \bar{g}_B \tag{18}$$

$$= \frac{4}{3}\pi R^3 \left(\frac{2\pi kT}{3m_e}\right)^{1/2} \frac{2^5 \pi e^6}{3hm_e c^3} \left(\frac{14}{13}\right)^2 n_e \frac{13}{14} n_e \bar{g}_B \tag{19}$$

$$= \left(\frac{2\pi kT}{3m_e}\right)^{1/2} \frac{1792 R^3 \pi^2 e^6}{117 hm_e c^3} n_e^2 \bar{g}_B \tag{20}$$

Knowing that most of the emission occurs in the X-ray, it is safe to assume that all of the emission is measured in the X-ray luminosity. Setting this equal to that which is measured allows us to solve for the average electron density. Doing so gives

$$n_e = \sqrt{L_x \left(\frac{3m_e}{2\pi kT}\right)^{1/2} \frac{117 hm_e c^3}{1792 R^3 \pi^2 e^6} \frac{1}{\bar{g}_B}} = \boxed{9.331 \times 10^{-5}\ cm^{-3} = n_e} \tag{21}$$

(c) **Consider Bremsstrahlung absorption and Thomson scattering and find at which frequency the gas becomes optically thick. (For combined scattering and absorption, p. 36 of Rybicki & Lightman might be useful).**

The optical depth for a spherical source with angle-independent absorption and scattering is given as

$$\tau_\nu = \int_0^R (\alpha_\nu + \sigma_\nu)\ dr = R(\alpha_\nu + \sigma_\nu) \tag{22}$$

For Bremsstrahlung absorption

$$\alpha_\nu^{\mathrm{ff}} = \frac{4e^6}{3m_e hc}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2} Z^2 n_e n_i \nu^{-3}(1-e^{-h\nu/kT})\bar{g}_{\mathrm{ff}} \tag{23}$$

$$= \frac{4e^6}{3m_e hc}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2}\left(\frac{14}{13}\right)^2 n_e \frac{13}{14} n_e \bar{g}_{\mathrm{ff}} \nu^{-3}(1-e^{-h\nu/kT}) \tag{24}$$

$$= \frac{56e^6}{39m_e hc}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2} n_e^2 \bar{g}_{\mathrm{ff}} \nu^{-3}(1-e^{-h\nu/kT}) \tag{25}$$

For Thomson scattering,

$$\sigma_\nu = \sigma_T n_e \tag{26}$$

Thus, for the optical depth we have that

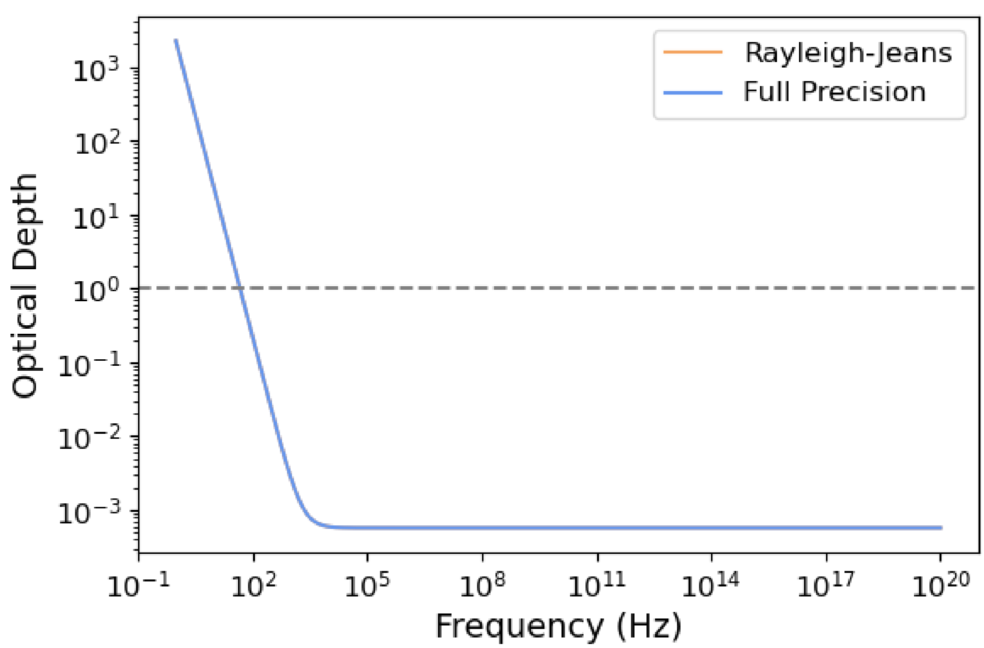
$$\tau_\nu = R\left(\frac{56e^6}{39m_e hc}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2} n_e^2 \bar{g}_{\mathrm{ff}} \nu^{-3}(1-e^{-h\nu/kT}) + \sigma_T n_e\right) \tag{27}$$

To solve for when the gas becomes optically thick, set this equal to 1 and solve for ν.

$$1 = R\left(\frac{56e^6}{39m_e hc}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2} n_e^2 \bar{g}_{\mathrm{ff}} \nu^{-3}(1-e^{-h\nu/kT}) + \sigma_T n_e\right) \tag{28}$$

This can be solved numerically, and doing so gives

$$\boxed{\nu_{\mathrm{cutoff}} = 47.299 \text{ Hz}} \tag{29}$$



(d) **What is the cooling time for this gas?**

The total cooling time for any gas is the total stored energy divided by the amount of energy lost per unit time. Assuming all of the energy is lost as X-ray flux, the cooling time is then, where the total thermal energy for one electron is $3/2kT$, and the total number of electrons is $n_e V$

$$t_{\text{cool}} = \frac{E}{L} = \frac{3kTn_eV}{2L_x} = \frac{12kTn_e\pi R^3}{6L_x} = \boxed{1.13 \times 10^{19}\ \text{s} = 358.1\ \text{Gyr}} \tag{30}$$

2. **The spectrum shown below is observed from a point source of unknown distance d. A model for this source is a spherical mass of radius R that is emitting synchrotron radiation in a magnetic field of strength B. The space between us and the source is uniformly filled with a thermal bath of hydrogen that emits and absorbs mainly by bound-free transitions, and it is believed that the hydrogen bath is unimportant compared to the synchrotron source at frequencies where the hydrogen bath is optically thin. The synchrotron source function can be written as**

$$S_\nu = A\left(\frac{B}{B_0}\right)^{-1/2}\left(\frac{\nu}{\nu_0}\right)^{5/2}. \tag{31}$$

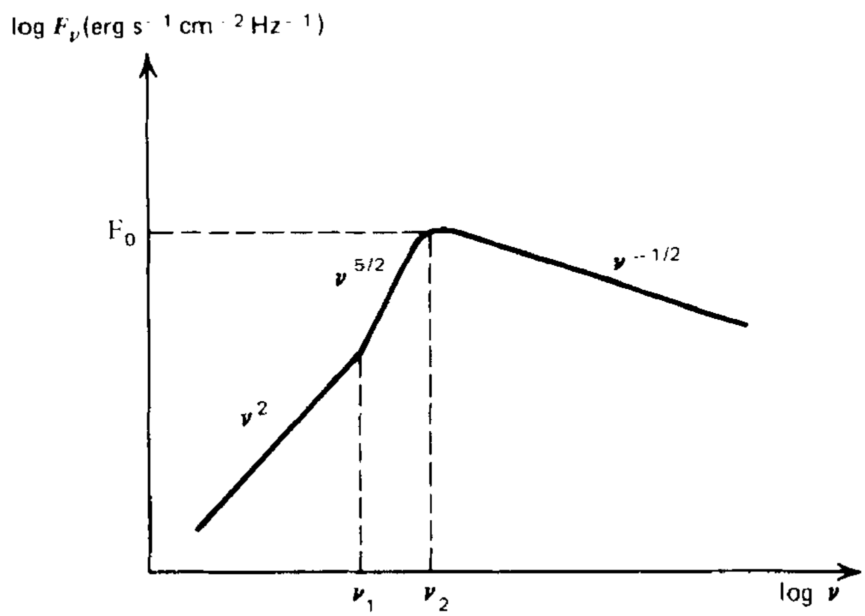
The absorption coefficient for synchrotron radiation is

$$\alpha_\nu^{\rm s} = C\left(\frac{B}{B_0}\right)^{(p+2)/2}\left(\frac{\nu}{\nu_0}\right)^{-(p+4)/2}, \tag{32}$$

and that for bound-free transitions is

$$\alpha_\nu^{\rm bf} = D\left(\frac{\nu}{\nu_0}\right)^{-3}, \tag{33}$$

where A, B_0, ν_0, C, and D are constants and p is the power law index for the assumed power law distribution of relativistic electrons in the synchrotron source.



(a) **Find the size of the source R and the magnetic field strength B in terms of the solid angle $\Omega = \pi(R^2/d^2)$ subtended by the source and the constants A, B_0, ν_0, C, D.**

We know that for large frequencies, synchrotron self-absorption becomes less and less efficient, specifically

$$\lim_{\nu\to\inf} \alpha_\nu^{\rm s} = \lim_{\nu\to\inf} \alpha_\nu^{\rm bf} = 0. \tag{34}$$

Assuming that there is no scattering, we then know that $\chi = \alpha = 0$, meaning

$$\mu \frac{\partial I_\nu}{\partial z} = \eta_\nu \implies \tag{35}$$

$$\mu I_\nu = \int_0^d \eta_\nu \, dz \tag{36}$$

Since we are looking directly at the point source, we also have that $\mu = 1$. Because we know the only emission is from the synchrotron radiation, we can calculate the emissivity using the source function.

$$\eta_\nu^{\mathrm{s}} = S_\nu \alpha_\nu^{\mathrm{s}} = A \left(\frac{B}{B_0}\right)^{-1/2} \left(\frac{\nu}{\nu_0}\right)^{5/2} C \left(\frac{B}{B_0}\right)^{(p+2)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p+4)/2} = AC \left(\frac{B}{B_0}\right)^{(p+1)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p-1)/2} \tag{37}$$

The intensity is now

$$I_\nu = \int_0^d \eta_\nu \, dz = \int_0^d AC \left(\frac{B}{B_0}\right)^{(p+1)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p-1)/2} dz \tag{38}$$

$$= AC \left(\frac{B}{B_0}\right)^{(p+1)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p-1)/2} \int_0^d dz \tag{39}$$

$$= ACd \left(\frac{B}{B_0}\right)^{(p+1)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p-1)/2}. \tag{40}$$

We can then calculate the flux measured at the observer as

$$F_\nu = \int I_\nu \hat{n} \, d\Omega = \int ACd \left(\frac{B}{B_0}\right)^{(p+1)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p-1)/2} \hat{n} \, d\Omega \tag{41}$$

$$= ACd \left(\frac{B}{B_0}\right)^{(p+1)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p-1)/2} \int \cos\theta \, d\Omega. \tag{42}$$

Assuming that the source of the synchrotron radiation is far away, we know that $R \ll d$, giving us the small angle approximation $\theta \approx 0$. The flux is finally

$$F_\nu = ACd \left(\frac{B}{B_0}\right)^{(p+1)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p-1)/2} \int d\Omega = \Omega ACd \left(\frac{B}{B_0}\right)^{(p+1)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p-1)/2}. \tag{43}$$

We also know from the provided spectrum that $F_\nu \propto \nu^{-1/2}$. Combining this with the expression above, we can solve for p.

$$F_\nu = \Omega ACd \left(\frac{B}{B_0}\right)^{(p+1)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p-1)/2} \propto \nu^{-1/2} \implies \tag{44}$$

$$-(p-1)/2 = -1/2 \implies p = 2 \tag{45}$$

Similarly, in the regime between ν_1 and ν_2, it must be true that the synchrotron emission is optically thick and the bound-free transitions are optically thin, as it is proportional to the synchrotron source function. Thus in this regime the intensity becomes

$$I_\nu = \int_0^{\tau_\nu} S_\nu e^{-(\tau'_\nu - \tau_\nu)/\mu} \, d\tau'_\nu \tag{46}$$

As we are measuring from the top of the synchrotron self-absorption atmosphere and there is no incoming, we want to know the intensity at optical depth $\tau_\nu = 0$. Additionally, since there is no incoming at the 'bottom' of the point source, $I_\nu(\tau_{\max} = 1) = 0$. We can then say

$$I_\nu(\tau_0 = 0) = \int_0^{\tau_{\max}} S_\nu e^{-\tau_\nu'}\, d\tau_\nu' \tag{47}$$

$$= A\left(\frac{B}{B_0}\right)^{-1/2}\left(\frac{\nu}{\nu_0}\right)^{5/2}\int_0^1 e^{-\tau_\nu'}\, d\tau_\nu' \tag{48}$$

$$= A\left(\frac{B}{B_0}\right)^{-1/2}\left(\frac{\nu}{\nu_0}\right)^{5/2}(1 - e^{-1}) \tag{49}$$

The transition between optically thick and optically thin occurs at the critical frequency ν_2. Assume this turnover occurs at exactly $\tau_\nu = 1$. Following the flux calculations from before, $F_\nu = \Omega I_\nu$, meaning at the turnover point

$$F_\nu(\nu = \nu_2) = F_0 = \Omega A\left(\frac{B}{B_0}\right)^{-1/2}\left(\frac{\nu_2}{\nu_0}\right)^{5/2}(1 - e^{-1}) \tag{50}$$

Solving this equation for B,

$$\boxed{B = \left(\frac{\nu_0}{\nu_2}\right)^{-5}\frac{\Omega^2 A^2 (1 - e^{-1})^2 B_0}{F_0^2}} \tag{51}$$

Similarly, knowing that the optical depth at the turnover point ν_2 is $\tau_\nu = 1$, which is strictly from synchrotron self-absorption, we can then say that

$$\tau_\nu = 1 = \int_0^R \alpha_\nu^{\mathrm{s}}(\nu_2)\, dr = \int_0^R C\left(\frac{B}{B_0}\right)^2\left(\frac{\nu_2}{\nu_0}\right)^{-3} dr \tag{52}$$

$$= C\left(\frac{B}{B_0}\right)^2\left(\frac{\nu_2}{\nu_0}\right)^{-3}\int_0^R dr \tag{53}$$

$$= RC\left(\frac{B}{B_0}\right)^2\left(\frac{\nu_2}{\nu_0}\right)^{-3} \tag{54}$$

Solving this for the radius of the synchrotron source,

$$R = \frac{1}{C}\left(\frac{B}{B_0}\right)^{-2}\left(\frac{\nu_2}{\nu_0}\right)^3 \tag{55}$$

Plugging in the value for the magnetic field from above,

$$R = \frac{1}{C}\left(\frac{1}{B_0}\left(\frac{\nu_0}{\nu_2}\right)^{-5}\frac{\Omega^2 A^2 (1 - e^{-1})^2 B_0}{F_0^2}\right)^{-2}\left(\frac{\nu_2}{\nu_0}\right)^3 \tag{56}$$

$$= \frac{1}{C}\left(\frac{\nu_0}{\nu_2}\right)^{10}\frac{F_0^4}{\Omega^4 A^4 (1 - e^{-1})^4}\left(\frac{\nu_2}{\nu_0}\right)^3 \tag{57}$$

$$= \boxed{\frac{1}{C}\left(\frac{\nu_0}{\nu_2}\right)^{7}\frac{F_0^4}{\Omega^4 A^4 (1 - e^{-1})^4} = R} \tag{58}$$

(b) **Now using D and ν_1, in addition to the previous constants, find the solid angle of the source and its distance.**

Similarly as before, we know that below ν_1 the hydrogen bath must be optically thick, as the provided power spectrum is proportional to the Rayleigh-Jeans tail of the blackbody curve, $F \propto \nu^2$. The point at which bound-free absorption becomes optically thick is thus the critical point ν_1. As before, this critical point occurs at the turnover when $\tau_\nu = 1$. Assuming that all of the absorption comes from bound-free transitions, we can then say that

$$\tau_\nu = 1 = \int_0^d \alpha_\nu^{\text{bf}}(\nu_1)\, dr = \int_0^d D\left(\frac{\nu_1}{\nu_0}\right)^{-3} dr = D\left(\frac{\nu_1}{\nu_0}\right)^{-3}\int_0^d dr = Dd\left(\frac{\nu_1}{\nu_0}\right)^{-3} \tag{59}$$

Solving for the distance to the source, we then have that

$$\boxed{d = \frac{1}{D}\left(\frac{\nu_1}{\nu_0}\right)^3} \tag{60}$$

From before we had that the solid angle towards the source is simply $\Omega = \pi(R^2/d^2)$. Using the relations for R and d from above, we have that the solid angle is

$$\Omega = \pi\left(\frac{R^2}{d^2}\right) = \pi\left(\frac{\left(\frac{1}{C}\left(\frac{\nu_0}{\nu_2}\right)^7 \frac{F_0^4}{\Omega^4 A^4(1-e^{-1})^4}\right)^2}{\left(\frac{1}{D}\left(\frac{\nu_1}{\nu_0}\right)^3\right)^2}\right) = \pi\left(\frac{\frac{1}{C^2}\left(\frac{\nu_0}{\nu_2}\right)^{14}\frac{F_0^8}{\Omega^8 A^8(1-e^{-1})^8}}{\frac{1}{D^2}\left(\frac{\nu_1}{\nu_0}\right)^6}\right) \tag{61}$$

$$= \pi\frac{D^2}{C^2}\left(\frac{\nu_0}{\nu_2}\right)^{14}\frac{F_0^8}{\Omega^8 A^8(1-e^{-1})^8}\left(\frac{\nu_0}{\nu_1}\right)^6 \implies \tag{62}$$

$$\Omega^9 = \frac{D^2}{C^2}\pi\left(\frac{\nu_0}{\nu_2}\right)^{14}\left(\frac{\nu_0}{\nu_1}\right)^6\frac{F_0^8}{\Omega^8 A^8(1-e^{-1})^8} \implies \tag{63}$$

$$\Omega = \boxed{\sqrt[9]{\frac{D^2}{C^2}\pi\left(\frac{\nu_0}{\nu_2}\right)^{14}\left(\frac{\nu_0}{\nu_1}\right)^6\frac{F_0^8}{\Omega^8 A^8(1-e^{-1})^8}} = \Omega} \tag{64}$$