LEC 5

Rosseland Approximation

Valid at high optical depths

It provides F as a function of the temperature gradient in a medium.

- If $I_\nu(\vec{r}) = B_\nu(T_0)$ in a $T_0 = \text{const.}$ medium $\rightarrow F_\nu = 0$

If T is the same everywhere radiation does not preferentially "flow" to a particular direction.

- Let's start w/ the RTE in 1D slab symmetry

$$\mu \frac{dI_\nu}{dz} = -\chi_\nu (I_\nu - S_\nu)$$

$$\Rightarrow \boxed{I_\nu = S_\nu - \frac{\mu}{\chi_\nu} \frac{dI_\nu}{dz}}$$

At $\tau \gg 1$, intensity changes slowly on the scale of the mfp.

Thus, to zeroth-order $\frac{dI_\nu^0}{dz} = 0$

$\Rightarrow I^{(0)}(z, \mu) = S_\nu^{(0)}$

and since we are isotropic at $\tau \gg 1$

$J_\nu^{(0)} = S_\nu^{(0)} = B_\nu(T) \leftarrow$

thus $I^{(0)}_\nu = B_\nu(T) \leftarrow$ radiation is thermal

But, $F^{(0)} \approx \int I \mu d\mu = 0$

So, to obtain first-order approx

We write $I^{(1)}_{(\nu)} = B_\nu(T) - \frac{\mu}{\chi_\nu} \frac{dB_\nu}{dz}$

then

$F_\nu = 2\pi \int_{-1}^{1} I_\nu^{(1)} \mu \, d\mu =$

$= -\frac{2\pi}{\chi_\nu} \frac{dB_\nu}{dz} \int_{-1}^{1} \mu^2 d\mu =$

$= -\frac{4\pi}{3\chi_\nu} \frac{dB_\nu}{dT} \frac{dT}{dz}$

For the total flux

Integrate over ν $\quad F = \int_0^\infty F_\nu \, d\nu$

$$= -\frac{4\pi}{3}\frac{dT}{dz} \int \frac{dB_\nu/dT \, d\nu}{\chi_\nu}$$

Introduce the Rosseland mean opacity

$$\frac{1}{\alpha_R} \equiv \frac{\int_0^\infty \frac{1}{\chi_\nu}\frac{dB_\nu}{dT}\, d\nu}{\int_0^\infty \frac{dB_\nu}{dT}\, d\nu}$$

then

$$F = -\frac{4\pi}{3}\frac{dT}{dz}\frac{1}{\alpha_R}\int_0^\infty \frac{dB_\nu}{dT}\, d\nu =$$

$$= -\frac{4\pi}{3}\frac{dT}{dz}\frac{1}{\alpha_R}\frac{d}{dT}\underbrace{\int_0^\infty B_\nu \, d\nu}_{\frac{\sigma_B T^4}{\pi}}$$

$$\Rightarrow \boxed{F = -\frac{16\sigma_B T^3}{3}\frac{\partial T}{\partial z}}$$

If we know $T(z)$, we get $F(z)$

Discussion on timescales, mfp, isotropy

I. Astrophysical problems have multiple timescales.

i) Propagation of rad. timescale

$t_R \sim \frac{\ell}{c}$ free-streaming

[ℓ: typical length scale of system]

$t_R \sim \frac{\lambda_{mfp}}{c}$, finite photon mfp

ii) fluid flow

$t_f \sim \frac{\ell}{v}$ (medium can move)

iii) ionization timescale t_I

iv) thermal timescale t_{th}

the timescale over which the ionization state or thermal properties of the medium change

When $t_R \ll t_f, t_I, t_{th}$ we can ignore time-dependence of I

II Mean-free-path: typical distance between interactions of a photon

$$\lambda_{\nu(mfp)} = \frac{1}{\chi_\nu}$$

Optical depth

$$\tau_\nu = \int_0^\ell \chi_\nu \, ds \approx \frac{\ell}{\lambda_{mfp}}$$ – average # of interactions

$$\xleftrightarrow{\quad\tau\quad}$$

$\tau \ll 1$ optically thin

large λ_{mfp}

free-streaming limit

$\tau \gg 1$

optically thick,

small λ_{mfp}

diffusion limit

Both τ_ν and S_ν determine the observation of a source

<u>Diffusion limit is a random walk</u>

~~[illegible]~~ $\langle \vec{r}^2 \rangle \neq 0$

in particular net displacement d_*^2

$$d_*^2 = \underset{\#\text{ of interactions}}{N}\ \underset{\text{mfp}}{d^2} \Longrightarrow d_* = \sqrt{N}\, d$$

For a photon to diffuse out from a $\tau \gg 1$ medium, we want $d_* = \ell$

$$d = \lambda_{mfp} \Rightarrow N = \frac{\ell^2}{\lambda_{mfp}^2} \sim \tau^2$$

Thus, the diffusion time is

$$t_d \sim \left(\frac{\ell}{\lambda_{mfp}}\right)^2 \underbrace{\left(\frac{\lambda_{mfp}}{c}\right)}_{\Delta t \text{ per step}} = \frac{\ell^2}{c\,\lambda_{mfp}} \Rightarrow$$

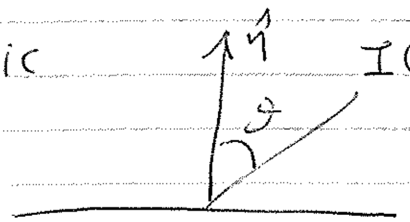
$$t_d \simeq \tau \cdot t_{cross}$$

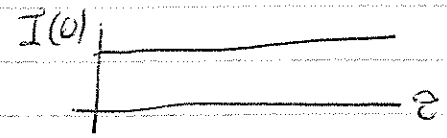
$$t_{cross} = \frac{\ell}{c}$$

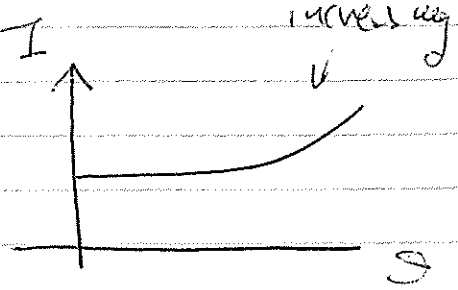
Since diffusion is a random walk high τ also generally isotropizes a radiation field.

More concepts on isotropy

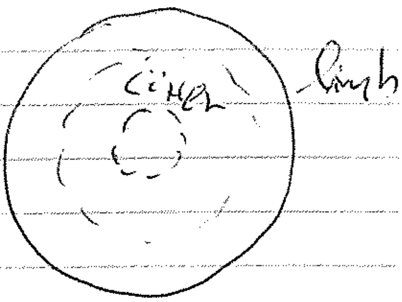
Isotropic



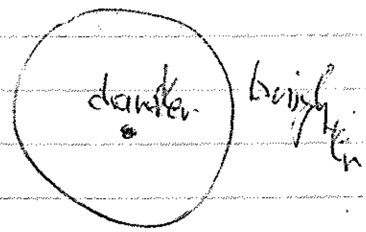




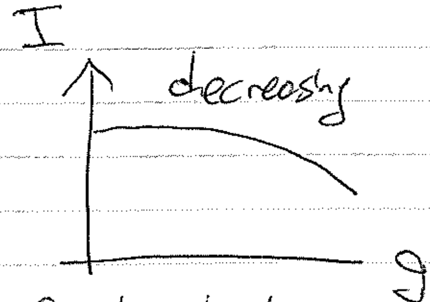
limb brightening



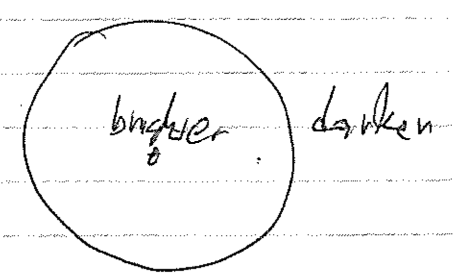
Appearance of a resolved source is uniform brightness from center to limb



Near the limb observer sees radiation away from normal ($\theta \sim 90°$) $\rightarrow$ higher I



limb darkening



$f =$ ~~a~~ average number of particles per phase space volume

The number ~~de~~ density then is

$$n(\vec{r},t) = \int f \, d^3p$$

And the total number

$$N = \int f \, d^3x$$

$$\frac{df}{dt} = 0 \Rightarrow \frac{\partial f}{\partial t} + c \cdot \hat{n} \nabla f + \frac{\partial f}{\partial p^i} \cancel{\frac{dp^i}{dt}}^{0} = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + c \cdot \hat{n} \cdot \bar{\nabla} f = 0 \Rightarrow$$

$$\frac{\partial}{\partial t}\int f \, d^3p + c \cdot \hat{n} \bar{\nabla} \int f \cdot d^3p = 0$$

$$\frac{\partial n}{\partial t} + c \hat{n} \vec{\nabla} n = 0$$

$$I = c h\nu \cdot n$$

BC's for RTE

We need 1 BC for each of the $\mu \times \nu$ ODEs in slab symmetry

$\tau = 0$

$\tau = \tau_{max}$

specify completely

$I(\tau_{max}, \mu, \nu) \quad , \; -1 \le \mu \le 1$

or

$\tau = 0 \quad , \; -1 \le \mu \le 1$

specify complete $\quad I(0, \mu, \nu)$

$\tau = \tau_{max}$

or <u>mix</u>

$\tau = 0 \quad , \; -1 \le \mu \le 0$

$I(0, \mu, \nu) \quad \downarrow I^-$

$\uparrow I^+ \quad \tau = \tau_{max} \quad , \; 0 \le \mu \le 1$

$I(\tau_{max}, \mu, \nu)$

HW 1

Prob. 1

Solve the time-ind. RTE to find $I_\nu(\mu)$

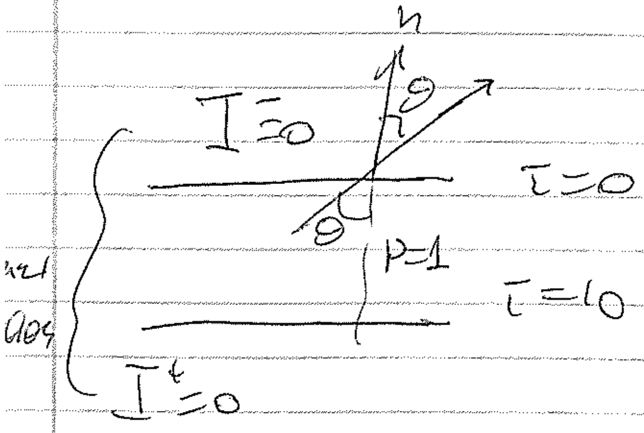
a. $S_\nu = B_\nu(T)$, $T(z) = \text{const}$

b. $S_\nu = B_\nu(T)$, $T(z) = T_0 \tau^{\xi}$

τ is the depth at $\mu = 1$, B_ν the Planck function.

You can take $\tau_{max} = 10$ for $\mu = 1$

B.Cs $I_\nu(\mu) = 0$, $-1 \le \mu < 0$ at $\tau = 0$, $I_\nu(\mu) = 0$ $0 \le \mu < 1$ at $\tau = \tau_{max}$



$$I_\nu(\tau_1, \mu) = I_\nu(\tau_2, \mu)\, e^{-(\tau_2 - \tau_1)/\mu} + \frac{1}{\mu} \int_{\tau_1}^{\tau_2} S_\nu(\tau')\, e^{-\frac{\tau' - \tau_1}{\mu}} d\tau'$$

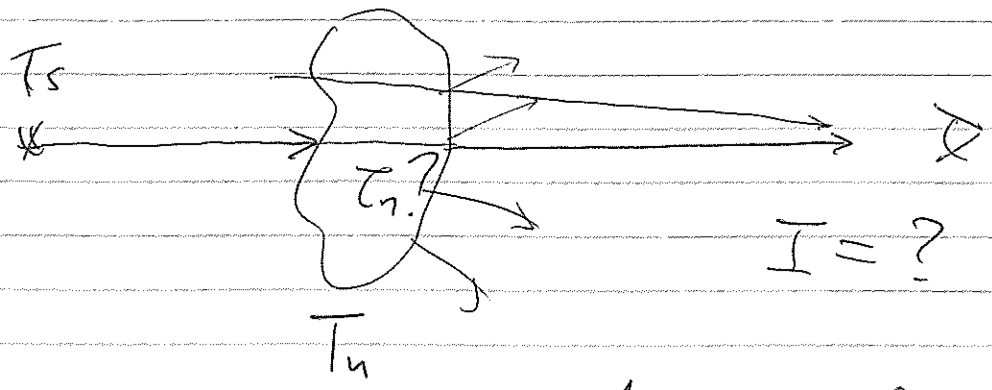
Formal solution

$$I_\nu(\tau = 0, \mu) = \overset{0}{\cancel{I_\nu(10, \mu)}}\, e^{-10\mu} + \frac{1}{\mu} \int_0^{10} S_\nu(\tau')\, e^{-\tau'/\mu} d\tau$$

<1

Prob. 2

treat it as a slab



Assume $S_\nu = B_\nu$

$T_s < T_n$