~~[illegible]~~ Lec 14

Sometimes you'll see

$$\underset{\uparrow}{\chi} = \frac{\eta}{\rho} = \alpha C_s \lambda$$

kinematic viscosity

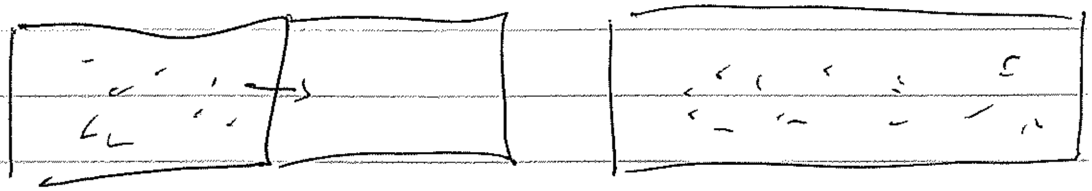
With σ_{ij} Euler's equ becomes

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = a_j - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{\text{pressure gradient}} + \underbrace{\frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_i}}_{\text{viscosity}}$$

Q's

1) What is the viscosity η, χ for a gas of non-interacting particles?

2) Is pressure gradient a force?



pressure gradient removing partition causes mean velocity

diffusion to re-equilibrate

The energy eq^n

2nd moment set $Q = v_j v_k$

Then

$$\frac{\partial}{\partial t}(\rho\langle v_j v_k\rangle) + \frac{\partial}{\partial x_i}(\rho\langle v_i v_j v_k\rangle) =$$

$$-\rho a_i\left\langle \frac{\partial v_k}{\partial v_i} v_j + \frac{\partial v_j}{\partial v_i} v_k\right\rangle = 0$$

But as before

$$\langle v_j v_k\rangle = \frac{\varphi_{jk}}{\rho} + u_j u_k$$

Now

$$\langle v_i v_j v_k \rangle = \langle (u_i + w_i)(u_j + w_j) \cdot (u_k + w_k) \rangle =$$

non-zero terms

$$= \langle w_i w_j w_k \rangle + u_i \langle w_j w_k \rangle + u_j \langle w_i w_k \rangle + u_k \langle w_i w_j \rangle + u_i u_j u_k$$

Define $\frac{Q_{ijk}}{\rho} = \langle w_i w_j w_k \rangle$

then

$$\langle v_i v_j v_k \rangle = \frac{Q_{ijk}}{\rho} + u_i \frac{\varphi_{jk}}{\rho} + u_j \frac{\varphi_{ik}}{\rho} + u_k \frac{\varphi_{ij}}{\rho} + u_i u_j u_k$$

Finally,

$$\left\langle \frac{\partial v_k}{\partial u_i} v_j + \frac{\partial v_j}{\partial u_i} v_k \right\rangle = \delta_{ik} u_j + \delta_{ij} u_k$$

[illegible]

Putting everything together

$$\frac{\partial}{\partial t}\left(\psi_{jk}+\rho u_j u_k\right)+$$
$$+\frac{\partial}{\partial x_i}\left(Q_{ijk}+u_i\psi_{jk}+u_j\psi_{ik}+ u_k\psi_{ij}+\rho u_i u_j u_k\right)$$
$$-\rho\left(a_k u_j + a_j u_k\right) = 0$$

Now we will take the trace of the eq. i.e $k=j$ and sum i.e. $A_{ii} = A_{11}+A_{22}+A_{33}$

$$\frac{\partial}{\partial t}\left(\psi_{jj}+\rho u_j u_j\right)+\frac{\partial}{\partial x_i}\left(Q_{ijj}+u_i\psi_{jj}+ u_j\psi_{ij}+u_j\psi_{ij} +\rho u_i u_j u_j\right)$$

~~But~~ $\cancel{\psi_{jj}} - 2\rho a_j u_j = 0$

But $\psi_{jj} = 3P$ (assuming isotropic pressure)

thus

$$\frac{\partial}{\partial t}\left(3P+\rho u^2\right)+\frac{\partial}{\partial x_i}\left[\left(3P+\rho u^2\right)u_i\right]$$

$$+\frac{\partial}{\partial x_i}\left[Q_{ijj}+2\Psi_{ij}u_j\right]$$

$$-2\rho a_j u_j = 0$$

$\frac{3}{2}P+\frac{1}{2}\rho u^2$ = internal + kinetic energy (per unit vol)

$\rho a_j u_j$ = work done by ext. forces

??

Using the continuity eqn, Euler's eqn and the def. of entropy

$$S=\frac{k}{\gamma-1}\ln\left(\frac{P}{\rho^{\gamma-1}}\right) \quad \gamma=\frac{5}{3},\ P=nkT$$

the energy eqn. can be written as

$$\rho T\frac{DS}{dt}=-\frac{\partial Q_{ijj}}{\partial x_i}+\frac{\sigma_{ij}\sigma^{ij}}{\rho}$$

First term heat flux $\frac{\partial F_i}{\partial x_i}$ or $\nabla \cdot F$

Second term entropy produced by viscosity

We need a model (closure) for F_i. To lowest order should depend on temperature gradient

$$F_i = -K \frac{\partial T}{\partial x_i} \quad \text{(Fick's law)}$$

heat conductivity coefficient

Full Eqn's

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \psi_{ij}}{\partial x_j} + a_i$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}P + \frac{1}{2}\rho u^2\right) +$$

$$+ \frac{\partial}{\partial x_i}\left[\left(\frac{3}{2}P + \frac{1}{2}\rho u^2\right)u_i\right] = \rho a_j u_j$$

$$- \frac{\partial}{\partial x_i}\left[\frac{1}{2} H_i + \psi_{ij} u_j\right]$$

Where

$$\psi_{ij} = P\delta_{ij} - \sigma_{ij}$$

$$\sigma_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) + \zeta\frac{\partial u_k}{\partial x_k}$$

$$H_i = -\kappa\frac{\partial T}{\partial x_i}$$

And $P = P(\rho, T)$ EOS

In total 5 PDEs
for 6 variables
$\rho, P, u_1, u_2, u_3, T$
+ an EOS $P = P(\rho, T)$

<u>Example: A perturbative solution</u>

How do perturbations propagate in a fluid?

Consider uniform medium and perturbation in 1 direction (e.g. z coordinate)

Let viscosity and heat conduction play no role and $T = $ const.

Then only need equs

$$\frac{\partial \rho}{\partial t} \quad \& \quad \frac{\partial u}{\partial t}$$

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial}{\partial z}(\rho_0 u_{z_0}) = 0$$

$$\frac{\partial u_{z_0}}{\partial t} + u_{z_0}\frac{\partial u_{z_0}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P_0}{\partial z} + a_z$$

$$= -\frac{1}{\rho_0}\frac{\partial\left(\frac{\rho_0}{m}KT_0\right)}{\partial z} + a_z$$

$$= -\frac{KT_0}{\rho_0 m}\frac{\partial \rho_0}{\partial z} + a_z$$

"o" conditions unperturbed

~~[illegible]~~

Now introduce perturb.

$$\left.\begin{aligned} \rho &= \rho_0 + \rho_1 \\ u_z &= u_{z_0} + u_1 \end{aligned}\right\} \xrightarrow{\text{keep 1st order terms}}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u_z) = 0$$

$$\frac{\partial u_z}{\partial t} + u_z\frac{\partial u_z}{\partial z} = -\frac{KT_0}{\rho m}\frac{\partial \rho}{\partial z} + a_z$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial z} + u_0 \frac{\partial \rho_1}{\partial z} = 0 \quad (1)$$

$$\frac{\partial u_1}{\partial t} + u_0 \frac{\partial u_1}{\partial z} = -\frac{KT}{\rho_0 m} \frac{\partial \rho_1}{\partial z} \quad (2)$$

If system is initially at rest $u_0 = 0$

$$\left.\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial z} &= 0 \\ \frac{\partial u_1}{\partial t} &= -\frac{KT}{\rho_0 m} \frac{\partial \rho_1}{\partial z} \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} \frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \frac{\partial^2 u_1}{\partial t \partial z} &= 0 \\ \frac{\partial^2 u_1}{\partial z \partial t} &= -\frac{KT}{\rho_0 m} \frac{\partial^2 \rho_1}{\partial z^2} \end{aligned}\right\} \quad \frac{\partial^2 \rho_1}{\partial t^2} = \frac{KT}{m} \frac{\partial^2 \rho_1}{\partial z^2}$$

Wave eqn!

Wave propagating

at speed $c_s = \sqrt{\frac{KT}{m}}$

More generally we will have

$$c_s^2 = \left(\frac{\partial p}{\partial e}\right)_s$$