Lect. 16

Hydrodynamical spherical accretion

Bondi accretion (1952)

Typical conditions in the interstellar medium, in matter exchange between stars, or accretion onto compact objects are such that the gas is hydrodynamical in nature, i.e. $\lambda_{mfp} << L$

Today's objective is.

- Useful when there* isn't enough angular momentum to force matter into a torus- or disk like geometry.

- Hydrodynamical cosmological simulations often employ Bondi accretion as a sub-grid model.

Hydro eqn's without non-ideal effects and constant entropy

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{u}) = 0 \quad (1)$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\{\vec{u}\} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi \quad (2)$$

- Spherical symmetry $\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta} \Rightarrow 0$

$u^{\theta,\varphi} = 0$

- Steady state : $\frac{\partial}{\partial t} \Rightarrow 0$
- $P = K\rho^{\Gamma}$

(1) $\Rightarrow$ $\nabla(\rho\bar{u}) = 0 \Rightarrow$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho u_r\right) = 0} \quad (3)$$

$\phi = -\frac{GM}{r}$

(2) $(\bar{u}\cdot\bar{\nabla})u = -\frac{1}{\rho}\bar{\nabla}P - \bar{\nabla}\phi \Rightarrow$

$$\boxed{u_r\frac{du_r}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2}} \quad (4)$$

(3) $\Rightarrow$ $\underbrace{r^2}_{\text{area}}\underbrace{\rho u_r}_{\text{mass flux}} = \text{const.} \Rightarrow \boxed{4\pi r^2\rho u_r = \dot{M}}$

must be const. at every r!

Hose analogy

$\underbrace{}_{\dot{M}}$

$$\frac{d}{dr}\left(\frac{1}{2}u_r^2\right) = -\frac{1}{\rho}\Gamma K \rho^{\Gamma-1}\frac{d\rho}{dr} + \frac{d}{dr}\left(\frac{GM}{r}\right)$$

$$\cancel{\Rightarrow \frac{d}{dr}\left(\frac{1}{2}u_r^2\right) = -\frac{\Gamma P}{\rho \rho}\frac{d\rho}{dr}}$$

$$\cancel{-\frac{1}{\rho}c_s^2\frac{d\rho}{dr}}$$

$$-\Gamma K \rho^{\Gamma-2}\frac{d\rho}{dr} = -\frac{\Gamma K}{\Gamma-1}\frac{d(\rho^{\Gamma-1})}{dr} =$$

$$= -\frac{1}{\Gamma-1}\frac{d}{dr}\left(\Gamma K \rho^{\Gamma-1}\right) =$$

$$= -\frac{1}{\Gamma-1}\frac{d}{dr}\left(\frac{dP}{d\rho}\right) = -\frac{1}{\Gamma-1}\frac{d}{dr}\left(c_s^2\right)$$

$$\frac{d}{dr}\left(\frac{1}{2}u_r^2 + \frac{1}{\Gamma-1}c_s^2 - \frac{GM}{r}\right) = 0 \Rightarrow$$

$$\boxed{\frac{1}{2}u_r^2 + \frac{1}{\Gamma-1}c_s^2 - \frac{GM}{r} = \text{const.} = \frac{1}{\Gamma-1}c_{s,\infty}^2}$$

Bernoulli eqn.

$$\boxed{\begin{aligned} &4\pi r^2 \rho u_r = \dot{M} \quad (5) \\ &\frac{1}{2}u_r^2 + \frac{1}{\Gamma-1}c_s^2 - \frac{GM}{r} = \frac{1}{\Gamma-1}c_{s,\infty}^2 \quad (6) \end{aligned}}$$

The flow is determined once $\dot{M}$, and $\rho(r)$, $u(r)$ are known.

- Bondi (1952) showed that different values of $\dot{M}$ lead to physically distinct classes of solutions for the same boundary condition at infinity. (6 classes)

- Here we are interested in that unique solution where $u_r \uparrow$ as $r \downarrow$ to free-fall velocity at small r, $u \to \left(\frac{2GM}{r}\right)^{\frac{1}{2}}$ as $r \to 0$. In fact, the relativistic equations at $r = \frac{2GM}{c^2}$ demand that we choose this solution to avoid singularities.

To calculate the required accretion rate, let's rewrite Eq. 3 as

$$\frac{\rho'}{\rho} + \frac{u_r'}{u_r} + \frac{2}{r} = 0 \quad (7) \quad \left(' = \frac{d}{dr} \right)$$

And Eq. 4 as

$$u_r u_r' + \frac{dP}{d\rho} \frac{\rho'}{\rho} + \frac{GM}{r^2} = 0 \Rightarrow$$

$$u_r u_r' + c_s^2 \frac{\rho'}{\rho} + \frac{GM}{r^2} = 0 \quad (8)$$

Solve for u_r', ρ' to obtain

$$u' = \frac{D_1}{D}, \quad \rho' = -\frac{D_2}{D} \quad (9)$$

$$D_1 = \frac{2c_s^2/r - GM/r^2}{\rho} \quad (10)$$

$$D_2 = \frac{2u_r/r - GM/r^2}{u} \quad (11)$$

$$D = \frac{u_r^2 - c_s^2}{u\rho} \quad (12)$$

Eq. (9) shows that to guarantee a smooth, monotonic increase in u with $r\downarrow$, ~~i.e.~~ and simultaneously avoid singularities (note $D \to 0 \Rightarrow u \to \infty$), the solution must pass through a "critical point", where

$$D_1 = D_2 = D = 0 \quad \text{at } r = r_s$$

$$D_1 = 0 \Rightarrow \frac{2c_{s,s}^2}{r_s} - \frac{GM}{r_s^2} = 0$$

$$D_2 = 0 \Rightarrow 2u_s^2/r_s - \frac{GM}{r_s^2} = 0$$

$$\Rightarrow \boxed{u_s^2 = c_{s,s}^2}$$

$$D = 0 \Rightarrow \boxed{u_s^2 = c_{s,s}^2}$$

Thus r_s is a <u>sonic point</u>

$$M = 1 = \frac{u}{c_{s,s}}$$

And $u_s^2 = c_{s,s}^2 = \frac{1}{2}\frac{GM}{r_s}$

r_s = sonic radius

For $r < r_s$ ~~the~~ $u > c_{s,s}$ supersonic

-"- $r > r_s$ $u < c_{s,s}$ sub-sonic

The flow is <u>transonic</u>

Now, the Bernouli equ at r_s becomes

$$\frac{1}{2}u_s^2 + \frac{1}{\Gamma-1}c_{s,s}^2 - \cancel{\frac{GM}{r_s}}^{\,2u_s^2} = \frac{1}{\Gamma-1}c_{s,\infty}^2$$

$$-\frac{3}{2}u_s^2 + \frac{1}{\Gamma-1}u_s^2 = \frac{1}{\Gamma-1}c_{s,\infty}^2$$

$$\frac{2-3\Gamma+3}{2\cancel{(\Gamma-1)}}u_{s,s}^2 = \frac{1}{\cancel{(\Gamma-1)}}c_{s,\infty}^2 \Rightarrow$$

$$\boxed{u_s^2 = c_{s,s}^2 = \frac{2}{5-3\Gamma}c_{s,\infty}^2} \quad (13)$$

Thus, $$\boxed{r_s = \frac{GM}{2u_s^2} = \left(\frac{5-3\Gamma}{4}\right)\frac{GM}{c_{s,\infty}^2}} \quad (14)$$

Thus, at the ~~top~~ sonic radius the grav.

potential $\frac{GM}{r_s}$ is comparable to the internal ambient thermal energy per unit mass (specific energy), $c_{s,\infty}^2$

Now we can compute the accretion rate from Eq. (5)

$$\dot{M} = 4\pi r_s^2 \rho_s u_s$$

We know r_s, u_s in terms of the sound speed at ∞

Now $c_s^2 = \frac{dP}{d\rho} = \Gamma K \rho^{\Gamma-1} \Rightarrow$

$$\left.\begin{aligned} \rho &= \left(\frac{1}{\Gamma K}\right)^{\frac{1}{\Gamma-1}} c_s^{2/\Gamma-1} \\ \rho_\infty &= \left(\frac{1}{\Gamma K}\right)^{\frac{1}{\Gamma-1}} c_{s,\infty}^{2/\Gamma-1} \end{aligned}\right\} \rho = \rho_\infty \left(\frac{c_s}{c_\infty}\right)^{\frac{2}{\Gamma-1}}$$

$$\rho_s = \rho_\infty \left(\frac{c_{s,s}}{c_\infty}\right)^{\frac{2}{\Gamma-1}}$$

$$\dot{M} = 4\pi \rho_\infty u_s r_s^2 \left(\frac{c_{s,s}}{c_\infty}\right)^{\frac{2}{\Gamma-1}} = 4\pi\lambda_s \left(\frac{GM}{c_{s,\infty}^2}\right)^2 \rho_\infty c_{s,\infty}$$

Where the non-dimensional eigenvalue λ_s is

$$\lambda_s = \left(\frac{1}{2}\right)^{\frac{\Gamma+1}{2(\Gamma-1)}} \left(\frac{5-3\Gamma}{4}\right)^{-\frac{5-3\Gamma}{2(\Gamma-1)}} \Bigg\}\ \text{depends only on } \Gamma!$$

For an ideal gas (pure hydrogen) we have

$$P = 2\rho\frac{kT}{m_u}, \quad c_s^2 = \frac{2\Gamma kT}{\mu m_u} = \Gamma\frac{P}{\rho}$$

Thus

$$T = T_\infty\left(\frac{\rho}{\rho_\infty}\right)^{\Gamma-1}$$

By specifying the conditions at infinity, ρ_∞, $c_{s,\infty}$, we immediately determine $\dot{M}$

For $\Gamma = \frac{5}{3}$ we obtain

$$\dot{M} = \underbrace{8.77\cdot 10^{-16}}_{\substack{\swarrow \\ 1.2\cdot 10^{10}\ g\, s^{-1}}} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\rho_\infty}{10^{-24}\,g\,cm^{-3}}\right) \left(\frac{c_{s,\infty}}{10\,km/s}\right)^{-3}$$

$M_\odot \cdot yr^{-1}$

If all the accretion power is converted to radiation, the

$L = \dot{M}c^2 \sim 1.1 \cdot 10^{31}$ erg $\cdot$ s^{-1}

Too dim

So, accretion from the interstellar medium not too promising for detection, especially at large distances