Lec. 10

Synchrotron emission

Radiation by charges in a magnetic field

Classical picture: Lorentz force on e^-

→ circular motion →

→ centrifugal acceleration

~~(1) $\frac{d}{dt}(\gamma m c^2) = 0$ no change in γ or E~~

~~$\frac{d}{dt}(\gamma$~~

$$F^\mu = \frac{q}{c} F^\mu{}_\nu U^\nu$$

antisymmetric

Note $F^\mu U_\mu = \frac{q}{c} F_{\mu\nu} U^\mu U^\nu = 0$

From $\frac{dp^\mu}{d\tau} = F^\mu$ →

$p^\mu = \gamma m (c, \vec{v})$

$$\frac{d}{dt}(\gamma m \vec{v}) = \frac{q}{c} F^i{}_\nu U^\nu = \frac{q}{c}\left(F^i{}_0 U^0 + F^i{}_j U^j\right)$$

$$= \frac{q}{c} \bar{v} \times \bar{B}$$

$$\frac{d}{dt}(\gamma m c) = \frac{q}{c} F^0{}_\nu U^\nu = \frac{q}{c} \cancel{F^0{}_i} U^i = 0$$

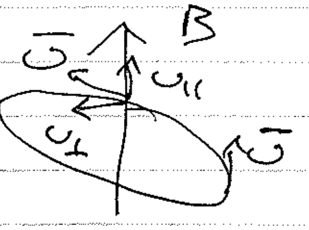
$$\frac{d}{dt}(\gamma mc) = 0$$

no change in γ
or E

$$m\gamma \frac{d\bar{v}}{dt} = e\frac{\bar{v}}{c} \times B$$

Consider $\bar{v}$ makes some small angle with B

then



then $$\frac{d\bar{v}_\perp}{dt} = \frac{e}{\gamma mc}\bar{v}_\perp \times \bar{B}$$

and $\dfrac{d\bar{v}_\parallel}{dt} = 0 \Rightarrow v_\parallel = \text{const.}$

but $|\bar{v}| = \text{const.}$ so $|\bar{V}_\perp| = \text{const.}$

thus $$|a_\perp| = \frac{e}{\gamma mc}|v_\perp| B$$

and the eq. of motion is

$$v_\perp(t) = v_\perp \begin{pmatrix} \sin\omega_B t \\ \cos\omega_B t \end{pmatrix}$$

ω_L = Larmor frequency

$$\omega_B = \frac{eB}{\gamma m c} = \text{synchrotron frequency}$$

Then the total radiated power is

$$P = \frac{2e^2}{3c^3}\gamma^4 a_\perp^2 = \frac{2}{3}\frac{e^4 v_\perp^2 B^2 \gamma^2}{m^2 c^5}$$

Using a pitch angle α between $\bar{v}$ and $\bar{B}$, and $\sigma_T = \frac{8\pi}{3}\frac{e^4}{m^2c^4}$

$$v_\perp = v\sin\alpha$$

$$P = \frac{1}{4\pi}\frac{v^2\sin^2\alpha B^2}{c}\gamma^2\sigma_T$$

We want the total power by a random isotropic distr. of e^-

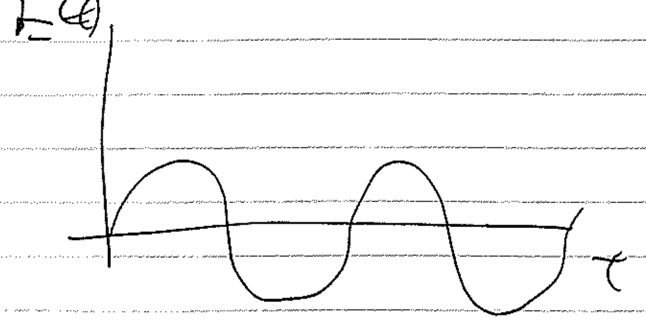
$$\langle \sin^2\alpha \rangle = \frac{\int \sin^2\alpha \, d\Omega}{\int d\Omega} = \frac{2}{3}$$

so

$$\langle P \rangle = \frac{1}{16\pi} \frac{\omega^2 B^2}{c} \gamma^2 \sigma_T$$

The energy spectrum is tedious

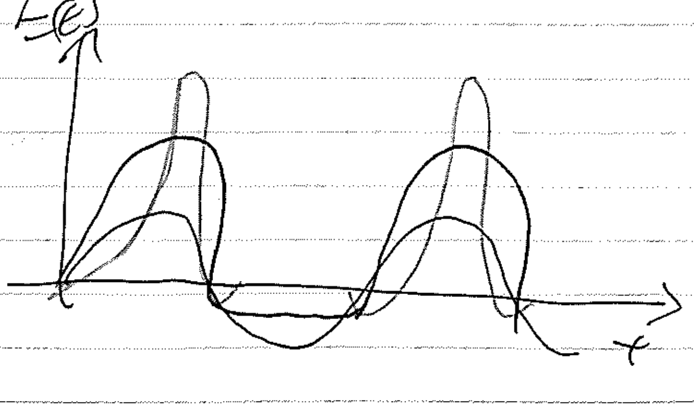
Qualitatively it looks like

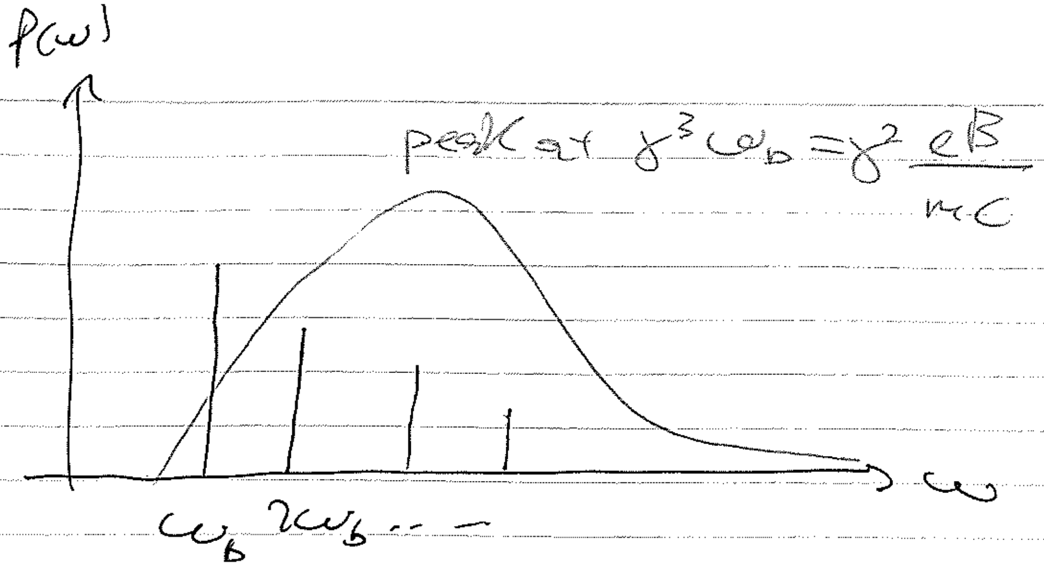


~~In the nonrel~~

For $\frac{v}{c} \ll 1$, e^- radiates dipole emission and the observer sees a sinusoidal E field

For $\frac{v}{c} \sim 1$, the width of the pulses go like $\frac{1}{\gamma^3} \omega_B$ due the Doppler effect





As $\frac{v}{c}\uparrow$, the E-field is distorted and beamed, and higher harmonics of ω_b start to con~~st~~ribute. The emission at each harmonic is given by

$$\eta_m = \frac{2e^2\omega_b^2\gamma^2}{c} \frac{(m+1)(m^{2m+1})}{(2m+1)!} \beta_\perp^{2m}$$

for $m\beta_\perp \ll 1$

As the e^- become fully relativist. $\gamma \gg 1$, the δ functions broaden and merge, and $P(\omega)$ can be approximated by modified Bessel functions

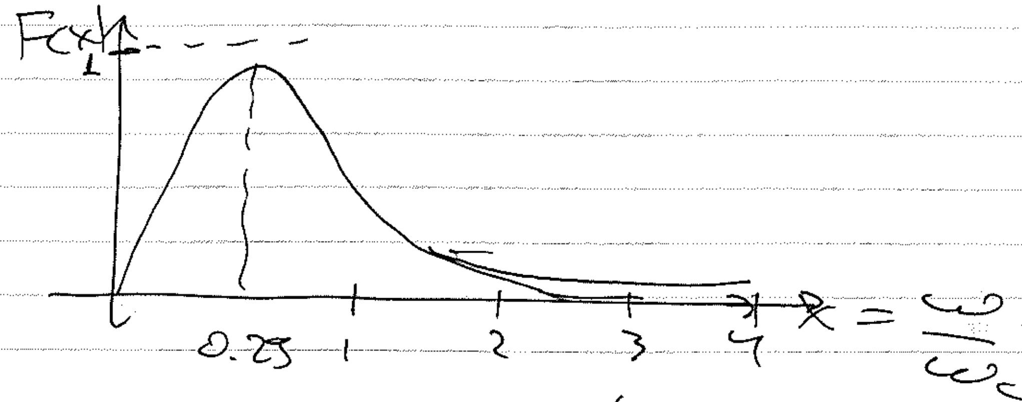
$$P(\omega) = \frac{\sqrt{3}\, e^3 B \sin\alpha}{2\pi m_e c^2} F(x)$$

with $x = \dfrac{\omega}{\frac{3}{2}\gamma^3 \omega_b \sin\alpha} = \dfrac{\omega}{\omega_c}$

and $F(x) = x \displaystyle\int_x^\infty K_{\frac{5}{3}}(\xi)\, d\xi$

modified Bessel function

The characteristic shape of the spectrum is determined by $F(x)$



The peak of synchrotron emission is at $\gamma^2 \omega_b$

ω_c = critical freq. above which radiation is negligible for all angles, and $P(\omega) \downarrow e^{-\omega}$

This was for 1 e^-

Need to integrate over vel. distribution of e^-

Two cases!

I) $N(\gamma) = C\gamma^{-p}$ power-law

$$P_{tot} = C\int_{\gamma_1}^{\gamma_2} p(\omega)\,\gamma^{-p}\,d\gamma \sim$$

$$\sim \int_{\gamma_1}^{\gamma_2} F\left(\frac{\omega}{\omega_c}\right)\gamma^{-p}\,d\gamma$$

Recall $\omega_c \sim \gamma^2$ so

for $\gamma_1 \simeq 1$ and $\gamma_2 \gg 1$

~~Probably~~

Assuming $F\left(\frac{\omega}{\omega_c}\right) \sim \delta\left(\omega - \gamma^2\omega_L\right)$

i.e emission centered on the highest harmonic (not bad)

$$P_{tot} \sim \int_{\gamma_1}^{\gamma_2} F.$$

$\omega' = \gamma^2\omega_L \Rightarrow \left(\frac{\omega'}{\omega_L}\right)^{\frac{1}{2}} = \gamma$ $\Big\}$ $\int F\,\omega'^{-\frac{p}{2}}$

$\gamma^{-p} \propto \left(\omega'\right)^{-\frac{p}{2}}$ ... $d\gamma \propto d\omega'^{\frac{1}{2}}$ ... $\Rightarrow$ ... $d\omega$

$$P_{tot}(\omega) \sim \omega^{-\frac{(p-1)}{2}}$$

II) thermal distr. ; relativistic Maxwellian

$$N(\gamma) = \frac{m_e c^2}{kT\, K_2\left(\frac{1}{\theta_e}\right)} \beta \gamma^2 \exp\left(-\frac{\gamma}{\theta_e}\right)$$

$$\theta_e = \frac{kT}{m_e c^2} \qquad \frac{E}{kT} = \frac{\gamma m c^2}{kT} = \frac{\gamma}{\theta_e}$$

normalized temperature

Bessel function $K_2\left(\frac{1}{\theta_e}\right)$ comes from the normalization of the distribution, s.t. $\int_1^\infty N(\gamma)d\gamma = 1$

In this case

$$P_{tot}(\omega) = \frac{e^2 \gamma \omega_b}{\sqrt{3}\pi c} \frac{\left(\omega / \gamma\omega_b\right)}{K_2\left(1/\theta_e\right)} I\left(\frac{X_M}{\sin\alpha}\right)$$

$$X_M = \frac{2\omega}{3\gamma\omega_b \theta_e^2} \quad \text{and} \quad I(X) = \frac{1}{X}\int_0^\infty z^2 \exp(-z)\, F\left(\frac{X}{z^2}\right) dz$$

see Mahadevan, Narayan & Yi 1996