Lec 13

$$\frac{\partial}{\partial t}\left(\rho\langle Q\rangle\right)+\frac{\partial}{\partial x^i}\left(\rho\langle v_i Q\rangle\right)$$

$$-\rho\left\langle a_i\frac{\partial Q}{\partial v^i}\right\rangle=0$$

Set $Q=1$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x^i}\left(\rho\langle v_i\rangle\right)=0$$

$$\Rightarrow\boxed{\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x^i}\left(\rho u_i\right)=0}$$

Continuity Eqn.

Describes rest-mass conservation

Let's integrate it over a vol.

$$\int\frac{\partial\rho}{\partial t}dV+\int\frac{\partial}{\partial x^i}\left(\rho u_i\right)dV=0$$

$$\Rightarrow\frac{\partial}{\partial t}\underbrace{\int\rho\,dV}_{\text{change in mass in }V}+\oint_S\underbrace{\rho\vec{u}}_{\text{mass flux}}\cdot\underbrace{d\vec{S}}_{\text{through surface enclosing }V}=0$$

Now $\quad Q = v_j \quad \left(\frac{\partial v_j}{\partial v_i} = \delta_{ij} \right)$

$$\frac{\partial}{\partial t}\left(\rho \langle v_j \rangle\right) + \frac{\partial}{\partial x^i}\left(\rho \langle v_i v_j \rangle\right)$$

$\vec{v} = \vec{u} + \vec{w}$

$$- \rho \langle a_i \delta_{ij} \rangle = 0$$

$$\langle v_i v_j \rangle = \langle (u_i + w_i)(u_j + w_j) \rangle$$

$$= u_i u_j + \underbrace{\langle w_i w_j \rangle}_{\text{random velocity}} + u_i \langle \cancel{w_j} \rangle^{0} + u_j \langle \cancel{w_i} \rangle^{0}$$

$$= \frac{\psi_{ij}}{\rho} + u_i u_j \quad , \quad \psi_{ij} = \rho \langle w_i w_j \rangle$$

or $\quad \psi_{ij} = \int_v w_i w_j f \, d^3 v$

Then, the moment eqn becomes

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}\left(\psi_{ij} + \rho u_i u_j\right) - \underbrace{\rho a_j}_{\text{external force}} = 0$$

Expand

$$\frac{\partial \rho}{\partial t} u_j + \rho \frac{\partial u_j}{\partial t} + \frac{\partial \psi_{ij}}{\partial x_i} + \frac{\partial \rho}{\partial x_i} u_i u_j$$

$$+ \rho \frac{\partial u_i}{\partial x_i} u_j + \rho u_i \frac{\partial u_j}{\partial x_i} - \rho a_j = 0$$

$$\Rightarrow \quad u_j \left(\frac{\partial \rho}{\partial t} + \cancelto{0}{u_i \frac{\partial \rho}{\partial x_i}} + \rho \frac{\partial u_i}{\partial x_i} \right) +$$

$$+ \rho \frac{\partial u_j}{\partial t} + \frac{\partial \psi_{ij}}{\partial x_i} + \rho u_i \frac{\partial u_j}{\partial x_i} - \rho a_j =$$

$$\Rightarrow \quad \frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = a_j - \frac{1}{\rho} \frac{\partial \psi_{ij}}{\partial x_i}$$

$$\Rightarrow \quad \frac{D u_j}{D t} = a_j - \frac{1}{\rho} \frac{\partial \psi_{ij}}{\partial_i}$$

i.e. $\ddot{r} = a + \text{stress}$

As in RTE, every time we write an eqn for the n^{th} moment, the eqn. involves the $(n+1)$st moment.

e.g. for ρ we need u_i

for u_i we need $u_i u_j$

and ψ_{ij}

Thus, we need to stop taking moments and add a "closure relation"

<u>What is ψ_{ij}?</u>

a) Diagonal terms

$\psi_{ii} = \int w_i^2 f d^3 v = P$

Recall $f d^3 v$ has units of density

mom. flux

Thus $\underbrace{\rho w_i}_{\text{momentum}} w_i = P$

We typically take

$$\Phi_{xx} = \Phi_{yy} = \Phi_{zz}$$

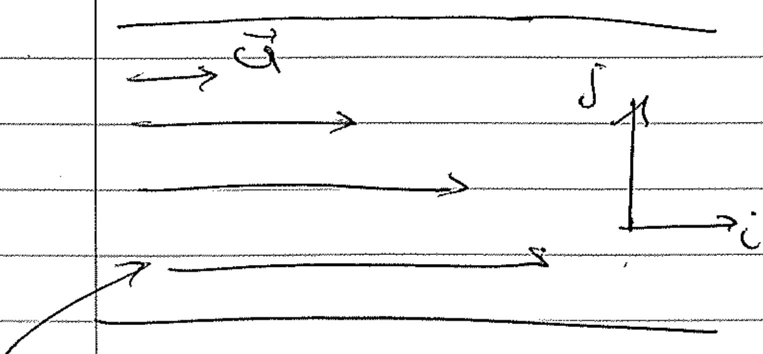
but not always true (e.g. in ionized plasmas)

b) off-diagonal terms

$$\Phi_{ij} = P\,\delta_{ij} - \sigma_{ij}$$

viscous stress tensor

~~flux~~ of ith component of momentum through j-direction



shear → transports momentum in ith direction in j direction

We need a phenomenological model for σ_{ij} in fluids!

What properties should σ_{ij} have?

- To lowest order, the flow of momentum should depend on the velocity gradients (gradient expansion)

$$\frac{\partial u_i}{\partial x^j}$$

- σ_{ij} must vanish if fluid is uniformly rotating, i.e.

$$\vec{u} = \vec{\omega} \times \vec{r} \quad \text{w/ const. } \omega$$

- $$\sigma_{ij} = \sigma_{ji}$$

The most general tensor that has these properties is usually written as

$$\sigma_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) + \zeta\,\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

$$\operatorname{tr}(\sigma) = \eta\left(2\frac{\partial u_k}{\partial x_k} - 2\frac{\partial u_k}{\partial x_k}\right) + \zeta\, 3\,\frac{\partial u_k}{\partial x_k}$$

$$= 3\zeta\,\frac{\partial u_k}{\partial x_k}$$

Experiment have shown that this form is a good phenomenological model and have ~~led to the~~ measured the η and ζ coefficients

shear viscosity — bulk viscosity

Let's do some dimensional analysis

$$[\sigma_{ij}] = [\text{density} \times \text{velocity}^2]$$

Thus $$= \left[\eta \cdot \frac{\text{velocity}}{\text{length}}\right] \Rightarrow$$

$$[\eta] = \text{density} \times \text{velocity} \cdot \text{length}$$

We can write η as

$$\eta = \alpha\, c_s\, \lambda \leftarrow \text{length scale}$$

α: parameter, c_s: sound speed

Sometimes you'll see

$$\chi = \frac{\eta}{\rho} = \alpha C_s \lambda$$

kinematic viscosity

With σ_{ij} Euler's equ becomes

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = a_j - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{\text{pressure gradient}} + \underbrace{\frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_i}}_{\text{viscosity}}$$

Q's

1) What is the viscosity η, χ for a gas of non-interacting particles?

2) Is pressure gradient a force?