$$J = \frac{1}{2}\int I_0 \, \delta(\mu-\mu_0)\, d\mu = \frac{I_0}{2}$$

$$H = \frac{1}{2}\int I_0 \, \mu \, \delta(\mu-\mu_0) = \frac{\mu_0 I_0}{2}$$

<u>Second moment:</u> Radiation Pressure Tensor

J is scalar, $\vec{H}$/$\vec{F}$ are vectors

P_{ij} is a tensor!

$$\boxed{P(\bar{r}, \nu, t) = \frac{1}{c}\int I(r, \hat{n}, \nu, t)\, \hat{n}\hat{n}\, d\Omega}$$

or $$\boxed{P_{ij} = \frac{1}{c}\int I \cdot n_i n_j \, d\Omega}$$

Lecture 3

ol: P_{ii} are related to energy density

eg. for a fluid $\int \rho\, v_i v_j \, d\Omega \Rightarrow$

~~radia~~ energy density / pressure

For rad. the trace of P_{ij}

$$\sum P_{ii} = P_{xx} + P_{yy} + P_{zz} =$$

$$= \frac{1}{c}\int I \, \underbrace{(n_x^2 + n_y^2 + n_z^2)}\, d\Omega$$

$= \frac{1}{c} \int I \, d\Omega = \mathcal{E}_R$

rad. energy density

If rad. field is isotropic:

$P_{ij} = 0 \quad , \; i \neq j$

$P_{ii} = P_{jj} = P_{kk} = \frac{1}{3} \mathcal{E}_R$ (EOS)

P_{ij} is rarely used in its all glory.
In slab ~~/~~ spherical symmetry,
we use KK (zz-comp) or rr.
Also, $\frac{1}{c}$ is typically dropped
and it's called "K", i.e.,
K_{rr}, K_{zz}

Let's look at J, H, K for rad. fields with different degree of (an)isotropy.

1. Isotropic $I(\mu) = I_0$

$$J = \frac{1}{4\pi}\int I \, d\mu \, d\phi = I_0$$

$$\bar{H} = \frac{1}{4\pi}\int I \mu \, d\mu \, d\phi = 0$$

$$K_{rr} = \frac{1}{4\pi}\int I \mu^2 d\mu \, d\phi = \frac{1}{2}\int_{-1}^{1} I_0 \mu^2 d\mu$$

$$= \frac{1}{3} I_0 = \frac{J}{3}$$

$\boxed{\text{for isotropic fields } \frac{K}{J} = \frac{1}{3}}$

ratio called the Eddington factor or Eddington ratio.

2. Almost isotropic

$$I_\nu(\tau,\mu) = a_\nu(\tau) + b_\nu(\tau)\,\mu$$

$$J = \frac{1}{2}\int_{-1}^{1} I\,d\mu = a$$

$$H = \frac{1}{2}\int_{-1}^{1} I\mu\,d\mu = \frac{b}{3}$$

$$K_{rr} = \frac{1}{2}\int_{-1}^{1} I\mu^2\,d\mu = \frac{a}{3}$$

$$\Rightarrow \frac{K}{J} = \frac{1}{3}$$, but $H \neq 0$, there's a net flux

3. Free-streaming limit

$$I(\mu) = I_0\,\delta(\mu-\mu_0)$$

$$J = \frac{1}{2}\int I_0\,\delta(\mu-\mu_0)\,d\mu = \frac{I_0}{2}$$

$$H = \frac{1}{2}\int I_0\,\mu\,(\delta(\mu-\mu_0))\,d\mu = \frac{\mu_0 I_0}{2}$$

$$K_{zz} = \frac{1}{2}\int I_0\,\mu^2\,\delta(\mu-\mu_0)\,d\mu = \frac{I_0\mu_0^2}{2}$$

$d\vec{r} = \vec{v}\,dt \quad , \quad \vec{r} \to \vec{r} + \vec{v}\,dt$

$d\vec{p} = \vec{F}\,dt \quad , \quad \vec{p} = \vec{p} + \vec{F}\,dt$

Liouville's theorem states:
For conservative forces and the absence of sources/sinks, phase space can be distorted but its volume remains unchanged!

For photons, "sources" and "sinks" are the emission, absorption, and scattering terms that describe interactions with external particles or fields. We generically call this the "collision" term.

So, Liouville's theorem says

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = 0$$

For collisions, we generalize to the Boltzmann equ. for photons $\frac{df}{dt} = \left(\frac{df}{dt}\right)_{coll}$ $\Rightarrow$

$$\frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial t} + \sum_j \frac{\partial f}{\partial p^j}\frac{\partial p_j}{\partial t}$$

$$= \left(\frac{df}{dt}\right)_{coll} \Rightarrow$$

$$\boxed{\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \vec{F}\cdot\vec{\nabla}_p f = \left(\frac{df}{dt}\right)_{coll}}$$

But, for photons $\vec{v} = c\vec{n}$ $F=0$ (no mass, no charge), so

$$\boxed{\frac{\partial f}{\partial t} + c\cdot\vec{n}\cdot\vec{\nabla} f = \left(\frac{df}{dt}\right)_{coll}}$$

Not rad transport eqn. yet

Let's take a closer look at the "collision" term. It includes

- emission - sources of new photons, denoted η
- extinction - $\chi = \underbrace{\alpha}_{\text{absorption}} + \underbrace{\sigma}_{\text{scattering}}$ } removes photons from the phase space

[Note: κ is also used for absorption coefficient; usually per unit mass]

* η is the energy emitted per unit Vol dV, into solid angle $d\Omega$, within freqs. $\nu, \nu+d\nu$ and time $t, t+dt$

$$\eta = \frac{dE}{dV\,d\Omega\,d\nu\,dt} \quad \left[\frac{\text{erg}}{\text{cm}^3 \cdot \text{sr} \cdot \text{Hz}\,\text{s}}\right]$$

* χ is the amount of energy removed from a beam with specific intensity I within vol. dV, into solid angle $d\Omega$

$$\chi \cdot I = \frac{dE}{dV\, d\Omega\, d\nu\, dt}$$

Note χ and η have different units! You can't remove something that doesn't exist, so $\chi \cdot I$ is the energy removed.

$$[\chi] = cm^{-1}$$

χ is the product of the absorption cross section $[cm^{2}]$ and the # density of absorbers cm^{-3}

$\frac{1}{\chi}$ is a measure of the distance a photon ~~travels~~ travels on average before it is removed from the beam. This is called mean free path (m.f.p) and we'll come back to it

Now we need relation between I and f.

$f_R = f\,d^3p$ is the number of photons propagating with $v=c$ in direction $\hat{n}$ into $d\Omega$ per unit vol

$$f = \frac{dE/h\nu}{\frac{\hat{n}\cdot d\vec{S}\cdot c\,dt}{dV}\cdot d\Omega\cdot d\nu}$$

thus $I = c\,h\nu \cdot f$

Now we can write the Boltzmann equ as

$$\frac{1}{ch\nu}\left[\frac{\partial I}{\partial t} + c\,\hat{n}\cdot\vec{\nabla} I\right] = \frac{\eta}{h\nu} - \frac{\chi I}{h\nu}$$

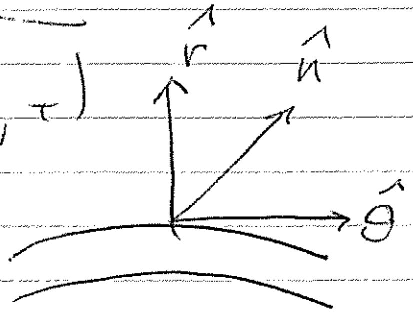
\# photons emitted

\# of photons absorbed or scattered

$$\boxed{\frac{1}{c}\frac{\partial I}{\partial t} + \hat{n}\vec{\nabla} I = \eta - \chi I}$$

Spherical geometry

$$I(\bar{r}, \hat{n}, \nu, t) \rightarrow I(r, \theta, \nu, t)$$



$$\bar{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}$$

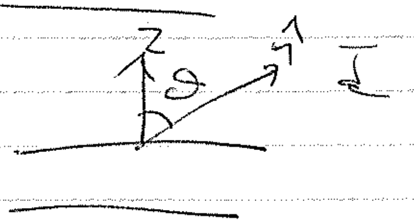
$$\hat{n} = (\cos\theta, \sin\theta)$$

$$\hat{n}\cdot\bar{\nabla} = \cos\theta \frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \Rightarrow$$

$$\boxed{\frac{1}{c}\frac{\partial I}{\partial t} + \left[\mu \frac{\partial}{\partial r} + \frac{1}{r}(1-\mu^2)^{\frac{1}{2}}\frac{\partial}{\partial \theta}\right] I = \eta - \chi I}$$

Still a PDE!

Planar (slab geometry)



$$\hat{n}\cdot\nabla = n_z \frac{\partial}{\partial z} = \mu \frac{\partial}{\partial z}$$

$$\boxed{\frac{1}{c}\frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial z} = \eta - \chi I} \quad \frac{\partial}{\partial x} = 0, \; \frac{\partial}{\partial y} = 0$$

Drop explicit t dependence?

When the properties of the medium do not change over timescales shorter than the propagation of the beam through the medium, we can drop t-dependence

$$\frac{\partial I}{\partial t} = 0$$

We will revisit this approx. in cases where it fails.

Thus, simplest rad. trans. equ.

$$\boxed{\mu \frac{\partial I}{\partial z} = \eta - \chi I} \quad \text{ODE!}$$

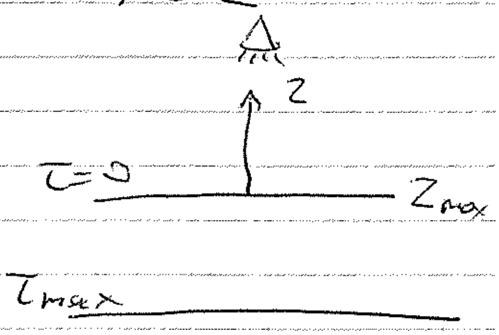
One for each μ, and each ν

Optical Depth & Source function

It is useful to introduce a dimensionless variable called optical depth:

$$d\tau_\nu = -\chi\, ds$$

In planar geometry: $d\tau_\nu = -\chi\, dz$



$$\tau_\nu = \int_{z_{min}}^{z_{max}} \chi_\nu(z')\,dz'$$

- sign convention

nce $\chi = \dfrac{1}{l_{mfp}}$

$$\tau_\nu = \int_{z_{min}}^{z_{max}} \frac{1}{l_{mfp}}\,dz \Rightarrow \tau = \#\text{ of mean free paths a photon travels before it escapes the medium}$$

Source function

$$S_\nu = \frac{\eta_\nu}{\chi_\nu}$$

So planar t-independent RTE →

$$\mu \frac{\partial I_\nu}{\partial \tau} = I_\nu - S_\nu$$ (note sign flip)

Ex. Special case of the source function in LTE.

Kirchoff's law for thermal emissio

$$\frac{\eta_\nu}{\alpha_\nu} = B_\nu(T) \text{ , but } \frac{\eta}{\alpha} = S$$

thus $S_\nu = B_\nu(T)$

!! Note $S_\nu = B_\nu$ does NOT imply $I_\nu = B_\nu$!!