Lec 22

Dynamics cont'd:

Last time: Conservation laws

(i) $\vec{P}_{cm} = 0$

ii) $E = T + U = \text{const.}$

$$T = \frac{1}{2}\sum_i m_i \underbrace{(\dot{\vec{r}}_i - \dot{\vec{R}}_{cm})}_{\dot{\vec{r}}_{i,cm}} \cdot (\dot{\vec{r}}_i - \dot{\vec{R}}_{cm}) = \frac{1}{2}\sum_i m_i v_{i,c}^2$$

iii) $\vec{J}_{cm} = \sum_i m_i \vec{r}_{i,cm} \times \dot{\vec{r}}_{i,cm} = \text{const.}$

Virial Theorem

Start with collisionless Boltzmann Eqn.

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0$$

and let $I = \int m f r^2 \, d^3\bar{r} \, d^3\bar{v}$

be the moment of inertia.

So, $\dot{I} = \int m \left(-v_i \frac{\partial f}{\partial x_i} + \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} \right) r^2 \, d^3\bar{r} \, d^3 v$

Since $\frac{\partial \phi}{\partial x^i}$ independent of v_i, and

$f \to 0$ when $v_i \to \pm\infty$

integration of the second term is

$$\int r^2 d^3 r \frac{\partial \phi}{\partial x_i} \left(\int \frac{\partial f}{\partial v_i} d^3 v \right) \to 0$$

The first term

$$\dot{I} = \int m \left(-v_i \frac{\partial f}{\partial x_i} \right) r^2 d^3 r d^3 v =$$

$$= -\int m v_i \cancel{\frac{d}{dx^i}\left(f r^2 \right)} d^3 r d^3 v + \int f m v_i \frac{\partial}{\partial x^i} (\vec{r} \cdot \vec{r}) d^3 r d^3 v$$

$$= 2 \int m f v_i r_j \cdot \frac{\partial r_j}{\partial x_i} d^3 r d^3 v$$

$$= 2 \int m f v_i r_j \cdot \delta_{ji} d^3 r d^3 v =$$

$$= 2 \int m f v_i r_i d^3 v d^3 r \Rightarrow$$

$$\underline{\dot{I} = 2 \int m \cdot \vec{r} \cdot \vec{v} f \cdot d^3 r d^3 v}$$

Taking another t-derivative and simplifying

$$\ddot{I} = 2 \int \left[m \left(v^2 - \vec{r} \nabla \phi \right) f \right] d^3 r d^3 v$$

kinetic energy is

$$T = \frac{1}{2}\int m v^2 f\, d^3r\, d^3\bar{v} \Rightarrow$$

$$\int m v^2 f\, d^3r\, d^3v = 2T$$

Potential energy is: $U = \int -\vec{r}\cdot\nabla\phi\, m f\, d^3r\, d^3v$

$$= \int \underbrace{\left(\int m f\, d^3v\right)}_{\rho} \phi\, d^3x$$

Thus $\frac{1}{2}\ddot{I} = 2T + U = 2(E-U) + U =$
$= 2E - U$

For a system in dynamical equil. $\ddot{I} = 0$

Thus, $\boxed{2T + U = 0}$ → Virial Theorem (VT)

Ex. simple application of V.T.

Assume all stars in cluster have the same mass. Then

$$2T = N m \langle v^2\rangle = M\langle v^2\rangle$$

$$U = -\frac{N(N-1)}{2}\frac{Gm^2}{R}$$, where

The "average radius" is

$$\frac{1}{\bar{R}} = \sum_{i<j} \frac{1}{|\bar{r}_i - \bar{r}_j|} \frac{2}{N(N-1)}$$

and $\langle v^2 \rangle$ is the mean square velocity of the cluster of stars. For $N \gg 1$, $N(N-1) \simeq N^2$, thus

$$U = -\frac{1}{2} \frac{GM^2}{\bar{R}}$$

$$\text{V.T.} \Rightarrow \quad \langle v^2 \rangle = \frac{1}{2} \frac{GM}{\bar{R}} \quad \Rightarrow$$

$$\langle v^2 \rangle = 4.6 \cdot 10^{-2} \left(\frac{M/M_\odot}{\bar{R}/1_{pc}} \right)^{\frac{1}{2}} \ \mathrm{Km \cdot s^{-1}}$$

For Pleiades cluster, $M = 300\, M_\odot$

$R = 3.5\ pc \Rightarrow \langle v^2 \rangle = 0.43\ \mathrm{Km \cdot s^{-1}}$ in good agreement with observations

** $\langle v^2 \rangle$, $\bar{R}$ are observable quantities hence M can be estimated from

$$M = \frac{2\bar{R} \langle v^2 \rangle}{G}$$

Often used in astronomical settings!

What about the escape of stars from the cluster?

The local escape velocity is

$$v_{esc}^2 = -2U \Rightarrow$$

$$\langle V_{esc}^2 \rangle = \frac{2GM}{\bar{R}}$$

Thus, $\boxed{\langle V_{esc}^2 \rangle = 4 \langle v^2 \rangle}$

Implications of the VT. for the stability of a cluster:

$$\ddot{I} = 4E - 2U$$

Since $U < 0$, if $E \geq 0 \Rightarrow \ddot{I} \geq 0$,

Thus, even if $\dot{I} < 0$, I will eventually grow and $\dot{I} \geq 0$. Given

$$I \sim \int m r^2 d^3 r \quad \text{or} \quad \sum m_i r_{ij}^2$$

then I will increase without limit!

Thus, necessary condition for stability is $E < 0$

Non-isolated Clusters

For clusters interacting with external bodies, additional effects arise, and the external grav. field will

- determine the motion of the cluster's c.o.m.
- exert tidal forces, which tend to disrupt the cluster and limit its radius

We define 3 macroscopic parameters for the spatial distribution of a a cluster in this case

1. The core radius R_c. In the core, stars are not affected much by external bodies and one can apply the VT within R_c.

2. Tidal radius, R_{tidal}, beyond which tidal forces disrupt the cluster

3. Cluster mass M.

These 3 paramaters suffice to describe the global dynamics of the cluster.

Calculation of Dynamical Timescale

1. Mixing time, τ_{mix}

From the def. of τ_{mix}, and using the VT for an average velocity we obtain

$$\tau_{mix} = \frac{2\bar{R}}{\langle v^2 \rangle^{1/2}} = \frac{2R}{\left(\frac{GM}{2\bar{R}}\right)^{1/2}} = \left(\frac{8\bar{R}^3}{GM}\right)^{1/2}$$

$$= \left(\frac{6}{\pi G \rho}\right)^{1/2} \qquad \rho = \frac{M}{\frac{4}{3}\pi \bar{R}^3}$$

similar to a free-fall time

Ex. globular cluster M3, $M = 10^5 M_\odot$
$\bar{R} = 15 pc$

$$\tau_{mix} = 7.8 \cdot 10^6 \text{ yr}$$

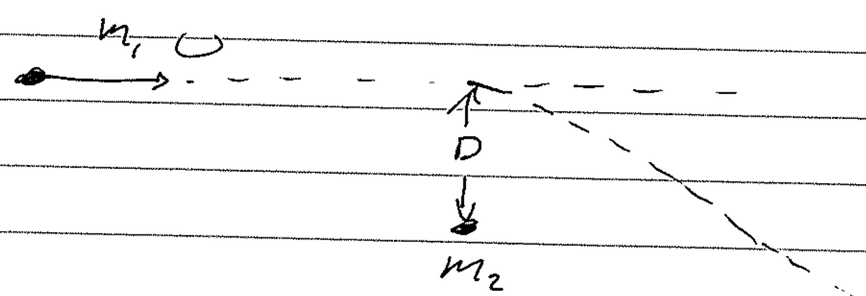
relatively short

Relaxation time

For a full derivation look at

Chandrasekhar's "Principles of Stellar dynamics" pp 51-67

Consider a binary encounter and assume the "collision" time is short compared to time between collisions.



Impulse $= m_1 \Delta V_\perp = F_\perp (\Delta t)_{coll} \simeq \dfrac{G m_1 m_2}{D^2} (\Delta t)_{coll}$

but $(\Delta t)_{coll} \simeq \dfrac{2D}{v}$ ~ the time over which the interaction is effective

Then $\rightarrow \Delta V_\perp \simeq \dfrac{2 G m_2}{D \cdot v}$ for a given D

Except In the case of very close 2-body encounters ~~where~~ we have $\Delta v_\perp \ll v$

If successive encounters are uncorrelated, then

$$\langle \Delta v_\perp \rangle = 0 \quad , \quad \text{but } \langle (\Delta v_\perp)^2 \rangle \neq 0$$

after N encounters

$$\langle v_\perp^2 \rangle = N \langle (\Delta v_\perp)^2 \rangle$$

Since τ_{rel} is s.t. the star loses memory of initiale conditions, we estimate τ_{rel} by setting $\langle v_\perp^2 \rangle \approx v^2$

Let's calculate $\langle v_\perp^2 \rangle$ by integrating over impact parameters.

The frequency of collisions in annulus $(D, D + \Delta D)$ is

$$d\nu_{coll}(D) = 2\pi D \cdot dD \cdot n \cdot v$$

So $\langle v_\perp^2 \rangle$ over some time interval τ is

$$\langle v_\perp^2 \rangle = \int_{D_{min}}^{D_{max}} dD \, 2\pi D \, n v \underbrace{\frac{4G^2 m_2^2}{D^2 v^2}}_{= (\Delta v_\perp)^2} \tau$$

$$= \frac{8\pi G^2 m_2^2 n \tau}{v} \ln\left(\frac{D_{max}}{D_{min}}\right)$$

An exact treatment shows that

$$D_{min} = \frac{G(m_1+m_2)}{v^2}$$

We take D_{max} = mean distance between stars

T_{rel} when $\langle v_\perp^2 \rangle \simeq v^2$, so we get

$$T_{rel} = \frac{v^3}{8\pi G m_2^2 n \Lambda}, \text{ where}$$

the gravitational logarith is defined as

$$\Lambda = \ln\left(\frac{D_0 v^2}{G(m_1+m_2)}\right)$$

In terms of typical numbers

$$T_{rel} = \frac{2\cdot 10^8}{(\Lambda/10)} \frac{(v/1\,km\,s^{-1})}{\left(\frac{m_2}{M_\odot}\right)^2 \left(n/1\,pc^{-3}\right)} \, yr$$

Because of the n dependence we expect denser parts of the cluster to relax faster!

To estimate Λ, we can use

$$v^2 = \frac{1}{2}\frac{GM}{\bar{R}}$$ and taking

$$N\left(\frac{p_0}{2}\right)^3 \simeq \bar{R}^3 \quad \text{w/ } m_1 = m_2$$

$$\boxed{\Lambda = \ln\left(\frac{1}{2}N^{2/3}\right)}$$ - a slowly varying function of N

N	10^2	10^3	10^5	10^{12}
Λ	2.4	3.9	7.0	17.7

Also τ_{rel} in terms of $N, m_2, \bar{R}$ (by eliminating v)

$$\tau_{rel} = \frac{1}{3(32G)^{1/2}} \frac{1}{\Lambda} \left(\frac{N\bar{R}^3}{m_2}\right)^{1/2}$$

- τ_{rel} is more sensitive to $\bar{R}$ than N, so clusters should be more relaxed than galaxies.

- τ_{rel} derived this way overestimates true τ_{rel} because it neglects all effects, but binary encounters.