Lec. 9

Compton Scattering

Last time we treated e-γ scattering not allowing for energy exchange during the interaction (elastic scattering). We also assumed the e^- was at rest.

More generally $\beta_e \doteq \frac{v_e}{c} \neq 0$.

Let's write the 4-momenta for γ and e^- as

$$p^{\mu}_{i,\gamma} = \frac{E_i}{c}(1, \hat{n}_i) \qquad p_{f,\gamma} = \frac{E_f}{c}(1, \hat{n}_f)$$

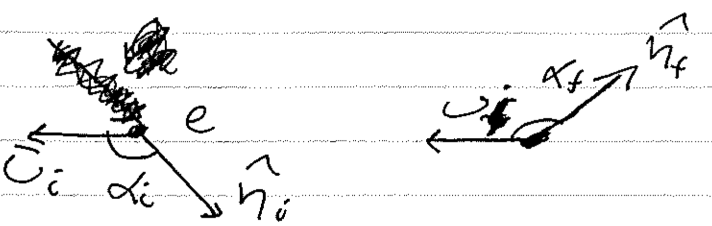
$$p^{\mu}_{i,e} = \gamma_i m (c, \bar{v}_i) \qquad p^{\mu}_{f,e} = \gamma_f m (c, \bar{v}_f)$$

$\bar{v}. \equiv$ three velocities

$m \equiv$ rest mass of e^-

$\hat{n} \equiv$ unit three-vectors $\qquad \hat{n}_i \cdot \hat{n}_f = \cos\theta$ (scat. angle)

Geometry



$$\hat{n}_i \cdot \bar{v}_i = v_i \cos\alpha_i \quad , \quad \hat{n}_f \cdot v_f = v_f \cos\alpha_f$$

four-mom. cons. $p^{\mu}_{i,\gamma} + p^{\mu}_{i,e} = p^{\mu}_{f,\gamma} + p^{\mu}_{f,e}$

squaring yields

$$\frac{E_f}{E_i} = \frac{1 - \beta_i \cos\alpha_i}{\underbrace{1 - \beta_i \cos\alpha_f}_{\text{energy gain from the electron}} + \underbrace{\frac{E_i}{\gamma mc^2}(1-\cos\theta)}_{\text{recoil term}}}$$

Simplest case $\beta_i = 0$ $(\gamma = 1)$ or in e^- rest frame

$$\frac{E_f}{E_i} = \frac{1}{1 + \frac{E_i}{mc^2}(1-\cos\theta)}$$

In terms of wavelength $\left(E = \frac{hc}{\lambda}\right)$

$$\lambda_f - \lambda_i = \lambda_c (1-\cos\theta)$$

$$\lambda_c \equiv \frac{h}{mc} \equiv \text{Compton wavelength}$$

for e^- $\lambda_c = 0.02426\ \text{Å}$

Now, since $0 \leq 1-\cos\theta$ then

$\frac{E_f}{E_i} \leq 1$, i.e., the photon always loses E due to the recoil of the e^-

Now, $\beta \neq 0$, but $\beta \ll 1$ and

$E_i \ll \gamma m c^2$ (non-relativistic)

- Expand to 1st-order in β:

$$\frac{E_f}{E_i} = 1 - \beta_i \cos\alpha_i + \beta_i \cos\alpha_f - \frac{E_i}{mc^2}(1-\cos\theta)$$

$$\Rightarrow \frac{\Delta E}{E_i} = -\frac{E_i(1-\cos\theta)}{mc^2} + \frac{v_i}{c}\left(\cos\alpha_f - \cos\alpha_i\right)$$

per scattering, γ may lose or gain E depending on which term dominates

But, on average for isotropic scattering $\int \cos\alpha_f - \cos\alpha_i = 0$

cause $\int \cos\theta = 0$

Thus $\left\langle \frac{\Delta E}{E_i} \right\rangle = -\frac{E_i}{mc^2}$

to 1st-order in $\frac{v}{c}$, photons don't gain or lose E due to electron motion β_i.

- Expand to 2nd-order in β:

$$\frac{E_f}{E_i} = 1 - \beta_i \cos\alpha_i + \beta_i \cos\alpha_f + \beta_i^2 \cos^2\alpha_f - \frac{E_i}{mc^2}(1-\cos\theta)$$

Average over angles $\langle \cos^2\alpha_f \rangle = \frac{1}{3}$

$$\frac{2\pi\int \cos^2\alpha_f \sin\alpha \, d\alpha}{2\pi\int \sin\alpha \, d\alpha}$$

$$\left\langle \frac{\Delta E}{E_i} \right\rangle = \underbrace{\frac{1}{3}\beta_i^2}_{\substack{E \text{ gain} \\ \text{from K.E.} \\ \text{of } e^-}} - \underbrace{\frac{E_i}{mc^2}}_{\substack{\text{loss due to} \\ e^- \text{ recoil}}}$$

If e^- are thermal, then

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} KT \implies$$

$$\beta_i^2 = \frac{3KT}{mc^2}$$

Then $\left\langle \frac{\Delta E}{E_i} \right\rangle = \frac{KT - E_i}{mc^2}$ } On average γ gain E when $KT > E_i$

γ lose E to e^- when $E_i > KT$

This is the basic physics behind "thermal bulk Comptonization"

When $kT > E$: thermal comptonization hardens the spectrum and brings $\langle E \rangle \to kT$

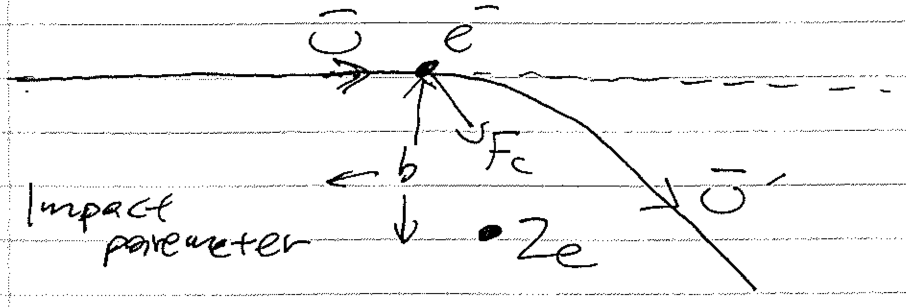
The Sunyaev-Zeldovich effect (thermal comptonization of CMB photons by hot gas in galaxy clusters) is a typical example.

Has two aspects: kinetic and thermal SZ

Bremsstrahlung or free-free emission/abs.

Following Rybicki & Lightman Ch. 5.

~~Ab~~ Free-free: Emission from a charge (eg. e^-) in the Coulomb field of another charge (ion, e^-). The charge undergoes acceleration, and a deviation from its path



$E \propto |\bar{a}(t)|^2$

accelerating charge

The dipole moment (position vector)

$$\bar{d} = -e\bar{R}, \quad \dot{\bar{d}} = -e\dot{\bar{R}}, \quad \ddot{\bar{d}} = -e\ddot{\bar{R}} = -e\dot{\vec{v}} = -e\vec{a}$$

$$E \propto |\ddot{\bar{d}}(t)|^2 \propto (\omega^2)^2 |\bar{d}(\omega)|^2$$

As before calculating power emitted involves integral

$$\bar{d}(\omega) \propto \int_{-\infty}^{\infty} e\dot{\bar{v}} e^{i\omega t} dt$$

But the Coulomb force is really effective on the e^- over a short period of time

$$\tau = \frac{b}{v} = \text{collision time}$$

When $\omega\tau \gg 1$, $e^{i\omega t}$ oscillates rapidly, and its contribution to the integral cancels out

so $\int_{-\infty}^{\infty} \dot{v}\,dt \simeq \Delta v$

Thus, the integral is prop. to $e\Delta v$

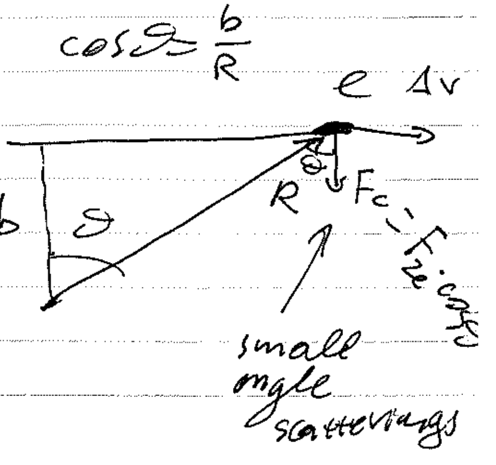
The energy emitted per frequency turns out to be

$$\frac{dE}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\bar{d}(\omega)|^2,$$ which becomes

$$= \frac{2e^2}{3\pi c^3}|\Delta v|^2$$

What is Δv?

$$m\frac{dv}{dt} = \frac{Ze^2}{R^2}\frac{b}{R},$$



$$R = (b^2+v^2t^2)^{\frac{1}{2}}$$

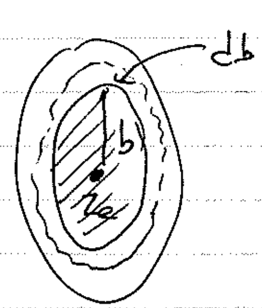
$$\Delta v = \frac{Ze^2}{m}\int_{-\infty}^{\infty}\frac{b\,dt}{(b^2+v^2t^2)^{\frac{3}{2}}} = \frac{2Ze^2}{mbv}$$

So $$\frac{dE}{d\omega} = \frac{8Z^2e^6}{3\pi c^3 m^2 v^2 b^2}$$

This is for a single e^- velocity and a single b.

We need to integrate over a dist. of electrons and b.

$$\frac{dE}{d\omega\, dV\, dt} = \underbrace{n_i}_{\#\text{ of ions}} \cdot \underbrace{n_e \cdot v}_{\text{flux of } e^- \text{ each ion "sees"}} \underbrace{\int 2\pi b\, db}_{\text{geometric cross section}} \cdot \underbrace{\frac{dE}{d\omega}}_{E \text{ per freq by } 1e^-}$$



$$= \frac{16e^6}{3c^3m^2v} n_e n_i Z^2 \underbrace{\ln\left(\frac{b_{max}}{b_{min}}\right)}_{\sim \text{Gaunt factor}}$$

$$g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{b_{max}}{b_{min}}\right)$$ which table look up

Now average over e^- velocities

$$\frac{dE}{d\omega dV dt} = \frac{\int_{v_{min}}^{\infty} \frac{dE(v,\omega)}{d\omega dV dt} v^2 e^{-\frac{mv^2}{2kT}} dv}{\int_{v_{min}}^{\infty} v^2 e^{-\frac{mv^2}{2kT}} dv}$$

So thermal Bremsstrahlung emission:

$$\eta_\omega = \frac{dE}{d\omega dV dt} = \frac{2^5 \pi e^6}{3mc^3} \left(\frac{2\pi}{3KM}\right)^{\frac{1}{2}} \cdot T^{-\frac{1}{2}} \cdot Z^2 n_e n_i e^{-h\nu/KT} \bar{g}_{ff}$$

Gaunt factor for thermal emission

Frequency integrated:

$$\frac{dE}{dV dt} = 1.4 \cdot 10^{-27} T^{\frac{1}{2}} n_e n_i Z^2 \bar{g}_B \quad \frac{erg}{s \cdot cm^3}$$

$\bar{g}_B = 1.1 - 1.5$, freq. & vel. averaged Gaunt factor

Thermal Bremsstrahlung absorption:

$$\eta_\nu^{ff} = \alpha_\nu^{ff} B_\nu(T) \quad \text{Kirchoff's law}$$

$$\alpha_\nu^{ff} = 3.7 \cdot 10^8 \, T^{-\frac{1}{2}} Z^2 n_e n_i \nu^{-3} \left(1 - e^{-\frac{h\nu}{kT}}\right) \bar{g}_{ff}$$

cm^{-1}

- Spectrum of optically thin free-free emission

$$I_\nu = I^+ + \frac{1}{\mu}\int_0^z \eta_\nu dz$$

Only freq. dependence of η_ν is the exponential. So optically thin thermal Bremss. produces a <u>flat</u> spectrum with an exponential cutoff at $h\nu \sim kT$

Notes last lecture

$$p_{i,\gamma}^{\mu} = \frac{E_i}{c}(1, \hat{n}_i)$$

$$p_{i,\gamma}^{\mu} + p_{i,e}^{\mu} - p_{f,\gamma}^{\mu} = p_{f,e}^{\mu} \Rightarrow$$

$$-\gamma_f^2 m^2 c^2\left(\cancel{1} - \frac{v_f^2}{c^2}\right) = -2 p_{i,\gamma}^{\mu} \cdot p_{f,\gamma\mu} +$$

$$+ 2 p_{i,\gamma}^{\mu} p_{i,e\mu} - 2 p_{i,e}^{\mu} p_{f,\gamma\mu} + p_{i,e}^2$$

$$\Rightarrow -m^2c^2 = -2\frac{E_i E_f}{c^2}(-1 + \hat{n}_i \cdot \hat{n}_f)$$

$$+2\frac{E_i}{c}\gamma_i m(-c + \hat{n}_i \cdot \bar{v}_i)$$

$$-2\frac{E_f}{c}\gamma_i m(-c + \hat{n}_f \cdot \bar{v}_i) - m^2c^2 \Rightarrow$$

$$-\cancel{m^2c^2} = 2\frac{E_i E_f}{c^2}(1 - \cos\theta)$$

$$-\cancel{2} E_i \gamma_i m(1 - \beta_i \cos\alpha_i)$$

$$+\cancel{2} E_f \gamma_i m(1 - \beta_i \cos\alpha_f)$$

$$-\cancel{m^2c^2}$$

$$\Rightarrow E_f \cdot \left[\frac{E_i}{\gamma_i m c^2}(1 - \cos\theta) + (1 - \beta_i \cos\alpha_f)\right]$$

$$= E_i (1 - \beta_i \cos\alpha_i) \Rightarrow$$

$$\frac{E_f}{E_i} = \frac{1 - \beta_i \cos\alpha_i}{1 - \beta_i \cos\alpha_f + \frac{E_i}{\gamma_i m c^2}(1 - \cos\theta)}$$

$\langle \cos^2\theta \rangle$ is over solid angles

$$\frac{\int \cos^2\theta \, d\Omega}{\int d\Omega} = \frac{1}{2}\frac{2\pi}{4\pi} \int \cos^2\theta \sin\theta \, d\theta =$$

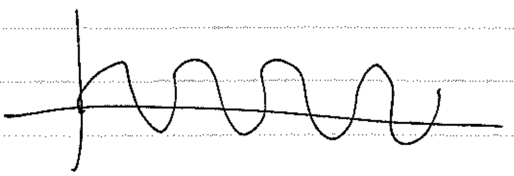
$$= -\frac{1}{2} \int \cos^2\theta \, d(\cos\theta) =$$

$$= -\frac{1}{2} \int \frac{1}{3} \, d(\cos^3\theta) =$$

$$= \frac{1}{2} \frac{1}{3} \cos^3\theta \Big|_{\pi}^{0} = \frac{1}{3}$$

Finally $\int \dot{v} e^{i\omega t} dt$

$$\sim \int \dot{v}(\cos\omega t) + i \sin\omega t$$



so for $\omega\tau \gg 1$

$0.$

For $\omega\tau \ll 1$ then $\int \dot{v} \cdot dt \sim$

$$\omega \frac{b}{v} \ll 1 \Rightarrow$$

$$\frac{c}{\lambda} \frac{b}{v} \ll 1 \to \frac{(b/\lambda)}{(v/c)} \ll 1 \Rightarrow$$

$$\frac{b}{\lambda} \ll \frac{v}{c} \ll 1$$