Lec 2

Ex. Let's look at what these ~~qu~~ quantities are in thermal equilib. (We haven't defined that, yet, but we'll see what it implies)

I is uniform (no $\vec{r}$ depend.)
isotropic (no $\hat{n}$ depend.)
~~time~~ stationary (no t-depend)

Given by Planck function

$$I_\nu(T) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1} = B_\nu(T)$$

$$J_\nu = B_\nu(T)$$

→ Monochromatic energy density is

$$\mathcal{E}_R(\nu) = \frac{4\pi}{c} J = \frac{4\pi}{c} B_\nu$$

→ Total energy density $\int d\nu\, \mathcal{E}_R(\nu)$

$$\mathcal{E} = \frac{8\pi h}{c^3}\int_0^\infty \left(e^{h\nu/kT} - 1\right)^{-1} \nu^3 d\nu = a T^4$$

$$a = \frac{8\pi^5 k^4}{15 c^3 h^3} = \text{rad. constant}$$

In spherical sym.

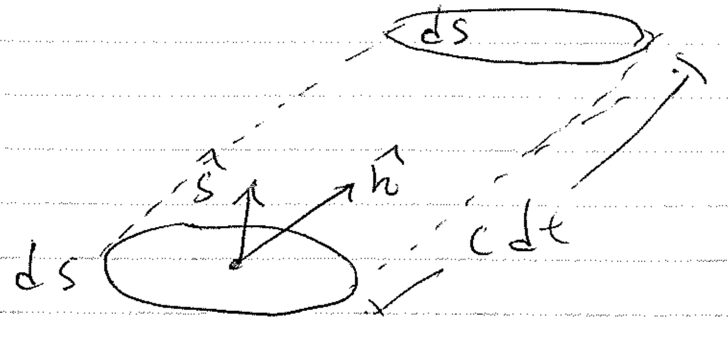
$$\bar{J}(r,\nu,t) = \frac{1}{2}\int_{-1}^{1} I(r,\mu,\nu,t)\,d\mu$$

The mean intensity is closely related to the energy density of radiation.

Consider a volume V. What is the energy contained in V?

What is the energy contained in volume dV?

Since $v=c$ for photons, they will cover a distance $c\,dt$ in time dt, so let's make up our volume by moving a surface in the direction of propagation



Then

$$dV = c\,dt\,\hat{n}\cdot ds$$
$$= c\,dt\,ds\cdot\cos\theta$$

Since

$$dE = I_\nu\, ds\cos\theta\, d\Omega\, d\nu\, dt =$$

$$= \frac{I_\nu}{c}\, d\Omega\, d\nu\, dV$$

First Moment: $\underset{\text{Energy}}{\text{Flux}}$

$$\vec{F} = \int I \hat{n} \, d\Omega$$

Is this the flux we defined earlier?

$$\vec{F} = \frac{dE}{dt\,dS\,d\nu} \hat{n} \qquad [F] = \frac{erg}{s\ cm^2\ Hz}$$

Recall: $dE = I_\nu \, dS \cdot \cos\theta \, d\Omega \, d\nu \, dt$

And $\qquad\qquad = I_\nu \cdot \hat{n} \cdot d\vec{S} \, d\Omega \, d\nu \, dt$

$$F = \frac{dE}{dt\,dS\,d\nu} \rightarrow \int I \hat{n} \cdot d\Omega = \vec{F}$$

All good!

Example: An isotropic rad. field is given by $I(\hat{n}) = I_0$ [same in every direction]

What is F = ? Solve in spherical coords.

$$F = \int I_0 \hat{n} \, d\Omega = I_0 \int \cos\theta \sin\theta \, d\theta \, d\varphi$$

$$= I_0 \, 2\pi \int_{-1}^{1} \mu \, d\mu = 0$$

But how can this be? In the centers of stars, I should be isotropic

How can there be flux from a star? Shouldn't there be a net outword flux?

Example: I is nearly isotropic, but not completely

What is F, J, if the rad. field has small deviation from isotropy that is connected to a variable called optical depth, as

$$I_\nu(\tau, \mu) = a_\nu(\tau) + b_\nu(\tau)\mu$$

$$J = \frac{1}{2}\int_{-1}^{1} I\, d\mu = a$$

$$\bar{F} = 2\pi\int_{-1}^{1} I\mu\, d\mu = 4\pi\frac{b}{3}$$

(in the $\hat{r}$ direction because τ changes with r
more on that later)

Notice that we defined J and F differently.

Why does one have $\frac{1}{4\pi}$ but not the other?

$$J = \frac{1}{4\pi}\int I\, d\Omega \quad , \quad \vec{F} = \int I\hat{n}\, d\Omega$$

We can define the "Eddington flux"

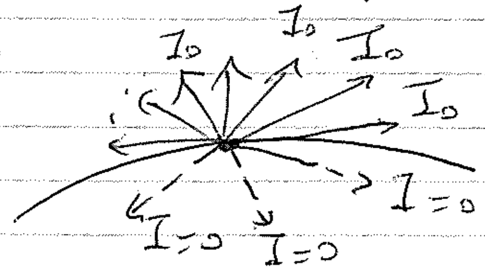
$$\vec{H} = \frac{1}{4\pi}\int I \hat{n}\, d\Omega$$

We'll use both $\vec{F}$ (the astrophysical flux) and $\vec{H}$.

In the above example

$$H = \frac{1}{2}\int_{-1}^{1} I \mu\, d\mu = \frac{b}{3}$$

Example: Consider radiation leaving from a boundary, isotropically



All outgoing rays have I_0. All inbound rays have $I=0$ "no radiation is coming into the domain from that boundary."

What is H?

$$H = \frac{1}{2}\int_{-1}^{1} I\mu\, d\mu = \frac{1}{2}\int_{0}^{1} I\mu\, d\mu =$$

$$= \frac{1}{2} I_0 \int_0^1 d\left(\frac{\mu^2}{2}\right) = \frac{1}{4} I_0$$

Or $F = 4\pi H = \pi I_0$

So, this is the case pertaining to the earlier problem with isotropic rad. all around: for an object of finite size radiating luminosity L "isotropically", on the surface $r=R$, there is no inward specific intensity, and at $r=R$, $F=\pi I$

For $r>R$ "free-streaming" starts, I stays constant, but $F \sim \frac{1}{r^2}$.

Example. Calculate J, H for a "free-streaming" rad field (specific intensity in one direction)

$I(\mu) = I_0 \delta(\mu - \mu_0)$

$$J = \frac{1}{2}\int I_0\,\delta(\mu-\mu_0)\,d\mu = \frac{I_0}{2}$$

$$H = \frac{1}{2}\int I_0\,\mu\,\delta(\mu-\mu_0) = \frac{\mu_0 I_0}{2}$$

Second moment: Radiation Pressure Tensor

J is scalar, $\vec{H}/\vec{F}$ are vectors

P_{ij} is a tensor!

$$\boxed{P(\bar{r},\nu,t) = \frac{1}{c}\int I(r,\hat{n},\nu,t)\,\hat{n}\hat{n}\,d\Omega}$$

or $\boxed{P_{ij} = \frac{1}{c}\int I\, n_i n_j\, d\Omega}$

Lecture 3

al: P_{ii} are related to energy density

eg. for a fluid $\int \rho\, v_i v_j\, d\Omega \Rightarrow$

~~radiation~~ energy density / pressure

For rad. the trace of P_{ij}

$$\sum P_{ii} = P_{xx} + P_{yy} + P_{zz} =$$

$$= \frac{1}{c}\int I\,(\underbrace{n_x^2 + n_y^2 + n_z^2})\,d\Omega$$