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Homework 5

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1. Consider the hydrostatic equilibrium equation in spherical symmetry of a fluid distribution

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}$$

- (a) **Integrate this equation over all volume V to prove the Virial theorem for static equilibrium fluid configurations**

$$3 \int P dV + W = 0$$

where W is the gravitational potential energy $W = - \int Gm(r)\rho(r)/r dV$.

Starting from the hydrostatic equilibrium equation, we can multiply by r and integrate both sides with respect to the volume

$$\int \frac{dP}{dr} r dV = - \int \frac{Gm(r)\rho(r)}{r^2} r dV \quad (1)$$

Assuming spherical symmetry, we know that $dV = 4\pi r^2 dr$. Thus

$$\int_0^\infty \frac{dP}{dr} 4\pi r^3 dr = - \int_0^\infty \frac{Gm(r)\rho(r)}{r^2} 4\pi r^3 dr \quad (2)$$

The left hand side can be integrated by parts, where we know that $\int dP/dr dr = P$. This gives

$$(4\pi r^3 P|_0^\infty - \int_0^\infty 3P 4\pi r^2 dr = - \int_0^\infty \frac{Gm(r)\rho(r)}{r} 4\pi r^2 dr \quad (3)$$

Clearly at $r = 0$ the left hand term is zero. Additionally, the boundary conditions for hydrostatic equilibrium state that at infinity, the pressure must go to zero, thus $P(r = \infty) = 0$, and the first term is simply zero.

$$-3 \int_0^\infty P 4\pi r^2 dr = - \int_0^\infty \frac{Gm(r)\rho(r)}{r} 4\pi r^2 dr \quad (4)$$

For both sides, we can substitute in the spherical symmetry volume element. Because we are integrating from $r = 0 \rightarrow r = \infty$, this corresponds to integrating from $V = 0 \rightarrow V = \infty$, i.e. over all volume. This gives

$$-3 \int_0^\infty P dV = - \int_0^\infty \frac{Gm(r)\rho(r)}{r} dV \quad (5)$$

The right hand side of this equation is simply the gravitational potential energy $W = - \int Gm(r)\rho(r)/r dV$. Finally we have that

$$\boxed{3 \int_0^\infty P dV + W = 0} \quad (6)$$

- (b) **Adopting an ideal gas equation of state and a constant density star, use the Virial theorem to estimate the characteristic (Virial) temperature for the sun assuming it's composed of pure hydrogen.**

For a constant density star, we know that $m(r)$ is simply the enclosed mass. That is,

$$m(r) = \int_0^r 4\pi r^2 \rho \, dr = 4\pi \rho \int_0^r r^2 \, dr = \frac{4}{3} \pi \rho r^3 \quad (7)$$

Thus, the virial theorem becomes

$$3 \int_0^R P 4\pi r^2 \, dr - \int_0^R \frac{Gm(r)\rho}{r} 4\pi r^2 \, dr = 12\pi \int_0^R P r^2 \, dr - 4\pi \rho G \int_0^R Gm(r)r \, dr \quad (8)$$

$$= 12\pi \int_0^R P r^2 \, dr - 4\pi \rho G \int_0^R \frac{4}{3} \pi \rho r^3 r \, dr \quad (9)$$

$$= 12\pi \int_0^R P r^2 \, dr - \frac{16}{3} \pi^2 \rho^2 G \int_0^R r^4 \, dr \quad (10)$$

$$= 12\pi \int_0^R P r^2 \, dr - \frac{16}{3} \pi^2 \rho^2 G \frac{1}{5} R^5 \quad (11)$$

$$= 3 \int_0^R P r^2 \, dr - \frac{4}{15} \pi G \rho^2 R^5 = 0 \quad (12)$$

For an ideal gas of pure hydrogen, we know that $P = nkT = \frac{\rho}{\mu m_p} kT$. Thus,

$$3 \int_0^R \frac{\rho}{\mu m_p} kT r^2 \, dr - \frac{4}{15} \pi G \rho^2 R^5 = 3 \frac{k}{\mu m_p} \int_0^R T r^2 \, dr - \frac{4}{15} \pi G \rho R^5 = 0 \quad (13)$$

We can define a virial temperature, T_{vir} , that is a sort of characteristic temperature, constant over radius, allowing us to write

$$3 \frac{kT_{\text{vir}}}{\mu m_p} \int_0^R r^2 \, dr - \frac{4}{15} \pi G \rho R^5 = 3 \frac{kT_{\text{vir}}}{\mu m_p} \frac{1}{3} R^3 - \frac{4}{15} \pi G \rho R^5 = \frac{kT_{\text{vir}}}{\mu m_p} - \frac{4}{15} \pi G \rho R^2 = 0 \quad (14)$$

For a sphere of constant density, we can define the density simply as $\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$. Thus

$$\frac{kT_{\text{vir}}}{\mu m_p} = \frac{4}{15} \pi G \frac{3M}{4\pi R^3} R^2 = \frac{1}{5} \frac{GM}{R} \quad (15)$$

Thus the virial temperature is

$$T_{\text{vir}} = \frac{1}{5} \frac{\mu m_p}{k} \frac{GM}{R} \quad (16)$$

Plugging in values for our sun assuming pure hydrogen, we get

$$T_{\text{vir}} = \frac{1}{5} \frac{1.67 \times 10^{-27}}{1.381 \times 10^{-23}} \frac{6.67 \times 10^{-11} \cdot 1.989 \times 10^{30}}{6.96 \times 10^5} = \boxed{4622044 \text{ K} = 4.622 \text{ MK} = T_{\text{vir}}} \quad (17)$$

- (c) Assuming an ideal gas equation of state, what is the physical interpretation of the $3 \int P dV$ term?

As from above, we know that for an ideal gas

$$P = nkT = \frac{\rho}{\mu m_p} kT \quad (18)$$

This gives

$$3 \int P dV = 3 \int nkT dV = 2 \int \frac{3}{2} nkT dV \quad (19)$$

The term $\frac{3}{2}kT$ is simply the internal energy of each particle, ε .

$$2\varepsilon \int n dV \quad (20)$$

The integral of the particle density over the entire volume simply gives the number of particles

$$2\varepsilon N \quad (21)$$

The energy of each particle times the number of particles, $N\varepsilon$, is simply the total internal energy of the system, U , which gives the classical expression of the virial theorem

$$3 \int P dV = \boxed{2U = -W} \quad (22)$$

Thus $3 \int P dV$ can be physically interpreted as twice the total internal energy of the system.

2. Take the inner edge of a thin accretion disk to be at the ISCO around a non-spinning black hole.

- (a) Calculate the maximum possible effective temperature as a function of the mass of the black hole and the mass accretion rate. What do you expect this temperature to be for accretion around a $10 M_\odot$ and a $10^9 M_\odot$ black hole at 10% of the critical mass accretion rate \dot{M}_{Edd} ?

The effective temperature of a thin accretion disk as a function of radius is given by

$$T_{\text{eff}} = \left(\frac{3GM}{4\pi\sigma} \right)^{1/4} \dot{M}^{1/4} \left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} R^{-3/4} \quad (23)$$

The maximum possible effective temperature occurs at R_{max} when

$$\frac{dT_{\text{eff}}}{dR} = \frac{d}{dR} \left[\left(\frac{3GM}{4\pi\sigma} \right)^{1/4} \dot{M}^{1/4} \left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} R^{-3/4} \right] = 0 \quad (24)$$

Assuming that all coefficient terms are constant with radius, this simplifies to

$$0 = \frac{dT_{\text{eff}}}{dR} = \left(\frac{3GM}{4\pi\sigma} \right)^{1/4} \dot{M}^{1/4} \frac{d}{dR} \left[\left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} R^{-3/4} \right] \quad (25)$$

$$= \left\{ R^{-3/4} \frac{d}{dR} \left[\left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} \right] + \left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} \frac{d}{dR} R^{-3/4} \right\} \quad (26)$$

$$= R^{-3/4} \frac{R_{\text{in}}}{8 \left(1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right)^{3/4} \left(\frac{R_{\text{in}}}{R} \right)^{1/2} R^2} - \left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} \frac{3}{4} R^{-7/4} \quad (27)$$

$$= R^{-11/4} \frac{R_{\text{in}}}{8 \left(1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right)^{3/4} \left(\frac{R_{\text{in}}}{R} \right)^{1/2}} - \left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} \frac{3}{4} R^{-7/4} \quad (28)$$

$$= \frac{\left(\frac{R_{\text{in}}}{R} \right)^{-1/2}}{8 \left(1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right)^{3/4}} \left(\frac{R_{\text{in}}}{R} \right) - \left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} \frac{3}{4} \quad (29)$$

$$= \frac{1}{6 \left(1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right)^{3/4}} \left(\frac{R_{\text{in}}}{R} \right)^{1/2} - \left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} \quad (30)$$

Solving this for R_{in}/R ,

$$\frac{1}{6 \left(1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right)^{3/4}} \left(\frac{R_{\text{in}}}{R} \right)^{1/2} = \left[1 - \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \right]^{1/4} \quad (31)$$

$$\left(\frac{R_{\text{in}}}{R} \right)^{1/2} = 6 - 6 \left(\frac{R_{\text{in}}}{R} \right)^{1/2} \quad (32)$$

$$\left(\frac{R_{\text{in}}}{R} \right)^{1/2} = \frac{6}{7} \quad (33)$$

$$\frac{R_{\text{in}}}{R} = \frac{36}{49} \quad (34)$$

This also implies that for T_{\max} , $R_{\text{in}} = \frac{49}{36}R_{\text{in}}$. This makes the maximum possible temperature

$$T_{\max} = \left(\frac{3GM}{4\pi\sigma}\right)^{1/4} \dot{M}^{1/4} \left[1 - \left(\frac{36}{49}\right)^{1/2}\right]^{1/4} \left(\frac{49}{36}R_{\text{in}}\right)^{-3/4} \quad (35)$$

$$= \left(\frac{3GM}{4\pi\sigma}\right)^{1/4} \dot{M}^{1/4} \left[1 - \frac{6}{7}\right]^{1/4} \frac{6\sqrt{42}}{49} R_{\text{in}}^{-3/4} \quad (36)$$

$$= \left(\frac{3GM}{4\pi\sigma}\right)^{1/4} \dot{M}^{1/4} \left[\frac{1}{7}\right]^{1/4} \frac{6\sqrt{42}}{49} R_{\text{in}}^{-3/4} \quad (37)$$

$$= \left(\frac{3GM}{4\pi\sigma}\right)^{1/4} \dot{M}^{1/4} 7^{-1/4} \frac{6\sqrt{42}}{49} R_{\text{in}}^{-3/4} \quad (38)$$

For an accreting, non-spinning black hole, this inner orbit is simply the inner most stable circular orbit (ISCO). For a non-spinning black hole, the ISCO is given by

$$R_{\text{ISCO}} = \frac{6GM}{c^2} \quad (39)$$

This makes the maximum temperature

$$T_{\max} = \left(\frac{3GM}{4\pi\sigma}\right)^{1/4} \dot{M}^{1/4} 7^{-1/4} \frac{6\sqrt{42}}{49} \left(\frac{6GM}{c^2}\right)^{-3/4} \quad (40)$$

$$= \frac{6\sqrt{42}}{49} \left(\frac{3}{28\pi\sigma}\right)^{1/4} \dot{M}^{1/4} \left(\frac{6}{c^2}\right)^{-3/4} (GM)^{-1/2} \quad (41)$$

$$= \frac{6}{49} \left(\frac{3}{28\pi\sigma}\right)^{1/4} \left(\frac{6}{c^2}\right)^{-3/4} \left(\frac{42}{GM}\right)^{1/2} \dot{M}^{1/4} \quad (42)$$

$$= \frac{1}{49} \left(\frac{18}{28\pi\sigma}\right)^{1/4} c^{3/2} \left(\frac{42}{GM}\right)^{1/2} \dot{M}^{1/4} \quad (43)$$

$$= \boxed{\frac{c^{3/2}}{49} \left(\frac{9\dot{M}}{14\pi\sigma}\right)^{1/4} \left(\frac{42}{GM}\right)^{1/2}} = T_{\max} \quad (44)$$

The Eddington luminosity for a black hole of mass M is given by

$$L = \varepsilon \dot{M} c^2 = \frac{4\pi GM m_p c}{\sigma_T} \quad (45)$$

This corresponds to an Eddington accretion rate of

$$\dot{M}_{\text{Edd}} = \frac{L}{c^2 \varepsilon} = \frac{4\pi GM m_p c}{\sigma_T \varepsilon} \frac{1}{c^2} = \frac{4\pi GM m_p}{\sigma_T c \varepsilon} \quad (46)$$

For a non-spinning black hole, the efficiency is

$$\varepsilon = \frac{1}{2} \frac{GM}{R_{\text{ISCO}} c^2} = \frac{1}{2} \frac{GM}{c^2} \frac{c^2}{6GM} = \frac{1}{12} \quad (47)$$

giving an accretion rate of

$$\dot{M} = 0.1 \dot{M}_{\text{Edd}} = \frac{24\pi GMm_p}{5\sigma_T c} \quad (48)$$

Plugging this into the maximum temperature gives

$$T_{\text{max}} = \frac{c^{3/2}}{49} \left(\frac{9}{14\pi\sigma} \frac{24\pi GMm_p}{5\sigma_T c} \right)^{1/4} \left(\frac{42}{GM} \right)^{1/2} \quad (49)$$

$$= \frac{c^{5/4}}{49} \left(\frac{27216m_p}{5GM\sigma\sigma_T} \right)^{1/4} \quad (50)$$

Finally, plugging in typical values for a black hole of mass $10 M_{\odot}$

$$\boxed{T_{\text{max}} = 5.2573 \text{ MK}} \quad (51)$$

and $10^9 M_{\odot}$

$$\boxed{T_{\text{max}} = 52573 \text{ K}} \quad (52)$$

- (b) **Given your answer to the previous part, at what wavelength would you have designed the Even Horizon Telescope in order to observe/image the inner accretion flow of an AGN? Is this what really happens? And how do you explain AGN observations in X-rays?**

Assuming pure blackbody radiation from the material in the disk, the peak wavelength for emission follows Wien's law.

$$\lambda_{\text{max}} = \frac{0.002898 \text{ m K}}{T} \quad (53)$$

To image the inner accretion flow where the temperature is maximum thus requires observing at a wavelength of approximately

$$\lambda_{\text{max}} = \frac{0.002898 \text{ m K}}{T_{\text{max}}} \quad (54)$$

Plugging in the answer for a black hole of mass $10 M_{\odot}$

$$\boxed{\lambda_{\text{max}} = 0.5512 \text{ nm}} \quad (55)$$

which corresponds to X-Rays, and for a mass of $10^9 M_{\odot}$

$$\boxed{\lambda_{\text{max}} = 55.12 \text{ nm}} \quad (56)$$

which corresponds to UV.

In theory, a wavelength of around 30 nm would be ideal for the EHT to observe the inner accretion flow of an AGN. This, however, is not what happens. While this is ideal for the innermost accretion flow, observing the larger disk means observing at lower temperatures. Additionally, more of the disk will emit at this wavelength, around 1.3 mm, meaning the resolution required to resolve the disk is much less. Moreover, observing at longer wavelengths allows for telescopes the size of the Earth with much greater resolving power than single individual telescopes at smaller diameters.

This purely thermal emission does not explain radiation from AGNs in the X-ray. However, thermal emission is not the only process of emission from AGNs. This higher frequency emission is created by reprocessed photons emitted from the disk, typically from Comptonization or synchrotron emission.

3. Consider an acoustic wave that propagates in a static, uniform, non-magnetized medium of adiabatic index γ . Estimate the time it takes (in units of the period of the wave) for the acoustic wave to steepen into a shock wave, as a function of the fraction perturbation in density. How far does the sound wave travel before it becomes a shock wave? Evaluate your answer for a 440 Hz sound wave that is propagating in dry air at a speed of about 350 m/s and corresponds to a density perturbation of about 10%. Why don't sounds in everyday life always turn into shock waves?

A shock occurs when the density perturbation wave becomes two-valued at some point. This point can be approximated as the point when the peak of the density wave overruns the trough of the density wave. Initially, the peak and trough of a sinusoidal wave are exactly half a wavelength, $\lambda/2$ away from each other. Thus, the peak of the wave must travel this far with respect to the trough of the wave. If the peak is moving at a velocity of v^+ , and the trough is moving at a velocity of v^- , the relative velocity between the two, i.e. the velocity the peak travels at while moving a total distance of $\lambda/2$, is simply $v^+ - v^-$.

If the initial wave is moving at a velocity of c_0 , then we know that

$$\lambda = Tc_0 \quad (57)$$

where T is the period of the wave. The time it takes for the wave peak to overtake the trough is thus

$$\tau_{\text{shock}} = \frac{\lambda/2}{v^+ - v^-} = \frac{Tc_0/2}{v^+ - v^-} \quad (58)$$

At any density perturbations, the wave at that point will travel faster or slower due to the difference in density, as a higher density medium has a higher sound speed. Ignoring the induced bulk motion of the medium, assuming the medium is initially at rest, and assuming an adiabatic equation of state, the speed at any point along the wave is

$$v = c_s = \sqrt{k\gamma\rho^{\gamma-1}} \quad (59)$$

where k is a constant. Let us define f as the fractional difference between the peak of the wave and the baseline, or equivalently the trough and the baseline, i.e.

$$f = \rho - \rho_0/\rho_0 = \Delta\rho/\rho_0 \quad (60)$$

We have that the shock time is now

$$\tau_{\text{shock}} = \frac{Tc_0/2}{v^+ - v^-} = \frac{Tc_0}{2(c_s^+ - c_s^-)} = \frac{T\sqrt{k\gamma\rho_0^{\gamma-1}}}{2\left(\sqrt{k\gamma\rho_+^{\gamma-1}} - \sqrt{k\gamma\rho_-^{\gamma-1}}\right)} = \frac{T\sqrt{\rho_0^{\gamma-1}}}{2\left(\sqrt{\rho_+^{\gamma-1}} - \sqrt{\rho_-^{\gamma-1}}\right)} \quad (61)$$

$$= \frac{T\rho_0^{(\gamma-1)/2}}{2\left(\rho_+^{(\gamma-1)/2} - \rho_-^{(\gamma-1)/2}\right)} = \frac{T}{2\left(\left(\frac{\rho_+}{\rho_0}\right)^{(\gamma-1)/2} - \left(\frac{\rho_-}{\rho_0}\right)^{(\gamma-1)/2}\right)} \quad (62)$$

From the above definition, we have that

$$\rho_+ = \rho_0 + \Delta\rho \quad (63)$$

$$\rho_- = \rho_0 - \Delta\rho \quad (64)$$

We then have that

$$\frac{\tau_{\text{shock}}}{T} = \frac{1}{2 \left(\left(\frac{\rho_0 + \Delta \rho}{\rho_0} \right)^{(\gamma-1)/2} - \left(\frac{\rho_0 - \Delta \rho}{\rho_0} \right)^{(\gamma-1)/2} \right)} = \frac{1}{2 \left(\left(1 + \frac{\Delta \rho}{\rho_0} \right)^{(\gamma-1)/2} - \left(1 - \frac{\Delta \rho}{\rho_0} \right)^{(\gamma-1)/2} \right)} \quad (65)$$

$$= \frac{1}{2 \left((1+f)^{(\gamma-1)/2} - (1-f)^{(\gamma-1)/2} \right)} \quad (66)$$

For a diatomic ideal gas, we know that $\gamma = 7/5$. Thus

$$\begin{aligned} \frac{\tau_{\text{shock}}}{T} &= \frac{1}{2 \left((1+f)^{(7/5-1)/2} - (1-f)^{(7/5-1)/2} \right)} = \frac{1}{2 \left((1+f)^{(2/5)/2} - (1-f)^{(2/5)/2} \right)} \\ &= \boxed{\frac{1}{2 \left((1+f)^{1/5} - (1-f)^{1/5} \right)}} = \frac{\tau_{\text{shock}}}{T} \end{aligned} \quad (67)$$

Similarly, if the wave propagates at speed c_0 for time τ_{shock} , the total distance traversed is

$$\boxed{d_{\text{shock}} = \frac{T c_0}{2 \left((1+f)^{1/5} - (1-f)^{1/5} \right)}} \quad (68)$$

For a sound wave of frequency 440 Hz, the period is simply

$$T = \frac{1}{f} = \frac{1}{440} \quad (69)$$

Assuming a density perturbation f of 10% and plugging in this value and the speed of sound $c_0 = 350$ m/s, we get that

$$\tau_{\text{shock}} = \frac{1/440}{2 \left((1+0.1)^{1/5} - (1-0.1)^{1/5} \right)} = 0.02834 \text{ s} = \boxed{28.34 \text{ ms} = \tau_{\text{shock}}} \quad (70)$$

and the distance traveled is

$$d_{\text{shock}} = 0.02834 \cdot 350 = \boxed{9.919 \text{ m} = d_{\text{shock}}} \quad (71)$$

Despite what this problem implies, typical sounds rarely turn into shock waves for numerous reasons. The main contributing factor is dissipative effects in typical mediums. In order for the wave to non-linearly steepen enough to form a shock, the non-linear term in the fluid equations of motion must dominate over the dissipative terms which we neglected. Dissipative effects can arise from viscosity, wave dispersion, diffusion, resistivity, friction, or any other dampening terms. For most fluids, these terms will dominate over the non-linear steepening. However, if there is a source of energy injection into the wave, such as a constant/traveling noise source, rather than just an isolated wave packet, a shock wave can be maintained.

4. In class, we derived the Rankine-Hugoniot jump conditions by assuming that γ is the same in the pre-shock and post-shock regions. This need not be the case in general. For example, a shock may dissociate hydrogen molecules and/or ionize hydrogen atoms, so that part of the pre-shock kinetic energy goes into dissociation and ionization.

From elementary thermodynamics, we know that

$$\gamma = 1 + \frac{2}{f} \quad (72)$$

where f is the number of internal degrees of freedom.

- (a) Ignoring radiation and magnetic fields, write down the one-dimensional jump conditions in the case $\gamma_1 \neq \gamma_2$.

The equations of fluid dynamics can be condensed for a 1D shock into three conservation equations corresponding to mass, momentum, and energy.

$$\rho_1 u_1 = \rho_2 u_2 \quad (73)$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \quad (74)$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \quad (75)$$

where, for a perfect gas with adiabatic equation of state $P = K\rho^\gamma$,

$$h = \varepsilon + \frac{P}{\rho} = \frac{\gamma}{\gamma - 1} \frac{P}{\rho} \quad (76)$$

When including the effects from ionization and dissociation, the energy equation must be modified. The ionization and dissociation take energy away from the post-shock material, or equivalently add energy to the pre-shock material. To be consistent with units, this must be a specific enthalpy added to the left-hand side of the energy conservation equation. The energy density is simply given by $\varepsilon_{\text{rad}} = n_{\text{HII}}(E_d + 2E_i)$. Dividing this quantity by the pre-shock density gives the pre-shock specific enthalpy added by dissociation and ionization, giving the new conservation equations

$$\rho_1 u_1 = \rho_2 u_2 \quad (77)$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \quad (78)$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} + \frac{\varepsilon_{\text{rad}}}{\rho_1} \quad (79)$$

where

$$\varepsilon_{\text{rad}} = n_{\text{HII}}(E_d + 2E_i) \quad (80)$$

Defining a mass flux $f = \rho u$, gives

$$\rho_1 u_1 = \rho_2 u_2 = f \quad (81)$$

Rewriting the second conservation equation with this new quantity,

$$f u_1 + P_1 = f u_2 + P_2 \implies \frac{f^2}{\rho_1} + P_1 = \frac{f^2}{\rho_2} + P_2 \quad (82)$$

Solving for f^2 ,

$$\frac{f^2}{\rho_1} - \frac{f^2}{\rho_2} = P_2 - P_1 \quad (83)$$

$$f^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = P_2 - P_1 \implies \quad (84)$$

$$f^2 = \frac{P_2 - P_1}{\frac{1}{\rho_1} - \frac{1}{\rho_2}} \quad (85)$$

Rewriting the energy conservation equation in terms of f^2 ,

$$h_1 + \frac{f^2}{2\rho_1^2} = \frac{\varepsilon_{\text{rad}}}{\rho_1} + h_2 + \frac{f^2}{2\rho_2^2} \quad (86)$$

$$\frac{\gamma_1}{\gamma_1 - 1} \frac{P_1}{\rho_1} + \frac{f^2}{2\rho_1^2} = \frac{\gamma_2}{\gamma_2 - 1} \frac{P_2}{\rho_2} + \frac{f^2}{2\rho_2^2} + \frac{\varepsilon_{\text{rad}}}{\rho_1} \quad (87)$$

$$\frac{f^2}{2\rho_1^2} - \frac{f^2}{2\rho_2^2} = \frac{\gamma_2}{\gamma_2 - 1} \frac{P_2}{\rho_2} - \frac{\gamma_1}{\gamma_1 - 1} \frac{P_1}{\rho_1} + \frac{\varepsilon_{\text{rad}}}{\rho_1} \quad (88)$$

$$\frac{1}{2} f^2 \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right) = \frac{\gamma_2}{\gamma_2 - 1} \frac{P_2}{\rho_2} - \frac{\gamma_1}{\gamma_1 - 1} \frac{P_1}{\rho_1} + \frac{\varepsilon_{\text{rad}}}{\rho_1} \quad (89)$$

Substituting back in our definition of f^2 ,

$$\frac{1}{2} \frac{P_2 - P_1}{\frac{1}{\rho_1} - \frac{1}{\rho_2}} \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right) = \frac{\gamma_2}{\gamma_2 - 1} \frac{P_2}{\rho_2} - \frac{\gamma_1}{\gamma_1 - 1} \frac{P_1}{\rho_1} + \frac{\varepsilon_{\text{rad}}}{\rho_1} \quad (90)$$

$$\frac{1}{2} (P_2 - P_1) \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) = \frac{\gamma_2}{\gamma_2 - 1} \frac{P_2}{\rho_2} - \frac{\gamma_1}{\gamma_1 - 1} \frac{P_1}{\rho_1} + \frac{\varepsilon_{\text{rad}}}{\rho_1} \quad (91)$$

Grouping ρ 's on each side,

$$\frac{P_2 - P_1}{2\rho_1} - \frac{\varepsilon_{\text{rad}}}{\rho_1} + \frac{\gamma_1}{\gamma_1 - 1} \frac{P_1}{\rho_1} = \frac{\gamma_2}{\gamma_2 - 1} \frac{P_2}{\rho_2} - \frac{P_2 - P_1}{2\rho_2} \quad (92)$$

$$\frac{1}{\rho_1} \left[\frac{P_2 - P_1}{2} - \varepsilon_{\text{rad}} + \frac{\gamma_1}{\gamma_1 - 1} P_1 \right] = \frac{1}{\rho_2} \left[\frac{\gamma_2}{\gamma_2 - 1} P_2 - \frac{P_2 - P_1}{2} \right] \quad (93)$$

Solving now for ρ_1/ρ_2 , the first Rankine-Hugoniot jump condition, gives

$$\frac{\rho_1}{\rho_2} = \frac{\frac{P_2 - P_1}{2} - \varepsilon_{\text{rad}} + \frac{\gamma_1}{\gamma_1 - 1} P_1}{\frac{\gamma_2}{\gamma_2 - 1} P_2 - \frac{P_2 - P_1}{2}} \quad (94)$$

$$\boxed{\frac{\rho_1}{\rho_2} = \frac{P_2 - P_1 - 2\varepsilon_{\text{rad}} + \frac{2\gamma_1}{\gamma_1 - 1} P_1}{\frac{2\gamma_2}{\gamma_2 - 1} P_2 - P_2 + P_1} = \frac{u_2}{u_1}} \quad (95)$$

Starting from Equation 91 and dividing by P_2 ,

$$\left(\frac{1}{2\rho_1} + \frac{1}{2\rho_2} \right) - \frac{P_1}{P_2} \left(\frac{1}{2\rho_1} + \frac{1}{2\rho_2} \right) = \frac{\gamma_2}{\gamma_2 - 1} \frac{1}{\rho_2} - \frac{\gamma_1}{\gamma_1 - 1} \frac{P_1}{P_2} \frac{1}{\rho_1} + \frac{\varepsilon_{\text{rad}}}{P_2 \rho_1} \quad (96)$$

Grouping and solving for P_1/P_2 ,

$$\frac{\gamma_1}{\gamma_1 - 1} \frac{P_1}{P_2} \frac{1}{\rho_1} - \frac{P_1}{P_2} \left(\frac{1}{2\rho_1} + \frac{1}{2\rho_2} \right) = \frac{\gamma_2}{\gamma_2 - 1} \frac{1}{\rho_2} + \frac{\varepsilon_{\text{rad}}}{P_2 \rho_1} - \left(\frac{1}{2\rho_1} + \frac{1}{2\rho_2} \right) \quad (97)$$

$$\frac{P_1}{P_2} \left[\frac{\gamma_1}{\gamma_1 - 1} \frac{1}{\rho_1} - \left(\frac{1}{2\rho_1} + \frac{1}{2\rho_2} \right) \right] = \frac{\gamma_2}{\gamma_2 - 1} \frac{1}{\rho_2} + \frac{\varepsilon_{\text{rad}}}{P_2 \rho_1} - \left(\frac{1}{2\rho_1} + \frac{1}{2\rho_2} \right) \quad (98)$$

$$\frac{P_1}{P_2} = \frac{\frac{\gamma_2}{\gamma_2 - 1} \frac{1}{\rho_2} + \frac{\varepsilon_{\text{rad}}}{P_2 \rho_1} - \left(\frac{1}{2\rho_1} + \frac{1}{2\rho_2} \right)}{\frac{\gamma_1}{\gamma_1 - 1} \frac{1}{\rho_1} - \left(\frac{1}{2\rho_1} + \frac{1}{2\rho_2} \right)} \quad (99)$$

Multiplying the right-hand side by $-2\rho_1\rho_2/-2\rho_1\rho_2$ gives the second Rankine-Hugoniot

$$\boxed{\frac{P_1}{P_2} = \frac{\rho_2 + \rho_1 - \frac{2\gamma_2}{\gamma_2 - 1} \rho_1 - \frac{2\varepsilon_{\text{rad}}\rho_2}{P_2}}{\rho_2 + \rho_1 - \frac{2\gamma_1}{\gamma_1 - 1} \rho_2}} \quad (100)$$

Assuming an ideal gas, the equation of state must follow

$$P = K\rho^\gamma = nkT = \frac{\rho}{\mu m_p} kT \quad (101)$$

Thus we can calculate the temperature jump condition as

$$\frac{P_1}{P_2} = \frac{\rho_1 \mu_2 m_p k T_1}{\rho_2 \mu_1 m_p k T_2} = \frac{1/2 \rho_1 T_1}{2 \rho_2 T_2} \implies \quad (102)$$

$$\boxed{\frac{T_1}{T_2} = \frac{4\rho_2 P_1}{\rho_1 P_2}} \quad (103)$$

- (b) **A supernova shock wave travels with speed $v_s = 10^3 \text{ km s}^{-1}$ into a purely molecular hydrogen cloud of density $N_{H_2} = 10^4 \text{ cm}^{-3}$ and $T_1 = 10 \text{ K}$. The molecules get completely dissociated by the shock and the resulting atoms get completely ionized. Given that the dissociation energy per molecule is $E_d = 4.5 \text{ eV}$ and the ionization energy per atom is $E_i = 13.6 \text{ eV}$, calculate the temperature T_2 and the electron density n_e behind the shock front.**

In order to calculate the density and temperature of the post-shock medium, we must convert the above equations into functions of only known variables (i.e. u_1, n_{HII}, T_1, E_d , and E_i). Starting with the temperature relation,

$$T_2 = T_1 \frac{\rho_1 P_2}{4\rho_2 P_1} = T_1 \frac{\rho_1}{4\rho_2} \frac{\frac{2\gamma_1}{\gamma_1 - 1} \rho_2 - \rho_2 - \rho_1}{\frac{2\gamma_2}{\gamma_2 - 1} \rho_1 - \frac{2\varepsilon_{\text{rad}}\rho_2}{P_2} - \rho_2 - \rho_1} \quad (104)$$

$$= T_1 \frac{\frac{2\gamma_1}{\gamma_1 - 1} \rho_1 \rho_2 - \rho_1 \rho_2 - \rho_1^2}{\frac{8\gamma_2}{\gamma_2 - 1} \rho_1 \rho_2 - \frac{8\varepsilon_{\text{rad}}\rho_2}{P_2} \rho_2 - 4\rho_2^2 - 4\rho_1 \rho_2} \quad (105)$$

As before, $\mu_2 = 1/2$ for fully ionized hydrogen, making the ideal gas equation of state for the post-shock region

$$P_2 = 2 \frac{\rho_2}{m_p} kT_2 \quad (106)$$

This then gives

$$T_2 = T_1 \frac{\frac{2\gamma_1}{\gamma_1-1} \rho_1 \rho_2 - \rho_1 \rho_2 - \rho_1^2}{\frac{8\gamma_2}{\gamma_2-1} \rho_1 \rho_2 - \frac{8\varepsilon_{\text{rad}} \rho_2}{2 \frac{\rho_2}{m_p} k T_2} \rho_2 - 4\rho_2^2 - 4\rho_1 \rho_2} \quad (107)$$

$$= T_1 \frac{\frac{2\gamma_1}{\gamma_1-1} \rho_1 \rho_2 - \rho_1 \rho_2 - \rho_1^2}{\frac{8\gamma_2}{\gamma_2-1} \rho_1 \rho_2 - \frac{4\varepsilon_{\text{rad}} m_p}{k T_2} \rho_2 - 4\rho_2^2 - 4\rho_1 \rho_2} \quad (108)$$

Creating a linear equation

$$\left(\frac{8\gamma_2}{\gamma_2-1} \rho_1 \rho_2 - \frac{4\varepsilon_{\text{rad}} m_p}{k T_2} \rho_2 - 4\rho_2^2 - 4\rho_1 \rho_2 \right) T_2 = T_1 \left(\frac{2\gamma_1}{\gamma_1-1} \rho_1 \rho_2 - \rho_1 \rho_2 - \rho_1^2 \right) \quad (109)$$

$$\frac{8\gamma_2}{\gamma_2-1} \rho_1 \rho_2 T_2 - \frac{4\varepsilon_{\text{rad}} m_p}{k} \rho_2 - 4\rho_2^2 T_2 - 4\rho_1 \rho_2 T_2 - \frac{2\gamma_1}{\gamma_1-1} \rho_1 \rho_2 T_1 - \rho_1 \rho_2 T_1 - \rho_1^2 T_1 = 0 \quad (110)$$

Because there are two unknowns in this equation, we must find another equation. Starting from the momentum conservation equation,

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 = \rho_2^2 u_2^2 \frac{1}{\rho_2} + P_2 \quad (111)$$

Combining this with the continuity equation,

$$\rho_1 u_1^2 + P_1 = \rho_1^2 u_1^2 \frac{1}{\rho_2} + P_2 \quad (112)$$

$$P_1 - P_2 = \frac{\rho_1^2}{\rho_2} u_1^2 - \rho_1 u_1^2 \quad (113)$$

Using the ideal gas equation of state

$$\frac{\rho_1}{2m_p} k T_1 - 2 \frac{\rho_2}{m_p} k T_2 = \frac{\rho_1^2}{\rho_2} u_1^2 - \rho_1 u_1^2 \quad (114)$$

$$\frac{\rho_1}{2m_p} k T_1 - 2 \frac{\rho_2}{m_p} k T_2 - \frac{\rho_1^2}{\rho_2} u_1^2 + \rho_1 u_1^2 = 0 \quad (115)$$

This gives a system of equations

$$\frac{\rho_1}{2m_p} k T_1 - 2 \frac{\rho_2}{m_p} k T_2 - \frac{\rho_1^2}{\rho_2} u_1^2 + \rho_1 u_1^2 = 0 \quad (116)$$

$$\frac{8\gamma_2}{\gamma_2-1} \rho_1 \rho_2 T_2 - \frac{4\varepsilon_{\text{rad}} m_p}{k} \rho_2 - 4\rho_2^2 T_2 - 4\rho_1 \rho_2 T_2 - \frac{2\gamma_1}{\gamma_1-1} \rho_1 \rho_2 T_1 - \rho_1 \rho_2 T_1 - \rho_1^2 T_1 = 0 \quad (117)$$

No analytical solution exists, but we can plug these into a numerical solver (i.e. mathematica thanks to Ningyuan) to get

$$\rho_2 = 4.01624 \rho_1 \quad (118)$$

$$T_2 = 1.133 \times 10^7 \text{ K} \quad (119)$$

Solving for $n_e = \frac{\rho_2}{m_p}$ gives

$$\boxed{n_e = 8.032 \times 10^4 \frac{1}{\text{cm}^3}} \quad (120)$$

$$\boxed{T_2 = 1.133 \times 10^7 \text{ K}} \quad (121)$$