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Physics for Astronomy (ASTR 589)

Homework 2

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1. **Consider a 1D atmosphere that is heated from below such that the temperature decreases from the bottom toward the surface. Assume that all extinction coefficients are angle-independent (i.e., no magnetic fields, anistropic scattering, etc). Using the formal solution, find out whether limb darkening, limb brightening, or both can occur in this atmosphere.**

   Hints:

   - **Express limb darkening/brightening as a condition on $dI_\nu/d\mu$.**
   - **An intermediate step you reach will probably be a condition on some derivative of the source function.**
   - **After that intermediate step, consider the case with no scattering. The general case with absorption plus scattering is more difficult but you may want to tackle it as a bonus.**

Limb darkening is defined as decreasing intensity with increasing view angle θ. Thus,

$$\frac{dI_\nu}{d\theta} < 0. \tag{1}$$

Since we know that $\mu = \cos\theta$, $d\theta = -\sin\theta$

$$\frac{dI_\nu}{d\theta} = -\frac{dI_\nu}{d\mu}\sin\theta < 0. \tag{2}$$

Because the viewing angle on the atmosphere is always from the top, we know that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and then that $\sin\theta > 0$. This then means that for limb darkening we must have that

$$\frac{dI_\nu}{d\mu} > 0. \tag{3}$$

The formal solution to the radiative transfer equation is

$$I_\nu(\tau_\nu, \mu) = I_\nu(\tau_0, \mu)e^{-(\tau_0-\tau_\nu)/\mu} + \frac{1}{\mu}\int_{\tau_0}^{\tau_\nu} S_\nu(\tau_\nu')e^{-(\tau_\nu'-\tau_\nu)/\mu}\, d\tau_\nu'. \tag{4}$$

Assuming that there is no incoming flux at the top of the atmosphere, $I_\nu(\tau_0 = 0) = 0$. Thus,

$$I_\nu(\tau_\nu, \mu) = \frac{1}{\mu}\int_{\tau_0}^{\tau_\nu} S_\nu(\tau_\nu')e^{-(\tau_\nu'-\tau_\nu)/\mu}\, d\tau_\nu'. \tag{5}$$

Because we are looking at the top of the atmosphere, the intensity at the top of the atmosphere must be integrated from the bottom of the atmosphere, $\tau_{\max}$ to the top of the atmosphere, $\tau_0 = 0$. Thus the formal solution becomes

$$I_\nu(0,\mu) = \frac{1}{\mu}\int_0^{\tau_{\max}} S_\nu(\tau'_\nu)e^{-\tau'_\nu/\mu}\,d\tau'_\nu. \tag{6}$$

To determine if the atmosphere has limb darkening or brightening, taking the derivative of the previous equation gives

$$\begin{aligned}
\frac{dI_\nu(0,\mu)}{d\mu} &= \frac{d}{d\mu}\left[\frac{1}{\mu}\int_0^{\tau_{\max}} S_\nu(\tau'_\nu)e^{-\tau'_\nu/\mu}\,d\tau'_\nu\right] && (7)\\
&= \frac{d}{d\mu}\left(\frac{1}{\mu}\right)\int_0^{\tau_{\max}} S_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu + \frac{1}{\mu}\frac{d}{d\mu}\left(\int_0^{\tau_{\max}} S_\nu(\tau'_\nu)e^{-\tau'_\nu/\mu}\,d\tau'_\nu\right) && (8)\\
&= \frac{-1}{\mu^2}\int_0^{\tau_{\max}} S_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu + \frac{1}{\mu}\int_0^{\tau_{\max}} \frac{d}{d\mu}\left(S_\nu(\tau'_\nu)e^{-\tau'_\nu/\mu}\right)\,d\tau'_\nu && (9)\\
&= \frac{-1}{\mu^2}\int_0^{\tau_{\max}} S_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu + \frac{1}{\mu}\int_0^{\tau_{\max}} \left(\frac{dS_\nu}{d\mu}e^{-\tau'_\nu/\mu} + S_\nu e^{-\tau'_\nu/\mu}\frac{d}{d\mu}\left(\frac{-\tau'_\nu}{\mu}\right)\right)\,d\tau'_\nu && (10)\\
&= \frac{-1}{\mu^2}\int_0^{\tau_{\max}} S_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu + \frac{1}{\mu}\int_0^{\tau_{\max}} \left(\frac{dS_\nu}{d\mu}e^{-\tau'_\nu/\mu} + S_\nu e^{-\tau'_\nu/\mu}\frac{-\tau'_\nu}{-\mu^2}\right)\,d\tau'_\nu && (11)\\
&= \frac{-1}{\mu^2}\int_0^{\tau_{\max}} S_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu + \frac{1}{\mu}\int_0^{\tau_{\max}} \frac{dS_\nu}{d\mu}e^{-\tau'_\nu/\mu} + \frac{1}{\mu}\int_0^{\tau_{\max}} S_\nu e^{-\tau'_\nu/\mu}\frac{\tau'_\nu}{\mu^2}\,d\tau'_\nu && (12)\\
&= \frac{-1}{\mu^2}\left[\int_0^{\tau_{\max}} S_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu - \mu\int_0^{\tau_{\max}} \frac{dS_\nu}{d\mu}e^{-\tau'_\nu/\mu} - \int_0^{\tau_{\max}} S_\nu e^{-\tau'_\nu/\mu}\frac{\tau'_\nu}{\mu}\,d\tau'_\nu\right] && (13)
\end{aligned}$$

If we assume that the atmosphere is in LTE meaning no angle dependent scattering, the source function is simply the blackbody function B_ν, which is not dependent on angle. This means that $dS_\nu/d\mu = dB_\nu/d\mu = 0$ and

$$\frac{dI_\nu(0,\mu)}{d\mu} = \frac{-1}{\mu^2}\left[\int_0^{\tau_{\max}} B_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu - \int_0^{\tau_{\max}} B_\nu e^{-\tau'_\nu/\mu}\frac{\tau'_\nu}{\mu}\,d\tau'_\nu\right] \tag{14}$$

The second integral can be broken up using integration by parts. We choose

$$\begin{aligned}
U = \frac{\tau'_\nu}{\mu}B_\nu, \quad dU = \frac{1}{\mu}S_\nu d\tau'_\nu + \frac{\tau'_\nu}{\mu}\frac{dS_\nu}{d\tau'_\nu}d\tau'_\nu\\
dV = e^{-\tau'_\nu/\mu}d\tau'_\nu, \quad V = -\mu e^{-\tau'_\nu/\mu}
\end{aligned} \tag{15}$$

We can then rewrite the derivative of the formal solution as

$$\begin{aligned}
\frac{dI_\nu(0,\mu)}{d\mu} &= \frac{-1}{\mu^2}\left[\int_0^{\tau_{\max}} B_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu - \left(\left(\frac{\tau'_\nu}{\mu}B_\nu(-\mu)e^{-\tau'_\nu/\mu}\right|_0^{\tau_{\max}} - \int_0^{\tau_{\max}}\left((-\mu)e^{-\tau'_\nu/\mu}\frac{1}{\mu}B_\nu d\tau'_\nu + \frac{\tau'_\nu}{\mu}\frac{dB_\nu}{d\tau'_\nu}\right)d\tau'_\nu\right)\right] && (16)\\
&= \frac{-1}{\mu^2}\left[\int_0^{\tau_{\max}} B_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu + \left(\tau'_\nu B_\nu e^{-\tau'_\nu/\mu}\right|_0^{\tau_{\max}} - \int_0^{\tau_{\max}}\left(e^{-\tau'_\nu/\mu}B_\nu d\tau'_\nu + \frac{\tau'_\nu}{\mu}\frac{dB_\nu}{d\tau'_\nu}\right)d\tau'_\nu\right) && (17)\\
&= \frac{-1}{\mu^2}\left[\int_0^{\tau_{\max}} B_\nu e^{-\tau'_\nu/\mu}\,d\tau'_\nu + \tau_{\max}B_\nu e^{-\tau_{\max}/\mu} - \int_0^{\tau_{\max}} e^{-\tau'_\nu/\mu}B_\nu d\tau'_\nu - \int_0^{\tau_{\max}} e^{-\tau'_\nu/\mu}\frac{\tau'_\nu}{\mu}\frac{dB_\nu}{d\tau'_\nu}d\tau'_\nu\right) && (18)\\
&= \frac{-1}{\mu^2}\left[\tau_{\max}B_\nu e^{-\tau_{\max}/\mu} - \int_0^{\tau_{\max}} e^{-\tau'_\nu/\mu}\frac{\tau'_\nu}{\mu}\frac{dB_\nu}{d\tau'_\nu}d\tau'_\nu\right] && (19)\\
&= \frac{-1}{\mu^2}\tau_{\max}B_\nu e^{-\tau_{\max}/\mu} + \frac{1}{\mu^3}\int_0^{\tau_{\max}} e^{-\tau'_\nu/\mu}\tau'_\nu\frac{dB_\nu}{d\tau'_\nu}d\tau'_\nu && (20)
\end{aligned}$$

Because the temperature depends on optical depth in the atmosphere, $dB_\nu/d\tau'_\nu$ cannot be taken out of the integral. Instead, we can break up the derivative.

$$\frac{dB_\nu}{d\tau'_\nu} = \frac{dB_\nu}{dT}\frac{dT}{d\tau'_\nu} \tag{21}$$

This makes the derivative

$$\frac{dI_\nu(0,\mu)}{d\mu} = \frac{-1}{\mu^2}\tau_{\max}B_\nu e^{-\tau_{\max}/\mu} + \frac{1}{\mu^3}\int_0^{\tau_{\max}} e^{-\tau'_\nu/\mu}\tau'_\nu\frac{dB_\nu}{dT}\frac{dT}{d\tau'_\nu}d\tau'_\nu \tag{22}$$

The right hand side of the above equation can be either positive or negative. The optical depth of the atmosphere, $\tau_{\max}$ is always a positive value, as is the source function B_ν and the scattering term $e^{-\tau_{\max}/\mu}$. Thus, the first term is always negative. Similarly, we are given that $\frac{dT}{d\tau'_\nu}$ is positive, as temperature decreases with height. Finally, the derivative of the Planck blackbody function with respect to temperature is always positive, making the second term of the equation positive. Depending on the exact strength/scale of the derivative of the temperature, $\frac{dT}{d\tau'_\nu}$, the intensity derivative could be either positive or negative, giving both limb darkening and limb brightening. However, taking the limit of the intensity derivative as the optical depth $\tau_{\max}$ goes to both 0 and infinity,

$$\lim_{\tau_{\max}\to 0}\frac{dI_\nu(0,\mu)}{d\mu} = 0 \tag{23}$$

$$\lim_{\tau_{\max}\to\infty}\frac{dI_\nu(0,\mu)}{d\mu} = \frac{1}{\mu^3}\int_0^{\infty} e^{-\tau'_\nu/\mu}\tau'_\nu\frac{dB_\nu}{dT}\frac{dT}{d\tau'_\nu}d\tau'_\nu > 0 \tag{24}$$

As expected, when the optical depth becomes zero there is neither limb darkening or brightening. In heigh optical depth atmospheres, such as a stellar atmosphere, the limit gives $\boxed{\text{limb darkening}}$. Thus, only in specific scenarios where the derivative of the temperature with respect to optical depth is small is $\boxed{\text{limb brightening}}$ possible, meaning limb darkening is much more likely in this atmosphere.

2. **According to the standard model of the sun, the central density is 153 g cm^{-3} and the Rosseland mean opacity at the center is 2.17 cm^2 g^{-1}.**

   (a) **Calculate the mean-free-path of a photon at the center of the sun.**

   The mean free path of a photon is calculated as

$$\ell = \frac{1}{\chi_\nu} = \frac{1}{n\sigma} = \frac{1}{\kappa_R \rho} = \frac{1}{2.17 \text{ g cm}^{-3} \cdot 153 \text{ cm}^2\text{g}^{-1}} = \boxed{0.003012 \text{ cm} = \ell} \tag{25}$$

   (b) **If this mean-free-path remained constant for the photon's journey to the surface, calculate the average time it would take for the photon to escape the sun.**

   The total average number of interactions a photon undergoes before exiting a medium is approximately

$$N = \left(\frac{l}{\ell}\right)^2 \tag{26}$$

   This means the total distance traveled is simply $N\ell$. The time required to travel this distance is then $N\ell/c$. Thus the total time required for a photon to escape the sun is approximately

$$\begin{aligned}\tau_{\text{diff}} &= \frac{N\ell}{c} = \frac{l^2\ell}{\ell^2 c} = \frac{l^2}{\ell c} = \frac{R_\odot^2}{\ell c} \\ &= \frac{(69.634 \times 10^9 \text{ cm})^2}{0.003012 \text{ cm} \cdot 3 \times 10^{10} \text{ cm s}^{-1}} = 5.370 \times 10^{13} \text{ s} = \boxed{1702816 \text{ yr} = 1.703 \text{ Myr} = \tau_{\text{diff}}}\end{aligned} \tag{27}$$

3. **Consider a pure hydrogen nebula surrounding a hot star of radius R. At some distance r from the star, the ionization equilibrium equation**

$$n_{H^0} \int_{v_0}^{\infty} \frac{4\pi J_\nu}{h\nu} \sigma_\nu \, d\nu = n_p n_e \alpha(T)$$

becomes

$$\frac{n_{H^0} R^2}{r^2} \int_{v_0}^{\infty} \frac{\pi F_\nu(R)}{h\nu} \sigma_\nu e^{-\tau_\nu} \, d\nu = n_p n_e \alpha(T),$$

where the optical depth is

$$\tau_\nu(r) = \int_0^r n_{H^0}(r') \sigma_\nu \, dr',$$

and the ionization cross section is approximately given by

$$\sigma_\nu = 6.3 \times 10^{-18} \left(\frac{\nu_0}{\nu}\right)^3 \text{ cm}^2,$$

where ν_0 corresponds to the ionization threshold for hydrogen at 912 Å. We will assume that the star emits like a blackbody. Define the ionization fraction x such that $n_{H^+} = xN_H$, $n_{H^0} = (1-x)n_H$.

(a) **Integrate numerically Equation 2 to plot the ionization fraction as a function of the distance from the star in parsecs. Repeat the exercise for these two cases: $T_{\text{eff}} = 45,000$ K, $R/R_\odot = 11$, and $T_{\text{eff}} = 40,000$ K, $R/R_\odot = 20$.**

Assume that $T_e = 10,000$ K in the nebular gas so that $\alpha = 2.59 \times 10^{-13}$ cm^3 s^{-1}, and $n_H = 10$ cm^{-3} throughout the nebula. You may assume that $\sigma_\nu = 6.3 \times 10^{-18}$ cm^2 in the calculation of the optical depth.

Inside of the nebula, the total number of electrons and protons is equal to the total number of ionized hydrogen, $n_p = n_e = n_{H^+} = xn_H$. Thus we have that

$$\frac{(1-x) n_H R^2}{r^2} \int_{v_0}^{\infty} \frac{\pi F_\nu(R)}{h\nu} \sigma_\nu e^{-\tau_\nu} \, d\nu = x n_H x n_H \alpha(T) = x^2 n_H^2 \alpha(T) \tag{28}$$

Rewriting, we get that

$$1 = \frac{(1-x) n_H R^2}{x^2 n_H^2 \alpha(T) r^2} \int_{v_0}^{\infty} \frac{\pi F_\nu(R)}{h\nu} \sigma_\nu e^{-\tau_\nu} \, d\nu \tag{29}$$

Similarly, the optical depth becomes

$$\tau_\nu(r) = \int_0^r n_{H^0}(r') \sigma_\nu \, dr' = \int_0^r (1-x) n_H \sigma_\nu \, dr' \tag{30}$$

Assuming that both the cross section and the ionization fraction are constant where $\sigma_\nu = \sigma_0$, we then have that

$$\tau_\nu(r) = (1-x)n_H\sigma_0 \int_0^r dr' = (1-x)n_H\sigma_0 r \tag{31}$$

The balance equation then becomes

$$\begin{aligned} 1 = \frac{(1-x)R^2}{x^2 n_H \alpha(T) r^2} \int_{v_0}^{\infty} \frac{\pi F_\nu(R)}{h\nu} \sigma_\nu e^{-(1-x)n_H\sigma_0 r}\, d\nu &= \frac{(1-x)R^2\sigma_0 e^{-(1-x)n_H\sigma_0 r}}{x^2 n_H \alpha(T) r^2} \int_{v_0}^{\infty} \frac{\pi F_\nu(R)}{h\nu} \left(\frac{\nu_0}{\nu}\right)^3 d\nu \\ &= \frac{(1-x)R^2\sigma_0\nu_0^3 e^{-(1-x)n_H\sigma_0 r}}{x^2 n_H \alpha(T) r^2} \int_{v_0}^{\infty} \frac{\pi F_\nu(R)}{h\nu^4}\, d\nu \end{aligned} \tag{32}$$

We know that for an isotropic source, the total flux through a boundary, assuming no incoming radiation, is simply $F_\nu = \pi I_\nu$. Thus we have that

$$1 = \frac{(1-x)R^2\sigma_0\nu_0^3 e^{-(1-x)n_H\pi^2\sigma_0 r}}{x^2 n_H \alpha(T) r^2} \int_{v_0}^{\infty} \frac{\pi^2 I_\nu(R)}{h\nu^4}\, d\nu = \frac{(1-x)R^2\sigma_0\nu_0^3\pi^2 e^{-(1-x)n_H\pi^2\sigma_0 r}}{x^2 n_H \alpha(T) r^2} \int_{v_0}^{\infty} \frac{\pi^2 I_\nu(R)}{h\nu^4}\, d\nu \tag{33}$$

Additionally, assuming the star is isotropic, $I_\nu = B_\nu$

$$\begin{aligned} 1 &= \frac{(1-x)R^2\sigma_0\nu_0^3\pi^2 e^{-(1-x)n_H\sigma_0 r}}{x^2 n_H \alpha(T) r^2} \int_{v_0}^{\infty} \frac{B_\nu(R)}{h\nu^4}\, d\nu \\ &= \frac{(1-x)R^2\sigma_0\nu_0^3\pi^2 e^{-(1-x)n_H\sigma_0 r}}{x^2 n_H \alpha(T) r^2} \int_{v_0}^{\infty} \frac{1}{h\nu^4} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}\, d\nu \\ &= \frac{2(1-x)R^2\sigma_0\nu_0^3\pi^2 e^{-(1-x)n_H\sigma_0 r}}{x^2 n_H \alpha(T) r^2 c^2} \int_{v_0}^{\infty} \frac{1}{\nu} \frac{1}{e^{h\nu/k_B T} - 1}\, d\nu \end{aligned} \tag{34}$$

Defining $u = h\nu/k_B T$, $du = h/k_B T\ du$, $\nu = k_B T u/h$, and $u_0 = h\nu_0/k_B T$, we can then say that

$$\begin{aligned} 1 &= \frac{2(1-x)R^2\sigma_0\nu_0^3\pi^2 e^{-(1-x)n_H\sigma_0 r}}{x^2 n_H \alpha(T) r^2 c^2} \frac{k_B T}{h} \int_{\nu_0}^{\infty} \left(\frac{h}{k_B T u}\right) \frac{1}{e^{h\nu/k_B T} - 1}\ \frac{h}{k_B T} d\nu \\ &= \frac{2(1-x)R^2\sigma_0\nu_0^3\pi^2 e^{-(1-x)n_H\sigma_0 r}}{x^2 n_H \alpha(T) r^2 c^2} \int_{u_0}^{\infty} \frac{1}{u} \frac{1}{e^u - 1}\, du \end{aligned} \tag{35}$$

Rewriting again into a form solvable numerically,

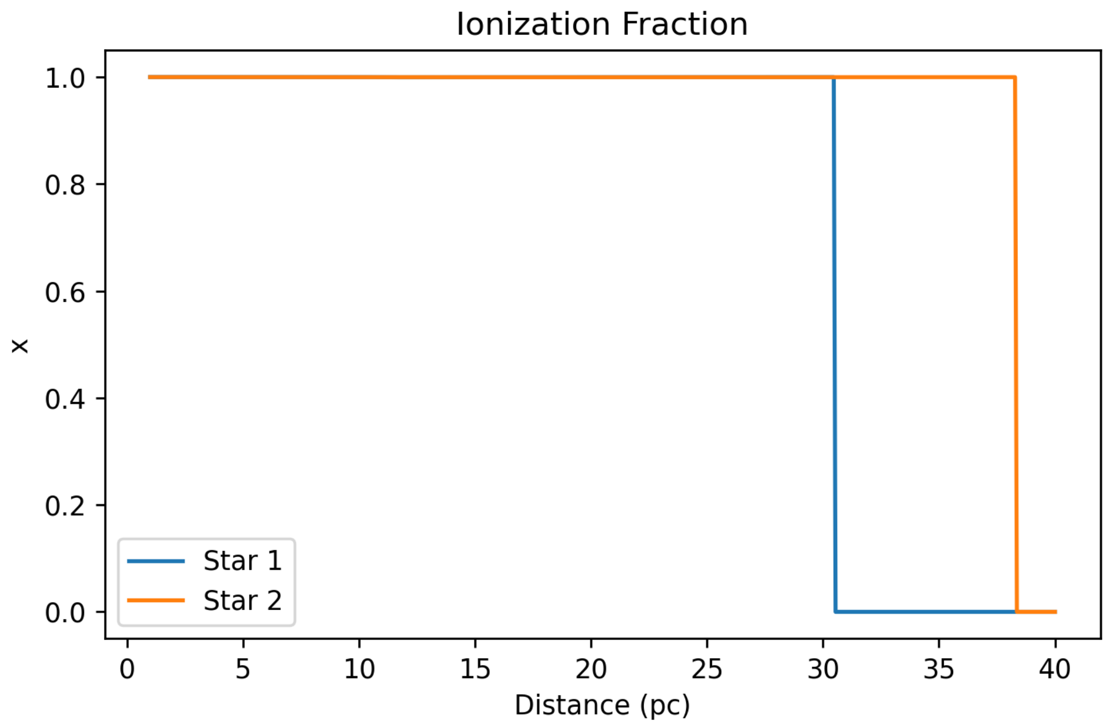
$$0 = \frac{2(1-x)R^2\sigma_0\nu_0^3\pi^2 e^{-(1-x)n_H\sigma_0 r}}{x^2 n_H \alpha(T) r^2 c^2} \int_{u_0}^{\infty} \frac{1}{u} \frac{1}{e^u - 1}\, du - 1 \implies \tag{36}$$

$$0 = \frac{2(1-x)R^2\sigma_0\nu_0^3\pi^2 e^{-(1-x)n_H\sigma_0 r}}{n_H \alpha(T) r^2 c^2} \int_{u_0}^{\infty} \frac{1}{u} \frac{1}{e^u - 1}\, du - x^2 \tag{37}$$

Similarly, we can write the derivative of this function to assist in numerical solving as

$$0 = \left[-e^{-(1-x)n_H\sigma_0 r} + (1-x)n_H\sigma_0 r e^{-(1-x)n_H\sigma_0 r}\right] \frac{2R^2\sigma_0\nu_0^3\pi^2}{n_H \alpha(T) r^2 c^2} \int_{u_0}^{\infty} \frac{1}{u} \frac{1}{e^u - 1}\, du - 2x \tag{38}$$

Doing this integration gives us



We can then extract the numerically calculated Stromgren radii, which for star one gives us

$$\boxed{R_s = 30.4649 \text{ pc}} \tag{39}$$

and for star two gives us

$$\boxed{R_s = 38.2806 \text{ pc}}. \tag{40}$$

From this we can see that the assumption that the ionization fraction drops off like a step function for the analytical solution for the Stromgren radius is an accurate assumption to make. Clearly the difference in the values comes from one of the other assumptions.

(b) **Calculate the number of ionizing photons for the two black bodies using**

$$Q(H^0) = \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu}\, d\nu.$$

With the same assumptions from before, this integral becomes

$$Q(H^0) = S_{uv} = \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu}\, d\nu = \int_{\nu_0}^{\infty} \frac{4\pi R^2 F_\nu}{h\nu}\, d\nu = 4\pi^2 R^2 \int_{\nu_0}^{\infty} \frac{I_\nu}{h\nu}\, d\nu = 4\pi^2 R^2 \int_{\nu_0}^{\infty} \frac{B_\nu}{h\nu}\, d\nu$$
$$= 4\pi^2 R^2 \int_{\nu_0}^{\infty} \frac{1}{h\nu} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}\, d\nu \tag{41}$$

This inner integral is the same as before, giving

$$S_{uv} = 4\pi^2 R^2 \int_{\nu_0}^{\infty} \frac{1}{h\nu} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}\, d\nu = \frac{8\pi^2 R^2}{c^2} \left(\frac{k_B T}{h}\right)^3 \int_{v_0}^{\infty} \frac{u^2}{e^u - 1}\, du \tag{42}$$

Numerically solving this integral as before, for the first star

$$\boxed{S_{uv} = 2.7443 \times 10^{49}\ s^{-1}} \tag{43}$$

And for the second star

$$\boxed{S_{uv} = 4.8905 \times 10^{49}\ \mathrm{s}^{-1}} \tag{44}$$

(c) **Calculate the Stromgren radius of both stars using $Q(H^0)$ from Part b.**

The analytical equation for the Stromgren radius is given by

$$R_s = \left(\frac{3S_{uv}}{4\pi\alpha(T)n_H^2}\right)^{1/3} \tag{45}$$

Thus plugging in the results for the first star,

$$\boxed{R_s = 20.4957\ \mathrm{pc}} \tag{46}$$

For the second star,

$$\boxed{R_s = 24.8488\ \mathrm{pc}} \tag{47}$$