So at. $R \gg R_{in}$

$$\boxed{U_r \simeq -\frac{3}{2}\frac{\nu}{R}} \quad (5)$$

So indeed $|U_r| \nearrow$ as $|\nu| \nearrow$

Lecture 18

Now, energy eqn

We had written

$$\rho T \frac{DS}{dt} = \frac{\sigma_{ij}\sigma^{ij}}{2\eta}$$

For thin disk, we need to include radiative cooling, because if the visc heat is not radiated the disk will puff up.

In this case

$$\underbrace{\Sigma T U_r \frac{dS}{dR}}_{\text{entropy advected}} = \underbrace{\nu \Sigma \left(R\frac{d\Omega}{dR}\right)^2}_{\substack{\text{Entropy generated}\\ \text{by viscosity}}} - F_c$$

cooling

~~radiative~~ radiative flux

If nothing is advected inwards and all energy is radiated away locally, then

$$F_c = \nu \Sigma \left(R \frac{d\Omega}{dR} \right)^2 \quad \xrightarrow{(4)}$$

$$F_c = \frac{\dot{M}}{3\pi} \left[1 - \left(\frac{R_{in}}{R} \right)^{\frac{1}{2}} \right] \left[-\frac{3}{2} R \left(\frac{GM}{R^3} \right)^{\frac{1}{2}} \right]^2$$

$$\Rightarrow \boxed{F_c = \frac{3GM\dot{M}}{4\pi R^3} \left[1 - \left(\frac{R_{in}}{R} \right)^{\frac{1}{2}} \right]} \quad (6)$$

This is the most important result in the theory of thin disks because

(I) Flux radiated away from each annulus is independent of viscosity!

Assuming black body emission we can define an effective temp.

$F_c = \sigma T_{eff}^4$ so (6) $\Rightarrow$

$$\boxed{T_{eff} = \left(\frac{3GM}{4\pi\sigma} \right)^{\frac{1}{2}} \dot{M}^{\frac{1}{4}} \left[1 - \left(\frac{R_{in}}{R} \right)^{\frac{1}{2}} \right]^{\frac{1}{4}} R^{-3/4}} \quad (7)$$

(II) In steady state, at $R \gg R_{in}$ the energy radiated is larger than the grav. potential energy released locally by a factor of 3!

~~(6) $R \gg R_{in}$~~

$$\cancel{\left(F_c \cdot 4\pi R^2\right) = 3\left(\frac{GM\dot{M}}{R}\right)}$$

Consider an annulus of width ΔR at $R \gg R_{in}$, and assume an amount of mass Δm, drifts from $R+\Delta R$ to R. The potential E released is

$$\Delta U = \Delta m\left[-\frac{GM}{R+\Delta R} + \frac{GM}{R}\right] =$$

$$= \Delta m\,\frac{GM}{R}\left[1 - \frac{1}{1+\frac{\Delta R}{R}}\right] =$$

$$= \frac{GM\Delta m}{R}\left(\cancel{1} - \cancel{1} - \frac{\Delta R}{R}\right) =$$

$$= -\frac{GM}{R^2}\,\Delta m \cdot \Delta R$$

If matter passes at rate $\dot{M}$,
then $\Delta m = \dot{M} \cdot \Delta t$

So, energy release rate is

$$\left|\frac{\Delta u}{\Delta t}\right| = \frac{GM\dot{M}}{R^2} \cdot \Delta R$$

Assuming that this energy is released locally, the flux of radiation would be

$$2 \cdot F \cdot 2\pi R \cdot \Delta R = \left|\frac{\Delta u}{\Delta t}\right| = \frac{GM\dot{M}}{R^2} \Delta R$$

$$\Rightarrow F = \frac{GM\dot{M}}{4\pi R^3}$$

Comparing to Eq. 6, we see that this is 3 times smaller.

Why? The inner B.C. plays an important role.

The total E released throughout the disk is

$$L = \int_{R_{in}}^{\infty} F_c \, 2\pi R \, dR = \frac{3}{2} G M \dot{M} \int_{R_{in}}^{\infty} R^{-2} \left[1 - \left(\frac{R_{in}}{R}\right)^{\frac{1}{2}}\right] dR$$

$$\Rightarrow \boxed{L = \frac{GM}{2R_{in}} \dot{M}}$$

So, half of the potential E released in the disk is radiated

If the central object is a star the other half is radiated as accreted material hits the surface or in a boundary layer.

If it's a BH it's advected into the BH

Let's define the radiative efficiency as

$$L = \epsilon \dot{M} c^2$$

So, $\varepsilon = \frac{1}{2}\frac{GM}{R_{in}c^2}$

The radiative efficiency in this disks depends on the inner edge of the accretion flow!!!

For a non-spinning BH

$$R_{in} = R_{isco} = \frac{6GM}{c^2} \Rightarrow \varepsilon = \frac{1}{12} \simeq 8.33\%$$

[$\varepsilon \simeq 6\%$ with GR effects]

For prograde accretion onto a maximally spinning BH

$$R_{in} = R_{isco} = \frac{GM}{c^2} \Rightarrow \varepsilon = 50\%$$

[$\varepsilon \simeq 42\%$ with GR

~~For a retrograde disk~~

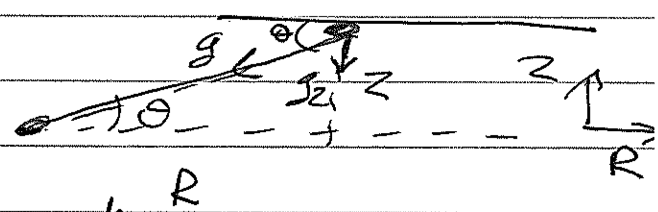
$$\cancel{R_{in} = R_{isco} = \frac{9GM}{c^2} \Rightarrow \frac{1}{18} \Rightarrow \varepsilon = 5.55\%}$$

Radiative efficiency in ADAF is very different because energy is <u>not</u> radiated away locally and the advection term dominates

Vertical structure of disk

Hydrostatic equilib.

$$\frac{1}{\rho}\frac{dP}{dz} = g_z$$



$$g_z = g\sin\theta = -\frac{GM}{R^2+z^2}\frac{z}{R} \overset{h/R \ll 1}{=} -\frac{GM}{R^2}\frac{z}{R}$$

$$\simeq \frac{GM}{R^3}z$$

so $$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GM}{R^3}z$$

Assume an isothermal disk

$$P = \frac{\rho}{m}kT$$

$$\frac{kT}{m}\frac{d\rho}{\rho\, dz} = -\frac{GM}{R^3}z \Rightarrow d\ln\rho = -\frac{GM}{R^3 c_s^2}z\,dz$$

c_s^2

$$\Rightarrow \rho = \rho_c e^{-\left(\frac{z}{2H}\right)^2} \quad \text{where}$$

$$\cancel{H^2 = \frac{R^3 c_s^2}{GM}} \qquad \boxed{H \equiv \frac{c_s}{\Omega}}$$

Scale height

We have so far assumed radiation forces are negligible

When is radiation important?

$$\frac{1}{2}\frac{\chi F_c}{c} \simeq \rho g_z \overset{R \gg R_{in}}{\Longrightarrow}$$

$$\frac{1}{2}\frac{3GM\dot{M}}{4\pi R^3}\frac{\chi}{c} \simeq \rho\frac{GM}{R^3}H$$

If dominant cross section is e^- scattering (as usually is the case in the inner parts of the accretions disks)

$$\left.\begin{aligned} \chi &= n_e\sigma_T \\ \rho &= m_p n_p \simeq m_p n_e \end{aligned}\right\} \quad \chi = \frac{\rho}{m_p}\sigma_T$$

$$\frac{1}{2}\frac{3\cancel{GM}\dot{M}}{4\pi \cancel{R^3}}\frac{\cancel{\rho}\sigma_T}{m_p c} = \cancel{\rho}\frac{\cancel{GM}}{\cancel{R^3}}H \Rightarrow$$

$$\dot{M}_{crit} = \frac{8\pi m_p c}{3\sigma_T}H$$

Total luminosity is $L = \varepsilon \cdot \dot{M} c^2$

so

$$L_{crit} = \frac{GM}{2R_{in}} \dot{M}_{crit} \Rightarrow$$

$$L_{crit} = \underbrace{\frac{4\pi G M m_p \cdot c}{\sigma_T}}_{L_{edd}} \underbrace{\left(\frac{H}{3R_{in}}\right)}_{\text{geometric factor}}$$

Eddington luminosity for spherical emission

Thus, the theory of thin accretion disks will break down as

$$L \simeq L_{edd} \Rightarrow \dot{M} \sim \dot{M}_{crit}$$

What about the temperature of thin accretion disks?

$$T_{eff} = \left(\frac{3GM}{4\pi\sigma}\right)^{1/4} \dot{M}^{1/4} \left[1 - \left(\frac{R_{in}}{R}\right)^{1/2}\right]^{1/4} R^{-3/4}$$

Max. temp.

$$\left.\frac{dT_{eff}}{dR}\right| = 0 \Rightarrow \boxed{R_{max} = 1.36\, R_{in}}$$

$$T_{eff}^{max} = \left(\frac{3GM}{4\pi\sigma}\right)^{1/4} \dot{M}^{1/4} \left[1 - 1.36^{-\frac{1}{2}}\right]^{\frac{1}{4}} (1.36)^{-3/4}$$
$$R_{in}^{-3/4}$$

$$\Rightarrow T_{eff}^{max} \sim M^{1/4} \dot{M}^{1/4} R_{in}^{-3/4}$$

for BH $R_{in} \sim M$ $\Rightarrow$

$$T_{eff}^{max} \sim M^{-\frac{1}{2}} \dot{M}^{1/4}$$

If we require $\dot{M} \le \dot{M}_{edd} = \frac{8\pi m_p c}{3\sigma_T}\left(\frac{H}{R_{in}}\right) R_{in}$

$$= \frac{8\pi m_p c}{3\sigma_T}\left(\frac{c_s}{v_{\varphi,in}}\right) R_{in}$$

$$\sim M$$

thus $T_{eff}^{max} \sim M^{-\frac{1}{2}} \cdot M^{\frac{1}{4}} \sim M^{-\frac{1}{4}}$

i.e. larger BH mass, the smaller max T in the disk