The Trouble with Heterogeneity: A Guide for Models in Macroeconomics and Finance

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Abstract

Heterogeneity is increasingly recognised as a central force shaping financial markets, yet existing models often rely on overly simplified two- or three-agent setups that fail to capture the diversity and granularity of modern economies. This paper examines the limitations of such coarse models and contrasts them with mean-field games (MFGs), commonly referred to as HANK, and large N-player games as alternative frameworks. While MFGs excel at representing rich distributions of heterogeneity, their reliance on exogenous aggregate risk makes them ill-suited to study the determinants of risk which is a drawback for macro-finance models. By contrast, granular N-player games (Gabaix 2011) allow idiosyncratic shocks to propagate into aggregate outcomes, providing microfoundations for aggregate risk. Without endogenising aggregate risk, a model cannot asses why aggregate shocks take the distribution they do, nor how they change when policies change. However, solving these models poses severe computational challenges due to market clearing and coupled policy functions across many agents. I evaluate recent advances in deep learning, showing that naïve Physics-informed neural networks (PINNs), as frequently proposed in the economics literature, struggle with diffusion control occurring in e.g. portfolio choice problems. I advocate for an actor-critic approach as a more robust alternative. Finally, I highlight the persistent difficulties of solving MFGs with aggregate risk and the limitations of perturbation and approximation methods in macro-financial contexts. By mapping the strengths and weaknesses of these approaches, I provide guidance for future research on heterogeneous agent models in finance.

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1 Introduction

Recent research highlights heterogeneity as a fundamental force shaping financial markets (e.g. Brunnermeier et al. (2021)). Three main sources are typically emphasised: preferences, beliefs, and labour income risk. Each generates distinct mechanisms that enrich the understanding of asset prices, volatility, and the term structure of interest rates, which are areas where representative agent models often have shortcomings.

Heterogeneity in *preferences* arises from differences in risk aversion and inter-temporal substitution. Because shocks redistribute wealth across investors, the market-dominant "marginal" agent changes over time. This shifting influence generates countercyclical variation in aggregate risk aversion, time-varying risk premia, and realistic term structure dynamics. Moreover, when only some households participate in equity markets, preference heterogeneity magnifies the equity premium by concentrating aggregate risk among a subset of investors. Overall, preference-based heterogeneity explains how endogenous changes in who prices risk drive volatility and long-run return dynamics.

Heterogeneity in *beliefs* emphasises disagreement about fundamentals, long-run risks, or future states of the economy. Because asset prices reflect the wealth-weighted views of investors, even modest differences in beliefs can lead to substantial fluctuations in valuations and premia. Disagreement generates both speculation and reallocation effects as wealth shifts toward optimistic or pessimistic agents, amplifying volatility and making premia countercyclical. In this way, belief heterogeneity explains phenomena such as excess volatility, return predictability, and high trading volumes in financial markets. Heterogeneity also arises from differences in income and endowment risk, often modelled using a continuum of households.

Heterogeneity in *labour income risk* due to idiosyncratic income shocks and borrowing constraints creates strong precautionary saving motives, which depress equilibrium risk-free rates. Beyond average consumption dynamics, the cross-sectional distribution of household risk, especially higher-order moments like skewness, becomes an important state variable for asset pricing. Moreover, shifts in income dispersion, participation costs, or borrowing limits affect not only the distribution of wealth but also interest rates, equity premia, and market participation.

For models in finance, heterogeneity is typically modelled with a two- or three-agent model. These models feature settings such as borrowers vs. savers (Huggett 1993), unconstrained vs. hand-to-mouth consumers (Galí et al. 2007), optimists vs. pessimists (Branger et al. 2019) or rational expectations vs. biased beliefs (Krishnamurthy & Li 2025).

However, these few-agent models are a highly coarse and likely inadequate representation of heterogeneity. While such setups can generate intuition, real-world economies are far more complex: income and wealth are distributed across a fat-tailed spectrum rather than concentrated in just two groups. Financial markets similarly comprise a diverse ecology of actors such as mutual funds, hedge funds, pension funds, insurance companies, broker-dealers, and banks of various sizes instead of simply a "leveraged intermediary" and a "household".

Diversification further highlights the limitations of few-agent models. Duarte et al. (2024) show that an economy with many Lucas trees (Lucas orchard) behaves very differently from one with only one or two. In the orchard, diversification reduces the sensitivity of interest rates to dividend yields and smooths out idiosyncratic volatility, whereas models with only a few assets overstate aggregate fluctuations. Capturing this transition from concentrated to diversified structures requires many assets, not just two or three.

In short, while few-agent models remain computationally tractable, they offer mainly stylised insights. Large N-player games, by contrast, preserve the diversity and granularity of modern economies, enabling the study of how heterogeneity, concentration, and diversification interact to generate aggregate fluctuations and systemic risk.

Over the past decade, the limit of a continuum of heterogeneous agents has been widely used in the field of macroeconomics (Auclert 2025). These models are called mean-field games (MFGs) as in Lasry & Lions (2007). In macroeconomics, these models are many times coined as HANK (Heterogeneous Agent New-Keynesian)[1]. In the mean-field limit ($N \to \infty$) agents are infinitesimal and cannot influence aggregates outcomes individually, whereas in finite-agent N-player games each agent has non-negligible mass and individual actions move prices and aggregates. This necessitates that only the evolving distribution of types must be tracked as opposed to each individual agent.

At first glance, MFGs may appear as the superior choice over N-player games as using (uncountably) more agents implies that a richer heterogeneity can be modelled. While this is true, N-player games have advantages over MFGs and both approaches excel in different questions.

MFGs excel when the objective is to capture the full distribution of heterogeneity in a tractable way. Because each agent is infinitesimal, one can model continuous variation in beliefs, income, or preferences. This makes MFGs particularly well-suited for studying inequality, redistribution, and long-run distributional dynamics. A notable example is Coimbra & Rey (2023) who analyse a macro-finance model with a continuum of financial intermediaries each featuring their own VaR constraint. They find that time-varying endogenous macroeconomic risk arises from the risk-shifting behaviour of the cross-sectional distribution of financial intermediaries.

Yet, MFGs have a structural limitation when applied to finance. Because each agent is infinitesimal, all idiosyncratic shocks wash out in the aggregate. This means that aggregate quantities such as prices, returns and volatility become deterministic unless one imposes aggregate shocks from the outside. In other words, MFGs cannot generate aggregate risk endogenously; they can only transmit it once it has been assumed.

This is especially problematic for finance, where a central goal is precisely to understand the origins of what makes asset returns risky or what drives time variation in risk premia? A framework that requires risk to be injected exogenously cannot answer these questions; it can only take risk as given.

Worse, this limitation runs against one of the primary motivations for moving beyond representative-agent models in the first place. Heterogeneity is induced to uncover how differences in beliefs, preferences, and exposures to idiosyncratic shocks generate volatility and premia at the aggregate level. But in MFGs, the very mechanism by which micro-level shocks accumulate into macro-level risk is shut down. As a result, MFGs are powerful for studying redistribution, inequality, and long-run structural dynamics, but they are ill-suited for questions at the core of asset pricing and financial stability.

By contrast, N-player games can provide a transmission mechanism from idiosyncratic to aggregate risk. In this case, shocks to large entities, such as dominant firms, major banks,

[1] MFGs are not exclusive to economics, see e.g. Bertucci et al. (2024).

or systemic intermediaries can propagate to the macroeconomy, giving rise to granular origins (Gabaix 2011) of aggregate fluctuations.[2] With a granular structure, large agents remain small enough to avoid representative-agent dynamics, yet large enough to transmit idiosyncratic shocks into aggregates. Gabaix (2011) shows that one third of U.S. GDP fluctuations can be traced to the 100 largest firms, with even stronger effects in less diversified economies. In MFGs agents are too fine to exhibit this behaviuor, whereas in two- or three-agent models the agents are too coarse. Thus, granularity is in the middle of these two frameworks, capturing the desirable properties of both: Due to many agents being present, a rich heterogeneity structure can be depicted and with granularity the idiosyncratic risk these individual agents face make aggregate quantities risky. Therefore, granular models provide a microfoundation of aggregate risk which allows finance to study the determinants of aggregate risk.

While idiosyncratic risk in a two- or three-agent model naturally translates into aggregate risk as well, the problem is that in models with very few agents, who by design must have a huge market share, there is no idiosyncratic risk as e.g. a shock to the representative agent representing the whole banking sector is a shock to the whole banking sector which is an aggregate shock. In the narrower sense, idiosyncratic risk cannot exist in few-agent models.

Granularity is therefore essential for studying systemic risk and crises. A coarse two-agent model may capture the idea that "banks matter", but it cannot differentiate the systemic importance of Lehman Brothers from that of a regional bank, or capture how sectoral concentration amplifies shocks. Large N-player games, in contrast, can represent the size distribution of intermediaries, allowing some to carry systemic weight while others remain small. This richer structure allows researchers to study both inequality in exposures and the system-wide consequences of large failures, which are central to understanding financial crises. By construction, MFGs cannot do this, since every agent is infinitesimal with no systematic impact and its fragility is irrelevant to the financial system.

For macroecomics, granular origins matter even if e.g. TFP variance in business cycle models is small, because the structure of firm concentration itself determines the distribution of macro (TFP) shocks and therefore the dynamics of the business cycle. Endogenising aggregate risk matters because stabilisation policy (monetary, macroprudential, or fiscal) interacts differently with aggregate shocks of granular origin than with "manna-from-heaven" TFP shocks. Without modelling granular origins, a business-cycle model cannot speak to why aggregate shocks take the distribution they do, nor how they change when the size distribution of firms/intermediaries changes.

Unfortunately, granularity marks a big conundrum for heterogeneous agent models in finance. Market clearing and coupled policy functions across many agents induce both the curse of dimensionality and severe numerical challenges. This explains why two- or three-agent models dominate the literature in finance.

Recently, researchers have highlighted the merits of deep learning (e.g. Fernández-Villaverde (2025), Fernández-Villaverde et al. (2024), Gu et al. (2024), Azinovic et al. (2022) or Maliar et al. (2021)). I show however that the Physics-informed neural networks (PINNs) as in Raissi et al. (2019), which the authors argue for, are not as straightforward and a naïve approach that simply approximates the value function with a neural network is inadequate in finance due to the role of diffusion control (portfolio choice). Unlike drift-control set-

[2] Very loosely speaking, out of the conventional models, OLG models best fit granular structures.

tings, portfolio choice requires an accurate approximation of the value function's Hessian, not just its gradients. Since the cited literature predominantly considers discrete time macroeconomic models without diffusion control, these issues are not discussed.

To highlight this point, I show that the naïve PINN is not able to adequately solve the Merton (1969) portfolio choice problem for small wealth levels, where sufficient non-linearities are present, and also struggles with large wealth levels where the value function is flat. To solve this problem, I resort to a trial solution, which is based on a separation approach, so that I exogenously fix the shape of the PINN in accordance to $U(a)$, where a is wealth. In this case, the PINN can adequately solve the portfolio choice problem. However, this is not a panacea as it is unclear how to extend these trial solutions to higher dimensions where an agent has more state variables than just his wealth or even multiple sources of wealth.

On top of that, naïve PINNs are highly susceptible to shape violations. If the gradient with respect to wealth is not strictly positive, consumption, which is derived from that gradient, becomes ill-defined, causing training to collapse due to evaluation errors. I show that this precludes mini-batching, since the network must repeatedly "re-learn" the shape of the value function for each new mini-batch. Since dynamic sampling from the ergodic set of the state variables is required to reduce the curse of dimensionality, this is a fatal drawback.

Therefore, I present an actor-critic approach as in Duarte et al. (2024), which offers a more robust alternative by separately approximating value and policy functions. This circumvents the shape violation constraints.

However, even beyond deep learning, the coupling of policy functions across many N players remains a formidable obstacle[3]. One possible path is to introduce heuristics, consistent with the idea that real-world institutions do not fully internalise the reactions of every other agent. This approach, suggested by Moll (2025), decouples the policy functions. However, it makes market clearing harder to enforce, since the coupling of policy functions determines the N-th player's action given the other $N-1$ players' actions via market clearing.

Given the difficulties of actually solving N-player games with granular origins, MFGs remain a practical fallback. Even though they do not exhibit granular origins, they still exhibit rich distributional dynamics which can be used to analyse research questions in finance as demonstrated by Coimbra & Rey (2023).

However, solving MFGs in finance is once again challenging since they must require aggregate risk in order for a risky asset to exist. Solving MFGs with aggregate risk is far from trivial. Even though the policy functions of the uncountably many agents are not coupled, each agent must know the distribution of agents and its law of motion in order to infer prices. For example, if an economy is populated with many borrowers, then an agent can infer that the interest rate will be higher than in the case if the economy is populated by many lenders. This gives rise to the Master Equation, which is also called the "Monster Equation" (Moll 2025) due to causing an extreme version of the curse of dimensionality.

A common simplification is the Krusell-Smith algorithm, which tracks aggregates using a guessed law of motion depending only on aggregates and not the full distribution of agents. However, its accuracy depends on functional-form assumptions with little guidance

[3] For example, Duarte et al. (2024) do not consider market clearing.

for complex settings. Neural networks promise flexibility (Fernández-Villaverde et al. 2023), but runtime and input-selection issues hinder their practicality. Finite-dimensional distributional approximations remove guesswork but introduce trade-offs in accuracy and tractability (Gu et al. 2024). A sequence-space approach (Azinovic & Žemlička 2025) has no guarantee of working either.

For this reason, in macroeconomics HANK models are mostly solved using perturbation methods. However, perturbation faces sharp limitations in finance. First-order perturbations imply certainty equivalence, eliminating risk premia; higher-order perturbations can restore them, but they still remain local solutions around a solution with no aggregate risk and are also considerably more difficult to solve in the context of heterogeneous agent models. Therefore, perturbation is not a viable option for finance as it requires both that agents internalise risk in their policy function and in general global solutions for e.g. jump- or disaster risk or any other high non-linearities.

By identifying these challenges, I aim to guide future research towards building more realistic models encompassing granular origins and working on the solution techniques as was done in the 80's and 90's with perturbation solutions of DSGE models. On top of that, my accompanying textbook, joint with Christian Schlag, seeks to provide a clear and systematic treatment of MFGs and their pitfalls, filling a gap in the current literature.

I proceed as follows: section 2 highlights in detail how heterogeneity has implications in finance and reviews the literature in more detail, section 3 is a shortened and edited excerpt from the textbook explaining how to solve MFGs with aggregate risk and section 4 details why solving heterogeneous agent models beyond two- or three-agent models in finance is difficult.

2 Heterogeneity in Finance

There are three main sources of heterogeneity studied in the literature: preferences, beliefs and (labour) income risk. Naturally, these elements needn't be studied in isolation and it is possible to study any combination of them. Preference differences highlight how marginal investors shape risk pricing. Belief differences show how disagreement fuels volatility and trading activity. Income risk with a continuum of agents reveals how distributional forces and inequality feed back into asset markets. In the following, I present a more detailed review of this literature.

2.1 Preferences

A common theme for this source of heterogeneity is that investors have different risk preferences or elasticity of inter-temporal substitution. For example, Weinbaum (2009) shows that heterogeneity in investors' risk preferences has major implications for financial markets. Even though complete markets allow for risk sharing, i.e. the formation of a representative agent, shocks to wealth redistribute risk between more and less risk-averse investors, causing aggregate risk aversion to change over time. This dynamic shift drives financial phenomena such as time-varying and persistent volatility (ARCH effects), leverage effects, implied volatility skews in options, return predictability, and large hedging demands. The main mechanism is as follows: when wealth shocks occur, less risk-averse investors, who hold more risky assets, are hit harder in downturns and benefit more in upturns. This shifts wealth (and thus influence) toward more risk-averse agents after bad shocks and toward less risk-averse ones after good shocks. As a result, aggregate risk aversion covaries with stock returns. This state-dependent variation in risk aversion can generate observed volatility patterns.

Schneider (2022) develops a heterogeneous-investor general equilibrium model to explain the U.S. real and nominal yield curves. The key insight is that differences in investors' elasticities of inter-temporal substitution (EIS), in addition to risk aversion, drive the dynamics of real and nominal rates. Heterogeneity produces a positive, time-varying term premium, an upward-sloping average term structure, and realistic volatility in long-term yields. These are outcomes that representative agent models struggle to match. Importantly, real shocks dominate in shaping both real and nominal yield curves, while inflation plays only a secondary role. The main mechanism is as follows: when a negative aggregate shock hits, leveraged, less risk-averse investors lose wealth, shifting market influence to more risk-averse, low-EIS investors. These marginal investors strongly smooth consumption over time (low EIS), which raises real interest rates, while their high risk aversion increases the price of risk. Together, this creates a higher term premium and an upward-sloping real and nominal yield curve. In contrast to representative agent models, the dynamics of borrowing, lending, and shifting wealth across heterogeneous agents endogenously generate persistent, non-linear real rates and realistic term structure properties.

Gârleanu & Panageas (2015) use an overlapping-generations (OLG) framework and examine how heterogeneity in risk aversion and the EIS affect asset prices. The key finding is that separating these two types of preference heterogeneity is crucial for understanding asset-pricing dynamics. The combination of different investor preferences and the OLG structure means that individual consumption growth becomes persistent even when aggregate consumption growth is not. This persistence, combined with recursive preferences, makes investors demand a higher compensation for risk. The model shows that if one were to try and infer the risk aversion of a representative agent, it would not be a simple average of the individual investors' risk aversions; it could be significantly higher than any single investor's. This effect, in turn, helps to explain variations in the equity premium and interest rates, and it improves the model's ability to match real-world asset-pricing data.

Guvenen (2009) studies asset pricing in a two-agent model in which all agents can invest into a riskless bond, but only some households participate in the stock market. Agents differ in their elasticity of inter-temporal substitution (EIS). This heterogeneity helps explain key financial puzzles such as the equity premium puzzle (Mehra & Prescott 1985). The reason is that non-stockholders (low EIS) rely heavily on the bond market to smooth their volatile labour income. Since aggregate shocks cannot be eliminated, this bond trading reallocates risk to stockholders (high EIS), making their consumption more volatile. As a result, stockholders require a large risk premium to hold equity.

2.2 Beliefs

Pohl et al. (2021) show that heterogeneity in beliefs about long-run risks has major implications for asset prices in consumption-based models with Epstein-Zin preferences. Introducing even small belief differences leads to time-varying consumption and wealth shares across agents. These fluctuations generate endogenous variation in risk premia and asset valuations, which helps explain key puzzles such as the high volatility of the price-dividend ratio, the countercyclical and time-varying equity premium, and the predictability of returns, especially their stronger predictability in recessions. Agents disagree about the persistence of long-run consumption growth shocks. Under Epstein-Zin preferences, this disagreement creates both a speculation motive (betting on different states) and a risk-sharing motive (agents with lower persistence beliefs insure those with higher persistence beliefs). The latter is crucial: during recessions, wealth shifts toward the pessimistic agents who demand higher risk premia, making risk premia countercyclical.

Ehling et al. (2017) develop an overlapping generations model in which investors form expectations about risk premia based on their own lifetime experiences rather than the full historical record. Such "learning-from-experience" biases generate systematic belief heterogeneity across cohorts: young agents extrapolate recent returns and act as trend chasers, while older, more experienced agents become contrarians. The cross-section of beliefs implies that asset prices are driven not by the true fundamentals but by the wealth-weighted "market view" of expected growth. This perspective provides an equilibrium explanation for key empirical regularities—such as the negative correlation between survey-based expectations and future returns, the extrapolative behaviour of the average investor, and the smoother but biased nature of consensus forecasts relative to the true risk premium. When a positive shock occurs, all agents revise expectations upward, but wealth reallocations amplify this effect because optimistic investors gain wealth and thus exert greater influence on the market view. This "overreaction" drives interest rates higher and risk premia lower. Young agents, who update aggressively, mistakenly perceive a high risk premium and increase risky investment precisely when the true premium is low, leading to slower wealth accumulation. In contrast, older agents act as contrarians, absorbing trades from the young. The resulting interplay across generations explains both persistent disagreement in beliefs and the countercyclical nature of risk premia. Unlike representative agent models or disagreement models with fixed types, this framework generates endogenous life-cycle transitions from extrapolators to contrarians

Xiong & Yan (2009) develop a dynamic equilibrium model of bond markets in which investors hold heterogeneous expectations about future economic conditions. Disagreement leads agents to trade against each other, producing endogenous wealth reallocations that, in turn, amplify bond yield volatility and generate time-varying bond premia. Even modest belief dispersion is sufficient to explain several longstanding puzzles, including the excess volatility of bond yields, the failure of the expectations hypothesis, and the ability of a tent-shaped linear combination of forward rates to predict bond returns. Two groups of investors rely on different learning models to estimate the unobservable inflation target, which drives future nominal short rates. Their disagreement creates speculative trading positions, and equilibrium bond prices reflect the wealth-weighted average belief about future short rates. Positive shocks favour optimistic investors, shifting wealth toward them and magnifying the effect of their beliefs on bond prices. This feedback loop generates excess volatility in bond yields and explains the failure of the expectations hypothesis: when the wealth-weighted belief is above the econometrician's belief, long-term yields and spreads are elevated, but long-term bond prices subsequently rise, producing falling yields.

Buraschi & Jiltsov (2006) develop a general equilibrium model of option pricing in which heterogeneous agents face model uncertainty and hold different beliefs about expected returns. Market incompleteness makes options non-redundant, while disagreement generates equilibrium trading volume and affects option-implied volatility. Using survey-based measures of belief dispersion, the authors show that their model matches key option market phenomena such as high option trading volumes, the volatility smile. Agents form heterogeneous posterior beliefs about dividend growth, leading optimists to demand out-of-the-money (OTM) calls and pessimists to demand OTM puts. These belief-driven hedging demands generate an endogenous volatility smile (OTM puts are more expensive than OTM calls) even when fundamental dividend volatility is constant. Differences in beliefs also make the stock's equilibrium volatility stochastic, since the wealth-weighted balance of optimistic vs. pessimistic agents changes over time. Importantly, the model links belief dispersion to option volume: as disagreement widens, option open interest rises, with a one-standard-deviation increase in their "Difference in Beliefs Index" raising option open interest by 20%. Belief heterogeneity also explains why option-implied volatility can exceed realised volatility.

2.3 Labour Income Risk/Continuum of Agents

Here, unlike in the previous two sections, I want to explicitly focus on the impacts of a *continuum* of heterogeneous agents. This can be achieved easily by incorporating labour income risk. In recent years in macroeconomics, HANK (Heterogeneous-Agent New Keynesian) models featuring a continuum of agents have become widely popular. Auclert (2025) reviews why this is the case and why there has been a resurgence of the early heterogeneous agent models from the 90's such as Krusell & Smith (1998).

Firstly, I want to emphasise the role of budget constraints in these models. A seminal paper for this is Huggett (1993) who addresses the risk-free rate puzzle which standard representative agent models failed to resolve. He introduces agents facing uninsurable idiosyncratic income shocks, who smooth consumption by holding a risk-free asset subject to borrowing constraints. In this incomplete markets setup, equilibrium risk-free rates are much lower than in representative agent benchmarks. The key mechanism is precautionary savings: agents save aggressively to buffer against adverse income realisations, as borrowing is constrained to one year's income. This excess savings drive down the equilibrium interest rate, discouraging further accumulation. Without borrowing constraints, agents could smooth consumption by relying on future income, weakening precautionary motives. Constraints amplify the effect by forcing self-reliance, leading to heightened savings and further lowering rates to clear the credit market. Thus, the combination of precautionary savings and borrowing limits explains the depressed equilibrium risk-free rate.

Secondly, I want to address the most obvious impact of a continuum of agents, namely their rich distribution. Constantinides & Ghosh (2017) study how heterogeneous household consumption risk, in particular the countercyclical left skewness in the distribution of household consumption growth, affects asset prices and helps explain longstanding puzzles in finance. Household consumption shocks are negatively skewed, persistent, and countercyclical. In downturns, households are more likely to face large negative shocks to consumption (e.g., job displacement). The risk-free rate and price-dividend ratio are pro-cyclical whereas the equity premium, expected market return, and its volatility are countercyclical. Importantly, household consumption risk explains the cross-section of stock returns (size, value, industry portfolios), performing as well as the Fama–French three-factor model. The main mechanism is as follows: Households face uninsurable idiosyncratic labour income shocks. These shocks translate into idiosyncratic consumption risk that does not wash out in equilibrium. The paper identifies the third central moment (skewness) of household consumption growth as the crucial state variable. During recessions, skewness becomes more negative (greater downside risk for households). With Epstein–Zin recursive preferences, households care about the timing of risk resolution. Negative, persistent, countercyclical skewness raises the aggregate marginal rate of substitution. This increases the price of risk, lowering risk-free rates and raising expected equity returns in bad times. Standard representative agent models fail to explain puzzles like the high equity premium or low risk-free rate. Importantly, a continuum of households is advantageous as the model can describe the entire distribution of household consumption growth (mean, variance, skewness, etc.). The third central moment (skewness) of this distribution is the key state variable driving asset prices. A two or three agent model would find it hard to realistically generate and track higher-order moments like skewness in the cross-section.

3 Modelling a Continuum of Agents

In this section I explain how to build a model that features a continuum of agents. This section is an edited and shortened excerpt from my textbook joint with Christian Schlag which thoroughly explains how MFGs work using concrete examples.

3.1 Idiosyncratic Risk

We will illustrate the incorporation of a continuum of heterogeneous agents step-by-step using the Standard Incomplete Markets Model, which is a slight extension of Huggett (1993) and serves as an import building block of many heterogeneous agent models featuring (labour) income risk.

Standard Incomplete Markets Model

- There is a continuum of households indexed by $h \in [0, 1]$ and each household has point mass zero.
- A household derives utility from consuming $c_t > 0$.
- In each period, the household receives income on its asset holdings $a_t \geq a_{min}$ with $\underline{a} < 0$.
- Households can lend and borrow the asset a_t to and from each other.
- The asset is in zero net supply and the constant interest rate r will ensure this by enforcing market clearing.
- Moreover, the household receives exogenous income $z_t > 0$. Let $log(z_t)$ follow an AR(1).
- Thus, each household h solves the following optimisation problem

$$
\begin{aligned}
V_0(z^{(h)}, a^{(h)}) = \max_{\{c_t^{(h)}>0\}_{t=0}^{\infty}} \; & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_t^{(h)}) \right] \\
\text{s.t.} \quad & a_{t+1}^{(h)} = (1+r)a_t^{(h)} + z_t^{(h)} - c_t^{(h)}, \\
& log(z_t^{(h)}) = \rho log(z_{t-1}^{(h)}) + \sigma w_t^{(h)}, \\
& w_t^{(h)} \overset{i.i.d}{\sim} N(0,1), \\
& a_t^{(h)} \geq \underline{a} \text{ with } \underline{a} < 0, \\
& z_0^{(h)}, a_0^{(h)} \text{ given and identical } \forall h, \\
& \text{Market Clearing.}
\end{aligned}
$$

One might wonder how heterogeneity arises in spite of households exhibiting identical preferences and sharing identical initial conditions (*ex ante* homogeneity). The idiosyncratic (income) shock $w_t^{(h)}$, experienced individually by each household, is the driver of household heterogeneity. Unlike aggregate shocks (absent in this model but discussed in the next section), $w_t^{(h)}$ is independently drawn for each household, leading to different trajectories of income $z_t^{(h)}$ across households.

For instance, a household that experiences multiple favourable income shocks $w_t^{(h)}$ will enjoy higher income $z_t^{(h)}$ and accumulate more assets $a_t^{(h)}$ over time. This accumulation leads to greater income from assets, given by $(1+r)a_t^{(h)}$, thereby widening the inequality gap relative to a less fortunate household that experiences unfavourable income shocks $w_t^{(h)}$.

Therefore, although households start with identical initial endowments and are fully characterised by the realisation of their state variables, they become heterogeneous *ex post*. This heterogeneity arises because the trajectory of income differs across households, which, in turn, affects the trajectory of asset accumulation, leading to the realisation of state variables differing over time across households.

Notice that there is no *aggregate risk*. In other words, there is no shock that affects all households simultaneously since the income shocks are on the individual household level.

Solving for the policy functions $c(z,a)$ and $a'(z,a)$ works as in any dynamic programming problem. In other words, as each household faces the same optimisation problem, they will exhibit the same policy function. Therefore, the consumption choice of household h is obtained by simply evaluating $c^{(h)} = c\big(z^{(h)}, a^{(h)}\big)$ and likewise for a'. In short, despite a continuum of heterogeneous agent, we do not need to solve an infinite number of Dynamic Programming problems (an infeasible task). Instead, we only need to solve one dynamic programming problem.

The difference to a representative agent model is that we need to keep track of the distribution of households. Without it, we can't compute any aggregate quantities. The distribution is defined on the state variables as each household is fully characterised by the realisation of its state variables. In the textbook we use discrete state space methods so that the value and policy function are approximated on $\mathcal{S} = \mathcal{G}_z \times \mathcal{G}_a$, where $\mathcal{G}_z$ are the gridpoints for the exogenous state and $\mathcal{G}_a$ for the endogenous state variable. For the sake of illustration, we will immediately define the distribution on this discretised state space $\mathcal{S}$ featuring point mass at all gridpoints. A general definition of the distribution can be found in Kirkby (2018).

Definition 3.1 (Distribution)**.** *For $(z,a) \in \mathcal{G}_z \times \mathcal{G}_a$, the distribution of households is denoted by $D_t(z,a)$ and defined as*

$$D_t(z,a) \equiv \mathbb{P}(z_t = z, a_t = a) \quad \textit{such that} \sum_{z\in\mathcal{G}_z}\sum_{a\in\mathcal{G}_a} D_t(z,a) = 1 \; \forall t \in \mathbb{N}.$$

Thus, $D_t(z_i, a_j)$ captures the mass of households from the unit interval that have income $z_i \in \mathcal{G}_z$ and asset level $a_j \in \mathcal{G}_a$ at time t.

Given $D_t(z,a)$, it is straightforward to compute aggregate quantities. For example, aggregate consumption $\mathcal{C}_t$ and aggregate asset supply $\mathcal{A}_t$ are computed by

$$\mathcal{C}_t = \sum_{z\in\mathcal{G}_z}\sum_{a\in\mathcal{G}_a} c(z,a)D_t(z,a) \qquad \text{and} \qquad \mathcal{A}_t = \sum_{z\in\mathcal{G}_z}\sum_{a\in\mathcal{G}_a} a'(z,a)D_t(z,a).$$

In contrast, a representative agent model assumes the entire mass of households is concentrated at a single point in the state space, i.e. a point mass of one at a single gridpoint and zero elsewhere. As a result, the consumption of the representative household is identical to aggregate consumption.

Using this, we can properly define the market clearing condition

Definition 3.2 (Market Clearing)**.** *The market clearing interest rate r^* is such that the following holds*

$$\mathcal{A}_t(r^*) = 0 \ \forall t.$$

Market clearing plays a pivotal role in heterogeneous agent models because it links the different agents together. This is the case because under market clearing the decisions of a group of households influence other households via their effects on factor prices (here: the interest rate r). Moreover, in this model the market clearing condition captures a key feature of heterogeneous agent models, namely that agents can lend to and borrow from one another, a dynamic that is absent in models with a single representative agent. Because the interest rate is determined by asset market clearing, and because this condition depends on the distribution D_t, the distribution D_t itself becomes a state variable. That is, households must know the distribution D_t in order to deduce the interest rate. Without the knowledge of D_t, no household can determine their optimal policy. For instance, if the distribution D_t displays a large mass of wealthy households, agents can infer a low interest rate, since many wealthy households are trying to lend.

From this mechanism MFGs derive their name. Households do not directly interact with one another, their interaction occurs indirectly via the market clearing condition and the distribution D_t (the "mean field"), which determine the equilibrium interest rate. In other words, household h_1 does not need to explicitly consider the policy decisions of another household h_2 because household h_2's actions are negligible in their impact on the aggregate asset supply $\mathcal{A}_t$. Consequently, the influence of household h_2 on the equilibrium interest rate r^* can safely be ignored by household h_1. What is significant to h_1, however, is the behaviour of the entire distribution of households, which collectively determines the equilibrium outcomes and in this case specifically the market clearing interest rate r^*.

In contrast, N-player games feature *explicit coupling* of policy functions: the policy of any household h generally depends on the policy functions of all other $N-1$ households. Each agent carries weight (point mass), so if some household $h_2 \neq h_1$ chooses to borrow heavily, this directly affects the interest rate—and household h_1 must internalise this. Solving such coupled systems can be complex, as illustrated by the Cournot oligopoly model in microeconomics[4].

Since the distribution is a state variable, we must know its Law of Motion. This is given by the following[5]

Proposition 3.1 (Law of Motion Distribution)**.** *Let $(z, a) \in \mathcal{G}_z \times \mathcal{G}_a$ and $(z', a') \in \mathcal{G}_z \times \mathcal{G}_a$. Moreover, z follows a finite-state Markov chain. Then, the distribution D satisfies the following Law of Motion*

$$D'(z', a') = \sum_{z \in \mathcal{G}_z} \sum_{a \in \mathcal{G}_a} D(z, a) \cdot \Pi_{z,z'} \cdot \chi\big(a'(z, a, D) = a'\big), \tag{3.1}$$

[4] This interdependence can imply that an N-player game is considerably more difficult to solve than its mean-field limit.

[5] In the textbook we derive this in detail.

where $\chi(\cdot)$ denotes the indicator function, $a'(z,a,D)$ the endogenous state's policy function and $\Pi_{z,z'}$ the transition probability from z to z'.

The Bellman Equation, which in this instance is also called the *Master Equation*, is given by the following

$$V(z,a,D) = \max_{c\in\Gamma(z,a,D)} \left\{ U\big(c(z,a,D)\big) + \beta\,\mathbb{E}_z\left[V\Big(z',a',D'\Big)\right]\right\} \tag{3.2}$$

$$\text{s.t.} \qquad \begin{aligned} a' &= [1+r^*\big(D\big)]a + z - c(z,a,D), \\ \mathcal{A}(r^*) &= \sum_{z\in\mathcal{G}_z}\sum_{a\in\mathcal{G}_a} a'(z,a,D)D(z,a) = 0, \\ D'(z',a') &= \sum_{z\in\mathcal{G}_z}\sum_{a\in\mathcal{G}_a} D(z,a)\cdot\Pi_{z,z'}\cdot\chi\big(a'(z,a,D)=a'\big). \end{aligned}$$

Notice the coupling, i.e. the value function depends on the distribution D and the distribution's Law of Motion depends on the policy function from the value function. The above problem is very hard to solve because we would have to solve it for all possible distribution functions D.

However, under the absence of aggregate risk we can restrict our attention to stationary equilibria and drastically simplify (3.2). In stationary equilibria, the distribution does not change over time and can be treated as a constant. In doing so, we decouple the value function from the distribution function. The stationary distribution is such that $D = D'$ while satisfying (3.1).[6]

We summarise the stationary equilibrium in the following definition.

Definition 3.3 (Stationary Equilibrium)**.** *Given parameter β, asset grid $\mathcal{G}_a$, income grid $\mathcal{G}_z$ with corresponding Markov Transition Matrix $\Pi_{z,z'}$, a Law of Motion $L(a,c,z)$ for the endogenous state and a set of feasible actions $\Gamma(z,a)$, the stationary equilibrium consists of the following.*

1. *A stationary value function $V(z,a)$ satisfying*

$$V(z,a) = \max_{c\in\Gamma(z,a)} \left\{U\big(c(z,a)\big) + \beta\sum_{z'\in\mathcal{G}_z}\Pi_{z,z'}V\Big(z',L\big(a,c(z,a),z\big)\Big)\right\}$$

   for all $(z,a)\in\mathcal{G}_z\times\mathcal{G}_a$.

2. *Policy functions $c(z,a)$ and $a'(z,a) = L\big(a,c(z,a),z\big)$.*

3. *A stationary distribution $D(z,a)$ satisfying*

$$D(z',a') = \sum_{z\in\mathcal{G}_z}\sum_{a\in\mathcal{G}_a} D(z,a)\cdot\Pi_{z,z'}\cdot\chi\Big(L\big(a,c(z,a),z\big)=a'\Big)$$

   for all $(z',a')\in\mathcal{G}_z\times\mathcal{G}_a$.

4. *Aggregate asset supply $\mathcal{A}(r)$ given by*[7]

$$\mathcal{A}(r) = \sum_{z\in\mathcal{G}_z}\sum_{a\in\mathcal{G}_a} a'(z,a)D(z,a).$$

[6] In the textbook we present an algorithm to determine the stationary distribution.
[7] By stationarity, one can equivalently use $\mathcal{A}(r) = \sum_{a\in\mathcal{G}_a}\sum_{z\in\mathcal{G}_z} aD(z,a)$.

5. A market clearing interest rate r^* such that

$$\mathcal{A}(r^*) = 0.$$

While the value function still depends on the distribution, since the distribution determines the market clearing interest rate r^*, which in turn influences the policy functions, the distribution is not treated as an argument of the value function in the stationary equilibrium because it is constant.

One can find r^* by guessing any reasonable r_0, then computing the stationary equilibrium and checking whether market clearing is fulfilled. If not, any root-finding algorithm can be used to iterate on the interest rate until market clearing is ensured.

It is important to once again stress the absence of aggregate shocks which are a necessity for a stationary equilibrium to exist. Thus, in stationary equilibria, all aggregate variables are stationary. This occurs because the distribution D of households does not change over time. In other words, while individual households experience changes in their income states and, accordingly, adjust their asset levels through their policies, the distribution D remains unchanged. This property implies that all aggregates such as aggregate asset supply and aggregate consumption remain constant over time.

3.2 Aggregate Risk

The inclusion of aggregate risk will induce dynamic behaviour in aggregate variables. However, it will significantly increase the complexity and can be a legitimate bottleneck.

To illustrate how aggregate risk can be introduced, assume, for example, that the income process for households were given by

$$\log\left(z_{t+1}^{(h)}\right) = \rho \log\left(z_t^{(h)}\right) + \sigma_1 w_t^{(h)} + \sigma_2 \zeta_t,$$

where ζ_t represents a white noise process.

In this framework, households face two types of income risk. Idiosyncratic income risk, driven by $w_t^{(h)}$, and aggregate income risk, induced by ζ_t. The term ζ_t is classified as aggregate income risk because it affects *all* households simultaneously, thereby making aggregate household income uncertain.

To explore the implications of aggregate risk, we present the model developed by Krusell & Smith (1998).

The economy consists of a continuum of households indexed by $h \in [0, 1]$, along with a single representative firm. Households face idiosyncratic unemployment risk, which determines their employment status, $e_t^{(h)}$. Employment status is represented as $e_t^{(h)} = 0$ for unemployed and $e_t^{(h)} = 1$ for employed. This status evolves according to a Markov chain, with transition probabilities that depend on the economy's aggregate state, Z_t, which is yet to be determined.

At the start of each period t, households supply their entire asset stock $a_t^{(h)}$ to the firm and employed households also supply one unit of labour. Assets depreciate at a constant rate δ.

Households earn income through two channels: a return r_t on their asset holdings and a wage w_t for their labour supply (if employed).

The representative firm uses a Cobb-Douglas production function to produce the output good, utilising aggregate household asset supply ($\mathcal{A}_t$) as capital and aggregate labour supply ($\mathcal{N}_t$) as inputs.

Aggregate risk is introduced through an exogenous Total Factor Productivity (TFP) shock, Z_t, which evolves according to a two-state Markov chain. The TFP shock has two possible states: a good state ($z_g = 1.01$) and a bad state ($z_b = 0.99$). This shock directly impacts the firm's production and therefore affects *all* households simultaneously[8].

The joint Markov chain of the TFP and employment shock are calibrated such that the average duration of both good and bad states is eight quarters. Furthermore, unemployment durations differ depending on the state. In good states, the average duration of unemployment is 1.5 quarters, while in bad states it increases to 2.5 quarters. The unemployment rate is 4% in good states compared to 10% in bad states. In short, we have two marginal Markov Transition Matrices, denoted by $\Pi_{e,e'|Z\to Z'}$ and $\Pi_{z,z'}$, where the notation emphasises that the transition probabilities of employment depend on the evolution of Z.[9]

Due to exogenous idiosyncratic employment, the aggregate labour supply is stationary and depends solely on Z_t. Due to the calibration of the Markov chains, labour supply $\mathcal{N}_t(Z_t)$ is given by

$$\mathcal{N}_t(Z_t) = \begin{cases} 0.96, & \text{if } Z_t = z_g \\ 0.9, & \text{if } Z_t = z_b \end{cases} \tag{3.3}$$

Given factor prices r_t and w_t and the parameters β, δ and risk aversion γ, every household $h \in [0,1]$ solves the following problem

$$\begin{aligned} \max_{\left\{c_t^{(h)}>0\right\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t^{(h)}) \\ \text{s.t. } & a_{t+1}^{(h)} = (1 + r_t - \delta)a_t^{(h)} + w_t e_t^{(h)} - c_t^{(h)}, \\ & a_t^{(h)} > 0 \ \forall t. \end{aligned}$$

Since a_t is a tangible good, it cannot be negative. On top of that, a household must always be able to consume so that assets must be strictly positive in this model.

The firm features the following profit function

$$Y_t - r_t K_t - w_t L_t \qquad\qquad s.t.\ Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}.$$

The firm is a price taker and pays competitive prices for capital and labour such that these are given by[10]

$$w_t^* = (1-\alpha)Z_t \left(\frac{K_t}{L_t}\right)^{\alpha} \qquad \text{and} \qquad r_t^* = \alpha Z_t \left(\frac{L_t}{K_t}\right)^{1-\alpha}. \tag{3.4}$$

Since capital and labour are supplied by households to the firm at the beginning of the period, the following holds

$$K_t = \int_0^1 a_t^{(h)} dh = \mathcal{A}_t \qquad \text{and} \qquad L_t = \int_0^1 e_t^{(h)} dh = \mathcal{N}_t. \tag{3.5}$$

[8]Specifically, households are affected via the resulting changes in r_t and w_t.

[9]This needn't be the case and is simply a property of the Krusell-Smith model.

[10]Using these, it can be immediately verified that the firm has zero profits.

In the Krusell-Smith model, all available capital and labour provided by households are fully used by the firm.

Thus, a defining feature of (3.4) is that it provides a closed-form solution for the factor prices. In other words, since K_t (aggregate capital) and L_t (aggregate labour supply) are known at the beginning of period t, determining factor prices does not require solving for them through root-finding methods; they are directly given by (3.4).

It is helpful to clarify the state variables. In order to make its decisions, the household must know the following

$$\mathcal{S}_t = (a_t, e_t; D_t, Z_t), \tag{3.6}$$

where $D_t(e, a)$ is the distribution over employment and assets (household-specific state variables).

Due to the coupling of the two exogenous state variables, the Law of Motion of the distribution is given by

$$D'(e', a'|Z \to Z') = \sum_{e \in \mathcal{G}_e} \sum_{a \in \mathcal{G}_a} D(e, a) \cdot \Pi_{e,e'|Z \to Z'} \cdot \chi\big(a'(\mathcal{S}) = a'\big). \tag{3.7}$$

A key observation about aggregate risk becomes immediately apparent: the household asset distribution will inherently evolve over time. This is evident since $\Pi_{e,e'|Z \to Z'}$ can vary across periods. For instance, an economy experiencing many good TFP shocks will tend to accumulate more wealth overall, resulting in its asset distribution exhibiting greater probability mass at higher asset levels compared to an economy subjected to many bad TFP shocks. Therefore, a model with aggregate risk will not admit a stationary distribution and therefore not a stationary equilibrium.

he Bellman Equation (Master Equation) is given by the following

$$\begin{aligned}
V(e, a; D, Z) = \max_{c>0} &\Big\{ U(c) + \beta \mathbb{E}_{e,Z}\Big[V(e', a'; D', Z')\Big]\Big\} \\
\text{s.t.} \quad & a' = (1 + r - \delta)a + we - c > 0, \\
& w \overset{(3.4)}{=} (1-\alpha) Z \left(\frac{K}{L}\right)^{\alpha}, \\
& r \overset{(3.4)}{=} \alpha Z \left(\frac{L}{K}\right)^{1-\alpha}, \\
& K = \sum_e \sum_a a D(e, a), \\
& L = (3.3)\ \&\ (3.5), \\
& D'(e', a') \overset{(3.7)}{=} \sum_e \sum_a D(e, a) \cdot \Pi_{e,e'|Z \to Z'} \cdot \chi\big(a'(e, a, D, Z) = a'\big), \\
& \Pi_{e,e'|Z \to Z'} \text{ and } \Pi_{z,z'}.
\end{aligned}$$

Once again, the value function and the distribution are coupled, but in the presence of aggregate risk we cannot induce a decoupling because only in the stationary equilibrium does it hold that $D = D'$. Therefore, we must solve the full above Bellman Equation. The central problem is that this is basically impossible given the current form with discrete state space methods. The reason is that $D(e, a)$ is a matrix of dimension $N_e \times N_a$ and for each of these values we'd require a grid in order to discretise the distribution. As an example, if the matrix consisted of 200 elements and for each of these elements we could define a grid with 10 values, the total number of possible discretised distributions would be 10^{200}. For comparison, the estimated number of atoms in the observable universe is roughly 10^{82}. Thus, explicitly discretising D is computationally intractable.

The approach proposed by Krusell & Smith (1998) addresses this issue by reducing the dimensionality of the state space $\mathcal{S}$. A careful inspection of the Bellman Equation reveals that households require knowledge of the entirety of the distribution $D(e, a)$ only to forecast next period's aggregate capital K'. Furthermore, the value of K' is necessary solely for deducing future factor prices r' and w'.

The key idea is to develop an approximate rule that enables households to forecast r' and w' with reasonable accuracy. Since forecasting these future factor prices ultimately hinges on forecasting K' in the model, households only need to predict K'. This simplification eliminates the need to include the full distribution D as an argument in the value function.

Therefore, let's consider a reduced Bellman Equation given by

$$\begin{aligned} V(e, a; K, Z) = \max_{c>0} &\Big\{ U(c) + \beta \mathbb{E}_{e,Z} \Big[V(e', a'; K', Z') \Big] \Big\} \\ \text{s.t. } & a' = (1 + r - \delta)a + we - c, \\ & w \overset{(3.4)}{=} (1 - \alpha) Z \left(\frac{K}{L} \right)^{\alpha}, \\ & r \overset{(3.4)}{=} \alpha Z \left(\frac{L}{K} \right)^{1-\alpha}, \\ & L = (3.3) \ \& \ (3.5), \\ & K' = M(K, Z), \\ & \Pi_{e,e'|Z \to Z'} \text{ and } \Pi_{z,z'}. \end{aligned} \tag{3.8}$$

Since K is just a number, we can discretise the state space as $\mathcal{S} = \mathcal{G}_e \times \mathcal{G}_a \times \mathcal{G}_K \times \mathcal{G}_Z$. While this increases the number of gridpoints compared to the previous model, this doesn't exceed the bounds of feasible computation.

However, the key challenge is to specify the Law of Motion of capital $M(K, Z)$. We did know how D evolves to D', but we don't know how K evolves to K' conditioning only on K and Z.

Krusell & Smith (1998) impose the following updating rule

$$log(K') = \begin{cases} a_0 + a_1 log(K), & \text{if } Z = Z_g \\ b_0 + b_1 log(K), & \text{if } Z = Z_b \end{cases} \tag{3.9}$$

with $a_0, a_1, b_0, b_1 \in \mathbb{R}$ that are to be determined. This is an educated guess on their behalf and there is no general approach to determine what the functional form of the Law of Motion $M(K, Z)$ ought to be.

Hence, finding the Law of Motion of K is reduced to a finite-dimensional problem governed by four parameters. In order to find these parameters, the Krusell-Smith algorithm starts with an initial guess for them and solves the Bellman Equation (3.8) under this guess. Then, the model is is simulated for T periods using (3.9). If the updating rule can accurately forecast K' given K and Z, the algorithm terminates[11].

[11] In the textbook we thoroughly explain the algorithm in great detail.

4 Trouble with Mean-field Games

From the previous section it is immediately clear that solving MFGs with aggregate risk is much more difficult than in the case of solely relying on idiosyncratic risk. Unfortunately, in finance it makes no sense to exclude aggregate risk because without it, aggregate quantities are not risky and there is little scope in financial research without a risky asset. One of the few exceptions is Huggett (1993) who could tackle the risk-free rate puzzle without relying on aggregate risk because he was considering a riskless asset. Therefore, financial research must rely on aggregate risk. In this section I highlight several practical problems that arise when trying to solve MFGs with aggregate risk which hinder their adoption in finance. More importantly, I elaborate on the conceptual limitations of MFGs when it comes to application in finance. When trying to resolve these conceptual limitations, the numerical solution becomes excruciatingly harder and solving MFGs with aggregate risk is already a complicated endeavour. Therefore, I analyse the role of deep learning and which (numerical) issues still have to be solved in order to adopt heterogeneous agent models with a rich distribution in finance.

4.1 Updating Rule

The most obvious obstacle when solving MFGs with aggregate risk is the guess of the updating rule (3.9). There is no general functional form one can rely on. Therefore, if the guessed functional form cannot reasonably approximate the true functional form, the Krusell-Smith algorithm will never converge. For this reason, Fernández-Villaverde et al. (2023) use a neural network, an extremely flexible functional form, for (3.9). However, there are two problems with this approach. Firstly, according to the GitHub repository of the paper, it states that the runtime is three weeks[12].

This is a severe problem given that their model is close to Krusell & Smith (1998). Being able to solve a model fast is not only important in order to quickly detect bugs in the code, but also for calibration and running robustness checks or trying alternative model specifications. Secondly, it is also unclear which inputs (3.9) requires. For example, it is not clear that aggregate capital K_t and aggregate technology Z_t are sufficient statistics to forecast next period capital K_{t+1}. Perhaps one also needs to include the interquartile range or volatility of capital as additional statistics. This can only be found with Trial & Error. The longer it takes to solve the model, the more arduous the process to find a sufficient set of inputs for (3.9).

Thus, if a model departs too much from previous models and previously successful updating rules are inadequate to forecast aggregate quantities, then there is fundamental uncertainty whether the model the researcher built is actually solvable.

Gu et al. (2024) try to circumvent the problem of guessing an appropriate updating rule by using finite-dimensional approximations of the distribution an incorporating this into the Master Equation. They explore three possibilities: Reducing the continuum of agents to a setting with a finite number of agents, discretising the state space and using the

[12] This runtime is likely an upper bound. One problem is that the code is completely written in MATLAB. Given the publication date of the working paper, it is likely that the code was written in 2017/2018. At this time period, MATLAB's capabilities for efficiently computing neural networks were limited. Nonetheless, Pytorch and CUDA support was available at that time and the authors could have made a different choice. Generally speaking, the reliance of macroeconomic models on MATLAB will likely be a bottleneck for more advanced solution techniques such as neural networks as the machine learning community predominantly uses Python.

Kolmogorov forward equation of the state variables, or projecting the cross-sectional distribution onto a finite set of basis functions[13].

Such distribution approximations have the advantage that the guess on the updating rule (3.9) is no longer required. Nevertheless, these methods introduce their own challenges. Finite-agent and discrete state space approximations can quickly become computationally burdensome, as the generator of the state variables may require a large number of parameters, making the resulting state vector high-dimensional and costly to handle numerically (curse of dimensionality). Similarly, finite-difference methods applied to the Kolmogorov forward equation scale poorly with dimensionality. Projection methods onto finite-dimensional bases are less computationally demanding, but their accuracy depends critically on the adequacy of the chosen basis to capture the true distribution.

One way to avoid the updating rule is to use a sequence-space approach (Azinovic & Žemlička 2025). Thus, for Z_t being the vector of aggregate exogenous state variables, then for simple enough economies a truncated history $Z_{t-T}, \ldots, Z_t$ will encode all necessary information such as the distribution which would both circumvent the problem of having to approximate the distribution and finding an appropriate updating rule. Whether this dimensionality reduction works is however highly model-specific.

4.2 Perturbation

Perturbation methods have become one of the most popular techniques in macroeconomics. Since the advent of `DYNARE`, perturbation has emerged as the standard approach for solving representative agent DSGE models with many state variables. In these settings, perturbation methods typically solve the deterministic version of the model, i.e. by setting the volatility parameters of all exogenous states to zero, and then derive a linearised solution around this deterministic equilibrium (Villemot 2011).

Extending perturbation techniques from standard representative agent DSGE models to heterogeneous agent models with aggregate risk has since become one of the most widely utilised computational strategies. The baseline approach was developed by Reiter (2009), whose method perturbs the model around the stationary equilibrium without aggregate risk, a setting that can be solved using techniques from section 3.1. In doing so, one circumvents the necessity to guess the updating rule (3.9), but this approach will run into the curse of dimensionality quickly.

Building on this contribution, more advanced and faster implementations of first-order perturbation have been developed, including Boppart et al. (2018), Bayer & Luetticke (2020), and Auclert et al. (2021). Notably, Auclert et al. (2021) introduce a Sequence-Space approach, which departs from recursive formulations where tomorrow's states are determined only by today's states. Instead, their method considers the full history of exogenous shocks, as this history encapsulates the relevant state variables at time t.[14] This approach, however, relies on perfect foresight shocks, limiting its suitability for financial applications. Bilal (2023) introduces a continuous-time perturbation method that relies on functional derivatives of the Master Equation. Ahn et al. (2018) use a linearisation of SDEs around the stationary equilibrium for a perturbation approach in continuous time.

A limitation of perturbation methods is that first-order perturbation implies certainty

[13] Compare e.g. Winberry (2018).

[14] An analogy would be an AR(1) process where the entire history of shocks also encodes the current value of the process.

equivalence, rendering the variance of aggregate shocks and thus the degree of uncertainty irrelevant. In particular, this results in zero risk premia, making first-order perturbation unsuitable for financial applications. While second-order perturbation yields positive risk premia, these are constant rather than time-varying. Third-order perturbation is required to obtain time-varying risk premia (Schmitt-Grohé & Uribe 2004). I showcase this in appendix B.

However, recently Caramp & Silva (2025) have developed a method of a first-order perturbation around the stationary equilibrium in a continuous-time MFG that attains time-varying risk-premia. This approach involves both rare disasters and heterogeneous beliefs and is quite cumbersome since a plethora of linearised model equations need to be explicitly derived. Approximating around a stationary equilibrium is mandatory for finance since approximating a portfolio choice problem around a deterministic (riskless) state is senseless[15], since this only adheres to no-arbitrage if the risky asset, that is made riskless, pays the same as the riskless asset.

Winberry (2018) proposes an approach that accommodates higher-order perturbations. Implemented in `DYNARE`, this method can, in principle, compute perturbations of any order n. However, doing so requires considerable coding overhead and I am not aware of any further research that used this method.

Regardless, perturbation methods always only provide local rather than global solutions. Fernández-Villaverde & Levintal (2018) find that even third-order perturbation methods can perform poorly compared to global solutions.

4.3 Rational Expectations

Moll (2025) argues that the assumption of rational expectations about equilibrium prices is both conceptually implausible and computationally infeasible[16]. In models such as Krusell & Smith (1998), wages and interest rates depend on the entire cross-sectional distribution of wealth and income. Under rational expectations, decision makers are assumed to forecast these prices indirectly by forecasting the whole future distribution. The results in the Master Equation which Moll coins as the "Monster equation", which represents an extreme version of the curse of dimensionality. He states that it is implausible to assume that real-world households or firms perform such calculations in forming expectations.

The main contribution of his line of critique is threefold. First, he challenges the rational expectations assumption on conceptual grounds: if even state-of-the-art computational methods cannot solve the Monster Equation efficiently, it is unreasonable to suppose that actual agents can. Second, he identifies the mechanism by which rational expectations generates intractability: forecasting prices through distributions rather than directly. Third, he proposes a new research agenda for expectations in heterogeneous-agent models, guided by three criteria: (1) computational tractability, (2) consistency with empirical evidence, and (3) at least partial immunity to the Lucas critique. By reframing the problem in this way, he argues that abandoning rational expectations can both simplify individual decision problems and open the door to richer macroeconomic dynamics, including boom–bust cycles that current RE-based models struggle to capture. This is possible because one could tackle global solutions of MFGs with aggregate risk without running into the problem of having to solve the Master Equation.

[15]This equally applies to a stationary equilibrium of an MFG

[16]Due to the issues presented in the previous two sections.

Replacing rational expectations raises several challenges. Chief among them is determining how agents actually form beliefs about prices without reverting to distribution forecasting. Alternative approaches must also restrict the vast "wilderness" of possible non-rational beliefs to ensure that models remain falsifiable and policy-relevant. Computationally, they should allow for global rather than local solution methods, and empirically they must be disciplined by evidence, such as survey data on expectations. Moreover, new approaches need to ensure that beliefs remain endogenous to policy changes, so as not to fall prey to the Lucas critique. At the same time, models should preserve the ability of heterogeneous-agent frameworks to generate non-linear aggregate dynamics rather than suppress them through oversimplification.

Moll highlights several directions[17]. One is the temporary equilibrium approach, which computes competitive equilibria conditional on subjective beliefs, linked to the notion of "internal rationality". Another is to use survey-based measures of expectations to build forecasting rules. Adaptive learning methods, such as least-squares learning from the economics literature, allow agents to update forecasting rules from observed data, while reinforcement learning methods from computer science offer a more flexible framework in which agents learn value functions in partially unknown environments. A further possibility is to assume that agents use heuristics or simplified forecasting models that avoid the complexity of distribution forecasting while remaining empirically plausible. To build the bridge to finance, Adam & Nagel (2023) and Brunnermeier et al. (2021) survey how subjective expectations can be used for asset pricing.

Taken together, these alternatives share a crucial feature: they allow agents to forecast prices directly rather than indirectly via cross-sectional distributions. This relaxes the rational expectations assumption in a way that greatly reduces computational complexity while still permitting heterogeneous-agent economies to evolve stochastically and non-linearly. The promise of such approaches is to enable tractable, empirically grounded models that capture expectation-driven phenomena such as booms and busts, while avoiding the unrealistic burden of the Monster Equation.

Moll (2025) and Moll & Ryzhik (2025) argue for MFGs without rational expectations. Their argument is that agents in the real world are unlikely to track the entire distribution of the underlying state variables or even know the underlying true transition probabilities. As an example, Moll (2025) argues that it is more natural for agents to forecast prices directly and feature updating rules of prices based on subjective beliefs[18].

5 Granular Origins

Despite the difficulties of solving MFGs with aggregate risk, these models remain attractive because they appear, at first glance, superior to N-player games. In the continuum limit ($N \to \infty$), MFGs can represent richer heterogeneity than finite-agent models.

However, this tractability comes at the cost of shutting off granular origins. As $N \to \infty$, each agent's weight approaches zero. With only idiosyncratic risk, all shocks cancel out in the aggregate, yielding riskless aggregate quantities and stationary equilibria. To make aggregate quantities, such as an asset all agents can invest in, risky, researchers must introduce artificial aggregate shocks. Yet, the nature of such shocks like total factor productivity (TFP) as in Krusell & Smith (1998) remains obscure, functioning as a residual

[17] Moll & Ryzhik (2025) provide a mathematical framework.

[18] Recall that the only reason Krusell & Smith (1998) forecast K' is to deduce future factor prices.

label for unexplained aggregate fluctuations[19]. This black-box treatment is particularly unsatisfactory in finance, where the source(s) of aggregate risk is central.

Granular Origins Granular origins (Gabaix 2011) directly address this black box by providing a microfoundation for aggregate fluctuations. He proposes that aggregate shocks, such as TFP, can stem from idiosyncratic shocks to large firms rather than from economy-wide exogenous disturbances. In this view, firm-level shocks do not always wash out but can scale up to the macro level, challenging the conventional assumption that they average out.

As an example, Gabaix (2011) shows that one third of GDP fluctuations of the U.S. economy can be attributed to the 100 largest firms. In other countries that are less diversified than the U.S., this effect is even more pronounced. Further empirical evidence is provided by Carvalho & Grassi (2019), Ben-David et al. (2021), Koijen & Gabaix (2024) and Jamilov & Monacelli (2025). Even more explicitly, Gabaix (2011) shows that for the approximately 10 million firms present in the U.S. and an estimated firm output (sales) volatility of 12%, without granular origins, i.e. without a few very large firms, GDP volatility would only be 0.012% which is far away from the roughly observed 1%. This forces models to use black box aggregate shocks to match the observed volatility and risk in the economy because they cannot be tracked back to observable firm-level events when neglecting granular origins.

Mechanism The mechanism of granular origins rests on the fat-tailed distribution of firm sizes, such as a Pareto distribution, in an N-player game. Modern economies follow a highly skewed distribution close to Zipf's law (Gabaix 2009), where a small set of very large firms dominates activity. Under such conditions, the law of large numbers weakens: aggregate volatility decays at $1/\ln(N)$ rather than $1/\sqrt{N}$. As a result, shocks to firms like Walmart, Boeing, or Nokia can shift GDP materially. The collapse of Lehman Brothers at the onset of the 2007–08 Global Financial Crisis illustrates this mechanism starkly.

Thus, unlike in MFGs where individual risk vanishes in the aggregate, granular origins preserve aggregate risk through idiosyncratic channels. This feature is crucial in finance, where identifying sources of risk is central. It is not a flaw of MFGs but a consequence of their design: with agents of point mass zero, MFGs cannot generate granularity.

Comparison Table 1 summarizes the contrast between N-player games and MFGs.

N-player games naturally capture granular origins, since a few large players (e.g. large firms, dominant traders) can drive macro dynamics. But this realism carries the curse of dimensionality: every agent's state variables and coupled policy functions must be tracked, making numerical computations extremely difficult.

By contrast, MFGs exploit the law of large numbers: individual impact vanishes, leaving an individual optimal control problem coupled to a population law of motion. Conceptually, MFGs are the analogue of perfect competition, allowing continuous heterogeneity across beliefs, preferences, and states (not just a few coarse types). This tractability makes them appealing for studying inequality and other distributional issues[20]. But in doing so, they rule out granular origins and must rely on artificial aggregate shocks.

[19] An exception to this would e.g. be exogenous natural disasters.

[20] For example, the world is not just made up of optimists and pessimists.

Table 1: N-player vs. MFGs

Feature	Finite-Agent Model	MFG (Continuum of Agents)
Number of Agents	Finite (N)	Infinite ($N \to \infty$)
Individual Impact	Non-negligible	Negligible
Computational Complexity	High (curse of dimensionality)	Can avoid the curse of dimensionality
Representation of Heterogeneity	Finite number of types	Continuous distribution of types
Equilibrium Concept	Nash Equilibrium	Approximate Nash Equilibrium[a]
Key Analogy	Strategy Game Theory	Perfect Competition

[a] Carmona & Delarue (2013) prove that MFGs approximate the solution of N-player games with large N if their point mass is small.

Status Quo MFGs have become widely popular in macroeconomics over the recent years (Auclert 2025), given by the rise of HANK models. However, their neglect of granular origins limits their use in finance. Consider a concrete example of an intermediary asset pricing or macro-finance model. In these models, the health and behaviour of financial intermediaries are crucial determinants of asset prices, particularly during periods of financial stress. The reason is that intermediaries channel funds between savers and borrowers, and constraints like capital requirements restrict their risk-bearing capacity, affecting asset prices and premia. Such mechanisms have been widely studied[21].

MFGs, by design, cannot capture these channels because an intermediary with point mass zero cannot affect aggregates. Only N-player games due to granular origins allow for this, but the computational burden typically forces researchers to restrict attention to two- or three-agent setups (Kargar 2021), which limits the richness of heterogeneity. This is an issue because one cannot vary the degree of diversification in the economy. For example, Duarte et al. (2024) extend Lucas tree models to Lucas orchard models and show stark differences of a diversified economy with many trees (orchard) compared to the standard models with only one or two trees.

This creates a central conundrum: we want models with rich in heterogeneity that also allow for granular origins, yet MFGs exclude the latter while large N-player games are borderline computationally intractable.

Synthesis One attempt at synthesis is Jamilov & Monacelli (2025). They develop a model with a continuum of heterogeneous banks, incomplete markets, and aggregate TFP shocks. Banks face uninsured idiosyncratic portfolio shocks, ex ante differences in efficiency, and exposure to common macro shocks. This generates a dynamic distribution of bank size (net worth, assets, deposits) that influences aggregate outcomes. On top of that, there is ex ante heterogeneity as banks differ permanently in their ability to earn returns (e.g., screening/monitoring efficiency). All banks share exposure to macroeconomic

[21] See e.g. He & Krishnamurthy (2013), Brunnermeier & Sannikov (2014), He et al. (2017), He & Krishnamurthy (2018), He & Krishnamurthy (2019), Elenev et al. (2021), Fernández-Villaverde et al. (2023).

shocks. Their model gives rise to a non-trivial dynamic distribution of bank size (net worth, assets, deposits), which can shape aggregate outcomes.

They ask how the size and income distribution of banks affect aggregate fluctuations. To do so, they supposedly utilise granular origins and find that shocks to the largest banks explain most of the aggregate variation due to idiosyncratic risk. However, genuine granular origins require discrete large entities with non-negligible mass, which cannot exist in a environment with a continuum of banks with negligible point mass.

This is why in their "too-big-to-fail" (TBTF) experiments they zero out the net worth of *all* banks of a given type. But this amounts to imposing group-level correlated shocks, not true granularity, which arises from sole idiosyncratic shocks. This makes their TBTF experiment effectively an aggregate shock.

To advance towards tackling granular origins in N-player games, one can build on Moll (2025). For instance, it is implausible that a large bank would internalise the responses of every individual actor in the financial market to each of its actions. Instead, banks are more likely to rely on heuristics that approximate their overall (market) impact on the financial system. In doing so, these heuristic approaches break the coupling of the policy functions and makes an N-player game much easier to solve. For example, in a completely unrelated context Jensen et al. (2024) adopt such a heuristic approach by estimating Kyle's Lambda to evaluate the market impact of an asset manager. When using such heuristics, market clearing has to be treated with extreme care as agents could easily implement policies that violate market clearing.

In the next section I illustrate how deep learning could be used to make advances towards solving models with granular origins by being able to tackle high-dimensional state spaces.

5.1 Deep Learning

One of the most promising avenues for solving high-dimensional N-player games, and thereby incorporating granular origins, is the use of deep learning methods. Solving economic models is inherently challenging. In the textbook, we illustrate how discrete state space methods can approximate global solutions. Projection methods with standard basis polynomials such as Chebyshev polynomials provide another alternative. However, both approaches face serious limitations: they do not generalise well to higher dimensions. As Fernández-Villaverde (2025) notes:

> "*The situation is frustrating: we end up working with the models we can solve, not with the models we would like to solve. Browsing through any recent volume of the top journals in economics reveals numerous passages in which authors apologize for the undesired simplifications they must impose to obtain numerical solutions.*"

In response to this problem, many authors have proposed deep learning as a remedy (e.g., Fernández-Villaverde (2025), Fernández-Villaverde et al. (2024), Gu et al. (2024), Azinovic et al. (2022), Maliar et al. (2021)). Essentially, they are arguing for the use of Physics-Informed Neural Networks (PINNs) as in Raissi et al. (2019).

In the following, I provide an example of these "undesired simplifications" to establish context for a broader audience. Next, I review the curse of dimensionality and explain how deep learning can be used to mitigate the curse of dimensionality. Importantly, I stress that deep learning on its own does not eliminate the curse of dimensionality.

Using the Merton (1969) portfolio choice problem as a benchmark, I demonstrate that deep learning, in the form proposed by the existing literature, does not directly yield successful applications in finance. The main challenge lies in diffusion control, which is considerably more demanding than drift control. I highlight the additional nuances that must be incorporated to deploy deep learning effectively in this context.

To illustrate a path forward, I present the work of Duarte et al. (2024), which shows how deep learning can be scaled to higher dimensions. However, critical issues remain unresolved. In particular, market clearing, which is a defining feature of heterogeneous agent models, and the coupled policy functions of N-player games are not yet tractably addressed with deep learning. Developing robust algorithms capable of handling these features in reasonable time is still an open challenge.

5.1.1 Example: Simplifying Model Assumptions

In this section, I present an exposition of Coimbra & Rey (2023) to illustrate the type of "*undesired simplifications*" that often arise in modelling. Our goal is not to criticise Coimbra & Rey (2023); instead, I use their contribution as a representative example of a broader challenge in the field. I choose their paper simply because it is a recent macro-finance application of an MFG.

Model Overview The model features a representative, risk-averse household; a continuum of heterogeneous, risk-neutral financial intermediaries; and a government/central bank that guarantees deposits. Households choose consumption and a savings portfolio, but cannot invest directly in risky capital, so all aggregate risk-taking is delegated to intermediaries. Instead, they allocate funds between: (i) a storage technology yielding a gross return of 1, and (ii) deposits in intermediaries paying a guaranteed gross return $R_t^D = 1 + r_t^D$. Government guarantees imply that defaults by intermediaries are covered through lump-sum taxes on the household.

Financial intermediaries are two-period-lived entities[22] endowed with constant equity ω. Each intermediary i chooses how much to invest in risky capital versus storage, funding its position with household deposits. Risk-taking is disciplined by an individual Value-at-Risk (VaR) constraint, parameterised by α_i, which represents the maximum probability of default tolerated by the intermediary (reflecting heterogeneity in risk appetite, regulation, or governance). Each intermediary i does the following:

- Invest in risky capital $k_{i,t}$, which yields a stochastic return

  $$R_{i,t+1}^K = \begin{cases} 1-\delta & \text{with prob. } \zeta, \\ \theta Z_{t+1} K_t^{\theta-1} + (1-\delta) & \text{with prob. } 1-\zeta, \end{cases}$$

  where Z_t is aggregate productivity (shock).
- Invest in the storage technology.
- Fund positions with household deposits $d_{i,t}$.

Intermediaries consume $\max(0, \pi_{i,t+1})$, where

$$\pi_{i,t+1} = R_{i,t+1}^K k_{i,t} + s_{i,t} - R_t^D d_{i,t}.$$

[22] Thus, in each period the intermediary resets and is endowed with equity ω. Therefore, no intermediary has to make decisions beyond the next period.

If $\pi_{i,t+1} < 0$, the intermediary defaults, consumption is zero, and the government repays depositors, so that there is *limited liability*.

Coimbra & Rey (2023) show that an intermediary that leverages will always do so up to its VaR constraint. The implied leverage ratio can be computed in closed-form and is given by

$$\lambda_{i,t} \equiv \frac{k_{i,t}}{\omega} = \frac{r_t^D}{r_t^D - r_t^{\alpha_i}},$$

is increasing in α_i (looser constraints) and decreasing in the cost of funds r_t^D. Hence, more risk-tolerant intermediaries lever up more, and leverage becomes highly sensitive to funding costs at low interest rates.

Lastly, aggregate capital K_t, which the household cannot invest in, is the sum of all intermediaries' holdings, determined by the distribution of α_i and the equilibrium deposit rate, i.e.

$$K_t = \int_0^1 k_{i,t} f(\alpha_i) di. \tag{5.1}$$

The government guarantees deposits. Defaults trigger lump-sum taxes T_t on the household but, in the baseline model, impose no deadweight losses (*costless default*). This parsimonious assumption allows tractability by avoiding complex contagion mechanisms. Later extensions introduce *costly default* in reduced form, where losses are proportional to the balance sheet of defaulting intermediaries.

Key Simplifications The analytical elegance of the framework comes at the cost of several restrictive assumptions, some of which Coimbra & Rey (2023) explicitly acknowledge:

- Continuum of intermediaries: Each intermediary is atomistic, ensuring that individual failures do not affect prices. This rules out "granular" effects of very large players that have been empirically documented.
- Constant equity endowment: Intermediaries are assumed to start each life with the same equity ω, abstracting from net worth dynamics that are central in models such as Brunnermeier & Sannikov (2014) or Bernanke et al. (1999). The authors note that this assumption is adopted to focus squarely on the heterogeneous leverage dynamics across intermediaries
- Risk-neutral intermediaries: By construction, intermediaries have no precautionary motives; all risk is absorbed by the VaR constraint. This eliminates potential interactions between risk aversion and balance sheet management.
- Representative household excluded from risky capital: Only intermediaries can invest in capital, a strong abstraction that removes any portfolio reallocation channel on the household side.
- Costless default (baseline): Government guarantees eliminate depositor discipline and, absent costly default, intermediary failure has no direct resource cost beyond lump-sum transfers.

These simplifications are not "flaws" but rather necessary devices to make the problem tractable. For example, the closed-form expression for the leverage ratio $\lambda_{i,t}$ hinges on constant intermediary equity and risk-neutrality. Introducing endogenous net worth dynamics would break the closed-form and require tracking the joint distribution of wealth across intermediaries. Moreover, the continuum assumption guarantees smooth leverage distributions. With finitely many intermediaries, large players could move markets, making aggregate outcomes non-differentiable and path-dependent. On top of that, allowing costly or contagious defaults would introduce nonlinearities in equilibrium (feedback loops between failures and prices), making the model much harder to solve and likely only tractable with simulation methods.

This is particularly relevant for the market clearing condition of K_t combined with the condition that the deposit rate r_t^D equates household savings to intermediary demand. Under the simplifying assumptions, the distribution of leverage across intermediaries can be expressed in terms of (r_t^D, Z_t), allowing the integral to be solved in closed form or with smooth numerical methods. If equity evolved endogenously, one would need to track the entire joint distribution of (α_i, ω_i) across intermediaries, massively increasing state dimensionality. If intermediaries were risk-averse or defaults were costly, expectations over future states would feed back into current decisions, making the market clearing fixed point high-dimensional and intractable analytically. If intermediaries were finite and large ("granular"), equilibrium prices would depend on individual balance sheet positions, requiring simulation of network or game-theoretic interactions.

In short, Coimbra & Rey (2023) exemplify the trade-off articulated in the opening quote: economists are often forced to work with models that can be solved, rather than with models that would fully capture the richness of financial dynamics. The simplifications such as risk neutrality, constant equity or costless defaults are concessions to tractability, not reflections of reality.

5.1.2 Curse of dimensionality

I now illustrate the curse of dimensionality and how deep learning can help mitigate it.

Suppose a model has d state variables, each discretised with n gridpoints. The resulting tensor-product grid contains n^d nodes which marks exponential growth in d. This exponential scaling is the essence of the curse of dimensionality.

The severity becomes clear with $d = 5$ and $n = 100$: the grid contains 10^{10} nodes. Storing a single double-precision value (8 bytes) per node requires roughly 80 GB of memory. The computational burden is even greater, since at each node one typically solves an optimisation problem or computes expectations.

The problem even worsens because higher dimensions demand denser coverage. In one dimension, a value function $V(x)$ varies only along a line. In two dimensions, $V(x, y)$ varies across a plane with infinitely many directions. As dimensions increase, potential local variations grow combinatorially, requiring exponentially finer grids. Thus, the curse reflects both the explosion in gridpoints and the difficulty of capturing high-dimensional variation.

Projection methods attempt to reduce complexity by approximating the value function with a linear combination of basis functions. The hope is that fewer coefficients suffice compared to naïve discretisation, since basis functions capture curvature. However, a tensor-product of m basis functions still requires m^d coefficients, and these coefficients are usually identified by solving equilibrium conditions on the gridpoints. This again scales poorly. Moreover, polynomial bases often violate economic shape restrictions such as monotonicity or concavity. These violations can cause first-order conditions to fail, rendering the model unsolvable—even in two dimensions.[23] This is why most textbook applications restrict attention to one-dimensional state spaces. Projection methods delay but do not eliminate the curse of dimensionality.

This is why researches often use projection methodsto reduce complexity by approximating the value function with a linear combination of basis functions. The hope is that fewer coefficients suffice compared to naïve discretisation, since basis functions capture curvature. However, a tensor-product of m basis functions still requires m^d coefficients, and these coefficients are usually identified by solving equilibrium conditions on the gridpoints. This again scales poorly. Moreover, polynomial bases often violate economic shape restrictions such as monotonicity or concavity. These violations can cause first-order conditions to fail, rendering the model unsolvable—even in two dimensions. This is why most textbook applications restrict attention to one-dimensional state spaces. Projection methods delay but do not eliminate the curse of dimensionality. Moreover, polynomial approximations may not respect economically necessary properties such as monotonicity or concavity. Violations of these shape restrictions can cause the root-finding procedure to enforce FOCs to fail, rendering the model unsolvable. This is already a problem if the underlying state space is two-dimensional.

Deep neural networks (DNNs) have emerged as flexible function approximators. They are universal function approximators (Hornik et al. 1989) and can capture highly non-linear relationships. Yet, naïve use of DNNs does not by itself overcome the curse: to train a network effectively, one still needs adequate coverage of the state space, which grows exponentially with dimension. To approximate the value function well, a neural network must be trained on an adequate coverage of the state space. If one attempts to cover the entire high-dimensional domain uniformly, the number of training points required grows exponentially, running once again into the curse of dimensionality.

The key advantage of DNNs lies in their ability to learn lower-dimensional representations of high-dimensional inputs. For instance, a network with 100 input nodes (representing a 100-dimensional state space) and a hidden layer of 32 nodes effectively learns a 32-dimensional representation of the input. This dimension reduction, similar in spirit to autoencoders (Bank et al. 2021), is a unique strength of neural networks compared to fixed basis expansions such as Chebyshev polynomials.

To tackle the curse of dimensionality even further, one must rely on sampling strategies that sample from the ergodic set of the state variable. For dynamic models, simulated policy functions typically generate ergodic distributions for the state variables. Although the theoretical state space $\mathcal{S} \subset \mathbb{R}^d$ may be large or unbounded, in equilibrium the process visits only a subset $\mathcal{S}_{ergodic} \subset \mathcal{S}$ with high probability. Approximating the value function accurately on $\mathcal{S}_{ergodic}$ suffices, since states outside it are rarely observed. Training neural networks on samples from this ergodic set focuses computational effort where it matters most. This approach can be e.g. found in Raissi (2018) and further refinements in Hu et al. (2024). Theoretically, any projection method can be trained on the ergodic set, but neural networks can capture non-linearities in high-dimensional spaces with fewer gridpoints in general much better than any other pre-defined functional form such as Chebyshev polynomials.

[23] For example, Krusell & Smith (1998) could only be solved by exploiting an artificial stationary equilibrium to initialise the algorithm close enough to the true solution.

However, DNNs also share limitations with projection methods. Standard architectures do not enforce monotonicity, concavity, or other properties implied by economic theory. Without architectural constraints or penalty terms, neural networks may produce value or policy functions that violate these restrictions, leading to unstable or invalid solutions. I return to this issue later.

In summary, the curse of dimensionality arises because both grid-based discretisation and tensor-product projection methods scale exponentially with the number of state variables. Projection methods and neural networks each mitigate but do not fully eliminate the problem: high-dimensional variation, interaction terms, and necessary shape constraints remain fundamental obstacles. A key strategy for practical tractability is to restrict approximation to the ergodic set of the state variables, where the model spends most of its time.

5.1.3 PINN for the Merton Model

Merton Model The Merton (1969) portfolio choice model provides a continuous-time framework for optimal portfolio selection and consumption under uncertainty. The investor seeks to maximise expected utility derived from both flow consumption and terminal wealth over a finite time horizon. I assume standard CRRA preferences with risk aversion $\gamma = 2$. Most importantly, a portfolio selection problem involves diffusion and not just drift control.

The financial market consists of

- A risk-free asset with constant rate of return r.
- A risky asset whose value S_t evolves according to the following geometric Brownian motion
$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t),$$
where $\mu > 0$ is the expected return, $\sigma > 0$ is the volatility and $W(t)$ is a standard Brownian motion.

The investor's asset wealth $a(t)$ evolves according to the SDE

$$da(t) = \big[ra(t) + \pi(t)(\mu - r)a(t) - c(t)\big]dt + \pi(t)\sigma a(t)\,dW(t),$$

where

- $\pi(t)$ is the share of wealth allocated to the risky asset at time t,
- $c(t)$ is the consumption rate at time t.

The investor maximises utility over consumption and terminal wealth on a finite horizon $[0, T]$. The value function for a $t \in [0, T]$ is

$$V(t, a) = \max_{\{c(s),\pi(s)\}_{s\in[t,T]}} \mathbb{E}_t\left[\int_t^T e^{-\rho(s-t)} U\big(c(s)\big)\,ds + e^{-\rho(T-t)} U\big(a(T)\big)\right],$$

subject to the wealth dynamics above, where ρ is the discount rate.

The Dynamic Programming Principle (DPP) implies that for any small $\Delta > 0$

$$V(t, a) = \max_{c(\cdot),\pi(\cdot)\in\mathcal{U}[t,T]} \mathbb{E}_t\Big[\int_t^{t+\Delta} e^{-\rho(s-t)} u(c(s))\,ds + e^{-\rho\Delta} V\big(t + \Delta, a(t + \Delta)\big)\Big]. \tag{5.2}$$

Now expand each term for a small $\Delta > 0$:

- The instantaneous reward term:

$$\int_t^{t+\Delta} e^{-\rho(s-t)} U\big(c(s)\big)\, ds = U\big(c(t)\big)\Delta + o(\Delta).$$

- The discount factor:

$$e^{-\rho\Delta} = 1 - \rho\Delta + o(\Delta).$$

- The continuation value. If $V \in C^{1,2}$, then by Ito's formula[24]

$$\mathbb{E}_t\big[V(t+\Delta, a(t+\Delta))\big] - V(t,a) = \mathbb{E}_t\Big[\int_t^{t+\Delta} \big(V_t + \mathcal{L}^{c(s),\pi(s)}V\big)(s, a(s))\, ds\Big],$$

  where the infinitesimal generator $\mathcal{L}^{c,\pi}$ (acting on the a variable) is

$$\mathcal{L}^{c,\pi}V = \big(ra + \pi(\mu - r)a - c\big)\, \partial_a V + 0.5(\pi\sigma a)^2\, \partial^2_{a^2} V.$$

  Hence, for small Δ,

$$\mathbb{E}_t\big[V(t+\Delta, a(t+\Delta))\big] = V(t,a) + \big(V_t + \mathcal{L}^{c(t),\pi(t)}V\big)(t,a)\, \Delta + o(\Delta).$$

Insert these expansions into (5.2) and keep terms up to order Δ:

$$\begin{aligned} V(t,a) &= \max_{c,\pi} \Big\{ u(c)\Delta + (1-\rho\Delta)\big[V(t,a) + (V_t + \mathcal{L}^{c,\pi}V)\Delta\big] + o(\Delta)\Big\} \\ &= \max_{c,\pi} \Big\{ V(t,a) + \big[u(c) - \rho V(t,a) + V_t + \mathcal{L}^{c,\pi}V\big](t,a)\, \Delta + o(\Delta)\Big\}. \end{aligned}$$

This gives the following intermediate expression

$$\boxed{V(t,a) \;=\; V(t,a) \;+\; \Big(\mathrm{HJB}(t,a,V)\Big)\, \Delta + o(\Delta)} \tag{5.3}$$

where

$$\boxed{\mathrm{HJB}(t,a,V) \;=\; \max_{c,\pi} \Big\{ U(c) - \rho V + V_t + \mathcal{L}^{c,\pi}V \Big\}.} \tag{5.4}$$

Evidently, by (5.3) it holds that $\mathrm{HJB}(t,a,V) = 0\ \forall t, a, V$ in the limit $\Delta \to 0$.

Due to the FOCs of the HJB, I obtain

$$c^*(t,a) = [U']^{-1}\big(\partial_a V(t,a)\big). \qquad \text{and} \qquad \pi^*(t,a) = -\frac{\mu - r}{\sigma^2 a}\, \frac{\partial_a V(t,a)}{\partial^2_{a^2} V(t,a)}. \tag{5.5}$$

Therefore, the system of equations that determine the solution to the portfolio choice problem are given by the following

$$\rho V = u\big(c^*\big) + \partial_t V + \big(ra + \pi^*(\mu - r)a - c^*\big)\partial_a V + 0.5\big(\pi^*\sigma a\big)^2 \partial^2_{a^2} V, \tag{5.6a}$$

$$c^* = \big[u'\big]^{-1}\big(\partial_a V\big), \tag{5.6b}$$

$$\pi^* = -\frac{\mu - r}{\sigma^2 a}\, \frac{\partial_a V}{\partial^2_{a^2} V}, \tag{5.6c}$$

$$V(T,a) = U\big(a\big). \tag{5.6d}$$

[24] The Brownian term with integrator $dW(t)$ disappears due to the integral being a martingale.

The previously mentioned sources all advocate for the use of Physics-informed neural networks (PINNs). This implies to use neural networks to represent the unknown functions in the system of equations (5.6a) - (5.6d), which is basically the same as standard projection methods.

In the remainder of this section, I showcase that PINNs are not a panacea and the naïve implementation as laid out in the sources is inadequate to deal with models in finance which typically feature a diffusion control, whereas models in macroeconomics typically do not feature diffusion control. I will detail all the steps that make the naïve implementation difficult After this, I will explain potential remedies according to Duarte et al. (2024). In doing so, I want to make clear that deep learning is not necessarily easy to implement and might not adequately solve the model either.

PINN. Since the value function govern the entire system (5.6a) -(5.6d), I will use

$$V(t,a) \approx NN\big(t,a|\boldsymbol{\theta}\big)$$

where $\boldsymbol{\theta}$ are the parameters of the neural network.

I employ a residual network (ResNet) as in He et al. (2015). More concretely,

- Input: two-dimensional vector (t, a).
- Hidden layers: three fully connected layers with 64 neurons each and tanh activation functions[25].
- Skip connection: a linear projection from the input directly to the second hidden layer, ensuring stability and improving gradient flow[26].
- Output: a single scalar $V(t, a)$.

I kept the architecture as simple as possible to follow the naïve PINN implementation suggested by the literature.

Weights are initialised using Xavier initialisation (Glorot & Bengio 2010) and biases are set to zero.

I also tried to implement dropout, but this harmed the performance of the neural net severely and ultimately lead to the network not being able to solve the model. The reason is that some of the neurons were pivotal to enforce the constraints on the shape of the value function (see below).

Shape Restriction. A very important aspect the previous literature doesn't mention is that the shape of the value function is critical. For instance, $c^* > 0$ only holds for positive gradients $\partial_a V$. CRRA utility is not defined for negative consumption values and will cause the training algorithm to fail if in any intermediate step the gradient is not strictly positive. Likewise, I require that the Hessian is negative definite. In this case this is easy since the Hessian $\partial^2_{a^2} V$ is simply a scalar and it is easy to enforce a scalar to be

[25] I tried other activation functions that are smooth ReLUs such as Softplus, but these didn't nearly work as well. Smoothness of the neural net is important as $V \in C^{1,2}$ is required. Hyperbolic tangent probably works well because it projects the inputs to a regularised hidden layer whose domain is in $[-1, 1]$. Machine learning is full of regularisation, as is also the case in neural networks with batch or layer normalisation.

[26] This might not be necessary considering that the network is still shadow. But the computational and implementation costs are were negligible and entirely outweighed by the potential benefits since e.g. the *tanh* function is susceptible to vanishing gradients.

negative. However, for a "proper" Hessian it is much more complicated to enforce negative definiteness. The fact that the shape of the value function is so important is why the naïve PINN approach does not generalise well and why other alternatives such as Duarte et al. (2024) exist.

The previously mentioned literature in section 5.1 primarily discusses discrete time applications where it is easily possible to derive an Euler equation relating contemporary marginal utility to future marginal utility. In other words, a system of equations that doesn't feature the value function can be derived. This is important because (consumption) policy functions are many times much more linear than the value function. Not only is the consumption policy therefore easier to learn, but also the marginal utility function will generate the necessary shape[27]. However, the neural network might select a negative value for consumption which still must be prevented in order from the training algorithm not to fail. I will come back to this issue. Lastly, the big advantage of continuous time is that for diffusion processes expectations can be computed in closed-form, whereas in discrete time one is always faced with the additional computational burden and imprecision of approximating expected values.

To ensure well-behaved derivatives, I enforce the following.

- $\partial_a V > 0$ is enforced by a `ReLU` transformation with small ε-shift.
- $\partial^2_{aa} V$ is enforced to be negative via $-(|V_{aa}| + \varepsilon)$ for $\varepsilon = 10^{-8}$.

Since $\partial_a V > 0$ ensures $c > 0$, only the portfolio share is left to be taken care of. An issue is that if the Hessian is degenerate, then π will explode and could cause the training algorithm to fail. Therefore, I regularise it by the following:

- π is bounded using a scaled tanh, i.e.
  $\pi = \pi_{max} \cdot \tanh(\pi_{\text{raw}}/\pi_{max})$ with $\pi_{max} = 5$ and π_{raw} is given by FOC (5.6c).

The transformation π is basically linear in $[-\pi_{max}, \pi_{max}]$, thereby presenting no distortion of FOC (5.6c) as long as $\pi_{raw} \in [-\pi_{max}, \pi_{max}]$. Notice that this does require some intuition about what the value of the portfolio weight should be. In other models with other variables this might not be as easy.

Loss Function. The training loss consists of three components:

$$L = w_{\text{pde}} \cdot L_{\text{PDE}} + w_{\text{term}} \cdot L_{\text{terminal}} + w_{\text{grad}} \cdot L_{\text{shape}},$$

where

- L_{PDE} penalises violations of the HJB equation residual, i.e. (5.6a)-(5.6c),
- L_{terminal} enforces the terminal condition (5.6d),
- L_{shape} penalises violations of monotonicity in V_a.[28]

Weights are chosen as $w_{\text{pde}} = 1.0$, $w_{\text{term}} = 5.0$, and $w_{\text{grad}} = 5.0$. It is important that the loss from the terminal condition is non-negligible. If it is, then it is possible that the neural network might learn a "cheat-solution" that satisfies the PDE up to numerical error, but not the terminal condition in any shape or form. In this instance, the neural

[27] E.g. if $NN(t, a) = c(t, a)$, then $U'(NN(t, a))$ will in all likelihood exhibit the necessary shape.

[28] One could also include violations of the negative definite Hessian, I simply implemented an easy baseline case.

network failed to learn the solution. Hence, the terminal condition serves as an anchor for the neural network.

Training Procedure. The network is trained using the Adam optimiser with learning rate 10^{-3}, weight decay 10^{-6}, and gradient clipping at norm 5.0.[29] A two-stage learning rate schedule is employed:

1. *Warm-up:* linear increase over the first 5% of steps.
2. *Cosine annealing:* smooth decay over the remaining steps.

The training is run for up to 30,000 epochs. I found that the learning rate scheduler was vital to achieve accuracy.

Sampling Strategy. Collocation points (t, a) are drawn from:

- *Interior domain:* $t \sim U[0, T]$, a sampled from an exponential grid over $[a_{\min}, a_{\max}]$ with $a_{\min} = 0.01$ and $a_{\max} = 10.0$, concentrating points near $a_{\min}$.

  I focus on $a_{\min}$ close to zero because in mean-field games important mechanisms arise from agents being highly constrained. Compare the discussion of Huggett (1993) in section 2.3.

  - I use $1,000$ sampled points.[30]
- *Terminal condition:* points at $t = T$ with a on the same exponential grid. I use 50 points.

I train the neural network on the entirety of the 1,050 sample points. I had tried mini-batching, but the problem is that when the training algorithm moves from mini-batch to mini-batch, the neural network violates the shape constraint. The training process lead to the neural net learning the correct shape, but thereby forgetting the progress it had made from fitting the PDE from the previous mini-batch. This ultimately led the training to fail and the model could not fit (5.6a)-(5.6d). This is a severe issue for generalising the naïve PINN approach to higher state spaces because one always have to sample the entire ergodic set which is computationally extremely burdensome. Next section deals with this issue in more detail.

Results. I firstly show that the PINN successfully fits the terminal condition. This is depicted in figure 1.

Next, and more importantly, I compare the numerical solution at $t = 0$ to the analytical solution. This is more important because it will show us how well the PINN could learn from the PDE. Consider the following three figures.

It is evident from figure 2 that the naïve PINN cannot adjust to the steepness of the value function. However, if I zoom into the figure slightly (right figure), we see that the PINN can successfully manage to fit the value function in where the curvature is not as extreme.

Likewise, figure 3 tells the same story. The PINN can fit the consumption policy (drift control) extremely well. However, this breaks down when considering the portfolio share

[29] Without the clipping norm, explosive updates of the parameters $\boldsymbol{\theta}$ are possible and the neural network will not be able to train properly.

[30] A robustness check with $10,000$ sample points yielded the same results.

(diffusion control) depicted in figure 4. Even when zooming in and considering a smaller range of wealth, the portfolio share still doesn't behave well. Only well into in the interior of the state space is the fit good. The reason is that since the Hessian appears in the denominator of the portfolio share, even slight inaccuracies can lead to the portfolio share blowing up and simply being off target. Figure 5 depicts the training loss. The spikes in the loss are due to the fact that I use adaptive momentum (ADAM), so that the optimiser regularly shoots out of local minima in order to get closer to the global minimum.

Since Merton (1969) is the easiest finance model one can come up with, it is safe to say that naïve PINNs cannot handle diffusion control well and are therefore are not generally suitable in finance as the previous literature suggested. This should be particularly clear since Merton (1969) does not require any *market clearing* which would make the whole model solution considerably more difficult. Figure 6 shows that the Merton model is not unlearnable because cubic splines perform much better than the naïve PINN. Only around the extremely small $a_{min} = 0.01$ do we see an explosion of the portfolio share, whereas the drift control consumption doesn't explode at all.

Trial Solution Lastly, I present a fix that enables a PINN to fit smaller wealth levels much better. I simply use

$$V(t, a) \approx NN(t, a|\theta)\left[1 + a^{1-\gamma}\right],$$

which somewhat resembles a separation approach one can use to separate out wealth from the value function. Using $a^{1-\gamma}$ as an anchor significantly helps the neural network to adequately fit the shape of the value function as figures 7 & 8 show. However, all other disadvantages such as mini-batching and the shape constraints remain.

However, it is of course unclear how trial solutions can be applied to general models. For more complicated models such well-known separation approaches may not be as easy. For example, a bank may have multiple sources of wealth such as non-financial and financial assets and multiple sources of debt such as deposits, long-term and short-term debt. Thus, it is not clear how this separation approach can be extended to give the PINN guidance for the shape of the value function.

5.1.4 Actor-Critic

Here I present an alternative neural network architecture which is more robust than the naïve PINN. As stated in the previous section, in discrete time a naïve PINN can be more easily implemented since discrete time FOC model equations usually only feature the policy functions (mostly without diffusion control) which are much more linear than the value function and less sensitive to shape violations.

To mitigate the problems of the naïve PINN, I present the work of Duarte et al. (2024) which tweaks the actor-critic approach in Lillicrap et al. (2015) to continuous time. A full mathematical treatment of the algorithm in Duarte et al. (2024) can be found in Cohen et al. (2025). In a nutshell, the actor-critic approach introduces two neural networks: one for the value function (critic) and one for the policy function (actor). The critic judges how well the actor performs since the critic displays the discounted value of future rewards under a policy function. The main benefit from this separation is that the value function and policy function are trained for separately. Thus, if the value function exhibits shape violations, these needn't necessarily immediately translate into an invalid policy function since the policy function is a separate neural network which doesn't hinge on the shape of the value function as with the naïve PINN due to (5.5).

Another key benefit is that mini-batching can be used which failed with the naïve PINN. In theory, to know how the parameters of a neural network have to be updated in order to minimise a loss criterion requires the full state space. Only with the full state space do we know the loss at every point and truly know how to update the parameters. However, this is infeasible. Therefore, we can only estimate this gradient. If the loss criterion is the mean-squared error, which is what is commonly used with PINNs, then when going from 100 observations to 10,000 is 100 times more costly, but the standard deviation of the gradient estimate will only decrease by a factor of 10. This is why mini-batching is preferred and why we do not want to use a dense grid for the entire state space, which is impossible anyway in high-dimensional spaces. On top of that, for general models we do not know the ergodic set of the state space. We can only start with a generic guess, refine the policy function and then simulate the state space forward in time to update the ergodic set. In other words, the model must be solved for at many different grids from many different subsets of the state space which is why it is disadvantageous that the naïve PINN basically forgets most of its previous progress by re-learning the proper shape of the value function.

Algorithm Below I present the algorithm of Duarte et al. (2024) by adapting it to the Merton model. Since it uses mini-batching, let the batch be denoted by $i = \{1, \ldots, I\}$ and the number of gridpoints in the batch by $N_B \geq 1$. I then denote the state variables by $s_i \in \mathbf{R}^{N_B \times 2}$, as the two state variables are time and wealth.

Firstly, the actor (policy neural network) is given by

$$\mathbf{u}(t, a|\boldsymbol{\theta}_C) \approx \begin{bmatrix} c(t,a) \\ \pi(t,a) \end{bmatrix},$$

where $\boldsymbol{\theta}_C$ are the respective parameters of the neural network. Since consumption must be strictly positive, one could either apply ReLU to the output node with some minimum ε bound, or one could apply the sigmoid function to the output node and then multiply it with wealth so that consumption is always denoted in the fraction of wealth. This helps to regularise the output node and could lead to faster training with overall better results. If imposed structures like these are remotely accurate, then these impositions greatly help the neural network in training.

As before, the value function is approximated by the following neural network

$$V\big(t, a|\boldsymbol{\theta}_V\big) \approx V(t,a).$$

The actor will be trained using (5.4) while the critic will be trained using (5.3). However, training the actor involves an optimisation step which is not only costly, but in general cannot be computed in closed-form for a neural network. Therefore, instead of gradient descent, one simply applies gradient ascent on (5.4).

Training the actor is called *policy improvement*. The goal is to find the policy that maximises the lifetime discounted reward. This works as follows:

- Start from the current policy network

$$\mathbf{u}_0(s_i) \equiv \mathbf{u}(s_i; \theta_C^{j-1}).$$

- Compute the control-gradient of the HJB with respect to the control vector $\boldsymbol{u}$ given the value function, i.e.

$$\nabla_{\mathbf{u}}\text{HJB}\Big(s_i; \mathbf{u}_0(s_i), V\big(\cdot;\theta_V^{j-1}\big)\Big) \; = \; \begin{bmatrix} \partial_c \text{HJB} \\ \partial_\pi \text{HJB} \end{bmatrix}\Bigg|_{\left(s_i, \mathbf{u}_0, V(\cdot;\theta_V^{j-1})\right)}.$$

which is known in closed-form.

- For gradient ascent with a full step-size, compute

$$\mathbf{u}_{1,i} \; = \; \mathbf{u}_0(s_i) \; + \; \nabla_{\mathbf{u}}\text{HJB}\Big(s_i; \mathbf{u}_0(s_i), V\big(\cdot;\theta_V^{j-1}\big)\Big). \tag{5.7}$$

- To update the parameters θ_C^{i-1}, compute

$$\theta_C^i = \arg\min_\theta \Big\{\mathcal{L}_C(\theta)\Big\} \qquad \text{where } \mathcal{L}_C(\theta_C) \; = \; \frac{1}{2I}\sum_{i=1}^{I} \big\|\mathbf{u}(s_i;\theta_C) - \mathbf{u}_{1,i}\big\|^2.$$

with gradient descent. That is, at θ_C^{i-1} go into the steepest direction that minimises the loss. This trains the policy network parameters $\boldsymbol{\theta}_C^i$ to match the target $\boldsymbol{u}_{1,i}$.

Updating the value function works by *policy evaluation*. This means that the critic is trained to properly assess the discounted lifetime reward associated to a given policy. This works as follows:

- Define a time-step Δt such as $\Delta_t = 10^{-3}$ and compute

$$V_{\text{target},i} = V(s_i;\theta_V^{j-1}) \; + \; \text{HJB}\Big(s_i; \mathbf{u}\big(s_i;\theta_C^j\big), V\big(\cdot;\theta_V^{j-1}\big)\Big)\,\Delta t. \tag{5.8}$$

- Compute

$$\theta_V^j = \arg\min_\theta \Big\{\mathcal{L}_V(\theta)\Big\} \qquad \text{with } \mathcal{L}_V(\theta_V) \; = \; \frac{1}{2I}\sum_{i=1}^{I} \big(V(s_i;\theta_V) - V_{\text{target},i}\big)^2$$

using gradient descent. That is, at $\theta_V = \theta_V^{j-1}$ go in the steepest direction that minimises the loss.

- As an alternative, one could directly minimise $(HJB(\theta_V|\boldsymbol{u}) - 0)^2$ given the policy functions as done in the naïve PINN. Duarte et al. (2024) state this is slower, but more stable than the above policy evaluation method.

To enforce the terminal condition, it is possible to train the boundary separately as in the naïve PINN or instead one can init

Results/Discussion Duarte et al. (2024) solve three different models. Most interestingly is their high-dimensional portfolio choice problem in section 3.3. Although they use a separation approach and can find a simpler function than the whole value function, it is clear from their results that the approach works well and in fact much better than the naïve PINN. However, their applications only involve partial equilibrium. Market clearing in particular is hard to solve for if the market clearing prices are not given in closed-form, which can be the case as in e.g. Huggett (1993). Therefore, it is not straightforward to extend this model to general equilibrium models. However, it provides a promising result as it mitigates several of the problems the naïve PINN had.

A further challenge for economic and financial models is the treatment of preferences. Most models adopt CRRA utility, but CRRA introduces steep non-linearities in the value function that neural networks struggle to approximate reliably. This is clear from the naïve PINN which struggles to fit the HJB for small wealth levels. In contrast, CARA utility can be more stable since it is bounded, which makes the learning problem more regular. Neural networks generally perform better when the target surface is stationary and regularised, whereas with CRRA the value function drops sharply for small asset levels, creating steep gradients that are difficult to capture. This instability is exacerbated by the fact that neural network training penalises large parameter weights through regularisation; such large weights might in principle be necessary to reproduce the steep curvature of CRRA utility. This highlights a broader point: much of neural network methodology such as batch normalisation, layer normalisation, softmax, and more relies on regularisation. Without some analogous form of regularisation of the value function, CRRA-based problems pose a hard challenge for PINNs, whereas CARA provides a more tractable alternative. This is also why the trial solution approach works well which can also be incorporated into the actor-critic algorithm.

6 Conclusion

Heterogeneity plays a pivotal role in shaping financial dynamics, thereby challenging traditional representative-agent frameworks. Our paper highlights the critical importance of incorporating granular origins, which reveal how micro-level shocks scale to the macroeconomy, enabling the study of the determinants of aggregate risk which, for instance, drives the risk premium. Traditional two- or three-agent models fail to capture the rich heterogeneity of real-world economies consisting of varying degrees of diversification and transmissions from idiosyncratic risk to aggregate risk, MFGs sacrifice granularity in favour of tractability. This creates a methodological gap for research questions in finance.

To incorporate granularity, large N-player games are required, but they are hindered by computational bottlenecks, such as the curse of dimensionality, coupled policy functions and market clearing. While deep learning has emerged as a potential tool to mitigate these issues, our exploration of both naïve PINNs and actor-critic approaches demonstrates the challenges of applying these techniques to finance. Diffusion-control problems as with portfolio choice, as common in finance, proved particularly difficult for naïve PINNs, highlighting the need for more robust architectures like actor-critic neural networks.

Looking ahead, we emphasize that solving granular N-player models necessitates innovations on two key fronts: computational methodologies and equilibrium concepts. On the computational side, robust methods such as deep learning actor-critic frameworks can provide promising results. In terms of equilibrium concepts, it is likely that the coupling of policy functions in large N-player games will have to be simplified by using heuristic approaches in which the agents use forecasting rules to assess their individual impact on market prices. These heuristics can also be used in MFGs to simplify their solution in the presence of aggregate risk.

Doing so could open significant opportunities to study systemic risk, financial crises, and the determinants of aggregate risk in a rigorously micro-founded yet computationally feasible manner. As with the development of DSGE models decades ago, progress will require equal parts innovation, empirical grounding, and numerical craftsmanship.

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A Figures

Figure 1: Terminal Condition (naive PINN)

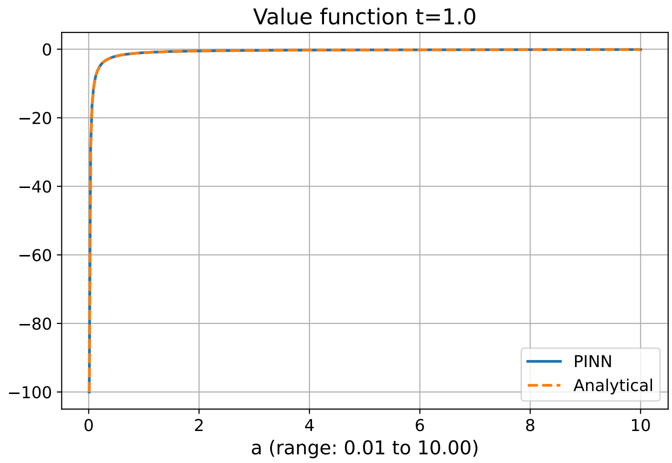


Figure 2: Value Function at $t = 0$ (naive PINN)

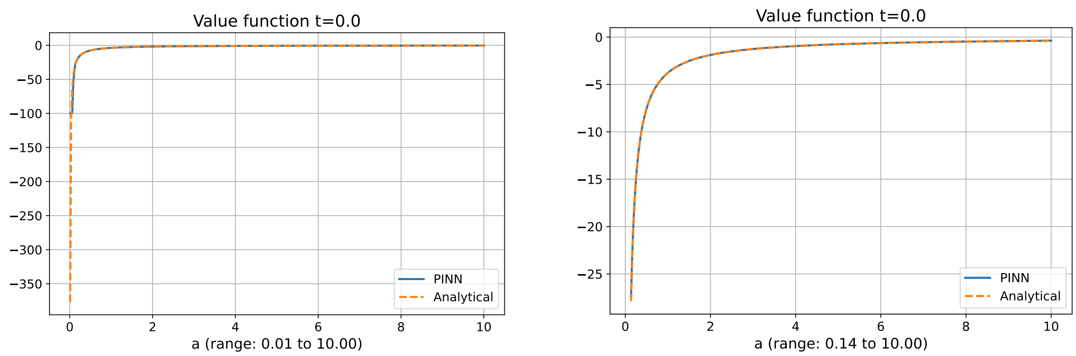


Figure 3: Consumption Policy at $t = 0$ (naive PINN)

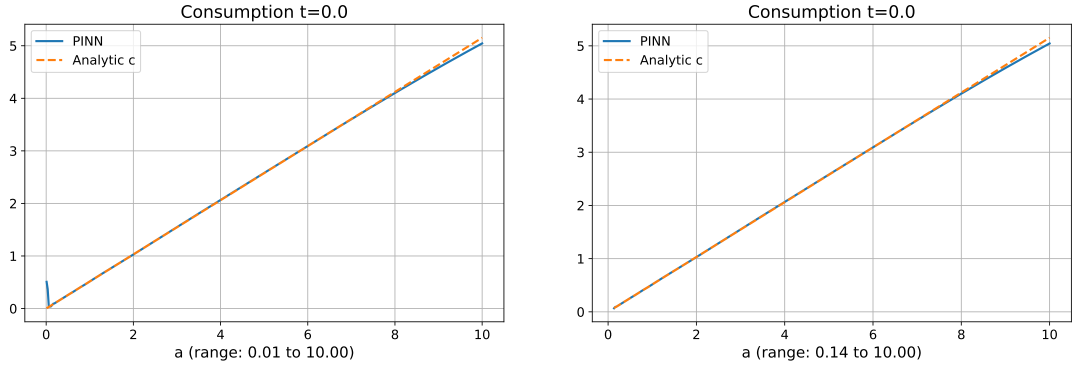


Figure 4: Portfolio Share at $t = 0$ (naive PINN)

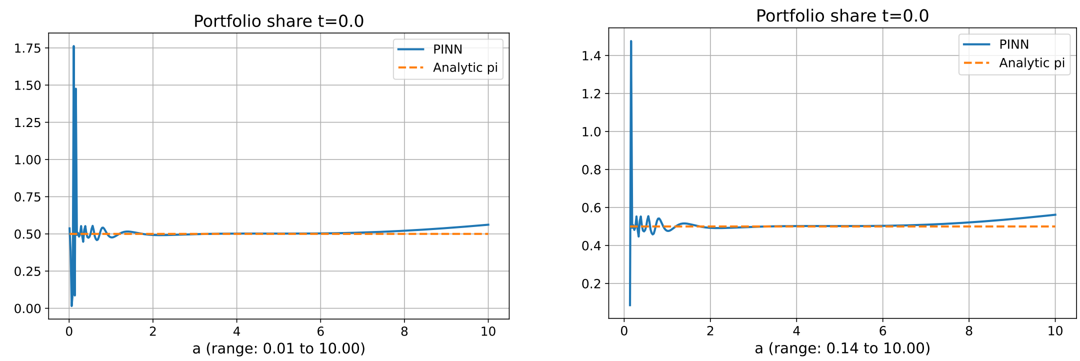


Figure 5: Training Loss (naive PINN)

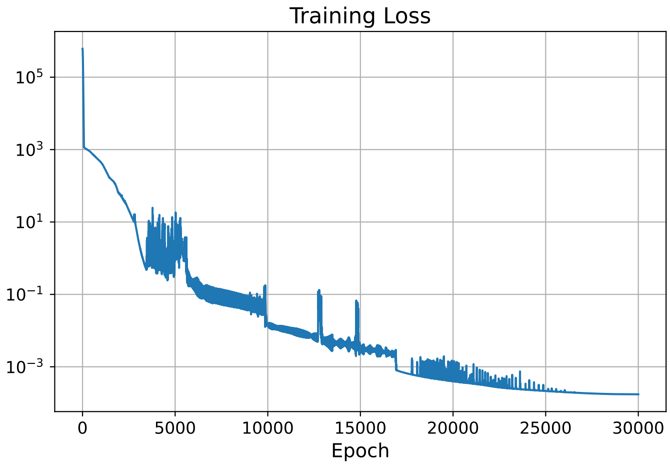


Figure 6: Policy Functions at $t = 0$ (Cubic Spline)

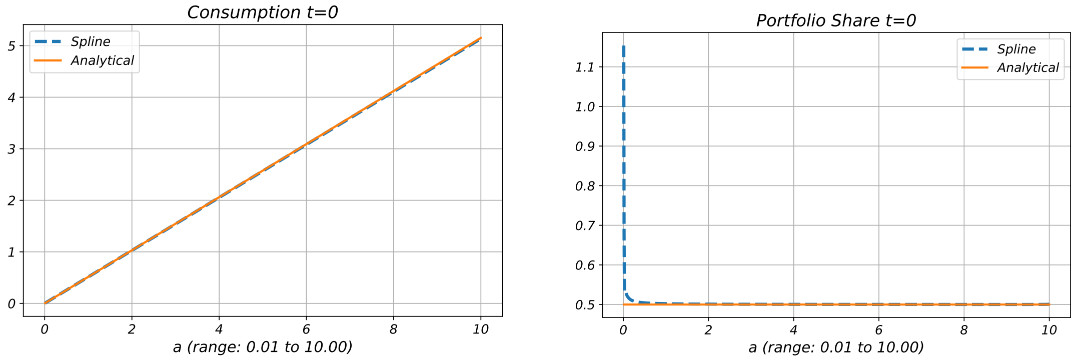


Figure 7: Value Function (Trial Solution)

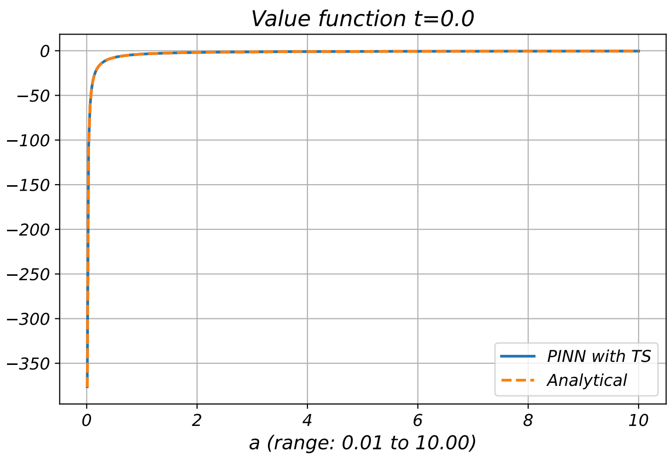


Figure 8: Policy Functions at $t = 0$ (Trial Solution)

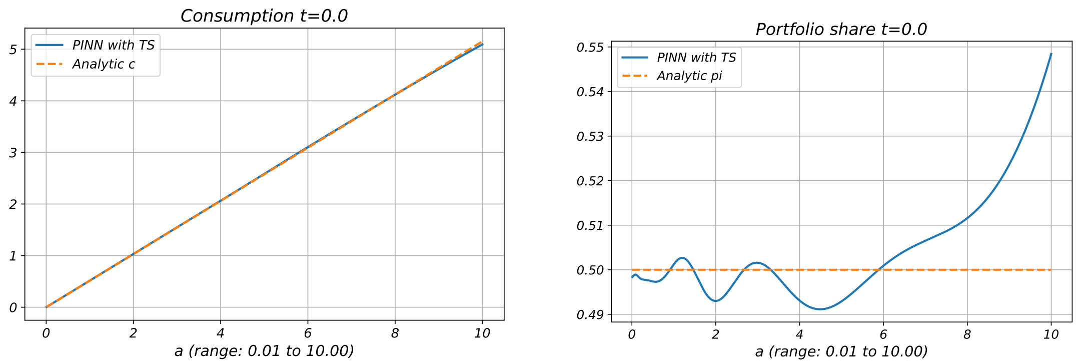
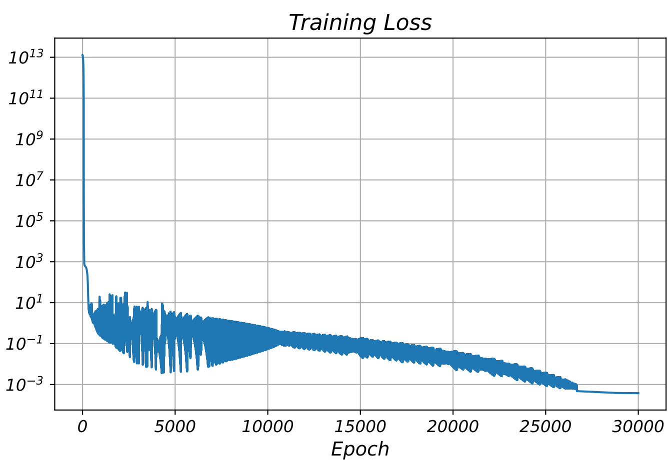


Figure 9: Training Loss (Trial Solution)



B Perturbation

Consider a general economic model in discrete time where the model equations are given by

$$\mathbb{E}\left[\boldsymbol{f}_\theta(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t, \boldsymbol{y}_{t+1}, \boldsymbol{u}_t)\middle|\Omega_t\right] = \boldsymbol{0},$$

where

- $\boldsymbol{y}_t$ are endogenous variables[31],
- $\boldsymbol{u}_t$ are exogenous shocks with $\boldsymbol{u}_t \sim WN(0, \Sigma_u)$,
- θ is the parameter vector,
- $\boldsymbol{f}_\theta(\cdot)$ represents the system of equilibrium conditions.

The information set Ω_t includes the model equations, parameters, $\boldsymbol{y}_{t-1}$, $\boldsymbol{y}_t$, and current shocks $\boldsymbol{u}_t$, but not the realations of future shocks.

We are considering an infinite-horizon model so that we are solving for a stationary solution that is independent of time.

The perturbation approach seeks an approximate solution for the policy function

$$\boldsymbol{y}_t = \boldsymbol{g}\big(\boldsymbol{y}_{t-1}, \boldsymbol{u}_t, \sigma\big),$$

where σ is a scaling parameter applied to the stochastic innovations, i.e.

$$\boldsymbol{u}_t = \sigma\boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim WN\big(\boldsymbol{0}, \Sigma_\varepsilon\big).$$

To make the stationarity explicit, if $\boldsymbol{y}_{t-1} = \boldsymbol{y}_{t+h-1}$ and $\boldsymbol{u}_t = \boldsymbol{u}_{t+h}$, then $\boldsymbol{y}_t = \boldsymbol{y}_{t+h}$.

The case $\sigma = 0$ corresponds to the non-stochastic steady-state, i.e. $\boldsymbol{y}_t = \overline{\boldsymbol{y}}\ \forall t$. We can solve for $\overline{\boldsymbol{y}}$ by solving

$$\boldsymbol{f}_\theta(\overline{\boldsymbol{y}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{y}}, \boldsymbol{0}) = \boldsymbol{0}.$$

Solving for the steady-state is generally possible since it corresponds to a system of non-linear equations, but usually an appropriate subset of equations can be found to get several low-dimensional root-finding problems[32].

Next, the policy function $\boldsymbol{g}$ is approximated with a first-order Taylor expansion around the steady-state such that

$$\boldsymbol{y}_t \approx \overline{\boldsymbol{y}} + \boldsymbol{g}_y(\boldsymbol{y}_{t-1} - \overline{\boldsymbol{y}}) + \boldsymbol{g}_u\boldsymbol{u}_t + \boldsymbol{g}_\sigma(\sigma - 0).$$

The partial derivatives are given by the following two Jacobian matrices and the gradient vector

$$\boldsymbol{g}_y = J_{\boldsymbol{y}_{t-1}}(\boldsymbol{g}), \quad \boldsymbol{g}_u = J_{\boldsymbol{u}_t}(\boldsymbol{g}), \quad \boldsymbol{g}_\sigma = \nabla\sigma(\boldsymbol{g}),$$

which are all evaluated at the steady-state $\big(\boldsymbol{y}_{t-1}, \boldsymbol{u}_t, \sigma\big) = (\overline{\boldsymbol{y}}, \boldsymbol{0}, \boldsymbol{0})$.

[31] This includes both the state variables and the control variables.

[32] If this isn't possible, one can linearise the entire system f_θ around the steady-state $\overline{\boldsymbol{y}}$ and compute the unknown quantities by solving the system of linear equations.

Deriving the Coefficients To solve for the partial derivatives $\boldsymbol{g}_y, \boldsymbol{g}_u, \boldsymbol{g}_\sigma$, the implicit function theorem is applied. To do so, firstly the auxiliary function $\boldsymbol{F}$ is defined by the following

$$\boldsymbol{F}(\boldsymbol{y}_{t-1}, \boldsymbol{u}_t, \boldsymbol{u}_{t+1}, \sigma) \equiv \boldsymbol{f}_\theta\Big(\boldsymbol{y}_{t-1}, \underbrace{\boldsymbol{g}(\boldsymbol{y}_{t-1}, \boldsymbol{u}_t, \sigma)}_{\equiv \boldsymbol{y}_t}, \underbrace{\boldsymbol{g}\Big(\boldsymbol{g}(\boldsymbol{y}_{t-1}, \boldsymbol{u}_t, \sigma), \boldsymbol{u}_{t+1}, \sigma\Big)}_{\equiv \boldsymbol{y}_{t+1}}, \boldsymbol{u}_t\Big).$$

In essence, $\boldsymbol{F}$ is simply $\boldsymbol{f}_\theta$ in which the policy function has been inserted.

As $\mathbb{E}_t[\boldsymbol{F}] = 0$, a first-order Taylor expansion of $\boldsymbol{F}$ at the steady-state ($\sigma = 0$) leads to the following system of conditions

$$\boldsymbol{F}_{\boldsymbol{y}_-}\widehat{\boldsymbol{y}}_- + \boldsymbol{F}_{\boldsymbol{u}}(\boldsymbol{u} - \boldsymbol{0}) + \boldsymbol{F}_{\boldsymbol{u}_+}\big(\mathbb{E}_t[\sigma \boldsymbol{\varepsilon}_{t+1}] - \boldsymbol{0}\big) + \boldsymbol{F}_\sigma(\sigma - 0) = \boldsymbol{0}. \tag{B.1}$$

where the hat indicates the deviation from steady-state. Furthermore, we drop the explicit time index and only distinguish between yesterday $(-)$, today and tomorrow $(+)$ since we are considering a stationary solution.

Due to $\mathbb{E}_t[\boldsymbol{\varepsilon}_{t+1}] = \boldsymbol{0}$ and (B.1) holding for all $\widehat{\boldsymbol{y}}_-, \boldsymbol{u}$ and σ, this entails the following system of equations

$$\begin{aligned} \boldsymbol{F}_{\boldsymbol{y}_-} &= \boldsymbol{0}, \\ \boldsymbol{F}_{\boldsymbol{u}} &= \boldsymbol{0}, \\ \boldsymbol{F}_\sigma &= \boldsymbol{0}, \end{aligned}$$

where the first two are matrices and the last object is a vector.

Next, we explain how to obtain the linearised policy function $\boldsymbol{g}$.

- Obtaining $\boldsymbol{g}_\sigma$
  Since

  $$\boldsymbol{F}_\sigma = \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{g}_\sigma + \boldsymbol{f}_{\boldsymbol{y}_+}\big(\boldsymbol{g}_{\boldsymbol{y}}\boldsymbol{g}_\sigma + \boldsymbol{g}_\sigma\big) = \boldsymbol{0},$$

  this entails that $\boldsymbol{g}_\sigma = \boldsymbol{0}$ because in general the other partial derivatives of the model $\boldsymbol{f}$ will be non-zero[33].

- Obtaining $\boldsymbol{g}_{y_-}$

  $$\begin{aligned} \boldsymbol{F}_{\boldsymbol{y}_-} &= \boldsymbol{f}_{\boldsymbol{y}_-} + \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{y}_-} + \boldsymbol{f}_{\boldsymbol{y}_+}\boldsymbol{g}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{y}_-} \\ 0 &= \boldsymbol{f}_{\boldsymbol{y}_-} + \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{y}} + \boldsymbol{f}_{\boldsymbol{y}_+}\boldsymbol{g}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{y}}, \end{aligned}$$

  where the second line follows from the fact that the function $\boldsymbol{g}$ is irrespective of time (stationary solution). Since the derivatives of the model $\boldsymbol{f}$ are known, the only unknown is the matrix $\boldsymbol{g}_{\boldsymbol{y}}$. However, no closed-form solution exists in general for this matrix equation. There exist several robust numerical procedures to compute $\boldsymbol{g}_{\boldsymbol{y}}$, see e.g. Meyer-Gohde (2021).

- Obtaining $\boldsymbol{g}_{\boldsymbol{u}}$

  $$\boldsymbol{F}_{\boldsymbol{u}} = \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{u}} + \boldsymbol{f}_{\boldsymbol{y}_+}\boldsymbol{g}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{u}} + \boldsymbol{f}_{\boldsymbol{u}}$$
  $$\Leftrightarrow$$
  $$\boldsymbol{g}_{\boldsymbol{u}} = -\Big[\boldsymbol{f}_{\boldsymbol{y}} + \boldsymbol{f}_{\boldsymbol{y}_+}\boldsymbol{g}_{\boldsymbol{y}}\Big]^{-1}\boldsymbol{f}_{\boldsymbol{u}}$$

[33] For example, when thinking of the Euler-Equation of consumption, changing contemporary or future consumption will impact the Euler-Equation.

Thus, we have obtained the linearised policy rule first-order approximation implies

$$\widehat{\boldsymbol{y}}_t = \boldsymbol{g_y}\widehat{\boldsymbol{y}}_{t-1} + \boldsymbol{g_u}\boldsymbol{u}_t + \overbrace{\boldsymbol{g}_\sigma}^{=0}\sigma,$$

The key drawback is *certainty equivalence*. Since $\boldsymbol{g}_\sigma = 0$, the size of the stochastic innovations σ does not affect the decision rule. In other words, although agents optime taking uncertainty into account, at the level of the first-order approximation the policy function is independent of the variance of shocks.

It is possible to generalise the above approach to an n-th order perturbation. It however immediately become cumbersome as matrices will turn into tensors.

Illustration

We consider the following problem

$$\begin{aligned} V_0(z_0, a_{-1}) = \max_{\{c_t, a_t\}_{t=0}^{\infty}} &\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t)\right] \\ \text{s.t.}\quad &a_t = (1+r)a_{t-1} + e^{z_t} - c_t, \\ &z_t = \rho z_{t-1} + \sigma w_t, \\ &w_t \overset{i.i.d}{\sim} N(0,1). \end{aligned}$$

Notice how we adopt a different timing convention which is the same timing convention as in `DYNARE`. Since assets are pre-determined, i.e. in period t we know what next period's assets are, we lag the time index of assets to reflect this. In other words, a_t is adapted to the filtration at time t. Moreover, in period t, we inherit the previous period's assets.

Thus, we denote $\boldsymbol{y}_{t-1} = [a_{t-1}, z_{t-1}, c_{t-1}]$, $\boldsymbol{y_t} = [a_t, c_t, z_t]$, $\boldsymbol{y}_{t+1} = [a_{t+1}, c_{t+1}, z_{t+1}]$ and $\boldsymbol{u}_t = \sigma w_t$

The Euler equation of consumption can be quickly derived. The system of equations is then given by

$$\boldsymbol{f}(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t, \boldsymbol{y}_{t+1}, \boldsymbol{u}_t) = \begin{pmatrix} U'(c_t) - \beta(1+r)U'(c_{t+1}) \\ a_t - (1+r)a_{t-1} - e^{z_t} + c_t \\ z_t - \rho z_{t-1} - \sigma w_t \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

such that $\mathbb{E}_t[\boldsymbol{f}(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t, \boldsymbol{y}_{t+1}, \boldsymbol{u}_t)] = 0$.

In steady-state we get $\overline{z} = 0$, $\beta(1+r) = 1$ and $r\overline{a} - 1 + \overline{c} = 0$. We can freely choose $\overline{c}$ since we have one excess degree of freedom. In first-order perturbation the deviations from steady-state will not be affected by the level of $\overline{c}$.

The policy functions are given by

$$\begin{aligned} a_t - \overline{a} &= a_a(a_{t-1} - \overline{a}) + a_c(c_{t-1} - \overline{c}) + a_z(z_{t-1} - \overline{z}) + a_u u_t + a_\sigma \sigma \\ c_t - \overline{c} &= c_a(a_{t-1} - \overline{a}) + c_c(c_{t-1} - \overline{c}) + c_z(z_{t-1} - \overline{z}) + c_u u_t + c_\sigma \sigma \\ z_t - \overline{z} &= z_a(a_{t-1} - \overline{a}) + z_c(c_{t-1} - \overline{c}) + z_z(z_{t-1} - \overline{z}) + z_u u_t + z_\sigma \sigma \end{aligned}$$

Writing this with the standard notation we have

$$\begin{pmatrix} a_t \\ c_t \\ z_t \end{pmatrix} = \underbrace{\begin{pmatrix} a_a & a_c & a_z \\ c_a & c_c & c_z \\ z_a & z_c & z_z \end{pmatrix}}_{\equiv \boldsymbol{g_y}} \begin{pmatrix} a_{t-1} \\ c_{t-1} \\ z_{t-1} \end{pmatrix}$$

Obtaining $\boldsymbol{g_u}$ This was given by

$$\boldsymbol{g_u} \;=\; -\Big[\boldsymbol{f_y} + \boldsymbol{f_{y+}}\boldsymbol{g_y}\Big]^{-1}\boldsymbol{f_u}.$$

At steady state ($z = 0$, $\beta(1+r) = 1$), the Jacobians are:

$$\boldsymbol{f_y} = \begin{pmatrix} 0 & U''(\bar{c}) & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{f_{y+}} = \begin{pmatrix} 0 & -U''(\bar{c}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{f_{y-}} = \begin{pmatrix} 0 & 0 & 0 \\ -(1+r) & 0 & 0 \\ 0 & 0 & -\rho \end{pmatrix}, \quad \boldsymbol{f_u} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

We now compute the following

$$\boldsymbol{M} = \boldsymbol{f_y} + \boldsymbol{f_{y+}}\boldsymbol{g_y} = \begin{pmatrix} -U''(\bar{c})\,c_a & U''(\bar{c}) & -U''(\bar{c})\,c_z\rho \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then obtain the following system of linear equations

$$\boldsymbol{M}\boldsymbol{g_u} = -\boldsymbol{f_u}$$

$$\begin{pmatrix} -U''(\bar{c})\,c_a & U''(\bar{c}) & -U''(\bar{c})\,c_z\rho \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_u \\ c_u \\ z_u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We can easily solve this system of linear equations and the solution is given by

$$\begin{pmatrix} a_u \\ c_u \\ z_u \end{pmatrix} = \begin{pmatrix} z_u(1 - c_z\rho)/(c_a + 1) \\ z_u(c_a + c_z\rho)/(c_a + 1) \\ 1 \end{pmatrix}$$

Obtaining $\boldsymbol{g_y}$ By solving the expression

$$\boldsymbol{0} \;=\; \boldsymbol{f_{y-}} + \boldsymbol{f_y}\,\boldsymbol{g_y} + \boldsymbol{f_{y+}}\,\boldsymbol{g_y}\,\boldsymbol{g_y},$$

one can obtain $\boldsymbol{g_y}$. However, solving the above is not a simple system of linear equations and cumbersome and doesn't further serve the illustration of the above abstract framework. However, $U''(\bar{c})$ will cancel out as well.

To solve the perturbation approach by hand, it makes more sense to insert the policy functions, then linearise $\boldsymbol{f}$ around the steady-state and solve for the unknown coefficients $\boldsymbol{g_y}$ and $\boldsymbol{g_u}$ using the method of undetermined coefficients.