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Asymptotics of Localized Solutions of the One-Dimensional Wave Equation with Variable Velocity. I. The Cauchy Problem

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Abstract. We present a systematic study of the construction of localized asymptotic solutions of the one-dimensional wave equation with variable velocity. In part I, we discuss the solution of the Cauchy problem with localized initial data and zero right-hand side in detail. Our aim is to give a description of various representations of the solution, their geometric interpretation, computer visualization, and illustration of various general approaches (such as the WKB and Whitham methods) concerning asymptotic expansions. We discuss ideas that can be used in more complicated cases (and will be considered in subsequent parts of this paper) such as inhomogeneous wave equations, the linear surge problem, the small dispersion case, etc. and can eventually be generalized to the 2- (and n-) dimensional cases.

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1. STATEMENT OF THE PROBLEM

The aim of this paper is to construct asymptotic solutions of the Cauchy problem for the one-dimensional wave equation

$$\Box_c u = 0, \qquad u = u(x,t), \qquad \Box_c = \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x}\Big(c^2(x)\frac{\partial}{\partial x}\Big), \tag{1.1}$$

with rapidly decreasing initial wave profile localized in a neighborhood of the point $x = a$ and with zero derivative $\partial u/\partial t$ for $t = 0$. We characterize localization by a small parameter μ and represent the initial condition in the form $V\big(x/\mu\big)$. We assume that the smooth function $V(y)$ decays sufficiently rapidly as $y \to \infty$; namely, $V(y) \leqslant C^0/y^n$ and $dV(y)/dy \leqslant C^1/y^{n+1}$ as $|y| \to \infty$, where C^0 and C^1 are some constants and $n \geqslant 3$ is an integer. As an illustration, one can take $V(y) = A/\cosh^2 \frac{y}{b}$ (a soliton), or $V(y) = Ae^{-\frac{y^2}{2b}}$ (a Gaussian exponential), or $V(y) = Ae^{-\frac{y^2}{2b}}\cos(ay+\phi)$ (a Gaussian exponential with oscillations), where $b > 0$, a, ϕ, and A are real parameters. We assume that the velocity $c(x)$ is a smooth function such that $0 < C_1 \leqslant c(x) \leqslant C_2$ as $-\infty < x < +\infty$, where C_1 and C_2 are some constants, and moreover, $c(x)$ tends to some positive constant C_∞ as $x \to \pm\infty$.

Thus, for Eq. (1.1), we consider the Cauchy problem with the initial conditions

$$u\Big|_{t=0} = V\Big(\frac{x-a}{\mu}\Big), \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0. \tag{1.2}$$

Equation (1.1) describes in particular long wave propagation in a narrow channel. The parameter μ characterizes the wave-to-channel length ratio [22].

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We wish to construct explicit asymptotic formulas for $\mu \ll 1$ generalizing the well-known D'Alembert formula to the case of variable velocity and give a rigorous justification of these formulas. This setting is very natural and has been often discussed in many physical and mathematical papers (e.g., see [2, 3, 10, 11, 17, 18, 22]). Moreover, the solution formulas can in principle be derived from the general formulas [4] for the solution of a similar problem in the n-dimensional case. Nevertheless, we have not been able to find a rigorous derivation and justification of such formulas directly for this rather simple problem in the literature; on the other hand, the transformation and simplification of the formulas in [4] is apparently more complicated than the straightforward computation in the one-dimensional case. We also note that although the definitive formulas have a very simple form, their proof is not completely elementary. Moreover, many elements of the proofs and constructions given in this paper arise in a more complicated form both in the one-dimensional case [if the velocity can vanish (in the linear theory of surge waves), in small dispersion problems, and in problems with sources on the right-hand side (which will be considered in subsequent parts of the paper)] and in multidimensional problems, for which some asymptotic solutions are announced in [5, 6, 7]. In this sense, the case studied here serves as a first step towards the solution of these more complicated problems, and the computations and argument given below are aimed precisely at this goal.

To solve problem (1.1)–(1.2), one can use various asymptotic constructions, e.g., asymptotic expansions of "Whitham type" [15, 22], asymptotic expansions "with respect to smoothness" [1, 12], and semiclassical expansions (WKB type expansions) [14], related to the parametrix technique well known in the theory of pseudodifferential equations [12, 19]. We shall discuss the possibility of applying all the above-mentioned approaches to this problem. Since the "semiclassical approach" seems to be most promising for generalizations to more complicated problems, we give it most of our attention and discuss the other approaches in the Appendices. The key though elementary techniques used in this paper are (1) the passage from a problem with rapidly decaying initial data to a problem on oscillating WKB asymptotics; (2) the use of boundary layer techniques [13, 16, 20] in view of the rapid decay of the initial wave profile.

The present (first) part of the paper is organized as follows. In Section 2, we state the main result (formulas (2.5) and (2.6) in the Theorem) and give simple illustrating examples. Auxiliary facts related to the theory of the Hamilton–Jacobi equation and its geometric interpretation are given in Section 3. The relationship between rapidly oscillating and localized solutions and some other useful analogies are discussed in Section 4. Section 5 contains formulas, based on WKB solutions, for corrections to the leading asymptotic term and estimates of the difference between the exact and asymptotic solutions. (The proof of the Theorem is given.) Appendix 1 deals with corrections based on the Whitham expansion, and their relationship with the expansion with respect to smoothness is discussed in Appendix 2.

2. STATEMENT OF THE RESULT

The asymptotics of the solution of problem (1.1)–(1.2) has a very simple form. Although the solution is unique, one can construct several asymptotics, which, of course, coincide neglecting small corrections. Let us write out two of such asymptotics. Let $X^{\pm}(t)$ be the solutions of the Cauchy problem

$$\dot{x} = \pm c(x), \tag{2.1}$$

$$x\big|_{t=0} = a, \tag{2.2}$$

specifying the trajectories on the x-axis, and let $c_0 = c(a)$. Equation (2.1) is integrable in quadratures; namely, $x = X^{\pm}(t)$ can be found from the equations

$$\tau(x) = \pm t, \qquad \text{where} \quad \tau(x) = \int_a^x \frac{d\xi}{c(\xi)}. \tag{2.3}$$

Thus the functions $x = X^{\pm}(t)$ are the inverses of the respective functions

$$\tau^{\pm}(x) = \pm\tau(x). \tag{2.4}$$

For either sign, the meaning of $\tau^{\pm}(x)$ is obvious. This is the time in which the trajectory issuing from the point a reaches the point x.

Theorem. *Under the conditions stated above, the following asymptotic formulas hold for the solution of the Cauchy problem* (1.1)–(1.2):

$$u(x,t) = \frac{1}{2}\sum_{\pm}\sqrt{\frac{c_0}{c\big(X^\pm(t)\big)}}\,V\left(\frac{c_0}{c\big(X^\pm(t)\big)}\frac{\big(x - X^\pm(t)\big)}{\mu}\right) + O(\mu|\log\mu|) \tag{2.5}$$

$$= \frac{1}{2}\sqrt{\frac{c_0}{c(x)}}\sum_{\pm}V\left(\frac{\Phi(x)\mp c_0 t}{\mu}\right) + O(\mu|\log\mu|). \tag{2.6}$$

Here the phase is given by the formula

$$\Phi(x) = c_0\tau(x). \tag{2.7}$$

These formulas are a straightforward, natural generalization of the D'Alembert formula for the wave equation with constant coefficients. At each time t, the solution $u(x,t)$ is localized in a neighborhood of the points $x = X^\pm(t)$, which readily follows from the representation (2.5). The points $x = X^\pm(t)$ are known as the *solution fronts*; they move in opposite directions at different (in general) velocities $c(X^\pm)$ and are the roots of the respective equations $\Phi(x)\mp c_0 t = 0$. The expression for the amplitudes expresses Green's law, well known in the theory of long waves in a channel; in this case, $c(x) = \sqrt{H(x)}$, where $H(x)$ is the channel depth. The shallower the channel, the larger the wave amplitude and the narrower the wave. Note that formulas (2.5) and (2.6) are of local character and the behavior of $c(x)$ far from the front does not affect the leading term of the asymptotic solution. However, it is convenient to assume for technical reasons that, as was already indicated above, $c(x)$ approaches a constant $c_\infty > 0$ as $x\to\pm\infty$. (The constant is the same for $+\infty$ and $-\infty$.) Hence, when specifying the function $c(x)$ in the examples below, *we assume that there exists a sufficiently large R such that $c(x)$ is defined by the corresponding analytical formulas for $|x| < R$ and is equal to some constant $c_\infty > 0$ for $|x| > 2R$.*

Let us give some examples.

Example a.1. Suppose that the reservoir depth is given by the function $H(x)=\left(\dfrac{2\sqrt{1+8e^x}}{1+\sqrt{1+8e^x}}\right)^2$. Then

$$c(x) = \frac{2\sqrt{1+8e^x}}{1+\sqrt{1+8e^x}},$$

and Eqs. (2.1) can be integrated in elementary functions:

$$X^\pm(t) = \log\frac{e^{\pm t}q(2+e^{\pm t}q)}{8}, \qquad q = \sqrt{1+8e^a} - 1. \tag{2.8}$$

Figure 1 illustrates the evolution of a wave packet given at the initial time by a solitary wave with profile $V(y) = 3e^{-y^2}$ for $a = -5$ and $\mu = 0.2$. The profiles $u(x,t)$ were computed according to the leading term of formula (2.5).

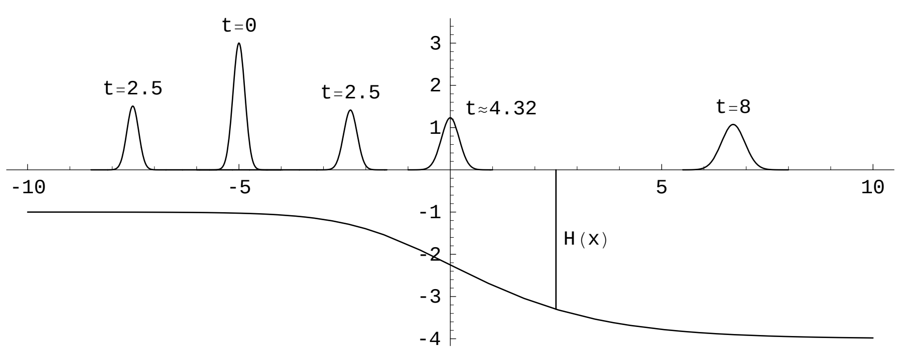

Fig. 1.

Figure 2 shows superimposed profiles $u(x,t)$ computed by the leading terms of formulas (2.5) and (2.6) at times $t \approx 4.32$ and $t = 8$ for $\mu = 0.5$. (We consider only the wave running to the right.) As is seen from Fig. 2 and the bottom profile in Fig. 1, the profiles almost coincide visually on the intervals where the reservoir depth is constant but can be visually distinguished from each other on the intervals where the bottom varies. The smaller the parameter μ, the weaker the difference.

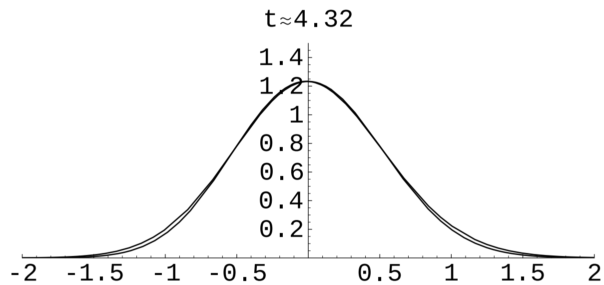

Fig. 2a.

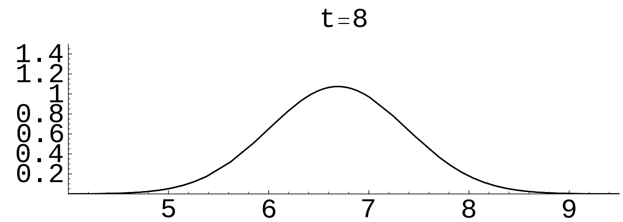

Fig. 2b.

Example b.1. Consider the wave propagation for the case in which $c^2(x) = \gamma^2 x$ on some interval $0 < a_1 < x < a_2$, where γ is a positive parameter and a_1 and a_2 are some constants. Suppose that the initial perturbation is localized in a neighborhood of some point $a \in (a_1, a_2)$. Just as in the preceding case, Eqs. (2.1) can be integrated in elementary functions:

$$X^{\pm}(t) = a\left(1 \pm \frac{\gamma}{2\sqrt{a}}t\right)^2. \tag{2.9}$$

We substitute (2.9) into (2.5), retain only the main term, and find the asymptotics of the solution in the form

$$u(x,t) = \frac{1}{2}\left\{\frac{1}{\sqrt{\left|1 - \frac{\gamma t}{2\sqrt{a}}\right|}} V\left(\frac{x - a\bigl(1 - \frac{\gamma t}{2\sqrt{a}}\bigr)^2}{\mu\bigl(1 - \frac{\gamma t}{2\sqrt{a}}\bigr)}\right) + \frac{1}{\sqrt{\left|1 + \frac{\gamma t}{2\sqrt{a}}\right|}} V\left(\frac{x - a\bigl(1 + \frac{\gamma t}{2\sqrt{a}}\bigr)^2}{\mu\bigl(1 + \frac{\gamma t}{2\sqrt{a}}\bigr)}\right)\right\}. \tag{2.10}$$

By formulas (2.7) and (2.3), we find

$$\Phi(x) = 2\bigl(\sqrt{ax} - a\bigr). \tag{2.11}$$

We substitute (2.9) and (2.11) into (2.6), retain only the main term, and get

$$u(x,t) = \frac{1}{2}\left\{\sqrt[4]{\frac{a}{x}}\, V\left(\frac{2\bigl(\sqrt{ax} - a\bigr) - \gamma\sqrt{a}t}{\mu}\right) + \sqrt[4]{\frac{a}{x}}\, V\left(\frac{2\bigl(\sqrt{ax} - a\bigr) + \gamma\sqrt{a}t}{\mu}\right)\right\}. \tag{2.12}$$

Figure 3 illustrates the evolution of a wave packet given at the initial time by a solitary wave with profile $V(y) = 2/\cosh^2 y$ for $\gamma = 0.2$, $a = 5$, and $\mu = 0.2$. The profiles $u(x,t)$ were computed modulo $O(\mu)$ according to formula (2.10).

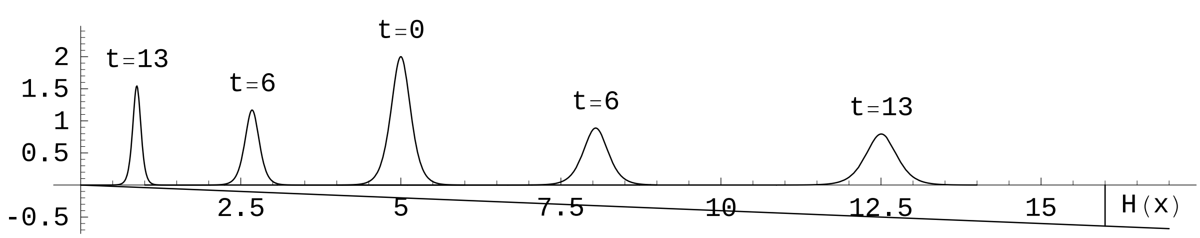

Fig. 3.

As is seen from Figs. 1 and 3, in both cases the initial solitary wave eventually splits into two solitary waves, one of which moves at the velocity $c\bigl(X^-(t)\bigr)$ to the left and the other at the velocity $c\bigl(X^+(t)\bigr)$ to the right. The deeper the reservoir, the larger the velocity and width and the smaller the amplitude of the solitary wave.

Figure 4 shows superimposed profiles $u(x,t)$, i. e., solitary waves running to the left (see Fig. 4a) and to the right (see Fig. 4b), computed by formulas (2.10) and (2.12) at time $t=6$ for $\mu=0.5$. Since the bottom varies linearly on the entire interval under consideration, the profiles are visually distinguishable from each other.

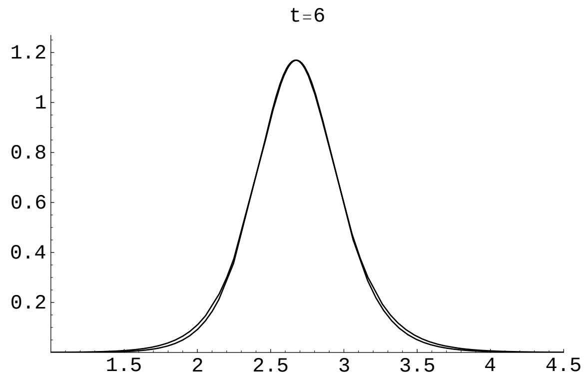

Fig. 4a.

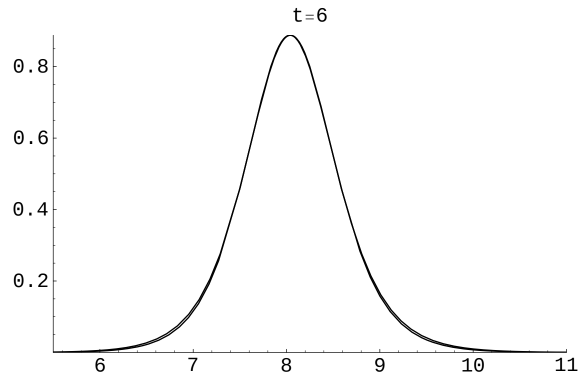

Fig. 4b.

3. THE LEADING ASYMPTOTIC TERM AND THE GEOMETRIC INTERPRETATION

The aim of the subsequent sections is to prove the Theorem in a form containing elements of the proof of a similar theorem in the multidimensional case. The change of variables $x\to x-a$ takes the point $x=a$ to the origin. To simplify the notation, *in what follows we assume that*

$$a=0 \tag{3.1}$$

in (1.2).

3.1. *The Eikonal Equations*

The solutions (2.5) and (2.6) can be represented as the sum of two solitary waves,

$$u=f^+\Bigl(\frac{S^+(x,t)}{\mu},x,t,\mu\Bigr)+f^-\Bigl(\frac{S^-(x,t)}{\mu},x,t,\mu\Bigr), \tag{3.2}$$

where the phases $S^\pm(x,t)$ and $f^\pm(\theta,x,t,\mu)$ are smooth functions and $f^\pm$ rapidly decay as $|\theta|\to\infty$. Solutions of the form $f^\pm\bigl(S^\pm(x,t)/\mu,x,t,\mu\bigr)$, describing solitary waves, have been studied especially thoroughly in the theory of nonlinear equations. Following the terminology adopted in that theory, we refer to them as Whitham solutions.

By substituting the functions $u^\pm=f^\pm\bigl(S^\pm(x,t)/\mu,x,t,\mu\bigr)$ into the original wave equation and the sum (3.2) into (1.2) and by carrying out the corresponding differentiations, we obtain

$$\Box_c u^\pm=\biggl(\frac{1}{\mu^2}\bigl({S_t^\pm}^2-c^2(x){S_x^\pm}^2\bigr)\frac{\partial^2 f^\pm}{\partial\theta^2}+\frac{1}{\mu}\hat{\Pi}^\pm\frac{\partial f^\pm}{\partial\theta}+\Box_c f^\pm\biggr)\bigg|_{\theta=S^\pm/\mu}=0, \tag{3.3}$$

where

$$\hat{\Pi}^\pm=2S_t^\pm\frac{\partial}{\partial t}+\Box_c S^\pm-2c^2(x)S_x^\pm\frac{\partial}{\partial x}, \tag{3.4}$$

and

$$f^+\Big|_{t=0} + f^-\Big|_{t=0} = A(x)V(\theta), \tag{3.5}$$

$$\frac{1}{\mu} f_\theta^+\Big|_{t=0} S_t^+\Big|_{t=0} + \frac{1}{\mu} f_\theta^-\Big|_{t=0} S_t^-\Big|_{t=0} + f_t^+\Big|_{t=0} + f_t^-\Big|_{t=0} = 0, \qquad \theta = \frac{S_0(x)}{\mu}. \tag{3.6}$$

The function $A(x)$ is equal to 1; later on, we shall need a different choice of $A(x)$. We use the standard asymptotic argument, match the coefficients of μ^{-2}, and assume that $f_{\theta\theta} \neq 0$, thus obtaining the following equation for the function $S^\pm$:

$$S_t^{\pm 2} - c^2(x) S_x^{\pm 2} = 0.$$

This equation splits into two *eikonal*, or *Hamilton–Jacobi, equations*

$$S_t^\pm \pm c(x) S_x^\pm = 0. \tag{3.7}$$

We assume that the signs $+$ and $-$ on $S^\pm$ correspond to the same signs in (3.7). (In fact, they were introduced in view of (3.7).) Recall that the solutions of these equations are called *phases*, or *actions.* The initial conditions (1.2) give rise to the initial condition

$$S^\pm\Big|_{t=0} = x \tag{3.8}$$

for (3.7). Having in mind a representation of the solution of the original problem in the form (2.6), consider the more general conditions

$$S^\pm\Big|_{t=0} = S_0(x) \tag{3.9}$$

assuming that

(i) $S_0(x)$ *is a smooth function equal to* 0 *only at the point* $x = 0$*; moreover,* $\dfrac{dS_0}{dx}(0) = 1.$

3.2. *Properties of Solutions of the Eikonal Equation and Characteristics*

To make the exposition self-contained, we recall well-known (and elementary) facts (e.g., see [8, 14]) concerning the integration of problems (3.7), (3.9). The solution can be reduced to finding one-parameter families of trajectories of the (bi)characteristic (Hamiltonian) systems

$$\dot p = \mp c'(x)p \equiv \frac{\partial \mathcal{H}^\pm}{\partial x}, \qquad \dot x = \pm c(x) \equiv \frac{\partial \mathcal{H}^\pm}{\partial p}, \tag{3.10}$$

$$p\Big|_{t=0} = S_0'(\alpha), \qquad x\Big|_{t=0} = \alpha, \tag{3.11}$$

where $c'(x) = \frac{dc}{dx}$, $S_0'(x) = \frac{dS_0'}{dx}$, $\alpha \in \mathbb{R}$ is a parameter, and the Hamiltonian is

$$\mathcal{H}^\pm(p,x) = \pm c(x)p. \tag{3.12}$$

The component x is called the *coordinate*, and the component p is called the *momentum.* We denote the solutions of (3.10) by $\mathcal{P}^\pm(\alpha,t)$, $\mathcal{X}^\pm(\alpha,t)$. Equations (2.1) split off from system (3.10), and hence the functions $\mathcal{X}^\pm(\alpha,t)$ are solutions of the Cauchy problem $x\big|_{t=0} = \alpha$ for Eq. (2.1); obviously, they are determined as solutions of the equations

$$\int_\alpha^x \frac{d\xi}{c(\xi)} = \pm t. \tag{3.13}$$

It is convenient to find the momentum components from the conservation law for the Hamiltonians $\mathcal{H}^{\pm}$ on the trajectories (3.10):

$$\mathcal{H}^{\pm}(\mathcal{P}^{\pm}(\alpha,t),\mathcal{X}^{\pm}(\alpha,t))=\mathcal{H}^{\pm}(\mathcal{P}^{\pm}(\alpha,0),\mathcal{X}^{\pm}(\alpha,0)) \quad\Longleftrightarrow\quad c(\mathcal{X}^{\pm}(\alpha,t))\mathcal{P}^{\pm}(\alpha,t)=c(\alpha)S_0'(\alpha). \tag{3.14}$$

It follows that

$$\mathcal{P}^{\pm}=\frac{c(\alpha)}{c(\mathcal{X}^{\pm})}S_0'(\alpha). \tag{3.15}$$

All trajectories $(\mathcal{P}^{\pm}(\alpha,t),\mathcal{X}^{\pm}(\alpha,t))$ corresponding to distinct S_0 satisfying property (**i**) obviously coincide with each other for $\alpha=0$ and coincide with the functions introduced in Section 2 for $a=0$:

$$(\mathcal{P}^{\pm}(\alpha,t),\mathcal{X}^{\pm}(\alpha,t))\big|_{\alpha=0}=(P^{\pm}(t),X^{\pm}(t)), \qquad P^{\pm}(t)=\frac{c_0}{c(X^{\pm}(t))}. \tag{3.16}$$

The Jacobians

$$J^{\pm}(\alpha,t)=\frac{\partial\mathcal{X}^{\pm}}{\partial\alpha} \tag{3.17}$$

play an important role in the integration of the eikonal (Hamilton–Jacobi) equations.

Lemma 1. *The Jacobian $J^{\pm}(\alpha,t)=\frac{\partial\mathcal{X}^{\pm}}{\partial\alpha}$ is nonzero for all t.*

Proof. By differentiating the equations $\dot{\mathcal{X}}^{\pm}(\alpha,t)=\pm c\big(\mathcal{X}^{\pm}(\alpha,t)\big)$ and $\mathcal{X}^{\pm}(\alpha,0)=\alpha$ with respect to the parameter α, we obtain

$$\dot{J}^{\pm}(\alpha,t)=\pm c'\big(\mathcal{X}^{\pm}(\alpha,t)\big)J^{\pm}, \qquad J\big|_{t=0}=1.$$

This equation can readily be integrated:

$$J^{\pm}(\alpha,t)=\frac{c\big(\mathcal{X}^{\pm}(\alpha,t)\big)}{c(\alpha)}, \tag{3.18}$$

whence it follows that $J^{\pm}\neq 0$ for all t.

The lemma implies that, for each of the signs $+$ and $-$, there exists a unique solution $\alpha^{\pm}(x,t)$ of the equation

$$\mathcal{X}^{\pm}(\alpha,t)=x \tag{3.19}$$

for α (i.e., Eq. (3.13) for α). According to the theory of Hamilton–Jacobi equations, for all t, there exists a unique solution of the eikonal equation with the initial condition $S^{\pm}\big|_{t=0}=S_0(x)$. It has the form

$$S^{\pm}=S_0\big(\alpha^{\pm}(x,t)\big). \tag{3.20}$$

Furthermore,

$$dS^{\pm}=\mathcal{P}^{\pm}d\mathcal{X}^{\pm} \quad\Longleftrightarrow\quad \begin{gathered} S^{\pm}(\mathcal{X}^{\pm}(\alpha,t),t)=S_0(\alpha),\\ \frac{\partial S^{\pm}}{\partial x}(\mathcal{X}^{\pm}(\alpha,t),t)=\mathcal{P}^{\pm}(\alpha,t)=\frac{c(\alpha)}{c(\mathcal{X}^{\pm})}S_0'(\alpha). \end{gathered} \tag{3.21}$$

In particular, if $S^{\pm}\big|_{t=0}=x$, then we obtain

$$S^{\pm}=\alpha^{\pm}(x,t), \qquad S^{\pm}(\mathcal{X}^{\pm}(\alpha,t),t)=\alpha, \qquad \mathcal{P}^{\pm}(\alpha,t)=\frac{c(\alpha)}{c(\mathcal{X}^{\pm})}. \tag{3.22}$$

These relations readily imply the following elementary assertion, which will be very useful later on in this paper.

Lemma 2. *The following relations hold*:

$$S^{\pm}(x,t)=0 \iff \alpha^{\pm}(x,t)=0 \iff x=X^{\pm}(t),$$
$$\frac{\partial S^{\pm}}{\partial x}(X^{\pm}(t),t)=P^{\pm}(t)=\frac{c(0)}{c(X^{\pm}(t))}, \qquad P^{\pm}(t)c(X^{\pm}(t))=c_0.$$

In other words, for each of the signs $+$ *and* $-$, *the zero set of the function* $S^{\pm}(x,t)$ *coincides with the respective trajectory* $X^{\pm}(t)$. *The functions* $x=X^{\pm}(t)$ *satisfy the equations*

$$\int_0^x \frac{d\xi}{c(\xi)}=\pm t. \tag{3.23}$$

For each of the signs $+$ and $-$, there are two distinguished solutions in the set of solutions of problems (3.7), (3.9). The first solution satisfies condition (3.8) and has already been described. The second solution is the function

$$S^{\pm}=S_0(x)\mp c_0t\equiv \Phi(x)\mp c_0t, \tag{3.24}$$

linearly depending on time t, where $\Phi(x)$ is defined in (2.7) and (2.3). The validity of (3.7) for such functions can be verified by elementary differentiation. Let us find the trajectories of system (3.10) corresponding to the solutions (3.24). First, note that it is often convenient to replace the parameter α characterizing the trajectories by a different parameter by setting $\alpha=\alpha(\tau)$, where τ is a new parameter. Needless to say, this is equivalent to the passage from conditions (3.11) to the conditions

$$p\Big|_{t=0}=\mathcal{P}_0^{\pm}(\tau), \qquad x\Big|_{t=0}=\mathcal{X}_0^{\pm}(\tau), \tag{3.25}$$

where $\mathcal{X}_0^{\pm}(\tau)$ and $\mathcal{P}_0^{\pm}(\tau)$ are smooth functions and $\mathcal{X}_0^{\pm}(\tau)$ determines a one-to-one correspondence between some simply connected open sets in $\mathbb{R}_\tau$ and $\mathbb{R}_x$. Without loss of generality, we assume that

$$\mathcal{X}_0^{\pm}(0)=0. \tag{3.26}$$

For the initial data (3.25) to correspond to (3.9), it is necessary and sufficient to have

$$d\Phi(\mathcal{X}_0^{\pm}(\tau))=\mathcal{P}_0^{\pm}(\tau)d\mathcal{X}_0^{\pm}(\tau) \iff \mathcal{P}_0^{\pm}(\tau)=\frac{d\Phi}{dx}(\mathcal{X}_0^{\pm}(\tau)).$$

If the functions $\mathcal{P}_0^{\pm}(\tau)$ and $\mathcal{X}_0^{\pm}(\tau)$ are given, then

$$\Phi(x)=\int_0^{\tau_0^{\pm}(x)}\mathcal{P}_0^{\pm}(\tau)d\mathcal{X}_0^{\pm}(\tau).$$

Here $\tau_0^{\pm}(x)$ is the solution of Eq. (3.24):

$$\mathcal{X}_0^{\pm}(\tau)=x.$$

Lemma 3. *The families of trajectories of system* (3.10) *corresponding to the solutions* (3.24) *are given by the formulas*

$$\mathcal{P}^{\pm}(\tau,t)=P^{\pm}(\tau+t), \qquad \mathcal{X}^{\pm}(\tau,t)=X^{\pm}(\tau+t); \tag{3.27}$$

moreover, relation (3.14) *for these trajectories has the form*

$$P^{\pm}(\tau+t)c(X^{\pm}(\tau+t))=c_0, \tag{3.28}$$

and the action functions are given by

$$s^{\pm}(\tau) = S^{\pm}(\mathcal{X}^{\pm}(\tau,t),t) = \pm c_0\tau. \tag{3.29}$$

Furthermore,

$$\Phi(x) \mp c_0 t = P^{\pm}(t)(x - X^{\pm}(t)) + O((x - X^{\pm}(t))^2). \tag{3.30}$$

Proof. The curves (3.27) are trajectories of system (3.10), since this system is autonomous. Relation (3.28) follows from Lemma 2. Let us express $\Phi(\mathcal{X}_0^{\pm}(\tau)) = S_0(\mathcal{X}_0^{\pm}(\tau))$ via τ. We have

$$\begin{aligned} S_0(\mathcal{X}_0^{\pm}(\tau)) = \int_0^{\tau} \mathcal{P}_0^{\pm}(\tau)d\mathcal{X}_0^{\pm}(\tau) &= \int_0^{\tau} \mathcal{P}_0^{\pm}(\tau)\dot{\mathcal{X}}_0^{\pm}(\tau)d\tau \\ &= \pm \int_0^{\tau} \mathcal{P}_0^{\pm}(\tau)c(\mathcal{X}_0^{\pm}(\tau))d\tau = \pm c_0 \int_0^{\tau} d\tau = \pm c_0\tau. \end{aligned}$$

By analogy with (3.21), from this and (3.25), we obtain (3.29). To pass from the functions $s^{\pm}(\tau)$ to the functions (3.24), one can solve the systems of equations

$$X^{\pm}(\tau + t) = x. \tag{3.31}$$

By (3.23), their solutions $\tilde{\tau}^{\pm}(x,t)$ have the form

$$\tilde{\tau}^{\pm}(x,t) = \tau^{\pm}(x) - t, \tag{3.32}$$

where

$$\tau^{\pm}(x) = \pm \int_0^x \frac{d\xi}{c(\xi)}. \tag{3.33}$$

By substituting (3.32) and (3.33) into (3.29), we obtain (3.24) and (2.7). Relations (3.30) can be obtained from Lemma 2.

The parameter τ is called the *intrinsic time* of the trajectory family (3.27). *In what follows, we consider either the families of trajectories* $(\mathcal{P}^{\pm}(\alpha,t), \mathcal{X}^{\pm}(\alpha,t))$ *found as solutions of problem* (3.10)–(3.12) *with* $S_0 = x$,

$$\mathcal{P}\Big|_{t=0} = 1, \qquad \mathcal{X}\Big|_{t=0} = \alpha, \tag{3.34}$$

or the families of trajectories $P^{\pm}(\tau + t)$, $X^{\pm}(\tau + t)$. *The parameter* τ *is always understood as intrinsic time.*

Let us write out formulas for some of the objects considered in this section as applied to Examples **a.1** and **b.1** (see Section 2).

Example a.2. Let us write out the trajectory families, actions, and Jacobians corresponding to $c(x)$ from Example **a.1**.

$$\begin{aligned} \mathcal{P}^{\pm}(\alpha,t) &= \frac{\eta(2 + e^{\pm t}(\eta - 1))}{(\eta + 1)(1 + e^{\pm t}(\eta - 1))}; \\ \mathcal{X}^{\pm}(\alpha,t) &= \log \frac{e^{\pm t}(\eta - 1)(2 + e^{\pm t}(\eta - 1))}{8}, \quad \eta = \sqrt{1 + 8e^{a+\alpha}}; \\ J^{\pm}(\alpha,t) &= \frac{(\eta + 1)(1 + e^{\pm t}(\eta - 1))}{\eta(2 + e^{\pm t}(\eta - 1))}; \\ S^{\pm}(x,t) &= \alpha^{\pm}(x,t) = \log \frac{e^{\pm 2t - a}\Big(4e^x + (e^{\pm t} - 1)\big(\sqrt{1 + 8e^x} - 1\big)\Big)}{4}. \end{aligned}$$

One can readily verify that $J^{\pm}\big|_{t=0} = 1$ and that the substitution of $x = X^{\pm}(t)$ of the form (2.8) into $S^{\pm}(x,t)$ gives $S^{\pm}\big|_{x=X^{\pm}(t)} = 0$. The intrinsic time is given by

$$\tau(x) = \frac{x - a}{2} + \frac{1}{2}\bigg(\log \frac{\sqrt{1 + 8e^x} - 1}{\sqrt{1 + 8e^x} + 1} - \log \frac{\sqrt{1 + 8e^a} - 1}{\sqrt{1 + 8e^a} + 1}\bigg).$$

Example b.2. For $c(x)$ from Example **b.1**, we have similar formulas

$$\mathcal{P}^{\pm}(\alpha,t)=1\Big/\Big(1\pm\frac{\gamma}{2\sqrt{a+\alpha}}t\Big),\qquad \mathcal{X}^{\pm}(\alpha,t)=(a+\alpha)\Big(1\pm\frac{\gamma}{2\sqrt{a+\alpha}}t\Big)^2;$$
$$J^{\pm}(\alpha,t)=1\pm\frac{\gamma}{2\sqrt{a+\alpha}}t,\qquad S^{\pm}(x,t)=\Big(\sqrt{x}\mp\frac{\gamma}{2}t\Big)^2-a.$$

Obviously, $J^{\pm}\big|_{t=0}=1$ and the substitution of $x=X^{\pm}(t)$ of the form (2.9) into $S^{\pm}(x,t)$ gives $S^{\pm}\big|_{x=X^{\pm}(t)}=0$. The intrinsic time is given by

$$\tau(x)=\frac{2}{\gamma}(\sqrt{x}-\sqrt{a}).$$

3.3. *The Geometric Interpretation and Lagrangian Manifolds*

The families of trajectories $\mathcal{X}^{\pm}(\alpha,t),\mathcal{P}^{\pm}(\alpha,t)$ and $X^{\pm}(\tau+t),P^{\pm}(\tau+t)$ have a simple, clear geometric interpretation. Namely, let $\mathbb{R}^2_{p,x}$ be the phase plane with coordinates p,x, and let $g^t_{\mathcal{H}^{\pm}}$ be the canonical transformations (phase flows) given by systems (3.10), i.e., the time t transformations taking each point p^0,x^0 to the solutions p^t,x^t of the Cauchy problem $p\big|_{t=0}=p^0$, $x\big|_{t=0}=x^0$ for systems (3.10). For each t, consider the curves

$$\Lambda^{\pm}_t=\{p=\mathcal{P}^{\pm}(\alpha,t),\ x=\mathcal{X}^{\pm}(\alpha,t)\mid\alpha\in\mathbb{R}\}. \tag{3.35}$$

Obviously, $\Lambda^{\pm}_t$ is obtained by shifts along the trajectories of system (3.10) from the straight line $\Lambda^{\pm}_0=\Lambda_0=\{p=1,x=\alpha\}$; in other words, $\Lambda^{\pm}_t=g^t_{\mathcal{H}^{\pm}}\Lambda_0$.

The parameter α on $\Lambda^{\pm}_t$ can be viewed as a coordinate on $\Lambda^{\pm}_t$. The coordinates on $\Lambda^{\pm}_t$ are not uniquely determined. The coordinate α provides a consistent parametrization of points of $\Lambda^{\pm}_t$: a point of $\Lambda^{\pm}_t$ has a coordinate α if it lies on the trajectory $\mathcal{P}^{\pm}(\alpha,t),\mathcal{X}^{\pm}(\alpha,t)$ of system (3.10) issuing at $t=0$ from the point $x=\alpha$ with momentum $p=1$. Such coordinates are called *Lagrangian coordinates*. Another natural coordinate is the original coordinate x. The passage from x to α is given by (3.19). Since $\mathcal{P}^{\pm}(\alpha,t)=\frac{\partial S^{\pm}}{\partial x}(\mathcal{X}^{\pm}(\alpha,t),t)$ (see (3.21)), it follows that the curves $\Lambda^{\pm}_t$ are represented in the x-coordinates in the form $\Lambda^{\pm}_t=\{p=\frac{\partial S^{\pm}}{\partial x}(x,t)\}$; for a given $+$ or $-$ sign, the formulas $s^{\pm}=\alpha$ and $S^{\pm}=\alpha^{\pm}(x,t)$ specify a same function written in different coordinates (α and x, respectively). These functions are actions on $\Lambda^{\pm}_t$. They satisfy the equation

$$dS^{\pm}=pdx\quad\Longleftrightarrow\quad ds^{\pm}=pdx\big|_{\Lambda^{\pm}_t}. \tag{3.36}$$

In the coordinates α, they are independent of t and equal to α. *In what follows, we always denote the action in the x-coordinates by an uppercase letter S and in other coordinates on $\Lambda^{\pm}_t$ (in particular, in the α-coordinates) by a lowercase letter s.* Note that the solution of Eq. (3.36) is not unique; it is determined neglecting some functions $g^{\pm}(t)$. For the corresponding functions (3.36) to be solutions of the eikonal equations, one should fix $g^{\pm}(t)$ by specifying $S^{\pm}$ on some trajectory (i.e., for some α), say, on $P^{\pm}(t),X^{\pm}(t)$ (i.e., for $\alpha=0$) by the formulas $s^{\pm}(\alpha,t)\big|_{\alpha=0}=s^{\pm}(\alpha,0)\big|_{\alpha=0}+\int_0^t\mathcal{L}^{\pm}dt$, where $\mathcal{L}^{\pm}=p\mathcal{H}^{\pm}_p-\mathcal{H}^{\pm}$ are the Lagrangians corresponding to $\mathcal{H}^{\pm}$. Since the Lagrangians are zero, it follows that $s^{\pm}(\alpha,t)\big|_{\alpha=0}=s^{\pm}(\alpha,0)\big|_{\alpha=0}=0$, whence $g^{\pm}(t)=0$ and $s^{\pm}=\alpha$. In other words, the actions are preserved on the trajectories of system (3.10).

All the preceding remains valid for different parametrizations of the curves $\Lambda^{\pm}_t$ as well as for the curves corresponding to problems (3.7), (3.9) with $S_0(x)$ different from x. We have already noted that the trajectory families (3.27) play an important role in applications. One can readily see that, for fixed signs, the corresponding curves on the phase plane coincide for all t. We denote these curves by $M^{\pm}$. As the canonical transformations $g^t_{\mathcal{H}^{\pm}}$ act on $M^{\pm}$, points of $M^{\pm}$ only move on $M^{\pm}$. Thus the curves $M^{\pm}$ are *invariant with respect to the action of* $g^t_{\mathcal{H}^{\pm}}$,

$$g^t_{\mathcal{H}^{\pm}} M^{\pm} = M^{\pm}. \tag{3.37}$$

The intrinsic time τ is a coordinate on $M^{\pm}$; for each time t, a point of $M^{\pm}$ has two coordinates, τ and $\tilde{\tau} = \tau - t$, where $\tilde{\tau}$ is the coordinate of the point of $M^{\pm}$ that moved along the trajectory of system (3.10) from $P^{\pm}(\tilde{\tau}), X^{\pm}(\tilde{\tau})$ to $P^{\pm}(\tau), X^{\pm}(\tau)$ in time t. The passage from the coordinate τ to the coordinate x is given by formula (3.33) and, at time t, by formula (3.32). The actions $s^{\pm}(\tau)$ and $S^{\pm}(x,t)$ are determined by relations (3.29) and (3.24). If $M^{\pm}$ is parametrized by the intrinsic time τ, then, in contrast to the parametrization by α, $s^{+}(\tau) \neq s^{-}(\tau)$. Finally, the Jacobians $J^{\pm} = J^{\pm}(\tau + t)$ are given by the formulas

$$J^{\pm}(\tau + t) = \frac{\partial X^{\pm}}{\partial \tau} \equiv \dot{X}^{\pm} \equiv \pm c(X^{\pm}(\tau + t)). \tag{3.38}$$

Now $J^{\pm}\big|_{t=0} = J^{\pm}(\tau) \neq 1$.

In the case under consideration, the projection of the curves $\Lambda^{\pm}_t$ and $M^{\pm}$ onto the x-axis is always one-to-one. In more general cases, say, if $c(x)$ may vanish, this one-to-one correspondence can fail, and then the phases $S^{\pm}(x,t)$ are multivalued functions (in contrast to the phases $s^{\pm}$ on $\Lambda^{\pm}_t$ and $M^{\pm}$). In the n-dimensional case, the curves $\Lambda^{\pm}_t$ and $M^{\pm}$ are replaced by n-dimensional surfaces, or, more precisely, n-dimensional manifolds known as *Lagrangian manifolds*. In this case, their use substantially clarifies the asymptotic constructions. Having in mind subsequent applications of the argument discussed here to more complicated problems as well as multidimensional problems, we refer to $\Lambda^{\pm}_t$ and $M^{\pm}$ as *Lagrangian* manifolds in what follows; next, $M^{\pm}$ will be called *curves* (*manifolds*) *invariant with respect to* $g^t_{\mathcal{H}^{\pm}}$, or simply *invariant curves* (*manifolds*). Coordinates like α can also be used on the manifolds $M^{\pm}$, but they prove to be less convenient than the intrinsic time τ.

Example a.3-b.3. Let us present the Lagrangian curves and trajectories for Examples **a.1** and **b.1** (Figs. 5 and 6, respectively). The Lagrangian curves $\Lambda^{\pm}_t$ for $t = 0, 2, 4, 6, 8, 10, 12$ in Example **a.1** and for $t = 0, 3, 6, 9, 12$ in Example **b.1** are shown by solid lines. Dotted lines depict the trajectories $\mathcal{X}^{\pm}(\alpha, t)$ for $\alpha = 0, 2, 4, 6, 8$ in Example **a.1** and for $\alpha = -4, -2, 0, 2, 4, 6, 8, 10$ in Example **b.1**. The trajectory $X^{\pm}(t)$ is shown by bold dotted line.

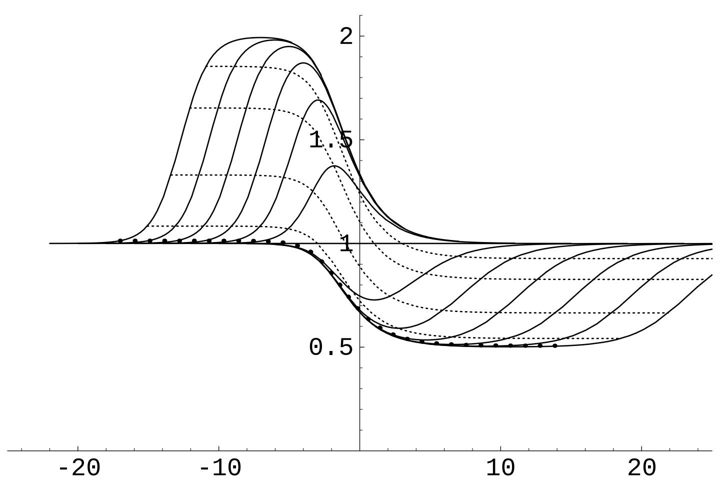

Fig. 5.

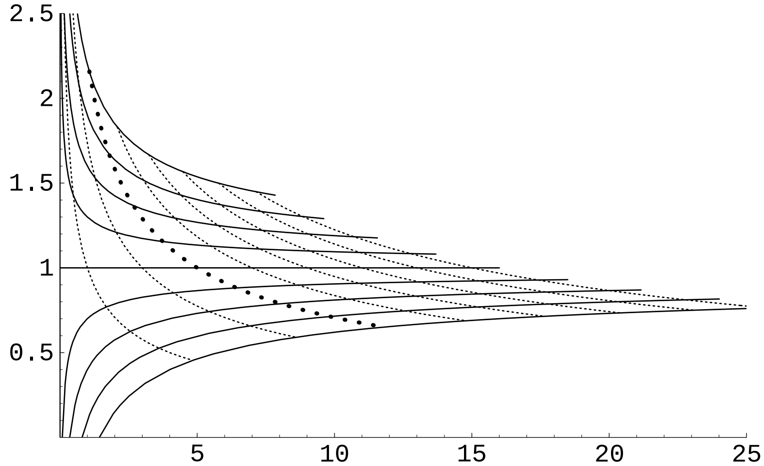

Fig. 6.

3.4. *Transport Equations and Their Solutions*

Now we should find the functions $f^{\pm}(\theta, x, t, \mu)$. One usually cannot find them precisely or even write out effective asymptotic expansions for these functions. However, one can effectively find the leading parts of these functions with respect to μ. These parts are independent of μ, and we denote them by $f_0^{\pm}(\theta, x, t)$. The requirement that Eqs. (3.7) be satisfied kills the terms of the order of $1/\mu^2$ in (3.3). To find the equations for $f_0^{\pm}(\theta, x, t, \mu)$, we assume that $f^{\pm}(\theta, x, t, \mu) = f_0^{\pm}(\theta, x, t, \mu) + O(\mu)$. Substituting this function into (3.3), we see that, in view of the eikonal equations, terms of the order of $1/\mu$ are killed if we require that the *eikonal equations*

$$\hat{\Pi}^{\pm} \frac{\partial}{\partial \theta} f_0^{\pm} = 0 \tag{3.39}$$

be satisfied. Arguing in a similar way, we find that the initial conditions (3.5), (3.6) induce the following initial conditions for the functions $f_0^{\pm}$ (we retain the function $A(x)$):

$$f_0^+\big|_{t=0} + f_0^-\big|_{t=0} = A(x)V(\theta), \qquad f_{0\theta}^+\big|_{t=0} S_t^+\big|_{t=0} + f_{0\theta}^-\big|_{t=0} S_t^-\big|_{t=0} = 0.$$

It follows from the eikonal equations (3.7) and conditions (3.9) that $S_t^+ = -S_t^-$. Hence the second equation can be simplified, and as a result we obtain

$$f_0^+\big|_{t=0} + f_0^-\big|_{t=0} = A(x)V(\theta), \qquad f_{0\theta}^+\big|_{t=0} - f_{0\theta}^-\big|_{t=0} = 0. \tag{3.40}$$

We additionally require that

$$f_0^{\pm} \to 0 \quad \text{as} \quad |\theta| \to \infty. \tag{3.41}$$

The system of characteristic equations for (3.39) with respect to the variables (x, t) coincides with (3.10). This remarkable fact permits one to obtain the following relation for an arbitrary differentiable function $g(x, t, \theta)$:

$$\left[\hat{\Pi}^{\pm} \frac{g(x, t, \theta)}{\sqrt{J^{\pm}(\alpha^{\pm}(x, t), t)}}\right]_{x=X^{\pm}(,t)} = \frac{2S_t^{\pm}(x, t)}{\sqrt{J^{\pm}(x, t)}}\bigg|_{x=X^{\pm}(\alpha,t)} \frac{d}{dt} g\big(X^{\pm}(\alpha, t), t, \theta\big), \tag{3.42}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} \pm c\big(X^{\pm}(\alpha, t)\big)\frac{\partial}{\partial x}$ is the derivative along the trajectories of system (3.10). The variable θ occurs in these relations as a parameter.

By representing $f_0^\pm$ in the form $f_0^\pm = \tilde f_0^\pm/\sqrt{J^\pm}$, for $\tilde f_0^\pm$ we obtain the equations $\frac{d}{dt}\frac{\partial \tilde f_0^\pm}{\partial\theta} = 0$. By integrating these equations in view of the initial conditions (3.40) and the relations $J^\pm\big|_{t=0} = 1$, we find the solution $f_0^\pm$ in the form

$$f_0^\pm = \frac{A(\alpha^\pm(x,t))V(\theta) \pm q(x,t)}{2\sqrt{J^\pm(\alpha^\pm(x,t),t)}} = \frac{A(\alpha^\pm(x,t))\sqrt{c(\alpha^\pm(x,t))}V(\theta) \pm q(x,t)}{2\sqrt{c(x)}}, \tag{3.43}$$

where $q(x,t)$ is a smooth function. Conditions (3.41) give $q = 0$, and we finally obtain

$$f_0^\pm = \frac{A(\alpha^\pm(x,t))\sqrt{c(\alpha^\pm(x,t))}}{2\sqrt{c(x)}}V(\theta). \tag{3.44}$$

The same argument in the case of the invariant manifold $M^\pm$ gives solutions of the transport equation in the form

$$f_0^\pm = \frac{A(\tau-t)\sqrt{c(X^\pm(\tau-t))}}{2\sqrt{c(x)}}\bigg|_{\tau=\tau^\pm(x)} V(\theta). \tag{3.45}$$

4. EXPANSIONS OF BOUNDARY LAYER TYPE AND THE RELATIONSHIP BETWEEN LOCALIZED AND RAPIDLY OSCILLATING SOLUTIONS

4.1. *Expansions of the Phase and the Amplitude and Lagrangian "Intervals"*

The formulas obtained above permit one to write out the leading term of the asymptotic solution of problem (1.1), (1.2):

$$u_0 = \frac{1}{2}(u_0^+ + u_0^-), \quad u_0^\pm = \frac{A(\alpha^\pm(x,t))\sqrt{c(\alpha^\pm(x,t))}}{2\sqrt{c(x)}}V\Big(\frac{S^\pm(x,t)}{\mu}\Big), \tag{4.1}$$
$$S^\pm(x,t) = \alpha^\pm(x,t).$$

This function u_0 differs from the functions (2.5) and (2.6). Let us show that they differ from each other by $O(\mu)$. Since the function $V(\theta)$ decays, it follows that the functions $V(S^\pm(x,t)/\mu)$ rapidly decay outside an $O(\mu)$-neighborhood of zeros of the functions $S^\pm(x,t)$. Hence we can employ the argument used in boundary layer theory [16, 20] and complex germ theory [13] and expand the phases and amplitudes in (4.1) in Taylor series. It suffices to retain terms linear in $x - X^\pm(t)$ in the phases and replace x by $X^\pm(t)$ in the amplitudes:

$$S^\pm(x,t) = P^\pm(t)(x - X^\pm(t)) + O((x - X^\pm(t))^2) \equiv \frac{c_0}{c\big(X^\pm(t)\big)}(x - X^\pm(t)) + O((x - X^\pm(t))^2),$$
$$A(\alpha^\pm(x,t)) = A(0) + O(x - X^\pm(t)), \qquad c(\alpha^\pm(x,t)) = c_0 + O(x - X^\pm(t)),$$
$$c(x) = c(X^\pm(t)) + O(x - X^\pm(t)). \tag{4.2}$$

The difference from the case of an "ordinary boundary layer" is that the points in whose neighborhoods the corresponding expansions are carried out are points of the fronts and move in the course of time. The possibility of using the expansions (4.2) is based on the following assertion.

Lemma 4. *Let $\sigma(y)$ be a smooth function vanishing only at the point $y = 0$, let $\frac{\partial\sigma}{\partial y}(0) = q \neq 0$, and let $\chi(y)$ be a smooth function bounded together with all derivatives. Then*

$$\chi(y)V\Big(\frac{\sigma(y)}{\mu}\Big) = \chi(0)V\Big(\frac{qy}{\mu}\Big) + \psi(y,\mu),$$

where $|\psi(y,\mu)| \leqslant K\mu$, $K = \mathrm{const}$.

Proof. One has $|\sigma(y)|/\mu \geqslant \mathrm{Const}\,\mu^{-1/4}$ outside an $O(\mu^{3/4})$-neighborhood of the point $y = 0$, and since the function $V(\theta)$ rapidly decays, it follows that the product $\chi(y)V(\sigma(y)/\mu)$ is $O(\mu^l)$

outside this neighborhood, where $l \geqslant 1$. By Taylor's formula, $\sigma(y) = qy + b_1y^2 + O(\mu^{9/4})$ and $\chi(y) = \chi(0) + b_2y + O(\mu^{3/2})$ in the $O(\mu^{3/4})$-neighborhood of $y = 0$. Now we apply Taylor's formula to the function $\chi(y)V(\sigma(y)/\mu)$, use the fact that V is bounded together with all derivatives, and obtain

$$\chi(y)V\Big(\frac{\sigma(y)}{\mu}\Big) = (\chi(0)+b_2y)\Big(V\Big(\frac{qy}{\mu}\Big)+\frac{\partial V}{\partial\theta}\Big(\frac{qy}{\mu}\Big)\frac{y^2}{\mu}\Big)+O(\mu)$$
$$= \chi(0)V(qy')+\mu\Big[b_2y'\Big(V(qy')+\mu\frac{\partial V}{\partial\theta}(qy')y'^2\Big)\Big]+O(\mu),$$

where the $o(\mu)$ estimate is valid in the norm of $\mathbf{C}(\mathbb{R}_x)$ and $y = \mu y'$. Since the function $V(qy')$ rapidly decays together with all derivatives, it follows that the expression in square brackets in the last relation is bounded for all y'. This proves the lemma.

Remark. 1. If the function $\chi(y)$ is represented as the product $\chi_1(y)\chi_2(y)$ of two bounded smooth functions, then one may well make the substitution $y \to 0$ in only one of the factors.

2. Two functions $\chi(y)V(\sigma(y)/\mu)$ and $\tilde\chi(y)V(\tilde\sigma(y)/\mu)$ in the above-mentioned class coincide modulo $O(\mu)$ if $\chi(0) = \tilde\chi(0)$, $\sigma(0) = \tilde\sigma(0)$, and $d\sigma/dy(0) = d\tilde\sigma/dy(0)$.

3. The assertion of the lemma and the preceding remarks remain valid if the functions $\chi(y)$ and $\sigma(y)$ smoothly depend on parameters ranging in a compact set.

Since the function $\chi(y)V(\sigma(y)/\mu)$ is essentially localized in an $O(\mu^{1-\varepsilon})$-neighborhood of $y = 0$, we can multiply it by a smooth cutoff function $\mathrm{e}(y)$, where $\mathrm{e}(y) = 1$ for $|y| \leqslant \delta/2$, $\mathrm{e}(y) = 0$ for $|y| \geqslant \delta$, and $\delta > 0$ is a small number independent of μ. (The function $\mathrm{e}(y)$ is shown in Fig. 7.) Then the difference is $O(\mu^k)$, $k > 0$, both in the norm of $\mathbf{C}(\mathbb{R}_x)$ and in the Sobolev norm in $\mathbf{H}_s(\mathbb{R}_x)$, $s > 0$, with some k depending on the behavior of V and σ at infinity. We restrict ourselves to the case $\sigma = y$, and for us it suffices to require that $|V| < C^0/|y|$ and $|dV/dy| < C^1/|y|^2$ as $|y| \to \infty$.

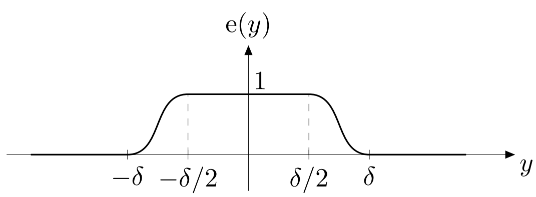

Fig. 7.

Lemma 5. *Under these assumptions,* $\|(1-\mathrm{e}(y))V(y/\mu)\|_{W_2^1} \leqslant K\mu$, *where* K *is some constant.*

Proof. The inequalities

$$\Big|V\Big(\frac{y}{\mu}\Big)\Big| < \mu\frac{C^0}{|y|}, \quad \Big|\frac{d}{dy}V\Big(\frac{y}{\mu}\Big)\Big| < \mu\frac{C^1}{|y|^2}$$

are valid for $|y| > \delta$. Hence

$$\Big\|(1-\mathrm{e}(y))V\Big(\frac{y}{\mu}\Big)\Big\|^2_{W_2^1} = \Big\|(1-\mathrm{e}(y))V\Big(\frac{y}{\mu}\Big)\Big\|^2_{L_2} + \Big\|\frac{d}{dy}\Big((1-\mathrm{e}(y))V\Big(\frac{y}{\mu}\Big)\Big)\Big\|^2_{L_2}$$
$$\leqslant 2\int_\delta^\infty \Big[V\Big(\frac{y}{\mu}\Big)^2\Big(1+\Big(\frac{d\mathrm{e}(y)}{dy}\Big)^2\Big)\Big]dy + 2\int_\delta^\infty \Big(\frac{d}{dy}V\Big(\frac{y}{\mu}\Big)\Big)^2 dy \leqslant \mu^2K_1\int_\delta^\infty\frac{dy}{y^2} = \mu^2\frac{K_1}{\delta},$$

where $K_1 > 0$ is a constant. This implies the desired estimate.

Let us apply the lemma to the functions (2.5), (2.6), and (4.1), setting

$$A(x) = \mathrm{e}(x) \tag{4.3}$$

in the last formula. By passing from x to the variable $y = x - X^{\pm}(t)$ and by taking into account the properties (see Subsection 3.2) of the functions $S^{\pm}(x,t)$ and $\Phi(x) \mp c_0t$ and the remarks to the lemma, we arrive at the following assertion.

Lemma 6. *The functions* (2.5)*,* (2.6)*, and* (4.1) *coincide modulo* $O(\mu)$*.*

In (4.1), we take A in the form (4.3). Then it suffices to know the phases $S^{\pm}$ also for $|\alpha^{\pm}| < \delta$. In turn, this means that, instead of the Lagrangian curves $\Lambda_t^{\pm}$, it suffices to have the Lagrangian "intervals"

$$\{p = \mathcal{P}^{\pm}(\alpha, t),\ x = \mathcal{X}^{\pm}(\alpha, t), \quad |\alpha| < \delta\}. \tag{4.4}$$

Essentially, these intervals also do not occur in the definitive formulas for the leading term of the asymptotics; the answer contains only the trajectories $X^{\pm}(t)$, $c(x)$, and $V(\theta)$. But technically these intervals prove very useful. To avoid complicated notation, in what follows, we denote Lagrangian "intervals" by the same symbol $\Lambda_t^{\pm}$.

In a similar way, one can introduce Lagrangian "intervals" corresponding to the invariant Lagrangian manifold $M^{\pm}$:

$$M_t^{\pm} = \{p = P^{\pm}(t+\tau),\ x = X^{\pm}(t+\tau), \quad |\tau| < \delta\}.$$

They have the property

$$g^t_{\mathcal{H}^{\pm}} M^{\pm}_{t_0} = M^{\pm}_{t_0+t}$$

for any t_0 and t.

Example a.4-b.4. Figures 8 and 9 show the Lagrangian intervals for Examples **b.1** and **a.1**, respectively. Dotted lines depict the "noninvariant intervals" $\Lambda_t^{\pm}$, and solid lines depict the intervals $M_t^{\pm}$ corresponding to the invariant Lagrangian manifold $M^{\pm}$.

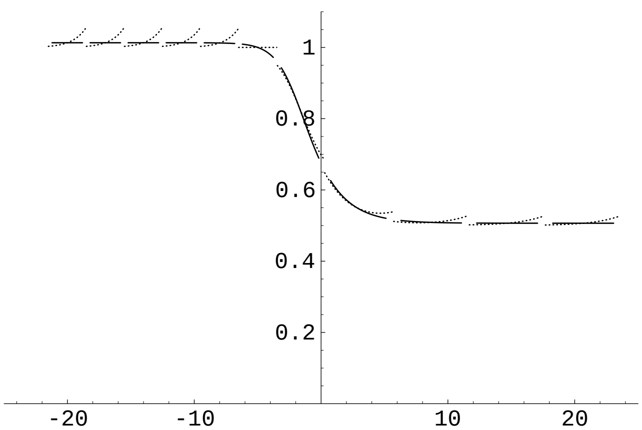

Fig. 8.

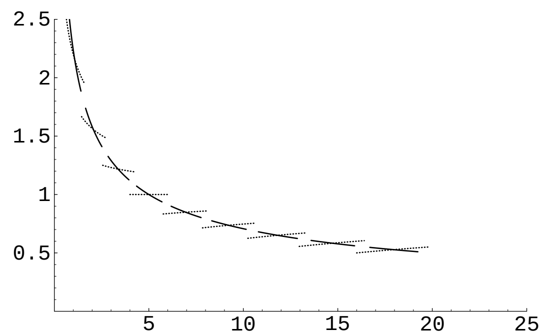

Fig. 9.

4.2. *Localized and Rapidly Oscillating Solutions*

In quantum and wave mechanics, the eikonal and transport equations, as well as the corresponding Hamiltonian systems, usually arise in the construction of rapidly oscillating WKB solutions

$$w^{\pm} = \varphi^{\pm}(x,t) e^{i\frac{S^{\pm}(x,t)}{h}}, \qquad \varphi^{\pm}(x,t) = \varphi_0^{\pm}(x,t) + O(h), \tag{4.5}$$

depending on a "semiclassical" small parameter h. *There is a very simple but important correspondence between the solutions* (4.1) *and* (4.5). Before we give the relevant formulas, let us recall elementary facts [2, 8, 14] concerning the solutions (4.5).

One obtains equations for the phases $S^{\pm}(x,t)$ and the amplitudes $\varphi^{\pm}(x,t)$ by substituting (4.5) into the original wave equation (1.1), by differentiating, and by matching the coefficients of h^{-1} and h^{-2}. One can obtain them by formally choosing $f = \varphi^{\pm}(x,t)e^{i\theta}$ in (3.3) and then by cancelling out $e^{i\theta}$. This readily gives the eikonal equations (3.7) and the transport equations

$$\hat{\Pi}^{\pm} \varphi_0^{\pm} = 0. \tag{4.6}$$

For the solutions we take the phases (3.20) and the amplitudes

$$\varphi_0^\pm = \frac{\mathrm{e}(\alpha^\pm(x,t))\sqrt{c(\alpha^\pm(x,t))}}{2\sqrt{c(x)}}.$$

We set $w_0^\pm = \varphi_0^\pm e^{i\frac{S^\pm(x,t)}{h}}$ and denote the Fourier transform of $V(\theta)$ by $\tilde V(\rho)$:

$$\tilde V(\rho) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} V(\theta)e^{-i\rho\theta}\,d\theta. \tag{4.7}$$

Now we set

$$h = \frac{\mu}{\rho}, \tag{4.8}$$

multiply the WKB solutions $w_0^\pm$ by $\tilde V(\rho)/\sqrt{2\pi}$, and integrate the result over ρ from $-\infty$ to ∞. This almost gives the desired correspondence between rapidly decaying and rapidly oscillating solutions:

$$u_0^\pm = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w_0^\pm\Big|_{h=\mu/\rho}\tilde V(\rho)d\rho \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi_0^\pm e^{i\frac{\rho}{\mu}S^\pm(x,t)}\tilde V(\rho)\,d\rho. \tag{4.9}$$

The parameter h is usually assumed to be positive; this assumption is needed in important computations and conclusions. Let us rewrite the last formula in a form in which h is only positive. We have

$$\begin{aligned}u_0^\pm = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\varphi_0^\pm e^{i\frac{\rho}{\mu}S^\pm(x,t)}\tilde V(\rho)\,d\rho + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\varphi_0^\pm e^{i\frac{\rho}{\mu}S^\pm(x,t)}\tilde V(\rho)\,d\rho\\ = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\varphi_0^\pm e^{i\frac{\rho}{\mu}S^\pm(x,t)}\tilde V(\rho)\,d\rho + \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\varphi_0^\pm e^{-i\frac{\rho}{\mu}S^\pm(x,t)}\tilde V(-\rho)\,d\rho.\end{aligned}$$

This, in conjunction with the relation $\tilde V(-\rho) = \overline{\tilde V(\rho)}$, which holds because $V(\theta)$ is real, implies the desired correspondence formula

$$u_0^\pm = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\int_0^{\infty}\varphi_0^\pm e^{i\frac{\rho}{\mu}S^\pm(x,t)}\tilde V(\rho)\,d\rho = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\int_0^{\infty} w_0^\pm\Big|_{h=\mu/\rho}\tilde V(\rho)\,d\rho. \tag{4.10}$$

Now let us make an important conclusion. Since the original equation is hyperbolic (and homogeneous in the derivatives), it follows that ρ is a parameter that is not involved in the integration of the eikonal and transport equations, and the multiplication by $\tilde V(\rho)$ and integration over ρ *commute* with the original wave operator. It is this fact that permits one first to construct oscillating semiclassical asymptotic solutions of the wave equation and then use integration to proceed to rapidly decaying solutions, taking into account elementary auxiliary considerations from boundary layer theory.

4.3. *Invariant Manifolds and Quasimode Expansions*

A correspondence similar to (4.9) can also be established for the solution representation (2.6) corresponding to invariant Lagrangian manifolds. This permits one to assign a clear physical meaning to (2.6). We have

$$\begin{aligned}\sum_\pm \frac12\sqrt{\frac{c_0}{c(x)}}\,V\Big(\frac{\Phi(x)\mp c_0t}{\mu}\Big) = \sum_\pm \frac12\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{c_0}{c(x)}}e^{i\frac{\rho}{\mu}\Phi(x)}e^{\mp i\frac{\rho c_0 t}{\mu}}\tilde V(\rho)\,d\rho\\ = \int_{-\infty}^{\infty}\Psi^-(x,\omega)e^{i\omega t}\Big(\mu\overline{\tilde V}\Big(\frac{\omega\mu}{c_0}\Big)\Big)d\omega, + \int_{-\infty}^{\infty}\Psi^+(x,\omega)e^{i\omega t}\Big(\mu\tilde V\Big(\frac{\omega\mu}{c_0}\Big)\Big)d\omega\end{aligned} \tag{4.11}$$

where

$$\Psi^\pm(x,\omega) = \frac{1}{2\sqrt{2\pi}}\sqrt{\frac{c_0}{c(x)}}e^{\pm i\frac{\omega}{c_0}\Phi(x)}, \qquad \omega = \frac{\rho c_0}{\mu}.$$

One can readily see that the functions $\Psi^{\pm}(x,\omega)$ satisfy the equation

$$-\frac{\partial}{\partial x}\Big(c^2(x)\frac{\partial}{\partial x}\Big)\Psi^{\pm}(x,\omega)=\omega^2\Psi^{\pm}(x,\omega) \mod O(1)$$

as $|\omega|\to\infty$ and are asymptotic generalized eigenfunctions of the operator $-\frac{\partial}{\partial x}\big(c^2(x)\frac{\partial}{\partial x}\big)$ corresponding to the eigenvalues ω^2. Thus, formula (2.6) is an expansion of the solution in asymptotic generalized eigenfunctions of this operator. Note that first, the functions $\Psi^{\pm}(x,\omega)$ can substantially differ from the true eigenfunctions (hence they are often called *quasimodes*) and second, the operator $-\frac{\partial}{\partial x}\big(c^2(x)\frac{\partial}{\partial x}\big)$ can apparently have discrete spectrum eigenfunctions, which are lacking in the integral representation (4.11). However, their contribution to the asymptotics of the solution of the original problem is at most $O(\mu)$. This can readily be explained from the physical viewpoint: the expansion (integral) (4.11) contains waves of various lengths, including "very long" waves corresponding to small ρ and nonsmall h. But the behavior of such waves (nonoscillating functions) cannot be described in the framework of the WKB method or geometric optics; they can only be studied as solutions of the full original wave equation or the corresponding Volterra equation (see Introduction in [12]). Since these components are small, they are of little interest to us. We estimate them by using *Petrovskii's estimates.*

5. ESTIMATE OF THE DIFFERENCE BETWEEN THE EXACT AND ASYMPTOTIC SOLUTIONS

5.1. *Petrovskii's Estimates*

We see that the construction of the leading term of the asymptotics (2.5), (2.6) is given by a relatively simple well-known procedure. It might seem that the corrections can be obtained fairly easily as well. Moreover, as a rule, the question concerning corrections is more of academic than of practical interest. Nevertheless, the situation in this problem is by no means elementary. Namely, attempts to construct corrections do not improve the accuracy of the asymptotic expansion. The first correction is indeed $O(\mu)$, but all subsequent corrections have the same order. This fact also reveals itself in formulas like (4.10) taking into account corrections. The corrections to the rapidly oscillating solutions $w_0^{\pm}$ are written out with respect to the parameter h, which fails to be small if the integration variable ρ is small; accordingly, the corrections to $u_0^{\pm}$ corresponding to these ρ are not arbitrarily small. Hence *we can constructively obtain only the leading asymptotic term of the solution of problem* (1.1),(1.2)*, and one can even say that the asymptotics exists but the asymptotic expansion does not.*

To estimate the difference between the exact and asymptotic solutions, we use Petrovskii's a priori estimates. It suffices to use the following (special) case of these estimates (see [9]). Let $w(x,t)$ be a solution of the Cauchy problem

$$\Box_c w=\mathcal{F}(x,t), \qquad w|_{t=0}=F_0(x), \qquad \frac{\partial w}{\partial t}\Big|_{t=0}=F_1(x). \tag{5.1}$$

Then w satisfies the inequalities

$$\|w\|_{W_2^1}+\Big\|\frac{\partial w}{\partial t}\Big\|_{W_2^1}\leqslant C\Big(\|F_0\|_{W_2^1}+\|F_1\|_{W_2^1}+\int_0^t\|\mathcal{F}\|_{L_2}\,dt'\Big), \tag{5.2}$$

where C is some constant (depending on $c(x)$) and $\|w\|_{W_2^1}=\big\|w\big\|_{L_2}^2+\big\|\partial w/\partial x\big\|_{L_2}^2$ is the norm in the Sobolev space W_2^1. By Sobolev's embedding theorem, which states that

$$\max_{x\in\mathbb{R}}|w|\equiv\|w\|_{\mathbb{C}}\leqslant\|w\|_{W_2^1},$$

this, together with (5.2), readily implies the desired estimate

$$\|w\|_{\mathbb{C}}\leqslant C\Big(\|F_0\|_{W_2^1}+\|F_1\|_{W_2^1}+\int_0^t\|\mathcal{F}\|_{L_2}\,dt'\Big). \tag{5.3}$$

Our subsequent argument is as follows. Let u_{as} be an asymptotic solution of problem (1.1), (1.2) satisfying (1.1) with remainder $\mathcal{F}$ and the initial conditions (1.2) with remainders F_0 and F_1. Let $w = u_{\mathrm{as}} - u$ be the difference between the exact and asymptotic solutions of problem (1.1), (1.2). Then the difference $w = u_{\mathrm{as}} - u$ obviously satisfies the estimate (5.3), and the proof of formulas (2.5) and (2.6) is reduced to proving that $\mathcal{F}$, F_0, and F_1 are small. For $u_{\mathrm{as}} = u_0$, where u_0 is defined in (4.1), one has $\mathcal{F} = O(\mu)$ and $F_2 = 0$ but $F_1 = O(1)$, and so the estimate (5.3) does not give the desired result for this choice. To obtain F_1 with small norm, one has to construct several (more precisely, two) corrections to the leading term. All these corrections have virtually the same order of smallness with respect to μ, and by including them in u_{as}, one does not change the order of $\mathcal{F}$ but reduces F_1. These corrections can be constructed by several methods. In one method, one seeks the corrections in "Whitham form," and the second method is based on WKB type solutions (see [4]). Although the first method seems to be more natural for the one-dimensional problem in question, it cannot be generalized to the multidimensional case. For completeness, we give both methods but pay more attention to the second method, having in mind the generalization to multidimensional problems. Let us start from the second method.

5.2. *Formulas for WKB Expansions*

Having in mind a representation of the asymptotic solution with corrections in a form similar to (4.9), we first construct certain asymptotic expansions of the WKB type. Consider the Cauchy problem

$$\Box_c u = 0, \qquad u|_{t=0} = A(x)e^{i\frac{S_0(x)}{h}}, \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0$$

for a function $u = u(x,t)$, where $A(x)$ and $S_0(x)$ are smooth functions and $h \ll 1$ is a small parameter. We seek particular asymptotic solutions of this problem in the form

$$u = u^+ + u^-, \qquad u^\pm = \varphi^\pm(x,t,h)e^{i\frac{S^\pm(x,t)}{h}}, \tag{5.4}$$

where the functions $\varphi^\pm(x,t,h)$ admit the expansion

$$\varphi^\pm(x,t,h) = \varphi_0^\pm(x,t) + h\varphi_1^\pm(x,t) + h^2\varphi_2^\pm(x,t) + \cdots. \tag{5.5}$$

By substituting the functions $u^\pm$ into the original wave equation, we obtain

$$\Box_c u = \Big(-\frac{1}{h^2}\big(S_t^{\pm 2} - c^2(x)S_x^{\pm 2}\big)\varphi + \frac{i}{h}\hat{\Pi}^\pm\varphi + \Box_c\varphi\Big)e^{i\frac{S^\pm(x,t)}{h}} = 0,$$

where the transport operators $\hat{\Pi}^\pm$ are defined in (3.4). (Formally, this relation follows from (3.3) if we set $f^\pm = \varphi^\pm(x,t,h)e^{i\theta}$ there.) By arguing just as in Subsections 3.1 and 3.4, by taking into account the expansion (5.5), and by matching the coefficients of like powers of h, we obtain the eikonal equations (3.7) for the phases $S^\pm$ and the chain

$$\hat{\Pi}^\pm\varphi_0^\pm = 0, \tag{5.6}$$

$$\hat{\Pi}^\pm\varphi_k^\pm = i\,\Box_c\varphi_{k-1}^\pm \tag{5.7}$$

of transport equations. By substituting (5.4) into the initial conditions, differentiating, taking into account the expansion (5.5), and matching the coefficients of like powers of h, we obtain the initial conditions (3.9) for the eikonal equations and the initial conditions

$$\varphi_0^+\Big|_{t=0} + \varphi_0^-\Big|_{t=0} = A(x), \qquad \varphi_0^+\Big|_{t=0}S_t^+\Big|_{t=0} + \varphi_0^-\Big|_{t=0}S_t^-\Big|_{t=0} = 0, \tag{5.8}$$

$$\varphi_k^+\Big|_{t=0} + \varphi_k^-\Big|_{t=0} = 0,$$

$$i\varphi_k^+\Big|_{t=0}S_t^+\Big|_{t=0} + i\varphi_k^-\Big|_{t=0}S_t^-\Big|_{t=0} + \varphi_{(k-1)t}^+\Big|_{t=0} + \varphi_{(k-1)t}^-\Big|_{t=0} = 0, \quad k = 1,2,\dots, \tag{5.9}$$

for the transport equations (5.6) and (5.7).

The solutions of the eikonal equations were constructed and studied in Subsection 3.1. The transport equations can be integrated with the use of the trajectories $(\mathcal{P}(\alpha,t),\mathcal{X}(\alpha,t))$ of system (3.10), (3.11) as well. By using relations (3.42) and the eikonal equations (3.7) for $t=0$ and by integrating with respect to t, we arrive at the following formulas for the general solutions of the transport equations:

$$\varphi_0^\pm = \frac{R_0^\pm(\alpha)}{\sqrt{J^\pm(\alpha,t)}}\bigg|_{\alpha=\alpha^\pm(x,t)}, \tag{5.10}$$

$$\varphi_k^\pm = \mp\left[\frac{i}{2c(\alpha)S_0'(\alpha)\sqrt{J^\pm(\alpha,t)}}\int_0^t \sqrt{J^\pm(\alpha,t^*)}\big[\Box_c\varphi_{k-1}^\pm(x,t^*)\big]_{x=\mathcal{X}^\pm(\alpha,t^*)}\,dt^*\right]_{\alpha=\alpha^\pm(x,t)}$$
$$+\frac{R_k^\pm(\alpha)}{\sqrt{J^\pm(\alpha,t)}}\bigg|_{\alpha=\alpha^\pm(x,t)}, \quad k=1,2,\dots \tag{5.11}$$

Here the Jacobians $J^\pm$ and the functions $\alpha^\pm(x,t)$ are defined in Subsection 3.2, and $R_k^\pm(\alpha)$ are yet unknown smooth functions. By substituting (5.10) and (5.11) into (5.8) and (5.9), in view of the relations $S_t^\pm\big|_{t=0} = \mp c(x)S_0'(x)$, we obtain the following systems for the unknown functions $R_k^\pm(x)$, $k=0,1,2,\dots$:

$$R_0^+ + R_0^- = A(x), \quad R_0^+ - R_0^- = 0,$$
$$R_k^+ + R_k^- = 0, \qquad R_k^- - R_k^+ = i\frac{\varphi^+_{(k-1)t}\big|_{t=0} + \varphi^-_{(k-1)t}\big|_{t=0}}{c(x)S_0'(x)}, \quad k=1,2,\dots$$

Hence we find

$$R_0^\pm(x) = \frac{A(x)}{2}, \quad R_k^\pm(x) = \mp i\frac{\varphi^+_{(k-1)t}\big|_{t=0} + \varphi^-_{(k-1)t}\big|_{t=0}}{2c(x)S_0'(x)}, \qquad k=1,2,\dots$$

Take $A(\alpha)=\mathrm{e}(\alpha)$. Now, by substituting the function found above into (5.10) and (5.11), we finally obtain

$$\varphi_0^\pm = \left[\frac{\mathrm{e}(\alpha)}{2\sqrt{J^\pm}}\right]_{\alpha=\alpha^\pm(x,t)}, \tag{5.12}$$

$$\varphi_k^\pm = \mp\frac{i}{2c(\alpha)S_0'(\alpha)\sqrt{J^\pm(\alpha,t)}}\bigg|_{\alpha=\alpha^\pm(x,t)}\bigg(\left[\int_0^t \sqrt{J^\pm(\alpha,t^*)}\big[\Box_c\varphi_{k-1}^\pm(x,t^*)\big]_{x=\mathcal{X}^\pm(\alpha,t^*)}\,dt^*\right]_{\alpha=\alpha^\pm(x,t)}$$
$$+\big[\varphi^+_{(k-1)t}+\varphi^-_{(k-1)t}\big]_{t=0}\bigg), \quad k=1,2,\dots \tag{5.13}$$

This, together with (5.4), (5.5), and the formulas for $S^\pm(x,t)$, gives the desired WKB-expansions.

For each t and for a fixed sign $+$ or $-$, the support of the function $\varphi_0^\pm$ given by (5.12) obviously coincides with the interval $D_t^\pm = \{x\in\mathbb{R}:\ |\alpha^\pm(x,t)|<\delta\}$, which is just the fixed interval $|\alpha|<\delta$ in the coordinates α, t. Clearly, the supports of the functions $\Box_c\varphi_0^\pm$ and $\varphi_{0t}^\pm$ have the same property. The integration over t in (5.13) obviously preserves this property. Thus, the support of each of the functions $\varphi_1^\pm$ is at least contained in the interval $D_t^\pm$. By reproducing the same argument for $k=2,3,\dots$, we find that *all the functions $\varphi_k^\pm$ are compactly supported and their supports are contained in $D_t^\pm$.*

The expression for $\varphi_1^\pm(x,t)$ can be slightly simplified. Namely, we have

$$\varphi_1^\pm = \mp\Bigg[\frac{i}{4c(\alpha)S_0'(\alpha)\sqrt{J^\pm(\alpha,t)}}\int_0^t \sqrt{J^\pm(\alpha,t^*)}\left[\Box_c\frac{\mathrm{e}\big(\alpha^\pm(x,t^*)\big)}{\sqrt{J^\pm\big(\alpha^\pm(x,t^*),t^*\big)}}\right]_{x=\mathcal{X}^\pm(\alpha,t^*)}dt^*$$
$$+\frac{\partial}{\partial t}\left(\frac{\mathrm{e}\big(\alpha^+(x,t)\big)}{\sqrt{J^+\big(\alpha^+(x,t),t\big)}}+\frac{\mathrm{e}\big(\alpha^-(x,t)\big)}{\sqrt{J^-\big(\alpha^-(x,t),t\big)}}\right)\bigg|_{t=0}\Bigg]_{\alpha=\alpha^\pm(x,t)}.$$

By differentiating (3.19) with respect to t, we obtain $\frac{\partial \mathcal{X}^\pm}{\partial \alpha}\big|_{t=0}\frac{\partial \alpha^\pm}{\partial t}\big|_{t=0}+\dot{\mathcal{X}}\big|_{t=0}=0$. Since $\frac{\partial \mathcal{X}^\pm}{\partial \alpha}\big|_{t=0}=1$ and $\dot{\mathcal{X}}\big|_{t=0}=\pm c(x)$, it follows that $\frac{\partial \alpha^\pm}{\partial t}\big|_{t=0}=\mp c(x)$. Hence

$$
\begin{aligned}
\frac{\partial}{\partial t}\bigg(\frac{\mathrm{e}\big(\alpha^+(x,t)\big)}{\sqrt{J^+\big(\alpha^+(x,t),t\big)}}+\frac{\mathrm{e}\big(\alpha^-(x,t)\big)}{\sqrt{J^-\big(\alpha^-(x,t),t\big)}}\bigg)\bigg|_{t=0}
&= \frac{1}{\sqrt{c(x)}}\,\frac{\partial}{\partial t}\Big[(\mathrm{e}\sqrt{c})\big(\alpha^+(x,t)\big)+(\mathrm{e}\sqrt{c})\big(\alpha^-(x,t)\big)\Big]\Big|_{t=0}\\
&= \frac{1}{\sqrt{c(x)}}\frac{\partial}{\partial \alpha}(\mathrm{e}\sqrt{c})(\alpha)\big[\alpha_t^+(x,t)+\alpha_t^-(x,t)\big]_{t=0}=0
\end{aligned}
$$

and

$$
\varphi_1^\pm=\mp\left[\frac{i}{4c(\alpha)S_0'(\alpha)\sqrt{J^\pm(\alpha,t)}}\int_0^t\sqrt{J^\pm(\alpha,t^*)}\bigg[\Box_c\frac{\mathrm{e}\big(\alpha^\pm(x,t^*)\big)}{\sqrt{J^\pm\big(\alpha^\pm(x,t^*),t^*\big)}}\bigg]_{x=\mathcal{X}^\pm(\alpha,t^*)}dt^*\right]_{\alpha=\alpha^\pm(x,t)}.
$$

Now let us write out the derivative $\partial\varphi_1^\pm/\partial t$ for $t=0$; this will be useful in what follows:

$$
\frac{\partial \varphi_1^\pm}{\partial t}\bigg|_{t=0}=\mp\frac{i}{4c(x)S_0'(x)}\Box_c\frac{\mathrm{e}\big(\alpha^\pm(x,t)\big)}{\sqrt{J^\pm\big(\alpha^\pm(x,t),t\big)}}\bigg|_{t=0}=\mp i\frac{2(\mathrm{e}cc'')(x)-4(\mathrm{e}'cc')(x)-\mathrm{e}(c')^2(x)}{16(cS_0')(x)}. \tag{5.14}
$$

Here the prime stands for the derivative d/dx.

5.3. *Asymptotic Expansion of Rapidly Decaying Solutions on the Basis of WKB Solutions*

Now we wish to use the oscillating asymptotic solutions $u^\pm$ and the integral representation (4.9) to construct the asymptotics of localized solutions. The mere replacement of the functions $\varphi_0^\pm$ in (4.9) by series with terms $h^k\varphi_k^\pm$ serves no good purpose, because, as was already mentioned, the parameter $h=\frac{\mu}{\rho}$ is not small for small ρ. But the contribution from these ρ proves to be small, and we eliminate it from the integral by introducing a cutoff function $\chi\big(\frac{\rho}{\mu}\big)$ (see Fig. 10) such that

$$
\chi(y)=\begin{cases}0, & \text{for}\quad |y|\leqslant 1,\\ 1, & \text{for}\quad |y|>2.\end{cases}
$$

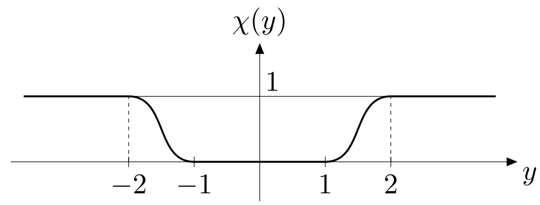

Fig. 10.

We shall see that to obtain the desired estimates, it suffices to retain only three terms in the expansion of $\varphi^\pm$, representing the asymptotic solution of the original problem (1.1)–(1.2) in the form

$$
u_{\mathrm{as}}(x,t)=\sum_\pm\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S^\pm(x,t)}{\mu}}\bigg(\varphi_0^\pm(x,t)+\frac{\mu}{\rho}\varphi_1^\pm(x,t)\chi\Big(\frac{\rho}{\mu}\Big)+\frac{\mu^2}{\rho^2}\varphi_2^\pm\chi\Big(\frac{\rho}{\mu}\Big)\bigg)\tilde V(\rho)\,d\rho, \tag{5.15}
$$

where the function $\tilde V(\rho)$ is defined in (4.7).

Let $u_{\mathrm{ex}}(x,t)$ be the exact solution of problem (1.1)–(1.2). Consider the function

$$
w=u_{\mathrm{as}}-u_{\mathrm{ex}}. \tag{5.16}
$$

Lemma 7. *The function w is the solution of problem* (5.1) *with*

$$\mathcal{F}(x,t)=\sum_{\pm}\left[\frac{\Box_c\varphi_2^{\pm}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S^{\pm}}{\mu}}\frac{\mu^2}{\rho^2}\chi\Big(\frac{\rho}{\mu}\Big)\tilde V\,d\rho+\frac{\Box_c\varphi_0^{\pm}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S^{\pm}}{\mu}}\Big(1-\chi\Big(\frac{\rho}{\mu}\Big)\Big)\tilde V\,d\rho\right],$$

$$\begin{aligned}F_0=(\mathrm{e}(x)-1)V\Big(\frac{x}{\mu}\Big),\qquad F_1&=\sum_{\pm}\left[\frac{\partial\varphi_2^{\pm}}{\partial t}\bigg|_{t=0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S_0}{\mu}}\frac{\mu^2}{\rho^2}\chi\Big(\frac{\rho}{\mu}\Big)\tilde V\,d\rho\right.\\&\left.+\frac{\partial\varphi_0^{\pm}}{\partial t}\bigg|_{t=0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S_0}{\mu}}\Big(1-\chi\Big(\frac{\rho}{\mu}\Big)\Big)\tilde V\,d\rho\right].\end{aligned}\tag{5.17}$$

Proof. The proof is by straightforward differentiation and uses the construction of the functions $S^{\pm}$ and $\varphi_k^{\pm}$ and the formulas for these functions.

5.4. *Petrovskii's Estimates for the Asymptotic Solution* (5.15)

By Lemma 5, the function F_0 satisfies the estimate $\|F_0\|_{W_2^1}=O(\mu)$. Let us prove that $\mathcal{F}$ and F_1 satisfy the estimates

$$\|\mathcal{F}\|_{L_2}=O(\mu),\qquad \|F_1\|_{W_2^1}=O(\mu)\tag{5.18}$$

as well. To this end, we establish several auxiliary estimates. Consider the functions

$$g_n\Big(\frac{\sigma}{\mu},\mu\Big)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho\sigma}{\mu}}\frac{\mu^n}{\rho^n}\chi\Big(\frac{\rho}{\mu}\Big)\tilde V(\rho)\,d\rho,\qquad n=1,2,\tag{5.19}$$

$$q_k\Big(\frac{\sigma}{\mu},\mu\Big)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\rho^k e^{\frac{i\rho\sigma}{\mu}}\Big(1-\chi\Big(\frac{\rho}{\mu}\Big)\Big)\tilde V(\rho)\,d\rho,\quad k=0,1,\tag{5.20}$$

where $\sigma(x,t)$ is either $S^{\pm}(x,t)$ or $S_0(x)$.

Lemma 8. *For each smooth compactly supported function $R(x,t)$ and each $t\in[0,T]$, where T is independent of μ, the following estimates hold*:

$$\Big\|R(x,t)g_n\Big(\frac{\sigma}{\mu},\mu\Big)\Big\|_{L_2}=O(\mu),\quad n=1,2,\qquad \Big\|R(x,t)q_k\Big(\frac{\sigma}{\mu},\mu\Big)\Big\|_{L_2}=O(\mu^{k+1}),\quad k=0,1.$$

Proof. Let us prove the first estimates. Let

$$M=\max_{x\in\mathbb{R},t\in[0,T]}\frac{R^2}{|\sigma_x|}.$$

We have

$$\begin{aligned}\Big\|R(x,t)g_n\Big(\frac{\sigma}{\mu},\mu\Big)\Big\|_{L_2}^2&=\left|\int_{-\infty}^{+\infty}R^2\Big|g_n\Big(\frac{\sigma}{\mu},\mu\Big)\Big|^2dx\right|\\&=\mu\left|\int_{-\infty}^{+\infty}\frac{R^2}{\sigma_x}\Big|g_n\Big(\frac{\sigma}{\mu},\mu\Big)\Big|^2\frac{\sigma_x}{\mu}dx\right|\leqslant\mu M\int_{-\infty}^{+\infty}\big|g_n(\theta,\mu)\big|^2d\theta,\end{aligned}$$

where $\theta=\sigma/\mu$. Since the norm of the Fourier transform of a function is equal to the norm of the function itself, we have

$$\begin{aligned}\mu M\int_{-\infty}^{+\infty}\big|g_n(\theta,\mu)\big|^2d\theta&=\mu^{2n+1}M\int_{-\infty}^{+\infty}\Big(\frac{\chi\tilde V}{\rho^n}\Big)^2d\rho\leqslant\mu^{2n+1}M\bigg(\int_{-\infty}^{-\mu}\frac{\tilde V^2}{\rho^{2n}}d\rho+\int_{\mu}^{+\infty}\frac{\tilde V^2}{\rho^{2n}}d\rho\bigg)\\&\leqslant 2\mu^{2n+1}M\max_{\rho\in\mathbb{R}}\tilde V^2\int_{\mu}^{+\infty}\frac{d\rho}{\rho^{2n}}=\mu^2\frac{2M}{2n-1}\max_{\rho\in\mathbb{R}}\tilde V^2.\end{aligned}$$

Likewise,

$$\Big\|R(x,t)q_k\Big(\frac{\sigma}{\mu},\mu\Big)\Big\|_{L_2}^2 \leqslant \mu M\int_{-\infty}^{+\infty}\big(\rho^k(1-\chi)\tilde V\big)^2\,d\rho$$
$$\leqslant \mu M\max_{\rho\in[-2\mu,2\mu]}\tilde V^2\int_{-2\mu}^{2\mu}\rho^{2k}\,d\rho=\mu^{2k+2}\frac{2^{2k+2}}{2k+1}M\max_{\rho\in[-2\mu,2\mu]}\tilde V^2,\quad k=0,1.$$

These inequalities are obviously equivalent to the estimates in the assertion of the lemma.

Lemma 9. *The functions $\mathcal F$ and F_1 defined in* (5.17) *satisfy the estimates* (5.18).

Proof. We have

$$\big\|\mathcal F(x,t)\big\|_{L_2}\leqslant\sum_{\pm}\Bigg[\bigg\|\frac{\Box_c\varphi_2^\pm}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S^\pm}{\mu}}\frac{\mu^2}{\rho^2}\chi\tilde V\,d\rho\bigg\|_{L_2}+\bigg\|\frac{\Box_c\varphi_0^\pm}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S^\pm}{\mu}}(1-\chi)\tilde V\,d\rho\bigg\|_{L_2}\Bigg].$$

The expressions whose norms occur here have the form $R(x,t)g_2\big(\frac{\sigma}{\mu},\mu\big)$ or $R(x,t)q_0\big(\frac{\sigma}{\mu},\mu\big)$, where $R(x,t)$ is a compactly supported function. The first estimate in (5.18) follows from Lemma 8. Next,

$$\begin{aligned}
\|F_1(x)\|_{W_2^1}^2&=\|F_1\|_{L_2}^2+\bigg\|\frac{\partial F_1}{\partial x}\bigg\|_{L_2}^2\\
&\leqslant\Bigg(\sum_{\pm}\Bigg[\bigg\|\frac{\partial\varphi_2^\pm}{\partial t}\bigg|_{t=0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S_0}{\mu}}\frac{\mu^2}{\rho^2}\chi\tilde V\,d\rho\bigg\|_{L_2}+\bigg\|\frac{\partial\varphi_0^\pm}{\partial t}\bigg|_{t=0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S_0}{\mu}}(1-\chi)\tilde V\,d\rho\bigg\|_{L_2}\Bigg]\Bigg)^2\\
&\quad+\Bigg(\sum_{\pm}\Bigg[\bigg\|\frac{\partial}{\partial x}\Big(\frac{\partial\varphi_2^\pm}{\partial t}\Big|_{t=0}\Big)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S_0}{\mu}}\frac{\mu^2}{\rho^2}\chi\tilde V\,d\rho\bigg\|_{L_2}\\
&\qquad+\bigg\|\frac{\partial}{\partial x}\Big(\frac{\partial\varphi_0^\pm}{\partial t}\Big|_{t=0}\Big)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S_0}{\mu}}(1-\chi)\tilde V\,d\rho\bigg\|_{L_2}\Bigg]\\
&\qquad+\sum_{\pm}\Bigg[\bigg\|\frac{\partial\varphi_2^\pm}{\partial t}\bigg|_{t=0}\frac{iS_0'}{\mu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S_0}{\mu}}\frac{\mu^2}{\rho}\chi\tilde V\,d\rho\bigg\|_{L_2}\\
&\qquad+\bigg\|\frac{\partial\varphi_0^\pm}{\partial t}\bigg|_{t=0}\frac{iS_0'}{\mu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{\frac{i\rho S_0}{\mu}}\rho(1-\chi)\tilde V\,d\rho\bigg\|_{L_2}\Bigg]\Bigg)^2.
\end{aligned}$$

Each of the expressions whose norms occur in this formula has the form $R(x,t)g_{1,2}\big(\sigma/\mu,\mu\big)$ or $R(x,t)q_{0,1}\big(\sigma/\mu,\mu\big)$, where $R(x,t)$ is a compactly supported function. By Lemma 8,

$$\|F_1\|_{W_2^1}^2=\big(O(\mu)\big)^2+\big(O(\mu)\big)^2=O(\mu^2).$$

This gives the second estimate in (5.18).

An application of Petrovskii's estimate (5.3) to the function w readily gives the following assertion.

Corollary. *One has*

$$w=O(\mu)\quad\Longleftrightarrow\quad u_{ex}=u_{\mathrm{as}}+O(\mu)$$

in the norm of $C(\mathbb R)$.

Proof of the Theorem. Now let us prove that the contribution of the second and third terms in formula (5.15) to u_{as} is $O(\mu\,|\log\mu|)$. To this end, we estimate the expressions

$$\int_{-\infty}^{+\infty}e^{\frac{i\rho S^\pm(x,t)}{\mu}}\Big(\frac{\mu}{\rho}\Big)^k\chi\Big(\frac{\rho}{\mu}\Big)\tilde V(\rho)\,d\rho,\quad k=1,2.\tag{5.21}$$

The function $\tilde V(\rho)$ decays at infinity faster than some negative power, and hence $|\tilde V(\rho)| \leqslant \frac{M_1}{1+|\rho|}$, where M_1 is some constant. Using this inequality and the definition of the function χ, we obtain

$$\Bigl|\int_{-\infty}^{+\infty} e^{\frac{i\rho S^{\pm}(x,t)}{\mu}}\frac{\mu}{\rho}\chi\Bigl(\frac{\rho}{\mu}\Bigr)\tilde V(\rho)\,d\rho\Bigr| \leqslant 2\mu M_1\int_\mu^\infty \frac{1}{\rho(1+\rho)}\,d\rho = 2\mu M_1\log\Bigl(\frac{1+\mu}{\mu}\Bigr) = O(\mu|\log\mu|)$$

and also

$$\Bigl|\int_{-\infty}^{+\infty} e^{\frac{i\rho S^{\pm}(x,t)}{\mu}}\Bigl(\frac{\mu}{\rho}\Bigr)^2\chi\Bigl(\frac{\rho}{\mu}\Bigr)\tilde V(\rho)\,d\rho\Bigr| \leqslant 2\mu^2\max_{\rho\in\mathbb{R}}\tilde V(\rho)\int_\mu^\infty \frac{1}{\rho^2}\,d\rho = 2\mu\max_{\rho\in\mathbb{R}}\tilde V(\rho) = O(\mu).$$

Hence we obtain

$$u_{\mathrm{as}}(x,t) = \sum_{\pm}\frac{\varphi_0^{\pm}(x,t)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\frac{i\rho S^{\pm}(x,t)}{\mu}}\tilde V(\rho)\,d\rho + O(\mu|\log\mu|) \equiv \sum_{\pm}\varphi_0^{\pm}(x,t)V\Bigl(\frac{S^{\pm}(x,t)}{\mu}\Bigr).$$

By substituting the expressions for $S^{\pm}(x,t)$ and $\varphi_0^{\pm}(x,t)$ given by (3.22) and (5.12) and by replacing $S^{\pm}(x,t)$ and $\varphi_0^{\pm}(x,t)$ according to formulas (4.2), we arrive at formula (2.5). An application of Lemma 4 gives formula (2.6) and completes the proof of the theorem.

6. APPENDIX 1. WHITHAM EXPANSIONS

6.1. *Formulas for the Expansion Coefficients*

As was already noted, one can obtain corrections to the leading asymptotic term by using representations similar to the leading term. We shall see that they even have the better estimate $O(\mu)$ rather than $O\bigl(\mu|\log\mu|\bigr)$. The disadvantage of the approach based on such expansions is that it does not admit a straightforward generalization to the multidimensional case (in contrast to the approach used above). To make the exposition complete, we describe this approach, mainly restricting ourselves to the initial function V of the special form $V(\theta) = 1/\cosh^2\theta$ so as to simplify the computations. Just as in the main text, for technical reasons, it is convenient to replace $V(\theta)$ in the initial condition by $\mathrm{e}(x)V(\theta)$, where, as before, $\mathrm{e}(x)$ is a cutoff function. As a result, we obtain initial conditions of the form

$$u|_{t=0} = \mathrm{e}(x)V\Bigl(\frac{S_0(x)}{\mu}\Bigr), \qquad u_t\Big|_{t=0} = 0. \tag{6.1}$$

We seek the solution in the form (3.2) assuming that the functions $f^{\pm}$ admit the expansions

$$f^{\pm} = f_0^{\pm}(\theta,x,t) + \mu f_1^{\pm}(\theta,x,t) + \mu^2 f_2^{\pm}(\theta,x,t) + \cdots. \tag{6.2}$$

By substituting the function $u^{\pm} = f^{\pm}\bigl(S^{\pm}(x,t)/\mu,x,t,\mu\bigr)$ into the original wave equation (1.1) and the initial conditions (6.1), we obtain relations (3.3) and (3.5), (3.6), respectively. We argue as in Subsections 3.1 and 3.4, take into account the expansion (6.2), match the coefficients of like powers of μ, and obtain the eikonal equations (3.7) for the phases $S^{\pm}$, whose solutions are described in Subsections 3.1 and 3.2, and the chain

$$\hat\Pi^{\pm}\frac{\partial}{\partial\theta}f_0^{\pm} = 0, \qquad \hat\Pi^{\pm}\frac{\partial}{\partial\theta}f_k^{\pm} = -\Box_c f_{k-1}^{\pm} \tag{6.3}$$

of transport equations, where the transport operators $\hat\Pi^{\pm}$ are defined in (3.4). From the initial conditions, we obtain the relations

$$f_0^+\Big|_{t=0} + f_0^-\Big|_{t=0} = \mathrm{e}(x)V(\theta), \qquad f_{0\theta}^+\Big|_{t=0}S_t^+\Big|_{t=0} + f_{0\theta}^-\Big|_{t=0}S_t^-\Big|_{t=0} = 0, \tag{6.4}$$

$$\begin{gathered} f_k^+\Big|_{t=0} + f_k^-\Big|_{t=0} = 0,\\ f_{k\theta}^+\Big|_{t=0}S_t^+\Big|_{t=0} + f_{k\theta}^-\Big|_{t=0}S_t^-\Big|_{t=0} + f_{(k-1)t}^+\Big|_{t=0} + f_{(k-1)t}^-\Big|_{t=0} = 0, \quad k=1,2,\dots \end{gathered} \tag{6.5}$$

In view of (3.42), we seek the solutions of the chain of transport equations in the form

$$f_k^\pm(\theta,x,t)=\frac{\tilde f_k^\pm(\theta,x,t)}{\sqrt{J^\pm(x,t)}},\qquad k=0,1,2,\dots$$

By using (3.42), for $\tilde f_k$ we obtain

$$\begin{aligned}
&\frac{\partial^2\tilde f_0^\pm(\theta,X^\pm(\alpha,t),t)}{\partial t\partial\theta}=0,\\
&\frac{\partial^2\tilde f_k^\pm(\theta,X^\pm(\alpha,t),t)}{\partial t\partial\theta}=-\left[\frac{\sqrt{J^\pm(x,t)}}{2S_t^\pm(x,t)}\Box_c f_{k-1}^\pm(\theta,x,t)\right]_{x=\mathcal{X}^\pm(\alpha,t)},\qquad k=1,2,\dots
\end{aligned}$$

By integrating these equations, we obtain

$$\begin{aligned}
&\tilde f_0^\pm(\theta,\alpha,t)=R_0^\pm(\theta,\alpha)+Q_0^\pm(\alpha,t),\\
&\tilde f_k^\pm(\theta,\alpha,t)=-\int_{-\infty}^{\theta}\int_0^t\left[\frac{\sqrt{J^\pm}}{2S_t^\pm}\Box_c f_{k-1}^\pm\right]_{x=\mathcal{X}^\pm(\alpha,t)}dt^*d\theta^*+R_k^\pm(\theta,\alpha)+Q_k^\pm(\alpha,t),\quad k=1,2,\dots,
\end{aligned}$$

where $R_k^\pm(\theta,\alpha)$ and $Q_k^\pm(\alpha,t)$, $k=0,1,2,\dots$, are arbitrary continuous functions. The functions $Q_k^\pm(\alpha,t)$ are independent of θ and do not affect the rapidly varying part of the solution. Hence we take $Q_k^\pm(\alpha,t)=0$, $k=0,1,2,\dots$, without loss of generality. By substituting the functions $f_k^\pm$ into (6.4) and (6.5) and by taking into account the eikonal equations (3.7), we obtain the following systems for $R_{k\theta}^\pm(\theta,x)$, $k=0,1,2,\dots$:

$$R_{0\theta}^+ + R_{0\theta}^- = \mathrm{e}(x)V'(\theta),\quad R_{0\theta}^+ - R_{0\theta}^- = 0, \tag{6.6}$$

$$R_{k\theta}^+ + R_{k\theta}^- = 0,\qquad R_{k\theta}^+ - R_{k\theta}^- = \frac{f_{(k-1)t}^+\big|_{t=0}+f_{(k-1)t}^-\big|_{t=0}}{c(x)S_0'(x)},\qquad k=1,2,\dots \tag{6.7}$$

Here $'$ stands for the derivative $d/d\theta$. By solving this system and by integrating with respect to θ under the assumption that $R_k^\pm\to0$ as $\theta\to-\infty$, we find

$$\begin{aligned}
R_0^\pm&=\frac{\mathrm{e}(\alpha)V(\theta)}{2}\Big|_{\alpha=\alpha^\pm(x,t)},\\
R_k^\pm(\theta,x)&=\pm\frac{1}{2c(x)S_0'(x)}\int_{-\infty}^{\theta}\left[f_{(k-1)t}^+(\theta^*,x,t)+f_{(k-1)t}^-(\theta^*,x,t)\right]_{t=0}d\theta^*,\quad k=1,2,\dots
\end{aligned}\tag{6.8}$$

From this, we obtain

$$\begin{aligned}
f_0^\pm&=\frac{\mathrm{e}(\alpha)V(\theta)}{2\sqrt{J^\pm}}\bigg|_{\alpha=\alpha^\pm(x,t)},\\
f_k^\pm&=\pm\Bigg[\frac{1}{2c(\alpha)S_0'(\alpha)\sqrt{J^\pm(\alpha,t)}}\int_0^t\sqrt{J^\pm(\alpha,t^*)}\left[\Box_c\int_{-\infty}^{\theta}f_{k-1}^\pm(\theta^*,\mathcal{X}^\pm(\alpha,t^*),t^*)d\theta^*\right]dt^*\\
&\qquad+\int_{-\infty}^{\theta}\left[f_{(k-1)t}^+(\theta^*,x,t)+f_{(k-1)t}^-(\theta^*,x,t)\right]_{t=0}d\theta^*\Bigg]_{\alpha=\alpha^\pm(x,t)},\quad k=1,2,\dots
\end{aligned}$$

By successively analyzing the dependence of the functions $f_k^\pm$ on θ, one can readily conclude that they can be represented as the products of $V_k(\theta)$ by functions depending on (x,t). More precisely, the following assertion can readily be proved by induction.

Lemma 10. *Let $V_k(\theta)$ be the functions defined by the recursion relations*

$$V_0(\theta) = V(\theta), \qquad V_k(\theta) = \int_{-\infty}^{\theta} V_{k-1}(\theta^*)d\theta^*.$$

Then

$$f_k^{\pm} = i^k \varphi_k^{\pm}(x,t)V_k(\theta). \tag{6.9}$$

Here $\varphi_k^{\pm}$ are defined in (5.12)–(5.13).

Since the function $V(\theta)$ rapidly decays as $|\theta| \to \infty$, it follows that $V_k(\theta)$ rapidly decay as $\theta \to -\infty$. However, we cannot claim the same thing about the behavior of $V_k(\theta)$ as $\theta \to +\infty$: in general, $V_k(\theta) = O(\theta^{k-2})$ as $\theta \to +\infty$. For example, if $V = 1/\cosh^2\theta$, then $V_1 = \tanh\theta + 1$, $V_2 = \log(1 + e^{2\theta})$, and as $\theta \to +\infty$, one has $V_1 \sim 2$ and $V_2 \sim 2\theta$.

Now let us return to the asymptotic expansion (3.2), (6.2). In view of the preceding, we have

$$u = \sum_{\pm} \Big\{ \varphi_0^{\pm}(x,t)V\Big(\frac{S^{\pm}}{\mu}\Big) + i\mu\varphi_1^{\pm}(x,t)V_1\Big(\frac{S^{\pm}}{\mu}\Big) + i^2\mu^2\varphi_2^{\pm}(x,t)V_2\Big(\frac{S^{\pm}}{\mu}\Big) + \cdots \Big\}. \tag{6.10}$$

By virtue of the properties of the functions V_k for $S^{\pm}(x,t) > \delta > 0$ as $\mu \to +0$, the estimates $\mu^k\varphi_k^{\pm}(x,t)V_k(S^{\pm}/\mu) \sim \mu$ hold for all $k = 1, 2, \dots$ Consequently, just as in the preceding sections, the series (6.10) (or (3.2), (6.2)) is not an asymptotic expansion. Nevertheless, just as before, one can use Petrovskii's estimates to prove that the first three terms of the series (6.10) give an asymptotic solution and $\varphi_0^+(x,t)V(S^+/\mu) + \varphi_0^-(x,t)V(S^-/\mu)$ is its leading term.

6.2. *Petrovskii's Estimates for the Solution of Whitham Type*

Let u_{ex} be the exact solution of problem (1.1)–(1.2). As in Subsection 5.3, consider the function

$$w = u_{\mathrm{as}} - u_{\mathrm{ex}},$$

where

$$u_{\mathrm{as}} = \sum_{\pm} \Big\{ \varphi_0^{\pm}(x,t)V\Big(\frac{S^{\pm}}{\mu}\Big) + i\mu\varphi_1^{\pm}(x,t)V_1\Big(\frac{S^{\pm}}{\mu}\Big) + i^2\mu^2\varphi_2^{\pm}(x,t)V_2\Big(\frac{S^{\pm}}{\mu}\Big) \Big\};$$

we assume that $S^{\pm}(x,0) \equiv S_0(x) = x$. By the construction of $\varphi_k^{\pm}$, the function w satisfies problem (5.1) with

$$\mathcal{F}(x,t) = \sum_{\pm} i^2\mu^2 V_2\Big(\frac{S^{\pm}(x,t)}{\mu}\Big)\Box_c\varphi_2^{\pm}(x,t),$$

$$F_0 = \big(\mathrm{e}(x) - 1\big)V\Big(\frac{x}{\mu}\Big), \qquad F_1 = \sum_{\pm} i^2\mu^2 \frac{\partial\varphi_2^{\pm}}{\partial t}\Big|_{t=0} V_2\Big(\frac{x}{\mu}\Big).$$

Lemma 11. *The functions $\mathcal{F}$, F_0, and F_1 satisfy the estimates*

$$\|\mathcal{F}\|_{L_2} = O(\mu), \qquad \|F_0\|_{W_2^1} = O(\mu), \qquad \|F_1\|_{W_2^1} = O(\mu). \tag{6.11}$$

Proof. The second estimate was proved earlier in Lemma 5. We prove the first and third estimates using the decaying function $V(\theta) = 1/\cosh^2\theta$ as an example. We have

$$\begin{aligned}
\Big\|\mu^2 V_2\Big(\frac{S^{\pm}(x,t)}{\mu}\Big)\Box_c\varphi_2^{\pm}(x,t)\Big\|_{L_2}^2 &= \mu^4 \int_{-\infty}^{+\infty} \big(\Box_c\varphi_2^{\pm}(x,t)\big)^2 \log^2\big(1 + e^{2\theta}\big)\Big|_{\theta = S^{\pm}(x,t)/\mu} dx \\
&= \int_{-\infty}^{+\infty} \mu^4\big(\Box_c\varphi_2^{\pm}(x,t)\big)^2 \frac{\log^2\big(1 + e^{2\theta}\big)}{\theta^2 + 1}\Big|_{\theta = S^{\pm}(x,t)/\mu} \Big(\frac{{S_0}^2\big(\alpha^{\pm}(x,t)\big)}{\mu^2} + 1\Big) dx \\
&\leqslant \mu^2 \int_{-\infty}^{+\infty} \big(\Box_c\varphi_2^{\pm}(x,t)\big)^2 \Big({S_0}^2\big(\alpha^{\pm}(x,t)\big) + \mu^2\Big) dx \leqslant C^2\mu^2,
\end{aligned}$$

since the function $\log^2(1+e^{2\theta})/(\theta^2+1)$ is bounded and the functions

$$\bigl(\Box_c\varphi_2^\pm(x,t)\bigr)^2,\quad \bigl(\Box_c\varphi_2^\pm(x,t)\bigr)^2{S_0}^2\bigl(\alpha^\pm(x,t)\bigr)$$

are smooth and compactly supported with respect to x. Here C is a positive constant. It follows that

$$\|\mathcal{F}\|_{L_2}\leqslant\sum_\pm\Bigl\|\mu^2V_2\Bigl(\frac{S^\pm(x,t)}{\mu}\Bigr)\Box_c\varphi_2^\pm(x,t)\Bigr\|_{L_2}=2C\mu\quad\Longrightarrow\quad\|\mathcal{F}\|_{L_2}=O(\mu).$$

Let us prove the third estimate. We have

$$\Bigl\|\mu^2\frac{\partial\varphi_2^\pm}{\partial t}(x,0)V_2\Bigl(\frac{x}{\mu}\Bigr)\Bigr\|^2_{W_2^1}=\Bigl\|\mu^2\frac{\partial\varphi_2^\pm}{\partial t}(x,0)V_2\Bigl(\frac{x}{\mu}\Bigr)\Bigr\|^2_{L_2}+\Bigl\|\mu^2\frac{\partial}{\partial x}\frac{\partial\varphi_2^\pm}{\partial t}(x,0)V_2\Bigl(\frac{x}{\mu}\Bigr)\Bigr\|^2_{L_2}.$$

Since the derivative $\frac{\partial\varphi_2^\pm}{\partial t}(x,0)$ is smooth and compactly supported with respect to x, we see that the first term on the right-hand side in this relation can be estimated by analogy with $\bigl\|\mu^2V_2\bigl(S^\pm(x,t)/\mu\bigr)\Box_c\varphi_2^\pm(x,t)\bigr\|^2_{L_2}$ and is also of the order of μ^2. Let us estimate the second term:

$$\begin{aligned}&\Bigl\|\mu^2\frac{\partial}{\partial x}\frac{\partial\varphi_2^\pm}{\partial t}(x,0)V_2\Bigl(\frac{x}{\mu}\Bigr)\Bigr\|_{L_2}\\&\quad=\Bigl\|\mu^2\Bigl(\frac{\partial}{\partial x}\frac{\partial\varphi_2^\pm}{\partial t}(x,0)\log\bigl(1+e^{2\theta}\bigr)\Big|_{\theta=S_0(x)/\mu}+\frac{\partial\varphi_2^\pm}{\partial t}(x,0)[\tanh\theta+1]\Big|_{\theta=S_0(x)/\mu}\frac{S_0'}{\mu}\Bigr)\Bigr\|_{L_2}\\&\quad\leqslant\Bigl\|\mu^2\frac{\partial}{\partial x}\frac{\partial\varphi_2^\pm}{\partial t}(x,0)\log\bigl(1+e^{2\theta}\bigr)\Big|_{\theta=S_0(x)/\mu}\Bigr\|_{L_2}+\Bigl\|\mu\frac{\partial\varphi_2^\pm}{\partial t}(x,0)[\tanh\theta+1]\Big|_{\theta=S_0(x)/\mu}S_0'\Bigr\|_{L_2}.\end{aligned}$$

Since the function $(\tanh\theta+1)$ is bounded, the functions $\frac{\partial}{\partial x}\frac{\partial\varphi_2^\pm}{\partial t}(x,0)$ and $\frac{\partial\varphi_2^\pm}{\partial t}(x,0)S_0'$ are smooth and compactly supported with respect to x, the resulting two terms can be estimated by analogy with $\Bigl\|\mu^2V_2\Bigl(\frac{S^\pm(x,t)}{\mu}\Bigr)\Box_c\varphi_2^\pm(x,t)\Bigr\|_{L_2}$. They are both of the order of μ. Hence

$$\Bigl\|\mu^2\tfrac{\partial}{\partial x}\frac{\partial\varphi_2^\pm}{\partial t}(x,0)V_2\Bigl(\frac{x}{\mu}\Bigr)\Bigr\|_{L_2}=O(\mu)\quad\text{and}\quad\Bigl\|\mu^2\frac{\partial\varphi_2^\pm}{\partial t}(x,0)V_2\Bigl(\frac{x}{\mu}\Bigr)\Bigr\|^2_{W_2^1}=O(\mu^2).$$

It follows that

$$\|F_1\|_{W_2^1}\leqslant\sum_\pm\Bigl\|\mu^2\frac{\partial\varphi_2^\pm}{\partial t}(x,0)V_2\Bigl(\frac{x}{\mu}\Bigr)\Bigr\|_{W_2^1}=O(\mu),$$

and the proof of the lemma is complete.

One can show that the assertion of the lemma remains valid for any arbitrary rapidly decaying function $V(\theta)$.

By applying Lemma 11 and formula (5.3), we obtain $\|w\|_{\mathbb{C}}=O(\mu)$.

Now let us estimate the corrections $f_k=i^k\mu^k\varphi_k^\pm(x,t)V_k\bigl(S^\pm(x,t)/\mu\bigr)$, $k=1,2$. As $\mu\to+0$, we have

$$\begin{aligned}f_1&=i\mu\varphi_1^\pm(x,t)[\tanh\theta+1]\Big|_{\theta=S^\pm(x,t)/\mu}\sim 2i\mu\varphi_1^\pm(x,t)=O(\mu),\\f_2&=-\mu^2\varphi_2^\pm(x,t)\log(1+e^{2\theta})\Big|_{t=S^\pm(x,t)/\mu}\sim-\mu^2\varphi_2^\pm(x,t)\frac{2S^\pm(x,t)}{\mu}=O(\mu),\end{aligned}$$

since $\varphi_k^\pm(x,t)(S^\pm)^{k-1}(x,t)$ are smooth and compactly supported with respect to x. Thus, the contribution of the corrections f_k, $k=1,2$, to the asymptotic solution $u_{\rm as}$ is $O(\mu)$.

This argument gives a *different proof of the theorem* with a slightly better estimate of the corrections.

7. APPENDIX 2. RELATIONSHIP BETWEEN ASYMPTOTIC EXPANSIONS OF RAPIDLY DECAYING SOLUTIONS WITH RESPECT TO A SMALL PARAMETER AND EXPANSIONS OF SINGULAR SOLUTIONS WITH RESPECT TO SMOOTHNESS

The asymptotic solution given in the Theorem can also be obtained from the asymptotics of the fundamental solution of the original wave equation with respect to smoothness. For completeness, let us show that this approach results in the Whitham expansion formulas given in Appendix 1.

Consider the Cauchy problem

$$\Box_c G = 0, \qquad G|_{t=0} = \delta\big(x-\xi\big), \qquad \tfrac{\partial G}{\partial t}\Big|_{t=0} = 0 \tag{7.1}$$

for the function $G = G(x,t)$, where $\delta\big(x-\xi\big)$ is the Dirac function and ξ is a parameter. Following [1, 12], we seek the asymptotic solution of this problem in the form $G = f^{+}\big(\Phi^{+}(x,t,\xi),x,t,\xi\big) + f^{-}\big(\Phi^{-}(x,t,\xi),x,t,\xi\big)$, where $\Phi^{\pm}(x,t,\xi)$ are smooth functions and $f^{\pm}(\theta,x,t,\xi)$ are represented by the smoothness expansions

$$f^{\pm} = \varphi_0^{\pm}(x,t,\xi)\delta(\theta) + i\varphi_1^{\pm}(x,t,\xi)\Theta(\theta) + i^2\varphi_2^{\pm}(x,t,\xi)\theta_+ + i^3\varphi_3^{\pm}(x,t,\xi)\frac{\theta_+^2}{2} + i^4\varphi_4^{\pm}(x,t,\xi)\frac{\theta_+^3}{3!} + \cdots . \tag{7.2}$$

Here $\Theta(\theta)$ is the Heaviside function, $\theta_+^k = \theta^k\Theta(\theta)$, and the factors i^k have been introduced for technical reasons. By substituting the functions $u^{\pm} = f^{\pm}\big(\Phi^{\pm}(x,t,\xi),x,t,\xi\big)$ into the original wave equation, we obtain

$$\Box_c u^{\pm} = \big({\Phi_t^{\pm}}^2 - c^2(x){\Phi_x^{\pm}}^2\big)\frac{\partial^2 f^{\pm}}{\partial\theta^2} + \hat\Pi\frac{\partial f^{\pm}}{\partial\theta} + \Box_c f^{\pm} = 0,$$

where $\hat\Pi$ is the transport operator (3.4) with $S^{\pm}(x,t)$ replaced by $\Phi^{\pm}$. Using the expansion (7.2), we obtain

$$\begin{aligned}\big({\Phi_t^{\pm}}^2 - c^2(x){\Phi_x^{\pm}}^2\big)&(\delta''\varphi_0^{\pm} + i\delta'\varphi_1^{\pm} + i^2\delta\varphi_2^{\pm} + i^3\Theta\varphi_3^{\pm} + \cdots)\\ &+ \Big(\delta'\hat\Pi\varphi_0^{\pm} + i\delta\hat\Pi\varphi_1^{\pm} + i^2\Theta\hat\Pi\varphi_2^{\pm} + \theta_+ i^3\hat\Pi\varphi_3^{\pm} + \cdots\Big)\\ &+ \Big(\delta\Box_c\varphi_0^{\pm} + i\Theta\Box_c\varphi_1^{\pm} + i^2\theta_+\Box_c\varphi_2^{\pm} + i^3\frac{\theta_+^2}{2}\Box_c\varphi_3^{\pm} + \cdots\Big) = 0.\end{aligned}$$

By matching the coefficients of the most singular terms containing the function δ'', we arrive at the eikonal equations (3.7) for the functions $\Phi^{\pm}$. The initial conditions in problem (7.1) induce the following conditions for the functions $\Phi^{\pm}$: $\Phi^{\pm}\big|_{t=0} = x-\xi$. Let $S^{\pm}(x,t)$ be the solutions of the eikonal equations (3.7) with the initial condition $S^{\pm}(x,0) = x$ for $t = 0$. Then, obviously, $\Phi^{\pm} = S^{\pm}(x,t) - \xi$, and $\Phi^{\pm}$ can be replaced by $S^{\pm}(x,t)$ in the transport operators. Next, by substituting the functions $f^{\pm}$ in the form of the series (7.2) into the last equation, by matching the coefficients of $\delta', \delta, \Theta, \theta_+, \theta_+^2/2, \ldots$, and by taking into account the eikonal equation, we arrive at Eqs. (5.6) and (5.7) for $\varphi_k^{\pm}$. The initial conditions for $\varphi_k^{\pm}$ have the form (5.8) and (5.9), and hence $\varphi_k^{\pm}$ are determined by relations (5.12) and (5.13) with e replaced by 1. Thus, $\varphi_k^{\pm}$ are independent of the parameter ξ. We eventually obtain

$$\begin{aligned}G \approx{}& \varphi_0^{+}(x,t)\delta(S^{+}(x,t)-\xi) + \varphi_0^{-}(x,t)\delta(S^{-}(x,t)-\xi)\\ &+ i\varphi_1^{+}(x,t)\Theta(S^{+}(x,t)-\xi) + i\varphi_1^{-}(x,t)\Theta(S^{-}(x,t)-\xi)\\ &+ i^2\varphi_2^{+}(x,t)(S^{+}(x,t)-\xi)_+ + i^2\varphi_2^{-}(x,t)(S^{-}(x,t)-\xi)_-\\ &+ \frac{i^3}{2!}\varphi_3^{+}(x,t)(S^{+}(x,t)-\xi)_+^2 + \frac{i^3}{2!}\varphi_2^{-}(x,t)(S^{-}(x,t)-\xi)_+^2 + \cdots .\end{aligned} \tag{7.3}$$

Let $G_N(x,t,\xi)$ be the truncated series (7.3) containing the powers $(S^+(x,t)-\xi)_+^k$ and $(S^-(x,t)-\xi)_+^k$ with $k \leqslant N$, and let $g(x,t,\xi) = G - G_N(x,t,\xi)$. The function g is a solution of some Volterra equation. This solution, however, cannot be determined as constructively as the functions $\varphi_k^\pm$. For us, it suffices to have in mind the following fact [12]: by choosing a sufficiently large N, one can ensure that g has as many derivatives as desired and is bounded on the plane $\mathbb{R}^2_{x,\xi}$ on a given interval $t \in [0,T]$. Let us return to the Cauchy problem (1.1), (1.2). Its solution satisfies

$$u = \int_{-\infty}^{\infty} G(x,t,\xi) V\big(\xi/\mu\big) d\xi \equiv \int_{-\infty}^{\infty} G_N(x,t,\xi) V\big(\xi/\mu\big) d\xi + \int_{-\infty}^{\infty} g(x,t,\xi) V\big(\xi/\mu\big) d\xi. \tag{7.4}$$

Since g is bounded, we see that the last term satisfies the estimate

$$\left| \int_{-\infty}^{\infty} g(x,t,\xi) V\big(\xi/\mu\big) d\xi \right| \leqslant \text{const} \int_{-\infty}^{\infty} \left| V\big(\xi/\mu\big) \right| d\xi = \mu \,\text{const} \int_{-\infty}^{\infty} |V\big(y\big)| dy = O(\mu). \tag{7.5}$$

The integral

$$\int_{-\infty}^{\infty} G_N(x,t,\xi) V\big(\xi/\mu\big) d\xi$$

obviously consists of the terms

$$\varphi_0^\pm(x,t) V\big(S^\pm(x,t)/\mu\big), \qquad \frac{i^{k+1}}{k)!} \varphi_{k+1}^\pm(x,t) \int_{-\infty}^{\theta} V\big(\xi/\mu\big)(\theta-\xi)^k d\xi \Big|_{\theta=S^\pm(x,t)}.$$

The first terms obviously give the leading asymptotic term described in the Theorem. Consider the term

$$\frac{1}{k!} \int_{-\infty}^{\theta} V\big(\xi/\mu\big)(\theta-\xi)^k d\xi \Big|_{\theta=S^\pm(x,t)}.$$

We make the change of variables $\xi = \mu y$ in the integral. It can be rewritten in the form

$$\frac{\mu^{k+1}}{k!} \int_{-\infty}^{\theta} V\big(y\big)(\theta-y)^k dy \Big|_{\theta=S^\pm(x,t)/\mu}.$$

By integrating k times by parts, we obtain, since $V(y)$ rapidly decays as $y \to -\infty$,

$$\frac{1}{k!} \int_{-\infty}^{\theta} V\big(y\big)(\theta-y)^k dy \Big|_{\theta=S^\pm(x,t)/\mu} = V_{k+1}(\theta)\big|_{\theta=S^\pm(x,t)/\mu},$$

where $V_k(\theta)$ are introduced in Lemma 10. It readily follows that *the representation of the asymptotic solution on the basis of* (7.4) *coincides with* (6.10). *The estimate* (7.5) *explains why the* $O(\mu)$ *correction in the asymptotic solution of the original problem cannot be refined. This is due to the presence of the function* g, *defined nonconstructively, in the expansion of the fundamental solution.*

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