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Asymptotic Solutions of the Cauchy Problem for the Nonlinear Shallow Water Equations in a Basin with a Gently Sloping Beach

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Abstract. Small amplitude solutions of the nonlinear shallow water equations in a one- or two-dimensional domain are considered. The amplitude is characterized by a small parameter ε. It is assumed that the basin depth is a smooth function whose gradient is nowhere zero on the set of its zeros (i.e., on the coastline in the absence of waves). A solution of the equations is understood to be a triple (time-dependent domain, free surface elevation, horizontal velocity) smoothly depending on ε and such that (i) the free surface elevation and the horizontal velocity are zero for $\varepsilon = 0$; (ii) the sum of the free surface elevation and the depth is positive in the domain and zero on the boundary; (iii) the free surface elevation and the horizontal velocity are smooth in the closed domain and satisfy the equations there. An asymptotic solution modulo $O(\varepsilon^N)$ is defined in a similar way except that the equations must be satisfied modulo $O(\varepsilon^N)$. We prove that, in this setting, the nonlinear shallow water equations with small smooth initial data have an asymptotically unique asymptotic solution modulo $O(\varepsilon^N)$ for arbitrary N. The proof is constructive (and leads to simple explicit formulas for the leading asymptotic term). The construction uses a change of variables (depending on the unknown solution and resembling the Carrier–Greenspan transformation) that maps the unknown varying domain onto the unperturbed domain. The resulting nonlinear system is within the scope of regular perturbation theory. The zero approximation is a Cauchy problem for a linear hyperbolic system with degeneracy on the boundary, whose unique solvability in the class of smooth functions is proved by lifting the problem to a closed 3-manifold (where the spatial part of the operator turns out to be hypoelliptic).

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1. INTRODUCTION

The problem on the run-up of long waves on a sloping beach plays an important role in the study of tsunami waves. Linear models do not suffice near the coast because they take into account neither uprush nor wave breaking. At the same time, observations of real tsunami waves show that wave breaking is a relatively rare phenomenon, while wave uprush and downrush can be fairly well described by the nonlinear shallow water equations in many cases (see, e.g., Pelinovskii [1]). Moreover, at least away from the coast, the unknown free surface elevation and horizontal velocities turn out to be small, and one can solve the problem by regular perturbation theory taking the linear approximation for the unperturbed problem. However, things are more complicated near the coast. First, the domain in which the solution is defined depends on the solution itself (the moving coastline is determined by the condition that the depth function plus the free surface elevation is zero), which renders the “naive” linearization procedure useless. Second, the hyperbolic linearized system degenerates on the boundary and hence needs additional analysis.

In the one- and quasi-one-dimensional cases, the first challenge was treated by many authors. For example, the famous Carrier–Greenspan paper [2] considers the case of a depth function linear in the coordinate and introduces a transformation that not only makes the domain independent of the solution but also *exactly* linearizes the system. As a result, the solution of the nonlinear problem is expressed parametrically via the solution of the linearized problem. This idea underlies a number of studies where, in various specific cases, exact or approximate solutions of the nonlinear system are constructed and estimates are obtained for the uprush value, the wave velocity, etc. (see, e.g., [3, 1, 4–7]). Nevertheless, one probably cannot hope to linearize the equations exactly for a general bottom profile even in the one- (let alone two-)dimensional case. This would be too good to be true. Therefore, it is natural to solve the problem in two steps: first, use a suitable transformation to ensure that the domain of the solution is fixed (in the one-dimensional case, a transformation of this kind was considered in [8]), and then solve the resulting transformed, but still nonlinear, system somehow (for example, viewing it as a perturbation of a linear system) in the new domain.

In the present paper, we study the Cauchy problem for the nonlinear shallow water equations in the two-dimensional case assuming that the basin depth is given by a smooth function whose gradient does not vanish on its zero set (i.e., on the coastline in the absence of waves). Under these assumptions, we define a transformation that makes the domain solution-independent and prove that the Cauchy problem for the nonlinear shallow water equations viewed as a small perturbation of the linearized equations (i.e., the unknown functions are of the order of a small parameter ε) have an asymptotically unique smooth asymptotic solution modulo $O(\varepsilon^n)$ for arbitrary N. The proof is based on a theorem stating that the Cauchy problem with smooth data for the degenerate hyperbolic linear approximation system has a unique solution, which is smooth. To prove that theorem, we use some results concerning the uniformization of degenerate equations [9] and hypoellipticity of second-order equations with nonnegative characteristic form [10].

The construction of the asymptotic solution used in the proof of the main result leads to an algorithm convenient for practical implementation: the main term of the asymptotic solution (which is the only one needed in practical calculations) is expressed by simple explicit formulas via the solution of the formally linearized problem. Efficient methods for approximately solving the latter based on semiclassical asymptotics are well known (see, e.g., [11] and the literature therein), and we do not dwell on this aspect in the present paper.

2. INFORMAL STATEMENT OF THE PROBLEM AND AN APPROXIMATE SOLUTION ALGORITHM

2.1. Original Equations

Consider the Cauchy problem for the shallow water equations

$$\eta_t + \langle \nabla, (D(x) + \eta)\mathbf{u}\rangle = 0, \quad \mathbf{u}_t + \langle \mathbf{u}, \nabla\rangle\mathbf{u} + g\nabla\eta = 0, \quad t \in [0, T], \tag{2.1}$$

$$\eta\,\big|_{t=0} = \eta^{(0)}(x), \quad \mathbf{u}\,\big|_{t=0} = \mathbf{u}^{(0)}(x), \tag{2.2}$$

where $D(x) = D(x_1, x_2)$ is a known function defining the basin bottom topography, g is the gravitational acceleration, $\eta = \eta(x, t)$ and $\mathbf{u} = \mathbf{u}(x, t) = (u_1(x, t), u_2(x, t))$ are the unknown functions (the free surface elevation and the horizontal velocity, respectively), and $\eta^{(0)}(x)$ and $\mathbf{u}^{(0)}(x)$ are their given initial values. In the tsunami wave propagation problem, the function $\eta^{(0)}(x)$ is usually concentrated in a neighborhood of some point (the tsunami source), and $\mathbf{u}^{(0)}(x) = 0$. (These initial data corresponding to the so-called *piston model* of tsunami generation [1].) As usual, the solution is assumed to be defined in the closure $\overline{\Omega}_t$ of a domain Ω_t depending on $t \in [0, T]$ such that the function $D(x) + \eta(x, t)$ is positive in Ω_t and zero on the boundary $\partial\Omega_t$. We impose the following condition on the function $D(x)$:

Condition 1. The function $D(x)$ is defined in a neighborhood of the closure $\overline{\Omega}_0$ of some bounded domain $\Omega_0 \subset \mathbb{R}^2$; it is smooth, positive in Ω_0 and negative outside $\overline{\Omega}_0$, and $\nabla D(x) \neq 0$ everywhere on $\partial\Omega_0$.

This condition means that the basin has a gently sloping beach near the coastline. The construction described below does not work for basins with vertical walls or zero beach sloping angle at some points of the coastline.

Prior to giving a rigorous analysis of "small" asymptotic solutions of problem (2.1), (2.2), we present an approximate solution algorithm (which actually provides the leading term of the asymptotic solution constructed later on in the paper).

2.2. Construction of an Approximate Solution of Problem (2.1), (2.2)

We assume (as is usually the case in the tsunami wave propagation problem) that the functions $\eta^{(0)}(x)$ and $\mathbf{u}^{(0)}(x)$ in the initial conditions (2.2) are supported in Ω_0 (i.e., vanish in a neighborhood of $\partial\Omega_0$).

Along with problem (2.1), (2.2), consider the *formally linearized problem*

$$N_t(y, t) + \langle \nabla_y, D(y)\mathbf{U}(y, t)\rangle = 0, \ \mathbf{U}_t(y, t) + g\nabla_y N(y, t) = 0, \quad (y, t) \in \overline{\Omega}_0 \times [0, T], \tag{2.3}$$

$$N\,\big|_{t=0} = \eta^{(0)}(y), \quad \mathbf{U}\,\big|_{t=0} = \mathbf{u}^{(0)}(y), \quad y \in \overline{\Omega}_0. \tag{2.4}$$

(Here it is convenient for us to use the variable $y = (y_1, y_2)$ instead of x.) Further, let $\rho(y)$ be a smooth cutoff function that is unity in a neighborhood of $\partial\Omega_0$ and zero outside some larger neighborhood such that $\nabla_y D(y) \neq 0$ on the support $\operatorname{supp}\rho$.

Algorithm for constructing an approximate solution.

Step 1. Construct a solution $(N(y,t),\mathbf{U}(y,t))$, $(y,t)\in\overline{\Omega}_0\times[0,T]$ (exact or approximate) of the formally linearized problem (2.3), (2.4).

Step 2. Specify an approximate solution $\eta=\eta(x,t)$, $\mathbf{u}=\mathbf{u}(x,t)$ of problem (2.1), (2.2) by the parametric formulas

$$x=y-N(y,t)\frac{\rho(y)\nabla_y D(y)}{|\nabla_y D(y)|^2},\quad \eta=N(y,t),\quad \mathbf{u}=\mathbf{U}(y,t). \tag{2.5}$$

Remarks. 1. Formulas (2.5) give the leading term of the expansion in powers of ε of the asymptotic solutions constructed below. They may or may not apply in practice depending on whether N and $\mathbf{U}$ can really be treated as small quantities (in particular, whether the first formula in (2.5) actually defines a diffeomorphism). Here we do not discuss this problem but only note that practical applications involve many characteristic quantities whose certain dimensionless combinations determine whether the approximation in use is adequate: Can the waves under consideration be treated as long ones? (That is, does the shallow water approximation apply?) Is it reasonable to take locally averaged bathymetry for the function $D(x)$ (see [12–14])? Can the problem indeed be regarded as a small perturbation of the linearized problem? (In practice, this is often the case as long as the amplitude is small enough that wave breaking does not occur.)

2. Formulas (2.5) depend on the choice of the cutoff function $\rho(y)$. However, if the solution is small indeed (otherwise the formulas do not give a correct answer anyway), then an increment of the order of N in the argument y only produces small changes in the functions N and $\mathbf{U}$ (long waves!), so that the effect of a specific choice of ρ on the approximate solution is negligible. In fact, the second term in the first formula in (2.5) is only significant in a neighborhood of the boundary, where it determines the uprush (and where the waves are higher), but it is just there that we have $\rho(y)=1$, and no problems are encountered. The boundary $\partial\Omega_t$ of the domain occupied by water at time t is given parametrically by the formula

$$x=y-N(y,t)\frac{\nabla_y D(y)}{|\nabla_y D(y)|^2},\qquad y\in\partial\Omega_0.$$

Further, it is the uprush and the solution near some specific points of the boundary that are of most interest in applications to tsunami wave propagation, and so formula (2.5) is in fact only needed in regions where $\rho=1$.

3. As was mentioned in the introduction, the construction of an approximate solution of the degenerate hyperbolic problem at Step 1 can be carried out by well-developed semiclassical methods [11]. Here one deals with another combination of characteristic quantities, which determines whether the long waves in question can be regarded as short ones for the sake of the semiclassical approximation (see, e.g., [15, 16]).

3. MATHEMATICAL STATEMENT OF THE PROBLEM

Let us now proceed to strict statements and proofs.

3.1. Passage to Equations with a Small Parameter

We are interested in "small" smooth solutions of problem (2.1), (2.2), which we construct asymptotically, viewing the problem as a perturbation of a linear problem. To this end, we formally introduce a small parameter $\varepsilon\geqslant 0$ and assume that the initial conditions have the form $\eta^{(0)}=\varepsilon\widetilde{\eta}^{(0)}(x,\varepsilon)$, $\mathbf{u}^{(0)}=\varepsilon\widetilde{\mathbf{u}}^{(0)}(x,\varepsilon)$, where $\widetilde{\eta}^{(0)}(x,\varepsilon)$ and $\widetilde{\mathbf{u}}^{(0)}(x,\varepsilon)$ are smooth functions. Accordingly, we seek the solution in the form $\eta=\varepsilon\widetilde{\eta}(x,t,\varepsilon)$, $\mathbf{u}=\varepsilon\widetilde{\mathbf{u}}(x,t,\varepsilon)$. From now on, we only deal with the new unknown functions $\widetilde{\eta}$ and $\widetilde{\mathbf{u}}$ and, to avoid awkward notation, omit the tildes (i.e., we denote them again by η and $\mathbf{u}$). Then for these new unknown functions we obtain the problem

$$\eta_t+\langle\nabla,D(x)\mathbf{u}\rangle+\varepsilon\langle\nabla,\eta\mathbf{u}\rangle=0,\quad \mathbf{u}_t+g\nabla\eta+\varepsilon\langle\mathbf{u},\nabla\rangle\mathbf{u}=0,\quad t\in[0,T], \tag{3.1}$$

$$\eta\,\big|_{t=0}=\eta^{(0)}(x,\varepsilon),\quad \mathbf{u}\,\big|_{t=0}=\mathbf{u}^{(0)}(x,\varepsilon), \tag{3.2}$$

or, in concise form,

$$\mathscr{L}\psi+\varepsilon b(\psi,\nabla\psi)=0,\quad t\in[0,T],\qquad \psi\,\big|_{t=0}=\psi^{(0)}, \tag{3.3}$$

where $\psi(x,t,\varepsilon)={}^{\mathrm{T}}(\eta(x,t,\varepsilon),\mathbf{u}(x,t,\varepsilon))$ is the unknown vector function and

$$\mathscr{L}=\begin{pmatrix}\partial_t & \nabla\circ D(x)\\ g\nabla & \partial_t\end{pmatrix},\qquad b(\psi,\nabla\psi)=\begin{pmatrix}\langle\nabla,\eta\mathbf{u}\rangle\\ \langle\mathbf{u},\nabla\rangle\mathbf{u}\end{pmatrix}.$$

3.2. Definition of Solution of Problem (3.3)

Let $\Omega(t,\varepsilon)\subset\mathbb{R}^2$ be a family of finite domains with smooth boundary defined for $t\in[0,T]$ and sufficiently small $\varepsilon\geqslant 0$ and smoothly depending on (t,ε), and let $\psi(x,t,\varepsilon)={}^{\mathrm{T}}(\eta(x,t,\varepsilon),\mathbf{u}(x,t,\varepsilon))$ be a vector function defined for $x\in\overline{\Omega}(t,\varepsilon)$ and smoothly depending[1] on (x,t,ε). We say that the pair $(\Omega,\psi)=(\Omega(t,\varepsilon),\psi(x,t,\varepsilon))$ is *admissible* if

$$D(x)+\varepsilon\eta(x,t,\varepsilon)>0\quad\text{for } x\in\Omega(t,\varepsilon),\qquad D(x)+\varepsilon\eta(x,t,\varepsilon)=0\quad\text{for } x\in\partial\Omega(t,\varepsilon).$$

We use the same terminology in the absence of t (i.e., if Ω and Ψ are independent of t).

Throughout the following, we assume that the initial function $\psi^{(0)}(x,\varepsilon)$ in problem (3.3) is defined in the closure of a domain $\Omega^{(0)}(\varepsilon)$ such that the pair $(\Omega^{(0)},\psi^{(0)})$ is admissible.

Definition 1. A *solution* of problem (3.3) is an admissible pair (Ω,ψ) such that (for $\varepsilon\geqslant 0$ for which the pair is defined) the initial conditions $(\Omega,\psi)|_{t=0}=(\Omega^{(0)},\psi^{(0)})$ hold and the function $\psi(x,t,\varepsilon)$ satisfies the equations in (3.3) in Ω.

To define asymptotic solutions, we need to compare admissible pairs defined, generally speaking, in different domains.

Definition 2. We say that admissible pairs (Ω_1,ψ_1) and (Ω_2,ψ_2) *coincide modulo* $O(\varepsilon^n)$ and write $(\Omega_1,\psi_1)\equiv(\Omega_2,\psi_2)\bmod O(\varepsilon^n)$ if there exists a family of diffeomorphisms[2] $f(\,\cdot\,,\varepsilon)=\mathrm{id}+O(\varepsilon^n)$ such that $f(\Omega_1)=\Omega_2$ and $\psi_1-f^*(\psi_2)=O(\varepsilon^n)$. (We use a similar terminology for the case in which the objects in question also depend on the parameter $t\in[0,T]$.)

Definition 3. An *asymptotic solution* of problem (3.3) *modulo* $O(\varepsilon^n)$ is an admissible pair (Ω,ψ) such that (for $\varepsilon\geqslant 0$ for which the pair is defined) the initial conditions

$$(\Omega,\psi)|_{t=0}\equiv(\Omega^{(0)},\psi^{(0)})\bmod O(\varepsilon^n)$$

hold and the function $\psi(x,t,\varepsilon)$ gives the discrepancy $O(\varepsilon^n)$ when substituted into the equations in (3.3).

4. CONSTRUCTION OF ASYMPTOTIC SOLUTIONS AND THE MAIN THEOREM

To construct asymptotic solutions of the Cauchy problem (3.3), we use a change of variables that transforms the variable domain $\Omega(t,\varepsilon)$, which depends on the solution itself, into Ω_0. Consider the domains

$$\Omega_\lambda=\{x: D(x)+\lambda>0\}. \tag{4.1}$$

The following proposition is a consequence of the properties of $D(x)$.

Proposition 1. *There exists a smooth family $\{F(\,\cdot\,,\lambda)\colon\mathbb{R}^2\to\mathbb{R}^2\}$ of diffeomorphisms defined for real λ with sufficiently small absolute value and satisfying*

$$F(\,\cdot\,,0)=\mathrm{id},\quad F(\,\cdot\,,\lambda)\,(\Omega_\lambda)=\Omega_0. \tag{4.2}$$

Proof. The inverse diffeomorphisms $G(\,\cdot\,,\lambda)=F^{-1}(\,\cdot\,,\lambda)$ can be constructed as follows. For $G(y,\lambda)$ one can take, say, the solution of the following Cauchy problem for an ordinary differential equation:

$$\frac{d}{d\lambda}G(y,\lambda)=-\frac{\nabla D(G(y,\lambda))}{|\nabla D(G(y,\lambda))|^2}\rho(y),\qquad G(y,0)=y,$$

where the smooth cutoff function $\rho(y)$ is unity in a sufficiently small neighborhood of $\partial\Omega_0$ and zero outside a larger neighborhood. □

Let us fix such a family and the inverse $G(\,\cdot\,,\lambda)$.

Let (Ω,ψ) be an arbitrary admissible pair. For sufficiently small ε, the mapping

$$y=F(x,\varepsilon\eta(x,t,\varepsilon)) \tag{4.3}$$

[1] As usual, a smooth function on a closed set is understood as a function that has a smooth continuation into a neighborhood of the set. For the domains considered here, this is equivalent to smoothness up to the boundary.

[2] Here id is the identity diffeomorphism and $O(\varepsilon^n)$ stands for a smooth function vanishing for $\varepsilon=0$ together with all derivatives of order $\leqslant n-1$ with respect to ε.

is a diffeomorphism of $\Omega(t,\varepsilon)$ onto Ω_0 for all $t\in[0,T]$. Set

$$\Psi(y,t,\varepsilon)=\psi(x,t,\varepsilon)\equiv\begin{pmatrix}N(y,t,\varepsilon)\\ \mathbf{U}(y,t,\varepsilon)\end{pmatrix},\qquad N(y,t,\varepsilon)=\eta(x,t,\varepsilon),\quad \mathbf{U}(y,t,\varepsilon)=\mathbf{u}(x,t,\varepsilon),\tag{4.4}$$

where x and y are related by the diffeomorphism (4.3). Then, for any sufficiently small $\varepsilon\geqslant 0$, the function Ψ is defined in[3] $\overline{\Omega}_0\times[0,T]$, and the change of variables inverse to (4.3) has the form

$$x=G(y,\varepsilon N(y,t,\varepsilon)).\tag{4.5}$$

Let us find out what problem is obtained from (3.3) for the vector function Ψ in (4.4) obtained from ψ by the change of variables (4.3).

Proposition 2. *There exists a smooth vector function $B(\xi,\zeta,\varepsilon)$, $\xi\in\mathbb{R}^3$, $\zeta\in\mathbb{R}^6$, defined for $|\varepsilon|<C(1+|\xi|+|\zeta|)^{-1}$, $C>0$, and satisfying*

$$\mathscr{L}\psi(x,t,\varepsilon)+\varepsilon b(\psi(x,t,\varepsilon),\nabla\psi(x,t,\varepsilon))=\mathscr{L}_y\Psi(y,t,\varepsilon)+\varepsilon B(\Psi(y,t,\varepsilon),\nabla_y\Psi(y,t,\varepsilon),\varepsilon)\tag{4.6}$$

for sufficiently small $\varepsilon\geqslant 0$, where x and y are related by the diffeomorphism (4.3) *and*

$$\mathscr{L}_y=\begin{pmatrix}\partial_t & \nabla_y\circ D(y)\\ g\nabla_y & \partial_t\end{pmatrix}.$$

The function $B(\xi,\zeta,\varepsilon)$ is independent of ψ (but depends on the choice of the family of diffeomorphisms $F(\,\cdot\,,\lambda)$).

Proof. The proof is by a straightforward computation. For convenience, we denote the arguments of the function $\Psi={}^{\mathrm{T}}(N,\mathbf{U})$ by (y,τ); these variables are related to (x,t) by the mapping (4.3) and the equality $\tau=t$. Consider the matrices

$$G_y(y,\lambda):=\begin{pmatrix}G_{1y_1}(y,\lambda) & G_{1y_2}(y,\lambda)\\ G_{2y_1}(y,\lambda) & G_{2y_2}(y,\lambda)\end{pmatrix},\qquad G_\lambda(y,\lambda)=\begin{pmatrix}G_{1\lambda}(y,\lambda)\\ G_{2\lambda}(y,\lambda)\end{pmatrix}$$

of partial derivatives, denote their transposes by $G_y^{\mathrm{T}}(y,\lambda)$ and $G_\lambda^{\mathrm{T}}(y,\lambda)$, and set

$$\mathscr{J}_0(y,\tau,\varepsilon)=G_y^{\mathrm{T}}(y,\varepsilon N(y,\tau,\varepsilon)),$$
$$\mathscr{J}_1(y,\tau,\varepsilon)=N_y(y,\tau,\varepsilon)G_\lambda^{\mathrm{T}}(y,\varepsilon N(y,\tau,\varepsilon)),\qquad \mathscr{J}_2(y,\tau,\varepsilon)=N_\tau(y,\tau,\varepsilon)G_\lambda^{\mathrm{T}}(y,\varepsilon N(y,\tau,\varepsilon)).$$

From now on, we omit the arguments of functions for brevity. It follows from (4.4) and (4.5) that

$$N_y=(\mathscr{J}_0+\varepsilon\mathscr{J}_1)\eta_x,\qquad N_\tau=\eta_t+\varepsilon\mathscr{J}_2\eta_x$$

(note that $\mathscr{J}_0$ is the identity matrix for $\varepsilon=0$), and these equations, in turn, imply that

$$\eta_x=(\mathscr{J}_0+\varepsilon\mathscr{J}_1)^{-1}N_y,\qquad \eta_t=N_t-\varepsilon\mathscr{J}_2(\mathscr{J}_0+\varepsilon\mathscr{J}_1)^{-1}N_y.$$

We use these expressions to obtain the expressions

$$\nabla_x=(\mathscr{J}_0+\varepsilon\mathscr{J}_1)^{-1}\nabla_y,\qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial\tau}-\varepsilon\mathscr{J}_2(\mathscr{J}_0+\varepsilon\mathscr{J}_1)^{-1}\nabla_y$$

for the derivatives ∇_x and $\partial/\partial t$ in the new coordinates. Now we are in a position to write a formula for B,

$$B(\Psi,\nabla_y\Psi,\varepsilon)=-\begin{pmatrix}\mathscr{J}_2(\mathscr{J}_0+\varepsilon\mathscr{J}_1)^{-1}N_y+\dfrac{\langle\nabla,D(y)\mathbf{U}\rangle-\langle(\mathscr{J}_0+\varepsilon\mathscr{J}_1)^{-1}\nabla,\big(D(G(y,\varepsilon N))+\varepsilon N\big)\mathbf{U}\rangle}{\varepsilon}\\ \mathscr{J}_2(\mathscr{J}_0+\varepsilon\mathscr{J}_1)^{-1}\nabla\mathbf{U}-\langle\mathbf{U},(\mathscr{J}_0+\varepsilon\mathscr{J}_1)^{-1}\nabla\rangle\mathbf{U}-\dfrac{(\mathscr{J}_0+\varepsilon\mathscr{J}_1)^{-1}-I}{\varepsilon}N_y\end{pmatrix}.$$

This completes the proof of the proposition. □ □

[3]Note that (Ω_0,Ψ) is no longer an admissible pair.

It follows from Proposition 2 that, to construct an asymptotic solution of problem (3.3) as $\varepsilon \to 0$, it suffices to construct an asymptotic solution of the problem

$$\mathscr{L}_y\Psi + \varepsilon B(\Psi, \nabla_y\Psi, \varepsilon) = 0, \quad (x,t) \in \overline{\Omega}_0 \times [0,T], \qquad \Psi|_{t=0} = \Psi^{(0)}, \quad x \in \overline{\Omega}_0, \tag{4.7}$$

where $\Psi^{(0)}(y,\varepsilon)$ is obtained from $\psi^{(0)}(x,\varepsilon)$ by the change of variables $y = F(x, \varepsilon\eta^{(0)}(x,\varepsilon))$, and then return to the original variables using formula (4.5). Let us solve (4.7) using regular perturbation theory. To this end, we write $\Psi^{(0)}$, Ψ, and B as the formal series

$$\begin{gathered}\Psi^{(0)} \sim \sum_{j=0}^{\infty} \Psi_j^{(0)}\varepsilon^j, \quad \Psi \sim \sum_{j=0}^{\infty} \Psi_j\varepsilon^j, \\ B(\Psi, \nabla\Psi, \varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j B_j(\Psi_0, \dots, \Psi_j, \nabla_y\Psi_0, \dots, \nabla_y\Psi_j)\end{gathered} \tag{4.8}$$

in powers of ε, substitute these expansions into problem (4.7), and obtain the chain of Cauchy problems

$$\mathscr{L}_y\Psi_0 = 0, \qquad \Psi_0|_{t=0} = \Psi_0^{(0)}, \tag{4.9}$$

$$\mathscr{L}_y\Psi_j = -B_{j-1}(\Psi_0, \dots, \Psi_{j-1}, \nabla_y\Psi_0, \dots, \nabla_y\Psi_{j-1}), \qquad \Psi_0|_{t=0} = \Psi_j^{(0)}, \quad j = 1, 2, \dots, \tag{4.10}$$

in the cylinder $\overline{\Omega}_0 \times [0,T]$ for the successive terms of the asymptotic solution.

Thus, the problem is reduced to solving a sequence of linear Cauchy problems with the operator $\mathscr{L}_y$. This is a hyperbolic operator degenerating on the boundary of the domain Ω_0. The following assertion holds.

Theorem 1. *Let v be a smooth vector function in the cylinder $\overline{\Omega}_0 \times [0,T]$, and let u_0 be a smooth function on $\overline{\Omega}_0$. Then there exists a unique smooth solution of the Cauchy problem $\mathscr{L}_y u = v$, $u|_{t=0} = u_0$ in $\overline{\Omega}_0 \times [0,T]$.*

The proof is given below in Sec. 5.

Theorem 1 permits successively constructing smooth solutions of problems (4.9) and (4.10) in $\overline{\Omega}_0 \times [0,T]$. Restricting ourselves to the first n problems, i.e., terminating the asymptotic series for Ψ at the nth step, we obtain an asymptotic solution modulo $O(\varepsilon^{n+1})$. Then we return to the original variables and obtain an asymptotic solution of the original problem (3.1), (3.2). Thus, we arrive at the following theorem.

Theorem 2. *The Cauchy problem* (3.1), (3.2) *has an asymptotic solution* (Ω, ψ) *modulo* $O(\varepsilon^{n+1})$ *for any* $n = 0, 1, 2, \dots$. *This solution is asymptotically unique. More precisely, if* (Ω_1, ψ_1) *and* (Ω_2, ψ_2) *are two such solutions, then* $(\Omega_1, \psi_1) \equiv (\Omega_2, \psi_2) \bmod O(\varepsilon^{n+1})$.

Proof. The existence of a solution is proved by the preceding constructions. It remains to prove the asymptotic uniqueness. To this end, we apply the change of variables (4.3) to each of the two solutions (Ω_1, ψ_1) and (Ω_2, ψ_2) mentioned in the theorem. The resulting functions Ψ_1 and Ψ_2 satisfy one and the same problem (4.7) $\bmod O(\varepsilon^{n+1})$. Therefore, the first n coefficients of their Taylor expansions in powers of ε satisfy the respective problems in the chain (4.9) and (4.10) with the same initial conditions; thus, it can readily be proved by induction that they coincide. Therefore, $\Psi_1 - \Psi_2 = O(\varepsilon^{n+1})$. Then the changes of variables (4.5) corresponding to Ψ_1 and Ψ_2 coincide $\bmod O(\varepsilon^{n+1})$; the composition of one of these changes of variables with the inverse of the other gives an almost identity family of diffeomorphisms as in Definition 2, thus establishing the coincidence $\bmod O(\varepsilon^{n+1})$ of the solutions (Ω_1, ψ_1) and (Ω_2, ψ_2). □

Since an exact solution is at the same time an asymptotic solution (for any n), we obtain

Corollary 1. *The solution of the Cauchy problem* (3.1), (3.2) *is unique* $\bmod O(\varepsilon^n)$ *for any* n.

5. PROOF OF THEOREM 1

In this section, we again denote the variable y by x. Thus, we must prove that if the function $D(x)$ satisfies Condition 1, then the problem

$$\eta_t + \langle \nabla, D(x)\mathbf{u} \rangle = f_1(x,t), \quad \mathbf{u}_t + g\nabla\eta = f_2(x,t), \tag{5.1}$$

$$\eta\,\big|_{t=0} = \eta^{(0)}(x), \quad \mathbf{u}\,\big|_{t=0} = \mathbf{u}^{(0)}(x) \tag{5.2}$$

in the cylinder $\overline{\Omega}_0 \times [0,T]$ with smooth initial data $\eta^{(0)}, \mathbf{u}^{(0)} \in C^\infty(\overline{\Omega}_0)$ and right-hand sides $f_1, f_2 \in C^\infty(\overline{\Omega}_0 \times [0,T])$ has a unique smooth solution $\eta, \mathbf{u} \in C^\infty(\overline{\Omega}_0 \times [0,T])$.

5.1. Reduction to a Wave Equation Degenerating on the Boundary of the Domain

Recall this well-known reduction. Differentiating the first equation in (5.1) with respect to t and expressing $\mathbf{u}_t$ from the second equation, we obtain the Cauchy problem for the wave equation

$$\eta_{tt} - \langle\nabla, gD(x)\nabla\eta\rangle = f_{1t} - \langle\nabla, D(x)f_2\rangle, \quad \eta\,\big|_{t=0} = \eta^{(0)}, \quad \eta_t\,\big|_{t=0} = f_1 - \langle\nabla, D\mathbf{u}^{(0)}\rangle \tag{5.3}$$

with smooth initial data and right-hand sides. Further, by the second equation in (5.1), the function $\mathbf{u}$ can be expressed via η in the form

$$u = u^{(0)} + \int_0^t (f_2 - g\nabla\eta)\,dt$$

and hence is smooth if so is η. Thus, it suffices to prove the existence and uniqueness of a smooth solution of the Cauchy problem (5.3) in $\overline{\Omega}_0 \times [0,T]$. The velocity $c(x) = \sqrt{gD(x)}$ in the wave equation in (5.3) vanishes on the boundary of the domain, and hence the existence and uniqueness of a solution and its smoothness up to the boundary require a proof.

5.2. Uniformization of Problem (5.3)

To prove this, we "lift" problem (5.3) to a closed 3-manifold M using the uniformization procedure suggested in [9]. The paper [9] deals with a class of second-order operators dubbed *operators with Bessel degeneration* on a manifold X with boundary. For the case of $X = \overline{\Omega_0}$, these are operators that, in any local coordinates $z = (z_1, z_2)$ such that $dz_1 \wedge dz_2 = dx_1 \wedge dx_2$, have the form $L = -\langle\nabla_z, B(z)\nabla_z\rangle$ with a smooth real symmetric matrix $B(z) = B^*(z) \geqslant 0$ satisfying the following conditions: (i) $B(z) > 0$ at the interior points of the domain; (ii) if the domain is given in the local coordinates by the inequality $z_1 > 0$, then the matrix $B(z)$ has the form $B(z) = z_1 A(z)$, where $A(z) = A^*(z) > 0$ everywhere including the boundary ($z_1 = 0$). Obviously, the operator

$$L = -\langle\nabla, gD(x)\nabla\rangle \tag{5.4}$$

occurring in (5.3) belongs to this class.

The construction in [9] gives a closed smooth 3-manifold M and a smooth (but not locally trivial!) bundle $\pi\colon M \to \overline{\Omega}_0$ such that (i) the group $\mathbb{S}^1$ acts on M, and π is the projection onto the quotient space; (ii) the restriction of π to the open set Ω_0 is isomorphic to the trivial bundle $\mathbb{S}^1 \times \Omega_0 \to \Omega_0$; (iii) over a neighborhood of an arbitrary point in $\partial\Omega_0$, the bundle π is described in suitable coordinates (z_1, z_2) where the domain is given by the inequality $\Omega_0 = \{z_1 > 0\}$ by the formulas

$$\mathbb{D}^2 \times \partial\Omega_0 \to \overline{\mathbb{R}_+} \times \partial\Omega_0, \quad (y, z_2) \mapsto \left(\frac{y_1^2 + y_2^2}{4}, z_2\right).$$

Here $\mathbb{D}^2 = \left\{(y_1, y_2) \in \mathbb{R}^2 : y^2 + y_2^2 \leqslant \delta\right\}$ is a circle of some radius $\delta > 0$ on which the group $\mathbb{S}^1$ acts by rotations. The subspace of $\mathbb{S}^1$-invariant functions in $C^\infty(M)$ (with respect to the natural action induced by the $\mathbb{S}^1$-action on M) is naturally identified with $C^\infty\left(\overline{\Omega}_0\right)$, while the subspace of $\mathbb{S}^1$-invariant functions in $L^2(M)$ (with respect to the measure $\pi^*(dx_1 \wedge dx_2) \wedge d\varphi$, where $d\varphi$ stands for the naturally defined form on the $\mathbb{S}^1$-orbits) is identified with $L^2(\overline{\Omega}_0)$. By Theorem 1 in [9], there exists an $\mathbb{S}^1$-invariant *lifting* of the operator L to M, which is an $\mathbb{S}^1$-invariant symmetric positive semidefinite second-order differential operator P whose restriction to $C^\infty\left(\overline{\Omega}_0\right)$ coincides with L. Let g, $v^{(0)}$, and $v^{(1)}$ be $\mathbb{S}^1$-invariant liftings to M of the functions $f_{1t} - \langle\nabla, D(x)f_2\rangle$, $\eta^{(0)}$, and $f_1 - \langle\nabla, D\mathbf{u}^{(0)}\rangle$, respectively, that is, their images under the natural embedding $C^\infty\left(\overline{\Omega}_0\right) \subset C^\infty(M)$. If η is a smooth solution of problem (5.3), then its $\mathbb{S}^1$-invariant lifting v is a smooth solution of the problem

$$v_{tt} + Pv = g, \quad v\,\big|_{t=0} = v^{(0)}, \quad v_t\,\big|_{t=0} = v^{(1)}. \tag{5.5}$$

Conversely, a smooth $\mathbb{S}^1$-invariant solution v of problem (5.5), when interpreted as a (time-dependent) element of the space $C^\infty\left(\overline{\Omega}_0\right)$, is a smooth solution of problem (5.3). Thus, it suffices to show that problem (5.5) always has a unique smooth solution (which, by the uniqueness, is automatically $\mathbb{S}^1$-invariant, since so are the operator and the data of the problem).

5.3. Analysis of the Uniformized Problem (5.5)

Theorem 1 in [9] also shows that the operator P is hypoelliptic (i.e., it follows from $Pv \in C^\infty(M)$ that $v \in C^\infty(M)$) by Theorem 2.6.2 in [10]. Since the operator P on $C^\infty(M)$ is also symmetric and positive semidefinite on the space $L^2(M)$, it follows that it is essentially self-adjoint. Therefore, the solution of the

Cauchy problem (5.5) in which the operator P is replaced by its closure exists and is unique in the space $L^2(M)$ and is well defined by the well-known formula

$$v(t) = \cos(Qt)v^{(0)} + \frac{\sin(Qt)}{Q}v^{(1)} + \int_0^t \frac{\sin(Q(t-\tau))}{Q}g(\tau)\,d\tau, \tag{5.6}$$

where $Q = P^{1/2}$. This implies the bound

$$\|v(t)\| \leqslant \|v^{(0)}\| + T\|v^{(1)}\| + \frac{T^2}{2}\sup_{\tau\in[0,T]}\|g(\tau)\|, \tag{5.7}$$

where $\|\cdot\|$ is the norm on the space $L^2(M)$. It is easily seen that the functions $P^k v(t)$, $k = 1, 2, \dots$, are solutions of the Cauchy problems obtained from (5.5) by applying the operator P^k to the right-hand side and the initial conditions. Therefore, estimates of the same type are valid for these functions, and since the initial conditions and the right-hand sides are smooth, we conclude that $\|P^k v(t)\| < \infty$ for all $k = 0, 1, 2, \dots$. The derivative $V = v_t$ of the solution of problem (5.5) with respect to time satisfies a problem of the form (5.5) with right-hand side g_t in the equation and with the initial conditions $V(0) = v^{(1)}$, $V_t(0) = g(0) - Pv^{(0)}$; now, by induction, one proves that

$$\sup_{t\in[0,T]}\|P^k\frac{\partial^j v(t)}{\partial t^j}\| < \infty, \quad k, j = 0, 1, 2, \dots \tag{5.8}$$

Let $\|\cdot\|_s$ be the norm on the Sobolev space $H^s(M)$, and so $\|\cdot\|_0 = \|\cdot\|$. By Theorem 2.6.2 in [10], taking into account the reasoning in the proof of Theorem 2.4.2 in [10], we see that the following hypoelliptic estimates hold for the operator P with any s:

$$\|w\|_{s+\delta} \leqslant C_s(\|Pw\|_s + \|w\|_0) \tag{5.9}$$

with some constants C_s, where $\delta > 0$ is a given number. (In fact, it follows from these theorems that one can take $\delta = 1/2$, but this is not essential for our purposes.) Without loss of generality, we can assume that $C_s \geqslant 1$ for all s. Then we obtain

$$\begin{aligned}
&\|w\|_\delta \leqslant C_0(\|Pw\|_0 + \|w\|_0),\\
&\|w\|_{2\delta} \leqslant C_\delta(\|Pw\|_\delta + \|w\|_0) \leqslant C_\delta C_0(\|P^2w\|_0 + \|Pw\|_0 + \|w\|_0),
\end{aligned}$$

and further, by induction, for all $k \in \mathbb{Z}_+$

$$\|w\|_{k\delta} \leqslant C_{(k-1)\delta}C_{(k-2)\delta}\cdots C_0(\|P^k w\|_0 + \|P^{k-1}w\|_0 + \dots + \|w\|_0). \tag{5.10}$$

It follows from (5.8) and (5.10) that

$$\sup_{t\in[0,T]}\|\frac{\partial^j v(t)}{\partial t^j}\|_{k\rho} < \infty, \quad j = 0, 1, 2, \dots, \tag{5.11}$$

for every $k \in \mathbb{Z}_+$, and thus $v \in C^\infty(M \times [0,T])$. This completes the proof of Theorem 1.

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