AA203 Final Project:
Optimal Control for Landing Rockets

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Motivation

The commercialization of space travel has caused a push towards rocket reusability. One promising scheme requires the precision powered-descent of a rocket's first stage. This paper reviews previous work on a numerically efficient, convex formulation of the powered-descent guidance problem and investigates an extension to the non-minimum phase first stage system. Results from previous work are reproduced for a two-dimensional point-mass system with non-convex thrust bounds, pointing constraints, and a glideslope boundary. A lossless convexification of the problem is formulated and solved.

The Optimization Problem

The general formulation of the point-mass powered descent problem is formulated as:

$$\begin{array}{llll} \text{minimize}_{\boldsymbol{T}(t)} & \int_0^{t_f} \|\boldsymbol{T}\| \, \mathrm{d}t & & \\ \text{subject to} & \dot{\boldsymbol{x}}(t) = A_{5,5}\boldsymbol{x} + B_{5,2}(\boldsymbol{g} + \frac{1}{m}\boldsymbol{T}) + C(\boldsymbol{x}) & \text{rocket dynamics} & \\ & 0 < \rho_1 \leq \|\boldsymbol{T}\| \leq \rho_2 & \text{thrust magnitude constraints} & \text{(Problem 1)} \\ & T_y \geq \|\boldsymbol{T}\| \cos\theta & \text{thrust pointing constraint} & \\ & \text{boundary and state conditions} & & \end{array}$$

$$\text{where} \quad \dot{\boldsymbol{x}}(t) = A_{5,5}\boldsymbol{x} + B_{5,2}(\boldsymbol{g} + \tfrac{1}{m}\boldsymbol{T}) + C(\boldsymbol{x}) \equiv \begin{bmatrix} O & I_2 & \mathbf{0} \\ O & O & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} O \\ I_2 \\ 0 \end{bmatrix} \left[\boldsymbol{g} + \tfrac{\boldsymbol{T}}{m}\right] + \begin{bmatrix} \mathbf{0}_{4\times 1} \\ -\alpha\|\boldsymbol{T}\| \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{r}^T & \dot{\boldsymbol{r}}^T & m \end{bmatrix}^T \qquad \boldsymbol{r} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$$

where	α	constant scalar describing mass consumption rate
	$\boldsymbol{T}$	2-dimensional controlled thrust vector
	ρ_1, ρ_2	bounds on thrust magnitude
	O, $\mathbf{0}$, 0	zero (square) matrix, zero vector, scalar zero

Note that the $\frac{\dot{m}}{m}$ term in Newton II: $\frac{1}{m}\boldsymbol{F} = \ddot{x} + \frac{\dot{m}}{m}\dot{x}$ has been neglected. Our goal is to apply discretization to convert the ODE into a difference equation in which $\boldsymbol{x}_k$ can be determined entirely by the control history (a vector). We will then use CVX to solve the problem for the control history vector.

Convexification

The first difficulty arises from the thrust constraint (derived from the physical limitations of rocket engines) which is non-convex: the feasible region is not simply connected. We formulate a relaxed version of the problem as follows, using a slack variable Γ.

$$\begin{array}{lll} \text{minimize}_{\boldsymbol{T}(t),\Gamma(t)} & \int_0^{t_f} \Gamma \,\mathrm{d}t & \\ \text{subject to} & \dot{\boldsymbol{x}}(t) = A_{5,5}\boldsymbol{x} + B_{5,2}(\boldsymbol{g} + \frac{1}{m}\boldsymbol{T}) + C(\boldsymbol{x}) & \text{rocket dynamics} \\ & \|\boldsymbol{T}\| \leq \Gamma & \text{thrust magnitude constraints} \\ & 0 < \rho_1 \leq \Gamma \leq \rho_2 & \text{thrust magnitude constraints} \\ & T_y \geq \Gamma \cos\theta & \text{thrust pointing constraint} \\ & \text{boundary and state conditions} & \end{array} \qquad \text{(Problem 2)}$$

In Appendix 1, we prove that any solution to (Problem 2) yields $\|\boldsymbol{T}\| = \Gamma$ throughout the trajectory, and therefore lies within the feasible set of (Problem 1). (Problem 2) now has convex constraints.

Change of Variables

The second difficulty arises from the non-linearity of the governing ODE (the second term contains $\frac{1}{m}$, where m is a member of the state variable). We overcome this with a change of variables:

$$\sigma \equiv \frac{\Gamma}{m} \qquad \boldsymbol{u} \equiv \frac{\boldsymbol{T}}{m} \qquad z = \ln m$$

This change of variables, combined with a slight approximation (see Appendix 2), allows us to formulate the problem in terms of a linear ODE with convex constraints:

$$\begin{array}{ll} \underset{t_f,\boldsymbol{u}(\cdot),\sigma(\cdot)}{\text{minimize}} & \displaystyle\int_0^{t_f} \sigma(t)\,\mathrm{d}t \\ \text{subject to} & \ddot{\boldsymbol{r}}(t) = \boldsymbol{u}(t) + \boldsymbol{g} \\ & \dot{z}(t) = -\alpha\sigma(t) \\ & \|\boldsymbol{u}(t)\| \leq \sigma(t) \\ & \mu_1(t)\left[1 - (z(t) - z_0(t)) + \dfrac{(z(t)-z_0(t))^2}{2}\right] \leq \sigma(t) \leq \mu_2\left[1 - (z(t) - z_0(t))\right] \\ & \ln(m_{wet} - \alpha\rho_2 t) \leq z(t) \leq \ln(m_{wet} - \alpha\rho_1 t) \\ & \|S_j\boldsymbol{x}(t) - \boldsymbol{v}_j\| + \boldsymbol{c}_j^T\boldsymbol{x}(t) + a_j \leq 0,\ j = 1,\ldots,n_S \\ & m(0) = m_{wet},\ r(0) = r_o,\ \dot{r}(0) = \dot{r}_o,\ r(t_f) = \dot{r}(t_f) = 0. \end{array} \qquad \text{(Problem 3)}$$

Discretization

(Problem 3) is sufficiently well-behaved that we can discretize it into a convex problem in $\mathbb{R}^q$ (where q is TBD).

Time is discretized as $t_k = k\Delta t, \quad k = 0,\ldots,N$, s.t. $N\Delta t = t_f$.

The control inputs are represented as piece-wise constant functions, $\phi_0,\ldots,\phi_{N-1}$, where

$$\phi_j(t) = \begin{cases} 1, & \text{when } t \in [t_{j-1}, t_j]; \\ 0, & \text{otherwise,} \end{cases} \quad , \quad j = 0,\ldots,N-1.$$

Then: $\begin{bmatrix} u(t) \\ \sigma(t) \end{bmatrix} = \sum_{j=0}^{N-1} \boldsymbol{p}_j\phi_j(t), \quad t \in [0, t_f]$ where $\boldsymbol{p}_j = \begin{bmatrix} \boldsymbol{u}(t_{j-1}) \\ \sigma(t_{j-1}) \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{u}_{j-1} \\ \sigma_{j-1} \end{bmatrix}$

We define: $\boldsymbol{\eta} = \begin{bmatrix} p_0^T & \ldots & p_{N-1}^T \end{bmatrix}^T, \quad \boldsymbol{x}_k = \begin{bmatrix} \boldsymbol{r}_k^T & \dot{\boldsymbol{r}}_k^T & z_k \end{bmatrix}^T, \quad \boldsymbol{y}_k = \begin{bmatrix} \boldsymbol{r}_k^T & \dot{\boldsymbol{r}}_k^T \end{bmatrix}^T, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_0 & \ldots & \sigma_{N-1} \end{bmatrix}^T$

Our ODE can be written: $\begin{bmatrix} \dot{\boldsymbol{r}} \\ \ddot{\boldsymbol{r}} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} O & I & \boldsymbol{0} \\ O & O & \boldsymbol{0} \\ \boldsymbol{0}^T & \boldsymbol{0}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \dot{\boldsymbol{r}} \\ z \end{bmatrix} + \begin{bmatrix} O \\ I \\ \boldsymbol{0}^T \end{bmatrix} \boldsymbol{g} + \begin{bmatrix} O & \boldsymbol{0} \\ I & \boldsymbol{0} \\ \boldsymbol{0}^T & -\alpha \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ \sigma \end{bmatrix}$ or $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{g} + C\boldsymbol{w}$

We can discretize this ODE, turning it into a difference equation:

$$A_d = e^{A\Delta t} \qquad B_d = \int_0^{\Delta t} e^{A(\Delta t - s)} B\,\mathrm{d}s \qquad C_d = \int_0^{\Delta t} e^{A(\Delta t - s)} C\,\mathrm{d}s$$

$$A_k = A_d^k \qquad B_k = B + \cdots + A_d^{k-1}B_d \qquad C_k = \begin{bmatrix} A_d^{k-1}C_d & \ldots & C_d & 0 & \ldots & 0 \end{bmatrix}$$

$$\dot{\boldsymbol{x}}_{k+1} = A_d\boldsymbol{x}_k + B_d\boldsymbol{g} + C_d\boldsymbol{w}_k \qquad \dot{\boldsymbol{x}}_k = A_k\boldsymbol{x}_0 + B_k\boldsymbol{g} + C_k\boldsymbol{\eta}$$

We approximate the cost of the trajectory as: $\int_0^{t_f} \sigma(t)\,\mathrm{d}t \approx \mathbf{1}^T \boldsymbol{\sigma}$

All inequality constraints are expressed as: $\tilde{g}(t_k, \eta) \leq 0, \quad k = 0, \ldots, N,$

for some function $\tilde{g} : \mathbb{R}^{3M+1} \to \mathbb{R}$. Note that the constraints are only satisfied at the temporal nodes but this is sufficient for our needs. Also, $\tilde{g}$ is a convex function of η if the original form is convex.

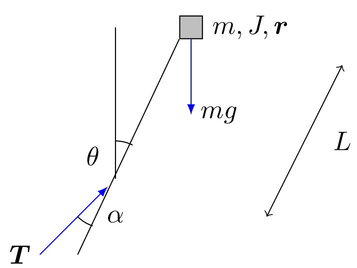
We now have a discretized approximation to (Problem 3):

$$
\begin{aligned}
\underset{N,\eta}{\text{minimize}} \quad & \mathbf{1}^T \boldsymbol{\sigma} \\
\text{subject to} \quad & \|\boldsymbol{u}_k\| \leq \sigma_k \\
& \mu_1(t_k) \left[1 - (z_k - z_0(t_k)) + \frac{(z_k - z_0(t_k))^2}{2}\right] \leq \sigma_k \leq \mu_2(t_k)[1 - (z_k - z_0(t_k))] \\
& \ln(m_{wet} - \alpha\rho_2 t_k) \leq z_k \leq \ln(m_{wet} - \alpha\rho_1 t_k) \\
& \|S_j \boldsymbol{y}_k - v_j\| + \boldsymbol{c}_j^T \boldsymbol{y}_k + a_j \leq 0, \quad \forall j = 1, \ldots, n_s \\
& \forall k = 1, \ldots, N \\
& \text{boundary conditions}
\end{aligned}
\qquad \text{(Problem 4)}
$$

(Problem 4) is a finite-dimensional second-order-cone program (SOCP). It can be readily solved given any N to the globally optimal solution. Note that N is related to the time of flight, $N = \frac{t_f}{\Delta t}$. Because minimizing fuel often is the same as minimizing time of flight, one can simply run a linear search starting with small N until a feasible solution is found.

Optimal Simultaneous Trajectory and Stability Control

In [1, 2], the point mass model is based on the assumption that the translational and attitude dynamics are decoupled, which "reduces the problem complexity considerably." Clearly this is not the case for a rocket propelled from its base and we turn our attention to the problem of stabilization. One option would be to design a linear feedback loop to stabilize the rocket, and command it to follow a trajectory computed from the solution of (Problem 4). However, $\boldsymbol{r}$ exhibits non-minimum phase dynamics (the rocket must first thrust to the right in order to move left), and non-minimum phase trajectory tracking is an active area of research. Instead, we concede to making a sequence of aggressive approximations to linearize an ODE to describe simultaneously the rocket's dynamics and trajectory. We consider a simplified version of the problem in which $\dot{m} = 0$. This model could be used in a finite-horizon optimal controller, and the inaccuracies resulting from the approximations could be corrected for at each step.



Consider the inverted pendulum, depicted above. We shall assume that $\frac{\pi}{2} \gg |\alpha| \gg |\theta|$.

Since $|\alpha| \gg |\theta|$: $\ddot{\boldsymbol{r}} \approx \frac{1}{m}\boldsymbol{T} + \boldsymbol{g}$ where $\boldsymbol{g} = \begin{bmatrix} 0 & -9.81 \end{bmatrix}^T$

Consider torques around m: $J\ddot{\theta} = -L\cos\theta T_x + L\sin\theta T_y$

Since θ is small: $J\ddot{\theta} \approx -LT_x + L\theta T_y$

We make the additional assumption that $T_y \approx mg$: $J\ddot{\theta} \approx -LT_x + Lmg\theta$

The resultant approximate ODE is:

$$\begin{bmatrix} \dot{\boldsymbol{r}} \\ \ddot{\boldsymbol{r}} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} O & I & 0 & 0 \\ O & O & 0 & 0 \\ \mathbf{0}^T & \mathbf{0}^T & 0 & 1 \\ \mathbf{0}^T & \mathbf{0}^T & \frac{Lmg}{J} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \dot{\boldsymbol{r}} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} O \\ I \\ \mathbf{0}^T \\ \mathbf{0}^T \end{bmatrix} \boldsymbol{g} + \begin{bmatrix} O \\ \frac{1}{m} I \\ \mathbf{0}^T \\ \begin{bmatrix} -\frac{L}{J} & 0 \end{bmatrix} \end{bmatrix} \boldsymbol{T} \qquad \text{or} \qquad \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{g} + C\boldsymbol{T}$$

We apply our usual discretization:

$$A_d = e^{A\Delta t} \qquad B_d = \int_0^{\Delta t} e^{A(\Delta t - s)} B \, \mathrm{d}s \qquad C_d = \int_0^{\Delta t} e^{A(\Delta t - s)} C \, \mathrm{d}s$$
$$A_k = A_d^k \qquad B_k = B + \cdots + A_d^{k-1} B_d \qquad C_k = \begin{bmatrix} A_d^{k-1} C_d & \dots & C_d & 0 & \dots & 0 \end{bmatrix}$$

We can then use: $\qquad \boldsymbol{x}_k = A_k \boldsymbol{x}_0 + B_k \boldsymbol{g} + C_k \boldsymbol{U} \qquad$ for $k \in \{0, ..., N\} \qquad$ (*)

where $\boldsymbol{U} = \begin{bmatrix} \boldsymbol{T}_0^T & \dots & \boldsymbol{T}_{N-1}^T \end{bmatrix}^T$

We can now write the linear optimization problem:

$$\begin{array}{ll} \text{minimize}_{\ \boldsymbol{G}, \{\boldsymbol{T}_k\}} & \mathbf{1}^T \boldsymbol{G} \\ \text{subject to} & \mathbf{0} \prec \rho_1 \mathbf{1} \preceq \boldsymbol{G} \preceq \rho_2 \mathbf{1} \\ & \|\boldsymbol{T}_k\| \preceq G_k \qquad \text{for } k \in \{0, ..., N-1\} \qquad\qquad \text{(Problem 5)} \\ & \text{difference equation (*)} \\ & \text{boundary conditions} \end{array}$$

Simulation Results

Simulations were run for three separate systems: a constant-mass point mass system, a variable-mass point mass system, and a constant-mass non-minimum phase system. Parameters for the point mass system make use of values as close to the Space X Falcon 9 v1.1 first stage as possible:

$$\begin{array}{rcl rcl} m_{\text{wet}} & = & 408 \text{ Mg}, & m_{\text{dry}} & = & 390 \text{ Mg}, \\ \rho_1 & = & 0.4 T_{\max}, & \rho_2 & = & 0.87 T_{\max} \\ T_{\max} & = & 5850 \text{ kN}, & \alpha & \approx & 3.6 \times 10^{-4} \text{ s/m} \end{array}$$

The initial state of the spacecraft relative to a goal state at the origin, in a surface-fixed guidance frame, is:

$$\boldsymbol{r}_o = \begin{bmatrix} 500 \\ 2000 \end{bmatrix} \text{m}, \qquad \dot{\boldsymbol{r}}_o = \begin{bmatrix} 100 \\ -100 \end{bmatrix} \text{m/s}$$

Simulation results are plotted in Appendix 3.

We find that in all cases the controller rides the rails, as expected in constrained problems. In the constant-mass point mass system, the controller is capable of following a minimum-throttle profile. This becomes infeasible in the variable-mass case, where the system must avoid violating the glide slope constraint.

Maximum and minimum time problems were solved for the variable-mass system. These were constrained primarily by available fuel. Results are interesting in terms of the warping of the (mass-normalized) thrust control bounds over time due to the decreasing mass. Bounds were determined by a linear search on flight time.

Simulations for the full-dynamics system were conducted with a separate set of input variables due to numerical instability with the Falcon 9 parameters. The non-minimum phase nature of the system is clearly visible in the data; the system first pushes itself in the opposite direction before curving back towards the origin. The system also follows a bang-bang control policy but exhibits chattering in the lower thrust regime.

Conclusion

Lossless convexification was successfully applied to the powered-descent problem for a variable-mass point mass system. A constant mass approximation allowed for the convexification of the non-minimal phase rocket problem. Further work on trajectory planning and tracking for non-minimum phase systems is required.

Appendix 1: Lossless Convexification

The following is adapted from [1] to our simplified problem and notation. Lemma 2 can be found in the paper. The proposed convexification of the problem can be written:

$$
\begin{array}{ll}
\text{minimize} & \int_0^{t_f} \Gamma(t)\,\mathrm{d}t \\
\text{subject to} & \dot{\boldsymbol{x}}(t) = A_{5,5}\boldsymbol{x} + B_{5,2}(\boldsymbol{g} + \frac{1}{m}\boldsymbol{T}) + C(\boldsymbol{x}) \text{ (see below)} \\
& \|\boldsymbol{T}(t)\| \leq \Gamma(t) \\
& 0 < \rho_1 \leq \Gamma(t) \leq \rho(2) \\
& T_y \geq 0 \qquad \text{(i.e. } \theta = \pi/2) \\
& m(t_f) \geq m_f \\
& \boldsymbol{x}_f = \begin{bmatrix} 0 & 0 & 0 & 0 & \text{free} \end{bmatrix},\ t_f \text{ is free.}
\end{array}
$$

Co-state conditions:

$$H = \Gamma + p_1\dot{x}_1 + p_2\dot{x}_2 + p_3\frac{T_x}{m} + p_4(-g + \frac{T_y}{m}) - p_5\alpha\,\|\boldsymbol{T}\|$$

$$-H_x = \begin{bmatrix} 0 \\ 0 \\ -p_1 \\ -p_2 \\ \frac{1}{m^2}(p_3T_x + p_4T_y) \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \end{bmatrix}$$

Transversality conditions: $\left.[\cancelto{0}{h_x} - p]^T\delta x_f + [H + \cancelto{0}{h_t}]\delta t_f\right|_{t_f} = 0$

Only the 5th component of δx_f is free. Whence: $p_5 = 0 \qquad H(t_f) = 0$

Now we have:

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

Equivalently, $\dot{\boldsymbol{\lambda}} = -R\boldsymbol{\lambda}$ where $\boldsymbol{\lambda} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix}^T$ and R is introduced by comparison.

Let $\boldsymbol{z}(t) = \begin{bmatrix} 0_{2\times 2} \\ I_2 \end{bmatrix} \boldsymbol{\lambda}(t) \equiv E\boldsymbol{\lambda}(t) \equiv \begin{bmatrix} p_3 \\ p_4 \end{bmatrix}$.

By Lemma 2, $\boldsymbol{z}(t)$ is either identically $\mathbf{0}$ or is zero at a countable number of points in $[0, t_f]$.

Suppose $\boldsymbol{z}(t) = \mathbf{0}$. This means $p_3 = p_4 = 0$. Combining with the above results: $H|_{t_f} = \Gamma(t_f) + p_1\cancelto{0}{\dot{x}_1(t_f)} + p_2\cancelto{0}{\dot{x}_2(t_f)} = \Gamma(t_f) = 0$ which cannot be the case, by our constraints.

Therefore $\begin{bmatrix} p_3 & p_4 \end{bmatrix}^T = \mathbf{0}$ on a countable (and therefore measure-zero) subset of $[0, t_f]$ (1).

By the second statement in Lemma 2, $\boldsymbol{z}(t) \neq -\alpha\hat{\boldsymbol{n}} = \alpha\hat{\boldsymbol{y}}$ (2).

Given (1), we apply the Pontryagin minimum principle without fear of singularities. Given Γ:

$$\begin{bmatrix} p_3^\star & p_4^\star \end{bmatrix} \begin{bmatrix} T_x^\star \\ T_y^\star \end{bmatrix} = \max_{\boldsymbol{T}:\|\boldsymbol{T}\|\leq\Gamma} \begin{bmatrix} p_3^\star & p_4^\star \end{bmatrix} \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

Because of (2) and since $\begin{bmatrix} p_3^\star & p_4^\star \end{bmatrix} \equiv \boldsymbol{z}^{\star T}$, we can always guarantee $\boldsymbol{z}^{\star T}\boldsymbol{T}^\star > 0$ and by straightforward geometric reasoning we conclude $\|\boldsymbol{T}\| = \Gamma$ always.

This result, adapted from the paper for our simpler problem set-up, shows that, given any Γ, the optimal control satisfies $\|\boldsymbol{T}\| = \Gamma$, and we can use the same convexification trick presented in the paper.

Appendix 2: Change of Variables

Consider (Problem 2), with the ODE rewritten and the state constraints made explicit:

$$\begin{aligned}
\underset{t_f, \boldsymbol{T}(\cdot), \Gamma(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \Gamma(t)\,\mathrm{d}t \\
\text{subject to} \quad & \ddot{\boldsymbol{r}}(t) = \boldsymbol{g} + \boldsymbol{T}(t)/m(t) \\
& \dot{m}(t) = -\alpha \Gamma(t) \\
& \|\boldsymbol{T}(t)\| \leq \Gamma(t) \\
& 0 < \rho_1 \leq \Gamma(t) \leq \rho_2 \\
& \|S_j \boldsymbol{x}(t) - \boldsymbol{v}_j\| + \boldsymbol{c}_j^T \boldsymbol{x}(t) + a_j \leq 0, \quad j = 1, \ldots, n_S \\
& m(0) = m_{wet},\ \boldsymbol{r}(0) = \boldsymbol{r}_0,\ \dot{\boldsymbol{r}}(0) = \dot{\boldsymbol{r}}_0,\ \boldsymbol{r}(t_f) = \dot{\boldsymbol{r}}(t_f) = \boldsymbol{0}
\end{aligned}$$

where $S_j,\ \boldsymbol{v}_l,\ a_j,\ \boldsymbol{c}_j$ parametrize additional state constraints (these are second order cone constraints)

We begin by introducing a change of variables: $\quad \sigma \equiv \frac{\Gamma}{m}, \quad$ and $\quad \boldsymbol{u} \equiv \frac{\boldsymbol{T}}{m}$

By substitution:

$$\ddot{\boldsymbol{r}}(t) = \boldsymbol{g} + \frac{\boldsymbol{T}(t)}{m(t)} \quad \to \quad \ddot{\boldsymbol{r}}(t) = \boldsymbol{u}(t) + \boldsymbol{g}$$
$$\dot{m}(t) = -\alpha \Gamma(t) \quad \to \quad \frac{\dot{m}(t)}{m(t)} = -\alpha \sigma(t)$$

Leading to:

$$m(t) = m_o \exp\left[-\alpha \int_0^{t_f} \sigma(\tau) d\tau\right].$$

We know that $\alpha > 0$, so minimizing fuel is equivalent to minimizing $\int_0^{t_f} \sigma(t)\,\mathrm{d}t$.

We can now express the control constraints in terms of the new variables

$$\|\boldsymbol{u}(t)\| \leq \sigma(t), \quad \forall t \in [0, t_f]$$
$$\frac{\rho_1}{m(t)} \leq \sigma(t) \leq \frac{\rho_2}{m(t)}, \quad \forall t \in [0, t_f]$$

Note that $\|\boldsymbol{u}^\star(t)\| = \sigma^\star(t), \forall t \in [0, t_f]$ is satisfied for any pair of optimal controls $u^\star$ and $\sigma^\star$. Given a prescribed $m(t)$, these inequalities are convex. We cannot use m as a variable (else the inequality is bilinear and non-convex); thus we introduce $z \equiv \ln m$.

$$\dot{z}(t) = -\alpha \sigma(t)$$
$$\rho_1 e^{-z(t)} \leq \sigma(t) \leq \rho_2 e^{-z(t)}, \quad \forall t \in [0, t_f]$$

The upper bound is non-convex. Introduce a second order cone of these inequalities using a Taylor series expansion of e^{-z}. We apply it to both ends for simplicity.

$$\mu_1(t)\left[1 - (z(t) - z_0(t)) + \frac{(z(t) - z_0(t))^2}{2}\right] \leq \sigma(t) \leq \mu_2\left[1 - (z(t) - z_0(t))\right], \quad \forall t \in [0, t_f]$$

where $\mu_1 \equiv \rho_1 e^{-z_0}$, $\mu_2 \equiv \rho_2 e^{-z_0}$, and $z_0(t) = \ln(m_{wet} - \alpha \rho_1 t)$.

We now obtain a linear approximation to the original problem:

$$
\begin{aligned}
\underset{t_f, \boldsymbol{u}(\cdot), \sigma(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \sigma(t)\,\mathrm{d}t \\
\text{subject to} \quad & \ddot{\boldsymbol{r}}(t) = \boldsymbol{u}(t) + \boldsymbol{g} \\
& \dot{z}(t) = -\alpha\sigma(t) \\
& \|\boldsymbol{u}(t)\| \leq \sigma(t) \\
& \mu_1(t)\left[1 - (z(t) - z_0(t)) + \frac{(z(t) - z_0(t))^2}{2}\right] \leq \sigma(t) \leq \mu_2\left[1 - (z(t) - z_0(t))\right] \\
& \ln(m_{wet} - \alpha\rho_2 t) \leq z(t) \leq \ln(m_{wet} - \alpha\rho_1 t) \\
& \|S_j \boldsymbol{x}(t) - \boldsymbol{v}_j\| + \boldsymbol{c}_j^T \boldsymbol{x}(t) + a_j \leq 0,\ j = 1, \ldots, n_S \\
& m(0) = m_{wet},\ r(0) = r_o,\ \dot{r}(0) = \dot{r}_o,\ r(t_f) = \dot{r}(t_f) = 0.
\end{aligned}
\tag{Problem 3}
$$

Appendix 3: Figures

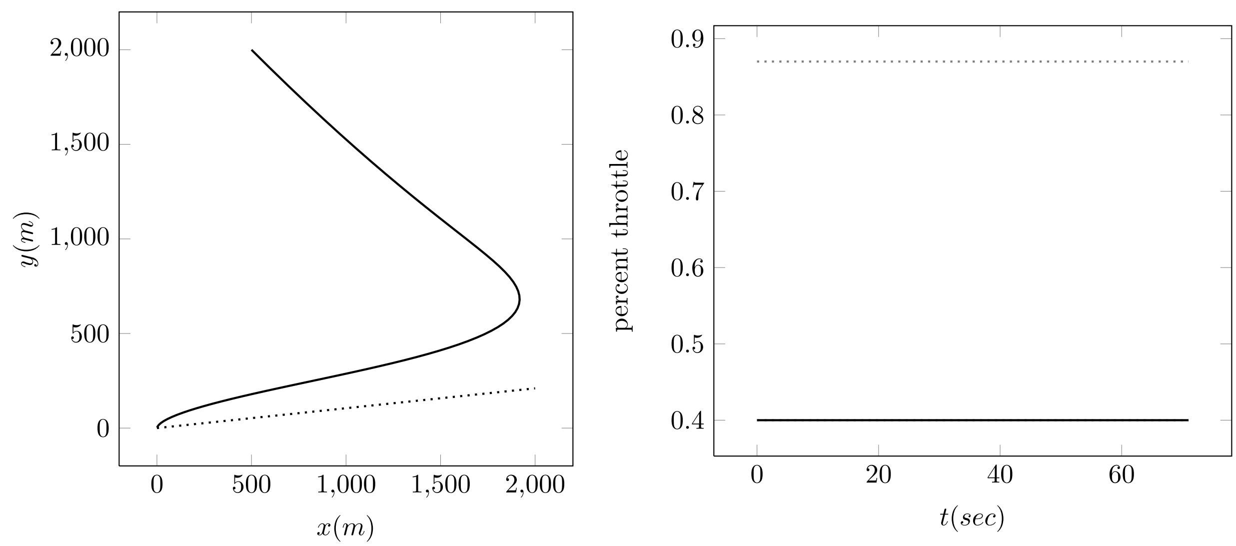


Figure 1: Trajectory and throttle profile for the constant mass system.

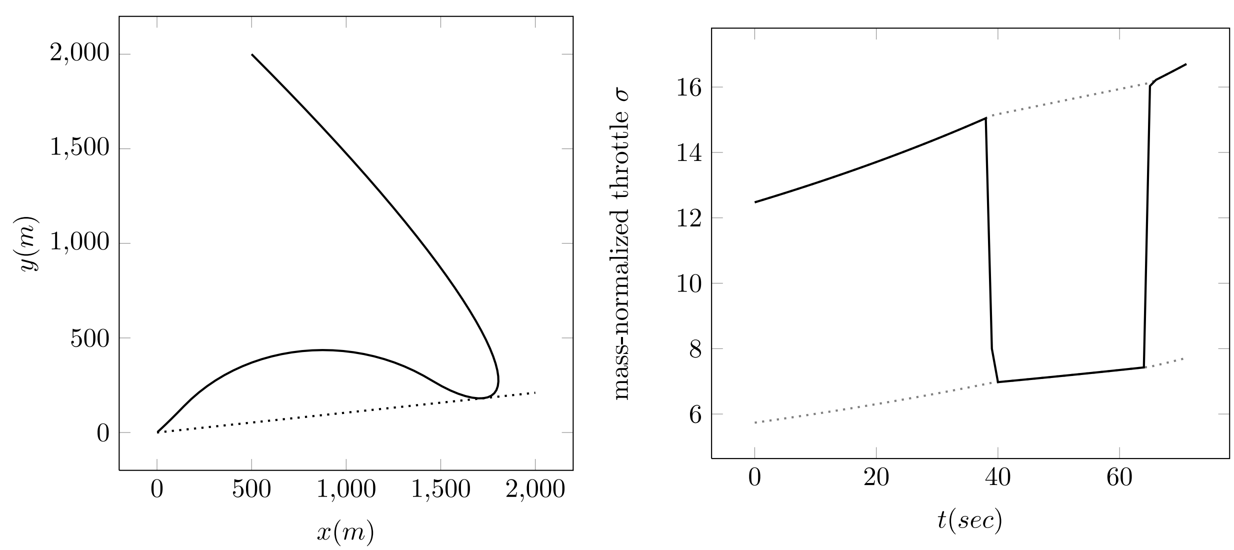


Figure 2: Trajectory and throttle profile for the variable mass system.

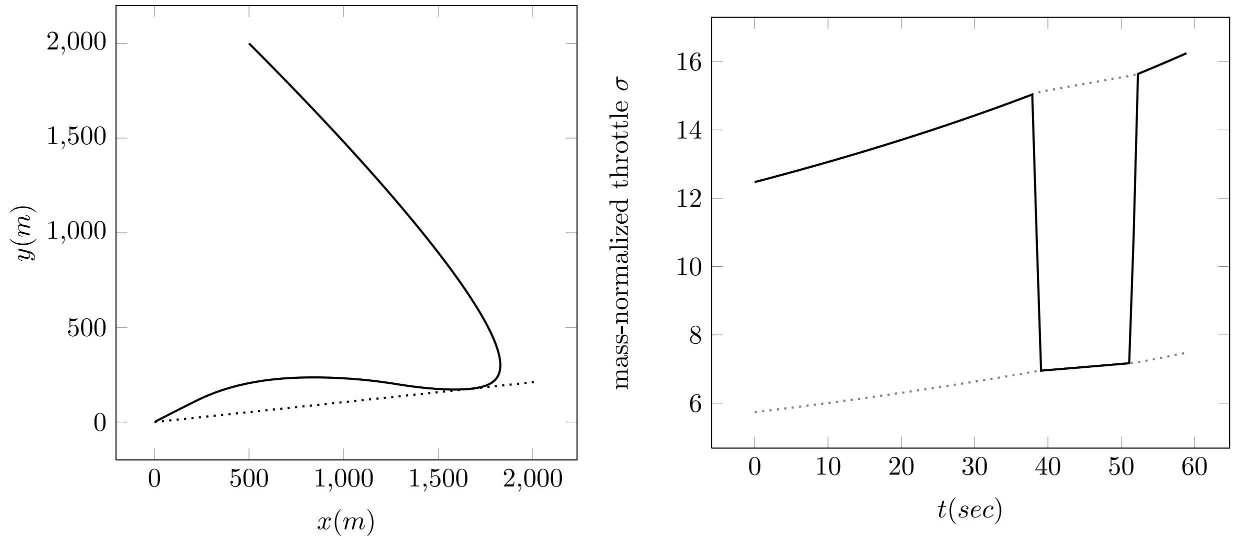


Figure 3: Minimum time trajectory and throttle profile for the variable mass system.

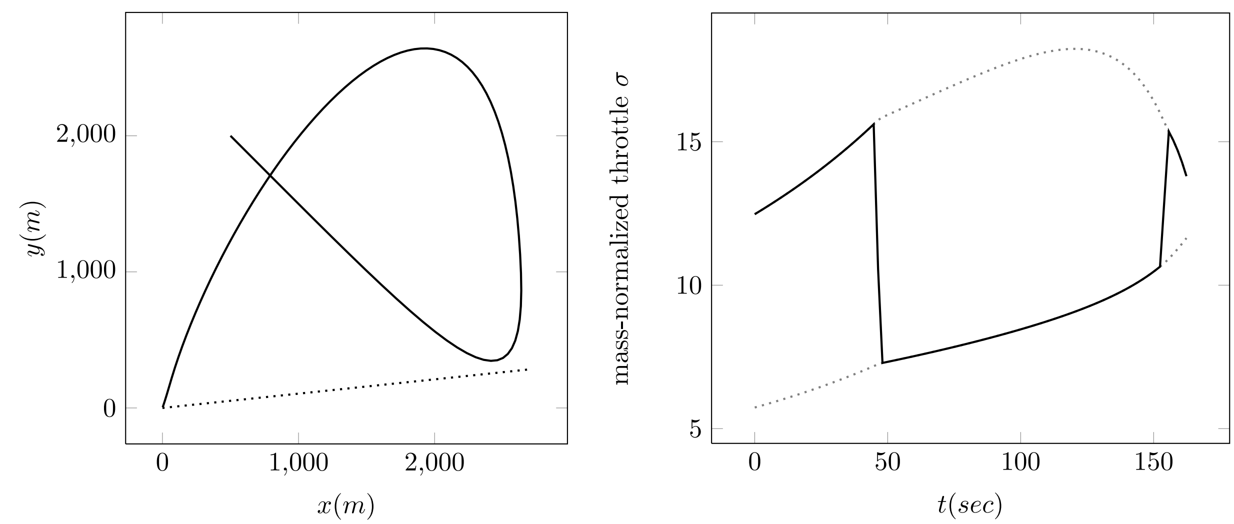


Figure 4: Maximum time trajectory and throttle profile for the variable mass system.

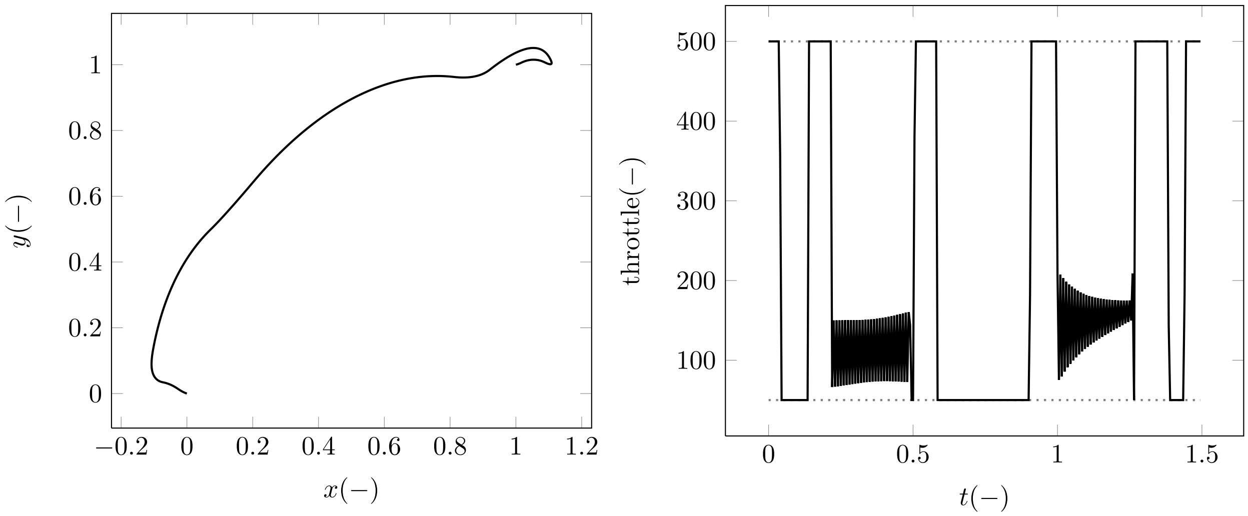


Figure 5: Trajectory and throttle profile for the non-minumum phase system. Notice the chatter.

References

[1] Behçet Açıkmeşe and Lars Blackmore. Lossless convexification of a class of optimal control problems with non-convex control constraints. *Automatica*, 47(2):341–347, 2011.

[2] Behçet Açıkmeşe and Scott R Ploen. Convex programming approach to powered descent guidance for mars landing. *Journal of Guidance, Control, and Dynamics*, 30(5):1353–1366, 2007.