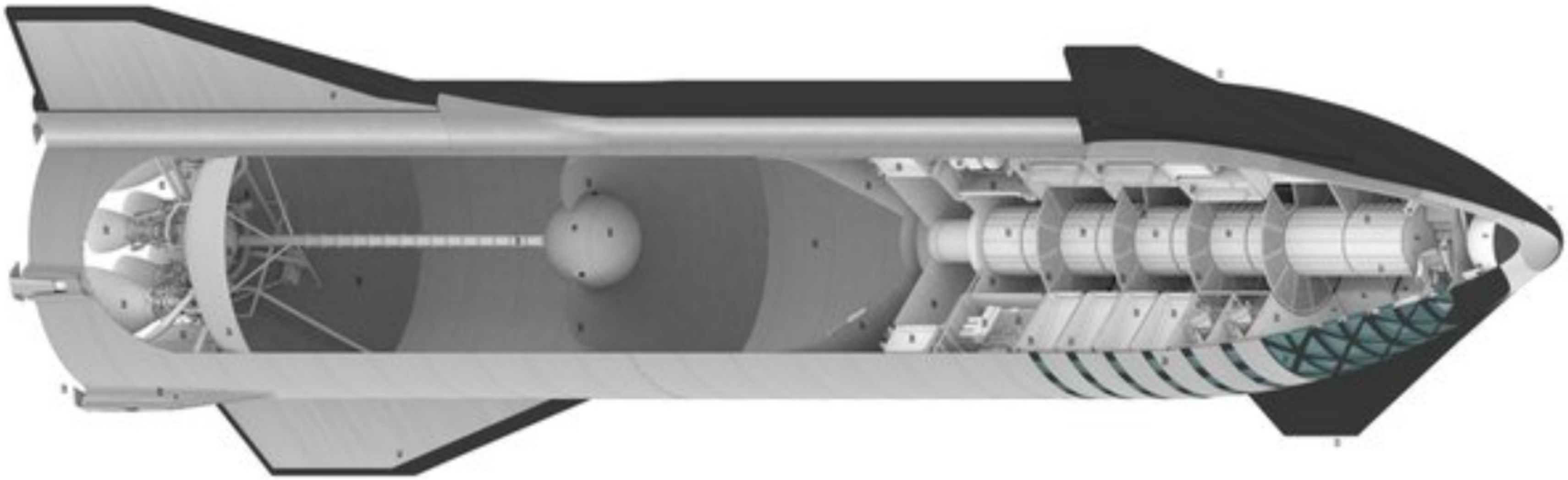
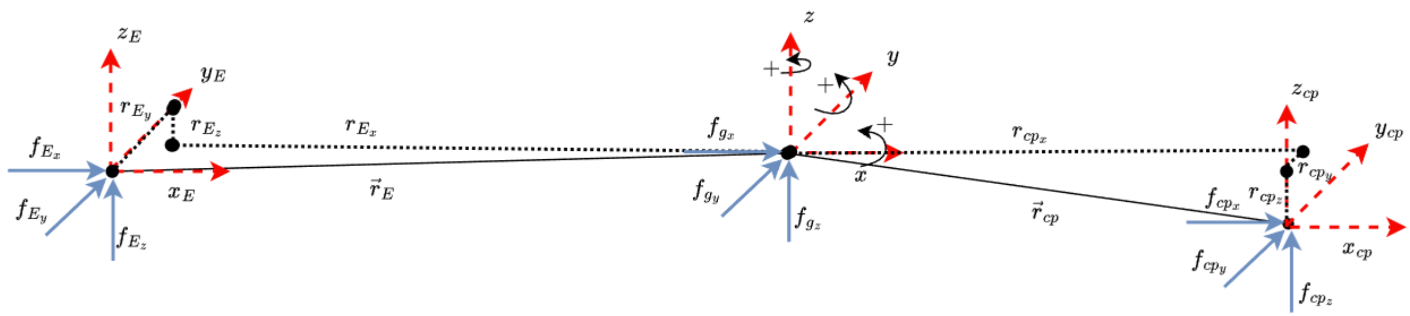
1 Diagram and Terms





First, let $F_E = \begin{bmatrix} f_{E_x} \\ f_{E_y} \\ f_{E_z} \end{bmatrix}$, $F_g = \begin{bmatrix} f_{g_x} \\ f_{g_y} \\ f_{g_z} \end{bmatrix}$, $F_{cp} = \begin{bmatrix} f_{cp_x} \\ f_{cp_y} \\ f_{cp_z} \end{bmatrix}$, $\vec{r}_E = \begin{bmatrix} r_{E_x} \\ r_{E_y} \\ r_{E_z} \end{bmatrix}$, and $\vec{r}_{cp} = \begin{bmatrix} r_{cp_x} \\ r_{cp_y} \\ r_{cp_z} \end{bmatrix}$. The total force is then,

$$F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = F_E + F_g + F_{cp} = \begin{bmatrix} f_{Ex} + f_{cpx} + f_{gx} \\ f_{Ey} + f_{cpy} + f_{gy} \\ f_{Ez} + f_{cpz} + f_{gz} \end{bmatrix} = \begin{bmatrix} f_{Ex} + f_{cpx} + gm\left(-2e_0e_2 + 2e_1e_3\right) \\ f_{Ey} + f_{cpy} + gm\left(2e_0e_1 + 2e_2e_3\right) \\ f_{Ez} + f_{cpz} + gm\left(e_0^2 - e_1^2 - e_2^2 + e_3^2\right) \end{bmatrix}.$$

The torque about the center of gravity is,

$$\tau = \vec{r}_E \times F_E + \vec{r}_{cp} \times F_{cp} = \begin{bmatrix} -f_{Ey}r_{Ez} + f_{Ez}r_{Ey} - f_{cpy}r_{cpz} + f_{cpz}r_{cpy} \\ f_{Ex}r_{Ez} - f_{Ez}r_{Ex} + f_{cpx}r_{cpz} - f_{cpz}r_{cpx} \\ -f_{Ex}r_{Ey} + f_{Ey}r_{Ex} - f_{cpx}r_{cpy} + f_{cpy}r_{cpx} \end{bmatrix} = \begin{bmatrix} -f_{Ey}r_{Ez} + f_{Ez}r_{Ey} - f_{cpy}r_{cpz} + f_{cpz}r_{cpy} \\ f_{Ex}r_{Ez} - f_{Ez}r_{Ex} + f_{cpx}r_{cpz} - f_{cpz}r_{cpx} \\ -f_{Ex}r_{Ey} + f_{Ey}r_{Ex} - f_{cpx}r_{cpy} + f_{cpy}r_{cpx} \end{bmatrix}.$$

The value of $\vec{r}_E$ is known to us (where we place the engines). To find $\vec{r}_{cp}$, use the following equation,

$$\vec{r}_{cp} = -\frac{(\sum_{\text{control surfaces}} F_i) \times (\sum_{\text{control surfaces}} M_i)}{\|(\sum_{\text{control surfaces}} F_i)\|^2} = -\frac{(\sum_{\text{control surfaces}} F_i) \times (\sum_{\text{control surfaces}} r_i \times F_i)}{\|(\sum_{\text{control surfaces}} F_i)\|^2}$$

the sums take into account the forces at each aerodynamic control surface.

2 Dynamics

From the UAV book page 32 and 34 we know that (these are in body frame),

$$m\left(\dot{V} + \omega \times V\right) = F \text{ and } J\dot{\omega} + \omega \times J\omega = \tau,$$

where $V = \begin{bmatrix} u & v & w \end{bmatrix}^T$, $\omega = \begin{bmatrix} p & q & r \end{bmatrix}^T$, J is the Mass Moment of Inertia tensor. First, handling the force term,

$$m\left(\dot{V} + \omega \times V\right) = F$$
$$\dot{V} + \omega \times V = \frac{1}{m}F$$
$$\dot{V} = -\omega \times V + \frac{1}{m}F$$
$$\boxed{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = -\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \frac{1}{m}F}.$$

Now for the torque term,

$$
\begin{aligned}
J\dot{\omega} + \omega \times J\omega &= \tau \\
J\dot{\omega} &= \tau - \omega \times J\omega \\
\dot{\omega} &= J^{-1}\left(\tau - \omega \times J\omega\right) \\
\dot{\omega} &= -J^{-1}\left(\omega \times J\omega\right) + J^{-1}\tau \\
\end{aligned}
$$

$$
\boxed{\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{bmatrix} = -J^{-1}\left(\begin{bmatrix}p\\ q\\ r\end{bmatrix} \times J\begin{bmatrix}p\\ q\\ r\end{bmatrix}\right) + J^{-1}\tau}.
$$

Before showing the equations, let $e = \begin{bmatrix}e_0 & e_1 & e_2 & e_3\end{bmatrix}^T$ represent the body's unit quaternion and $P_g = \begin{bmatrix}p_n & p_e & p_d\end{bmatrix}^T$ represent the body's inertial position. Using the quaternion representation of dynamics from the back of the book results in and substituting in forces results in,

$$
\begin{aligned}
\dot{P}_g &= \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & -2e_0e_3 + 2e_1e_2 & 2e_0e_2 + 2e_1e_3 \\ 2e_0e_3 + 2e_1e_2 & e_0^2 - e_1^2 + e_2^2 - e_3^2 & -2e_0e_1 + 2e_2e_3 \\ -2e_0e_2 + 2e_1e_3 & 2e_0e_1 + 2e_2e_3 & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} V \\
\dot{V} &= -\omega \times V + \frac{1}{m}F \\
\dot{e} &= \begin{bmatrix} 0 & -\frac{p}{2} & -\frac{q}{2} & -\frac{r}{2} \\ \frac{p}{2} & 0 & \frac{r}{2} & -\frac{q}{2} \\ \frac{q}{2} & -\frac{r}{2} & 0 & \frac{p}{2} \\ \frac{r}{2} & \frac{q}{2} & -\frac{p}{2} & 0 \end{bmatrix} e \\
\dot{\omega} &= -J^{-1}\left(\omega \times J\omega\right) + J^{-1}\tau.
\end{aligned}
$$

3 Linearizing

For this section, we will use the general form of the dynamics equations. Let $x = \begin{bmatrix}\dot{p}_n & \dot{p}_e & \dot{p}_d & \dot{u} & \dot{v} & \dot{w} & \dot{e}_0 & \dot{e}_1 & \dot{e}_2 & \dot{e}_3 & \dot{p} & \dot{q} & \dot{r}\end{bmatrix}^T$. First, we need to find equilibrium. So,

$$\dot{x} = f(x,u) \implies \begin{bmatrix}\dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \\ \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{p} \\ \dot{q} \\ \dot{r}\end{bmatrix} = \begin{bmatrix} u\left(e_0^2+e_1^2-e_2^2-e_3^2\right)+v\left(-2e_0e_3+2e_1e_2\right)+w\left(2e_0e_2+2e_1e_3\right) \\ u\left(2e_0e_3+2e_1e_2\right)+v\left(e_0^2-e_1^2+e_2^2-e_3^2\right)+w\left(-2e_0e_1+2e_2e_3\right) \\ u\left(-2e_0e_2+2e_1e_3\right)+v\left(2e_0e_1+2e_2e_3\right)+w\left(e_0^2-e_1^2-e_2^2+e_3^2\right) \\ -\frac{e_1p}{2}-\frac{e_2q}{2}-\frac{e_3r}{2} \\ \frac{e_0p}{2}+\frac{e_2r}{2}-\frac{e_3q}{2} \\ \frac{e_0q}{2}-\frac{e_1r}{2}+\frac{e_3p}{2} \\ -qw+rv+\frac{f_{Ex}+f_{cpx}+gm(-2e_0e_2+2e_1e_3)}{m} \\ pw-ru+\frac{f_{Ey}+f_{cpy}+gm(2e_0e_1+2e_2e_3)}{m} \\ -pv+qu+\frac{f_{Ez}+f_{cpz}+gm\left(e_0^2-e_1^2-e_2^2+e_3^2\right)}{m} \\ -\frac{q(-J_{xz}p-J_{yz}q+J_{zz}r)-r(-J_{xy}p+J_{yy}q-J_{yz}r)}{J_{xx}}+\frac{-f_{Ey}r_{Ez}+f_{Ez}r_{Ey}-f_{cpy}r_{cpz}+f_{cpz}r_{cpy}}{J_{xx}} \\ -\frac{-p(-J_{xz}p-J_{yz}q+J_{zz}r)+r(J_{xx}p-J_{xy}q-J_{xz}r)}{J_{yy}}+\frac{f_{Ex}r_{Ez}-f_{Ez}r_{Ex}+f_{cpx}r_{cpz}-f_{cpz}r_{cpx}}{J_{yy}} \\ -\frac{p(-J_{xy}p+J_{yy}q-J_{yz}r)-q(J_{xx}p-J_{xy}q-J_{xz}r)}{J_{zz}}+\frac{-f_{Ex}r_{Ey}+f_{Ey}r_{Ex}-f_{cpx}r_{cpy}+f_{cpy}r_{cpx}}{J_{zz}} \end{bmatrix}.$$

The following subsections handle generating the state for the descent phase and landing phase.

3.1 Descent

$$\begin{bmatrix} 0&0&0&0&0&0&-1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2} \\ 0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0 \\ 0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0 \\ 0&0&0&0&2g&0&0&0&0&0&0&0 \\ 0&0&0&0&0&2g&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ \frac{1}{m}&0&0&\frac{1}{m}&0&0&0&0&0 \\ 0&\frac{1}{m}&0&0&\frac{1}{m}&0&0&0&0 \\ 0&0&\frac{1}{m}&0&0&\frac{1}{m}&0&0&0 \\ 0&-\frac{r_{Ez}}{J_{xx}}&\frac{r_{Ey}}{J_{xx}}&0&0&0&0&0&\frac{gm}{J_{xx}} \\ \frac{r_{Ez}}{J_{yy}}&0&-\frac{r_{Ex}}{J_{yy}}&0&0&0&0&0&0 \\ -\frac{r_{Ey}}{J_{zz}}&\frac{r_{Ex}}{J_{zz}}&0&0&0&0&-\frac{gm}{J_{zz}}&0&0 \end{bmatrix} u$$

References:

https://www.intechopen.com/chapters/64567