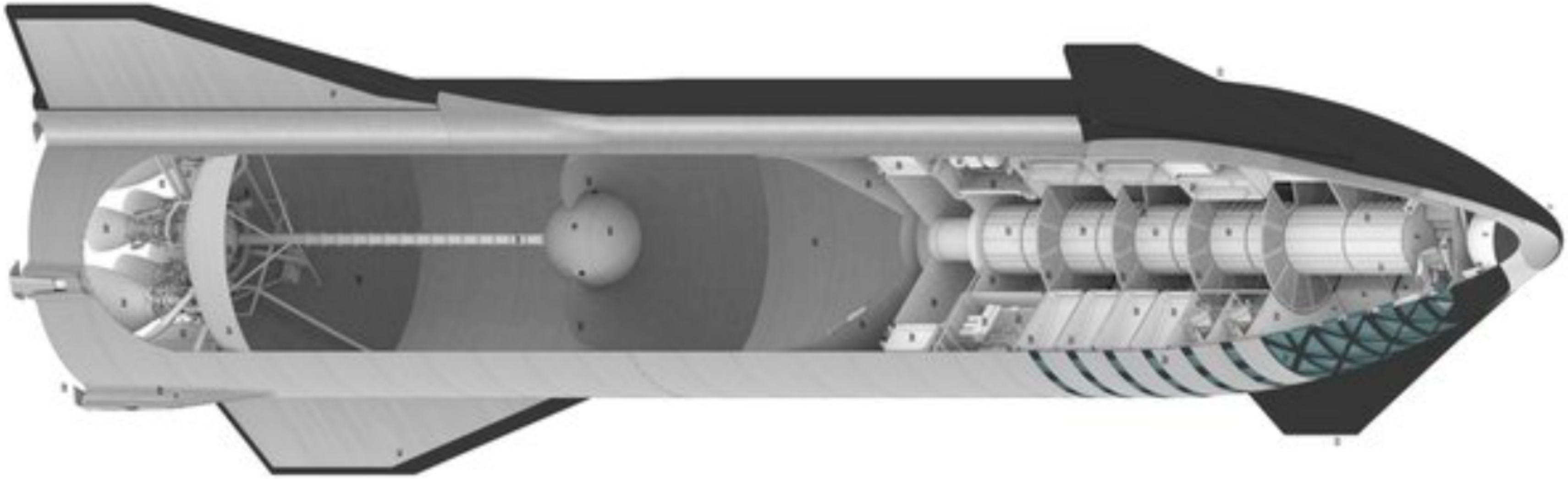
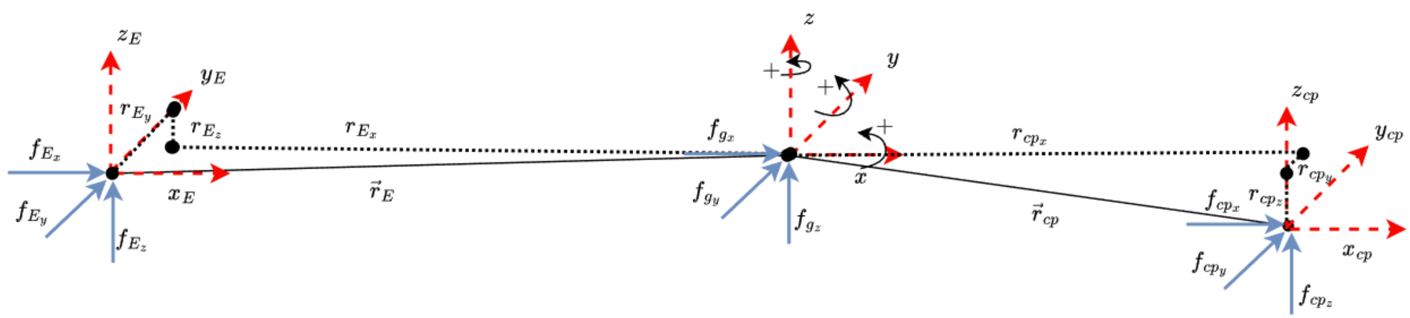
1 Diagram and Terms





First, let $F_E = \begin{bmatrix} f_{E_x} \\ f_{E_y} \\ f_{E_z} \end{bmatrix}$, $F_g = \begin{bmatrix} f_{g_x} \\ f_{g_y} \\ f_{g_z} \end{bmatrix}$, $F_{cp} = \begin{bmatrix} f_{cp_x} \\ f_{cp_y} \\ f_{cp_z} \end{bmatrix}$, $\vec{r}_E = \begin{bmatrix} r_{E_x} \\ r_{E_y} \\ r_{E_z} \end{bmatrix}$, and $\vec{r}_{cp} = \begin{bmatrix} r_{cp_x} \\ r_{cp_y} \\ r_{cp_z} \end{bmatrix}$. The total force is then,

$$F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = F_E + F_g + F_{cp} = \begin{bmatrix} f_{Ex} + f_{cpx} + f_{gx} \\ f_{Ey} + f_{cpy} + f_{gy} \\ f_{Ez} + f_{cpz} + f_{gz} \end{bmatrix}.$$

The torque about the center of gravity is,

$$\tau = \vec{r}_E \times F_E + \vec{r}_{cp} \times F_{cp} = \begin{bmatrix} -f_{Ey}r_{Ez} + f_{Ez}r_{Ey} - f_{cpy}r_{cpz} + f_{cpz}r_{cpy} \\ f_{Ex}r_{Ez} - f_{Ez}r_{Ex} + f_{cpx}r_{cpz} - f_{cpz}r_{cpx} \\ -f_{Ex}r_{Ey} + f_{Ey}r_{Ex} - f_{cpx}r_{cpy} + f_{cpy}r_{cpx} \end{bmatrix}$$

The value of $\vec{r}_E$ is known to us (where we place the engines). To find $\vec{r}_{cp}$, use the following equation,

$$\vec{r}_{cp} = -\frac{(\sum_{\text{control surfaces}} F_i) \times (\sum_{\text{control surfaces}} M_i)}{\|(\sum_{\text{control surfaces}} F_i)\|^2} = -\frac{(\sum_{\text{control surfaces}} F_i) \times (\sum_{\text{control surfaces}} r_i \times F_i)}{\|(\sum_{\text{control surfaces}} F_i)\|^2}$$

the sums take into account the forces at each aerodynamic control surface.

2 First Formulation of Dynamics

Using the quaternion representation of dynamics from the back of the book results in and substituting in forces results in,

$$\begin{bmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{bmatrix} = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & -2e_0e_3 + 2e_1e_2 & 2e_0e_2 + 2e_1e_3 \\ 2e_0e_3 + 2e_1e_2 & e_0^2 - e_1^2 + e_2^2 - e_3^2 & -2e_0e_1 + 2e_2e_3 \\ -2e_0e_2 + 2e_1e_3 & 2e_0e_1 + 2e_2e_3 & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} F + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tau$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -qw + rv \\ pw - ru \\ -pv + qu \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} F + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tau$$

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{p}{2} & -\frac{q}{2} & -\frac{r}{2} \\ \frac{p}{2} & 0 & \frac{r}{2} & -\frac{q}{2} \\ \frac{q}{2} & -\frac{r}{2} & 0 & \frac{p}{2} \\ \frac{r}{2} & \frac{q}{2} & -\frac{p}{2} & 0 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} F + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tau$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 \left(p^2 - r^2\right) \\ -\Gamma_1 qr + \Gamma_7 pq \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} F + \begin{bmatrix} \Gamma_3 & 0 & \Gamma_4 \\ 0 & \frac{1}{J_{yy}} & 0 \\ \Gamma_4 & 0 & \Gamma_8 \end{bmatrix} \tau.$$

Simplifying,

$$\begin{bmatrix}\dot{p}_n\\ \dot{p}_e\\ \dot{p}_d\end{bmatrix} = \begin{bmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & -2e_0e_3+2e_1e_2 & 2e_0e_2+2e_1e_3\\ 2e_0e_3+2e_1e_2 & e_0^2-e_1^2+e_2^2-e_3^2 & -2e_0e_1+2e_2e_3\\ -2e_0e_2+2e_1e_3 & 2e_0e_1+2e_2e_3 & e_0^2-e_1^2-e_2^2+e_3^2\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}$$

$$\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix} = \begin{bmatrix}-qw+rv\\ pw-ru\\ -pv+qu\end{bmatrix} + \begin{bmatrix}\frac{1}{m} & 0 & 0\\ 0 & \frac{1}{m} & 0\\ 0 & 0 & \frac{1}{m}\end{bmatrix} F$$

$$\begin{bmatrix}\dot{e}_0\\ \dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix} = \begin{bmatrix}0 & -\frac{p}{2} & -\frac{q}{2} & -\frac{r}{2}\\ \frac{p}{2} & 0 & \frac{r}{2} & -\frac{q}{2}\\ \frac{q}{2} & -\frac{r}{2} & 0 & \frac{p}{2}\\ \frac{r}{2} & \frac{q}{2} & -\frac{p}{2} & 0\end{bmatrix}\begin{bmatrix}e_0\\ e_1\\ e_2\\ e_3\end{bmatrix}$$

$$\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{bmatrix} = \begin{bmatrix}\Gamma_1 pq-\Gamma_2 qr\\ \Gamma_5 pr-\Gamma_6\left(p^2-r^2\right)\\ -\Gamma_1 qr+\Gamma_7 pq\end{bmatrix} + \begin{bmatrix}\Gamma_3 & 0 & \Gamma_4\\ 0 & \frac{1}{J_{yy}} & 0\\ \Gamma_4 & 0 & \Gamma_8\end{bmatrix}\tau.$$

2.1 State Space

We now need a state space. Let

$$x = \begin{bmatrix}p_n & \dot{p}_n & p_e & \dot{p}_e & p_d & \dot{p}_d & u & \dot{u} & v & \dot{v} & w & \dot{w} & e_0 & \dot{e}_0 & e_1 & \dot{e}_1 & e_2 & \dot{e}_2 & e_3 & \dot{e}_3 & p & \dot{p} & q & \dot{q} & r & \dot{r}\end{bmatrix}^T.$$

We now need to convert the equation above into state space form. Below is the general format it needs to take,

$$\dot{x} = Ax + B\begin{bmatrix}F\\ \tau\end{bmatrix}.$$

3 General Formulation

From the UAV book page 32 and 34 we know that (these are in body frame),

$$m\left(\dot{V}+\omega\times V\right) = F \text{ and } J\dot{\omega}+\omega\times J\omega = \tau,$$

where $V = \begin{bmatrix} u & v & w \end{bmatrix}^T$, $\omega = \begin{bmatrix} p & q & r \end{bmatrix}^T$, J is the Mass Moment of Inertia tensor. First, handling the force term,

$$m\left(\dot{V} + \omega \times V\right) = F$$
$$\dot{V} + \omega \times V = \frac{1}{m}F$$
$$\dot{V} = -\omega \times V + \frac{1}{m}F$$
$$\boxed{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = -\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \frac{1}{m}F}.$$

Now for the torque term,

$$J\dot{\omega} + \omega \times J\omega = \tau$$
$$J\dot{\omega} = \tau - \omega \times J\omega$$
$$\dot{\omega} = J^{-1}\left(\tau - \omega \times J\omega\right)$$
$$\dot{\omega} = -J^{-1}\left(\omega \times J\omega\right) + J^{-1}\tau$$
$$\boxed{\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = -J^{-1}\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times J\begin{bmatrix} p \\ q \\ r \end{bmatrix}\right) + J^{-1}\tau}.$$

Before showing the equations, let $e = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix}^T$ represent the body's unit quaternion and $P_g = \begin{bmatrix} p_n & p_e & p_d \end{bmatrix}^T$ represent the body's inertial position. Using the quaternion representation of dynamics from the back of the book results in and substituting in forces results in,

$$
\begin{aligned}
\dot{P}_g &= \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & -2e_0e_3 + 2e_1e_2 & 2e_0e_2 + 2e_1e_3 \\ 2e_0e_3 + 2e_1e_2 & e_0^2 - e_1^2 + e_2^2 - e_3^2 & -2e_0e_1 + 2e_2e_3 \\ -2e_0e_2 + 2e_1e_3 & 2e_0e_1 + 2e_2e_3 & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} V \\
\dot{V} &= -\omega \times V + \frac{1}{m} F \\
\dot{e} &= \begin{bmatrix} 0 & -\frac{p}{2} & -\frac{q}{2} & -\frac{r}{2} \\ \frac{p}{2} & 0 & \frac{r}{2} & -\frac{q}{2} \\ \frac{q}{2} & -\frac{r}{2} & 0 & \frac{p}{2} \\ \frac{r}{2} & \frac{q}{2} & -\frac{p}{2} & 0 \end{bmatrix} e \\
\dot{\omega} &= -J^{-1} \left(\omega \times J\omega \right) + J^{-1}\tau.
\end{aligned}
$$

References:

https://www.intechopen.com/chapters/64567