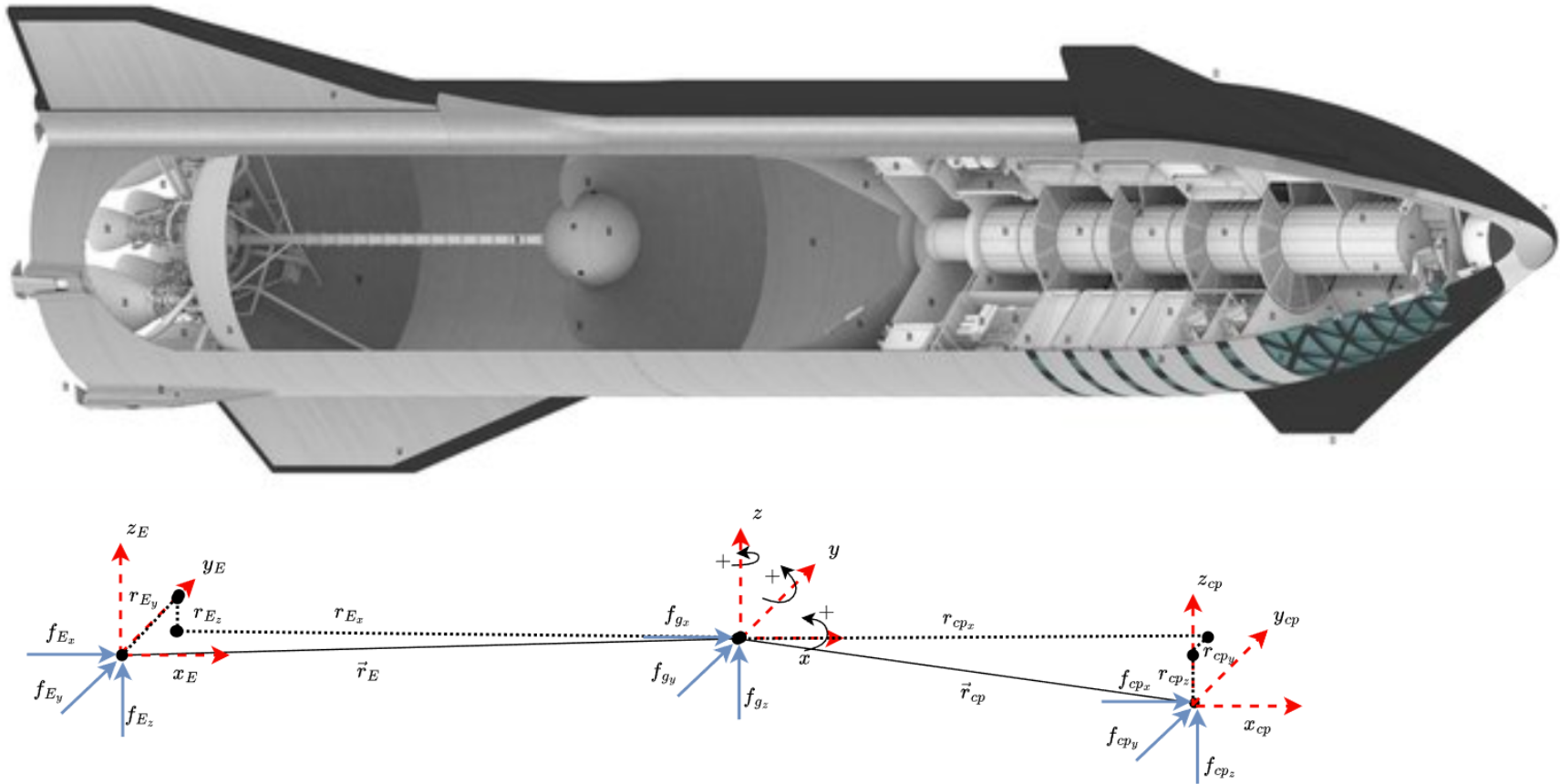


1 Diagram and Terms



First, let $F_E = \begin{bmatrix} f_{E_x} \\ f_{E_y} \\ f_{E_z} \end{bmatrix}$, $F_g = \begin{bmatrix} f_{g_x} \\ f_{g_y} \\ f_{g_z} \end{bmatrix}$, $F_{cp} = \begin{bmatrix} f_{cp_x} \\ f_{cp_y} \\ f_{cp_z} \end{bmatrix}$, $\vec{r}_E = \begin{bmatrix} r_{E_x} \\ r_{E_y} \\ r_{E_z} \end{bmatrix}$, and $\vec{r}_{cp} = \begin{bmatrix} r_{cp_x} \\ r_{cp_y} \\ r_{cp_z} \end{bmatrix}$. The total force is then,

$$F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = F_E + F_g + F_{cp} = \begin{bmatrix} f_{E_x} + f_{cp_x} + f_{g_x} \\ f_{E_y} + f_{cp_y} + f_{g_y} \\ f_{E_z} + f_{cp_z} + f_{g_z} \end{bmatrix}.$$

The torque about the center of gravity is,

$$\tau = \vec{r}_E \times F_E + \vec{r}_{cp} \times F_{cp} = \begin{bmatrix} -f_{Ey}r_{Ez} + f_{Ez}r_{Ey} - f_{cpy}r_{cpz} + f_{cpz}r_{cpy} \\ f_{Ex}r_{Ez} - f_{Ez}r_{Ex} + f_{cpz}r_{cpz} - f_{cpz}r_{cpz} \\ -f_{Ex}r_{Ey} + f_{Ey}r_{Ex} - f_{cpz}r_{cpy} + f_{cpy}r_{cpz} \end{bmatrix}$$

The value of \vec{r}_E is known to us (where we place the engines). To find \vec{r}_{cp} , use the following equation,

$$\vec{r}_{cp} = -\frac{(\sum_{\text{control surfaces}} F_i) \times (\sum_{\text{control surfaces}} M_i)}{\|(\sum_{\text{control surfaces}} F_i)\|^2} = -\frac{(\sum_{\text{control surfaces}} F_i) \times (\sum_{\text{control surfaces}} r_i \times F_i)}{\|(\sum_{\text{control surfaces}} F_i)\|^2}$$

the sums take into account the forces at each aerodynamic control surface.

2 First Formulation of Dynamics

Using the quaternion representation of dynamics from the back of the book results in and substituting in forces results in,

$$\begin{aligned} \begin{bmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{bmatrix} &= \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & -2e_0e_3 + 2e_1e_2 & 2e_0e_2 + 2e_1e_3 \\ 2e_0e_3 + 2e_1e_2 & e_0^2 - e_1^2 + e_2^2 - e_3^2 & -2e_0e_1 + 2e_2e_3 \\ -2e_0e_2 + 2e_1e_3 & 2e_0e_1 + 2e_2e_3 & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} F + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tau \\ \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} &= \begin{bmatrix} -qw + rv \\ pw - ru \\ -pv + qu \end{bmatrix} [1] + \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} F + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tau \\ \begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} &= \begin{bmatrix} 0 & -\frac{p}{2} & -\frac{q}{2} & -\frac{r}{2} \\ \frac{p}{2} & 0 & \frac{r}{2} & -\frac{q}{2} \\ \frac{q}{2} & -\frac{r}{2} & 0 & \frac{p}{2} \\ \frac{r}{2} & \frac{q}{2} & -\frac{p}{2} & 0 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} F + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tau \\ \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ -\Gamma_1 qr + \Gamma_7 pq \end{bmatrix} [1] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} F + \begin{bmatrix} \Gamma_3 & 0 & \Gamma_4 \\ 0 & \frac{1}{J_{yy}} & 0 \\ \Gamma_4 & 0 & \Gamma_8 \end{bmatrix} \tau. \end{aligned}$$

Simplifying,

$$\begin{aligned}
 \begin{bmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{bmatrix} &= \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & -2e_0e_3 + 2e_1e_2 & 2e_0e_2 + 2e_1e_3 \\ 2e_0e_3 + 2e_1e_2 & e_0^2 - e_1^2 + e_2^2 - e_3^2 & -2e_0e_1 + 2e_2e_3 \\ -2e_0e_2 + 2e_1e_3 & 2e_0e_1 + 2e_2e_3 & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\
 \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} &= \begin{bmatrix} -qw + rv \\ pw - ru \\ -pv + qu \end{bmatrix} + \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} F \\
 \begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} &= \begin{bmatrix} 0 & -\frac{p}{2} & -\frac{q}{2} & -\frac{r}{2} \\ \frac{p}{2} & 0 & \frac{r}{2} & -\frac{q}{2} \\ \frac{q}{2} & -\frac{r}{2} & 0 & \frac{p}{2} \\ \frac{r}{2} & \frac{q}{2} & -\frac{p}{2} & 0 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} \\
 \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ -\Gamma_1 qr + \Gamma_7 pq \end{bmatrix} + \begin{bmatrix} \Gamma_3 & 0 & \Gamma_4 \\ 0 & \frac{1}{J_{yy}} & 0 \\ \Gamma_4 & 0 & \Gamma_8 \end{bmatrix} \tau.
 \end{aligned}$$

2.1 State Space

We now need a state space. Let

$$x = [p_n \quad \dot{p}_n \quad p_e \quad \dot{p}_e \quad p_d \quad \dot{p}_d \quad u \quad \dot{u} \quad v \quad \dot{v} \quad w \quad \dot{w} \quad e_0 \quad \dot{e}_0 \quad e_1 \quad \dot{e}_1 \quad e_2 \quad \dot{e}_2 \quad e_3 \quad \dot{e}_3 \quad p \quad \dot{p} \quad q \quad \dot{q} \quad r \quad \dot{r}]^T.$$

We now need to convert the equation above into state space form. Below is the general format it needs to take,

$$\dot{x} = Ax + B \begin{bmatrix} F \\ \tau \end{bmatrix}.$$

3 General Formulation

From the UAV book page 32 and 34 we know that (these are in body frame),

$$m(\dot{V} + \omega \times V) = F \quad \text{and} \quad J\dot{\omega} + \omega \times J\omega = \tau,$$

where $V = [u \ v \ w]^T$, $\omega = [p \ q \ r]^T$, J is the Mass Moment of Inertia tensor. First, handling the force term,

$$\begin{aligned}
 m(\dot{V} + \omega \times V) &= F \\
 \dot{V} + \omega \times V &= \frac{1}{m}F \\
 \dot{V} &= -\omega \times V + \frac{1}{m}F \\
 \boxed{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} &= -\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \frac{1}{m}F.}
 \end{aligned}$$

Now for the torque term,

$$\begin{aligned}
 J\dot{\omega} + \omega \times J\omega &= \tau \\
 J\dot{\omega} &= \tau - \omega \times J\omega \\
 \dot{\omega} &= J^{-1}(\tau - \omega \times J\omega) \\
 \dot{\omega} &= -J^{-1}(\omega \times J\omega) + J^{-1}\tau \\
 \boxed{\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} &= -J^{-1} \left(\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times J \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right) + J^{-1}\tau.}
 \end{aligned}$$

Before showing the equations, let $e = [e_0 \ e_1 \ e_2 \ e_3]^T$ represent the body's unit quaternion and $P_g = [p_n \ p_e \ p_d]^T$ represent the body's inertial position. Using the quaternion representation of dynamics from the back of the book results in and substituting in forces

results in,

$$\dot{P}_g = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & -2e_0e_3 + 2e_1e_2 & 2e_0e_2 + 2e_1e_3 \\ 2e_0e_3 + 2e_1e_2 & e_0^2 - e_1^2 + e_2^2 - e_3^2 & -2e_0e_1 + 2e_2e_3 \\ -2e_0e_2 + 2e_1e_3 & 2e_0e_1 + 2e_2e_3 & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} V$$

$$\dot{V} = -\omega \times V + \frac{1}{m}F$$

$$\dot{e} = \begin{bmatrix} 0 & -\frac{p}{2} & -\frac{q}{2} & -\frac{r}{2} \\ \frac{p}{2} & 0 & \frac{r}{2} & -\frac{q}{2} \\ \frac{q}{2} & -\frac{r}{2} & 0 & \frac{p}{2} \\ \frac{r}{2} & \frac{q}{2} & -\frac{p}{2} & 0 \end{bmatrix} e$$

$$\dot{\omega} = -J^{-1}(\omega \times J\omega) + J^{-1}\tau.$$

4 Linearizing

For this section, we will use the general form of the dynamics equations. Let $x =$

$$\begin{bmatrix} u \\ v \\ w \\ \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}$$

First, we need to find equilibrium. So,

$$\dot{x} = f(x, u) \implies \begin{bmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \\ \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} =$$

$$\begin{aligned} & -qw + rv + \frac{f_{Ex} + f_{Ey}}{2} \\ & pw - ru + \frac{f_{Ex} + f_{Ey}}{2} \\ & -pv + qu + \frac{f_{Ex} + f_{Ey}}{2} \\ & -\frac{e_1 p}{2} - \frac{e_0 p}{2} + \frac{e_0 q}{2} - \frac{e_0 r}{2} + \frac{e_1 r}{2} \\ & -\frac{(J_{xy}J_{yz} + J_{xz}J_{yy})(p(-J_{xy}p + J_{yy}q - J_{yz}r) - q(J_{xx}p - J_{xy}q - J_{xz}r))}{J_{xx}J_{yy}J_{zz} - J_{xx}J_{yz}^2 - J_{xy}^2J_{zz} - 2J_{xy}J_{xz}J_{yz} - J_{xz}^2J_{yy}} + \frac{(J_{xy}J_{yz} + J_{xz}J_{yy})(-f_{Ex}r_{Ey} + f_{Ey}r_{Ex} - f_{cpx}r_{cpy} + f_{cpy}r_{cpx})}{J_{xx}J_{yy}J_{zz} - J_{xx}J_{yz}^2 - J_{xy}^2J_{zz} - 2J_{xy}J_{xz}J_{yz} - J_{xz}^2J_{yy}} - \frac{(J_{xy}J_{zz} + J_{xz}J_{yy})(-p(-J_{xz}p - J_{yz}q + J_{zz}r) + r(J_{xx}p - J_{xy}q - J_{xz}r))}{J_{xx}J_{yy}J_{zz} - J_{xx}J_{yz}^2 - J_{xy}^2J_{zz} - 2J_{xy}J_{xz}J_{yz} - J_{xz}^2J_{yy}} \\ & -\frac{(J_{xx}J_{yz} + J_{xy}J_{xz})(p(-J_{xy}p + J_{yy}q - J_{yz}r) - q(J_{xx}p - J_{xy}q - J_{xz}r))}{J_{xx}J_{yy}J_{zz} - J_{xx}J_{yz}^2 - J_{xy}^2J_{zz} - 2J_{xy}J_{xz}J_{yz} - J_{xz}^2J_{yy}} + \frac{(J_{xx}J_{yz} + J_{xy}J_{xz})(-f_{Ex}r_{Ey} + f_{Ey}r_{Ex} - f_{cpx}r_{cpy} + f_{cpy}r_{cpx})}{J_{xx}J_{yy}J_{zz} - J_{xx}J_{yz}^2 - J_{xy}^2J_{zz} - 2J_{xy}J_{xz}J_{yz} - J_{xz}^2J_{yy}} - \frac{(J_{xx}J_{zz} - J_{xz}^2)(-p(-J_{xz}p - J_{yz}q + J_{zz}r) + r(J_{xx}p - J_{xy}q - J_{xz}r))}{J_{xx}J_{yy}J_{zz} - J_{xx}J_{yz}^2 - J_{xy}^2J_{zz} - 2J_{xy}J_{xz}J_{yz} - J_{xz}^2J_{yy}} \\ & -\frac{(J_{xx}J_{yy} - J_{xy}^2)(p(-J_{xy}p + J_{yy}q - J_{yz}r) - q(J_{xx}p - J_{xy}q - J_{xz}r))}{J_{xx}J_{yy}J_{zz} - J_{xx}J_{yz}^2 - J_{xy}^2J_{zz} - 2J_{xy}J_{xz}J_{yz} - J_{xz}^2J_{yy}} + \frac{(J_{xx}J_{yy} - J_{xy}^2)(-f_{Ex}r_{Ey} + f_{Ey}r_{Ex} - f_{cpx}r_{cpy} + f_{cpy}r_{cpx})}{J_{xx}J_{yy}J_{zz} - J_{xx}J_{yz}^2 - J_{xy}^2J_{zz} - 2J_{xy}J_{xz}J_{yz} - J_{xz}^2J_{yy}} - \frac{(J_{xx}J_{yz} + J_{xy}J_{xz})(-p(-J_{xz}p - J_{yz}q + J_{zz}r) + r(J_{xx}p - J_{xy}q - J_{xz}r))}{J_{xx}J_{yy}J_{zz} - J_{xx}J_{yz}^2 - J_{xy}^2J_{zz} - 2J_{xy}J_{xz}J_{yz} - J_{xz}^2J_{yy}} \end{aligned}$$

Note: the three last entries are not 0, they are just huge.

$$\begin{bmatrix} 0 & \sqrt{2}g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \sqrt{2}g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}g & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{r_{E_y}(J_{xy}J_{yz}+J_{xz}J_{yy})}{J_{xx}J_{yy}J_{zz}-J_{xx}J_{yz}^2-J_{xy}^2J_{zz}-2J_{xy}J_{xz}J_{yz}-J_{xz}^2J_{yy}} + \frac{r_{E_z}(J_{xy}J_{zz}+J_{xz}J_{yz})}{J_{xx}J_{yy}J_{zz}-J_{xx}J_{yz}^2-J_{xy}^2J_{zz}-2J_{xy}J_{xz}J_{yz}-J_{xz}^2J_{yy}} & -\frac{r_{E_x}(J_{xy}J_{yz}+J_{xz}J_{yy})}{J_{xx}J_{yy}J_{zz}-J_{xx}J_{yz}^2-J_{xy}^2J_{zz}-2J_{xy}J_{xz}J_{yz}-J_{xz}^2J_{yy}} \\ -\frac{r_{E_y}(J_{xx}J_{yz}+J_{xy}J_{xz})}{J_{xx}J_{yy}J_{zz}-J_{xx}J_{yz}^2-J_{xy}^2J_{zz}-2J_{xy}J_{xz}J_{yz}-J_{xz}^2J_{yy}} + \frac{r_{E_z}(J_{xx}J_{zz}-J_{xz}^2)}{J_{xx}J_{yy}J_{zz}-J_{xx}J_{yz}^2-J_{xy}^2J_{zz}-2J_{xy}J_{xz}J_{yz}-J_{xz}^2J_{yy}} & -\frac{r_{E_x}(J_{xx}J_{yz}+J_{xy}J_{xz})}{J_{xx}J_{yy}J_{zz}-J_{xx}J_{yz}^2-J_{xy}^2J_{zz}-2J_{xy}J_{xz}J_{yz}-J_{xz}^2J_{yy}} \\ -\frac{r_{E_y}(J_{xx}J_{yy}-J_{xy}^2)}{J_{xx}J_{yy}J_{zz}-J_{xx}J_{yz}^2-J_{xy}^2J_{zz}-2J_{xy}J_{xz}J_{yz}-J_{xz}^2J_{yy}} + \frac{r_{E_z}(J_{xx}J_{yz}+J_{xy}J_{xz})}{J_{xx}J_{yy}J_{zz}-J_{xx}J_{yz}^2-J_{xy}^2J_{zz}-2J_{xy}J_{xz}J_{yz}-J_{xz}^2J_{yy}} & -\frac{r_{E_x}(J_{xx}J_{yy}-J_{xy}^2)}{J_{xx}J_{yy}J_{zz}-J_{xx}J_{yz}^2-J_{xy}^2J_{zz}-2J_{xy}J_{xz}J_{yz}-J_{xz}^2J_{yy}} \end{bmatrix}$$

References:

<https://www.intechopen.com/chapters/64567>