ECE 565
Estimation Detection and Filtering
Project Assignment: Dithering

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1. Introduction and motivation

This project deals with dithering of an analog signal for quantization. You are interested in finding the value A through N binary observations from a comparator. Let

$$y(n) = \text{sign}(A + v(n)), \qquad n = 1, 2, \ldots, N \tag{1}$$

where A is an unknown analog value, $v(n)$ is a zero mean uniform noise process with variance of σ^2. Consider two options: (i) $v(n) \sim \mathcal{N}(0, \sigma^2)$ and (ii) $v(n) \sim U[-\sqrt{3}\sigma, \sqrt{3}\sigma]$. You are interested in finding A.

What is Dithering?

Dither is a signal processing technique that add noise to a signal before reducing the sample word size or resolution of the sample. It is done to preserve information that would be lost due to distortion from reducing resolution.

Dither is usually intentionally applied to randomize quantization error by adding noise. Dither technology is widely used in signal processing of digital audio, video data, and image processing. For instance, dithering is often one of the last stages of mastering audio information to a CD.

How does adding noise help preserving information?

Another common application of dither is preserving image quality when truncating the image size. For example, we start with an 8-bit grayscale image showing different percentages of black from 15% to 85%. Next, we truncate the sample size to a single bit. As a result, we can two main phenomena: not only we lost image's black level below 50% completely, but all of the black levels above 50% are now 100% black now.

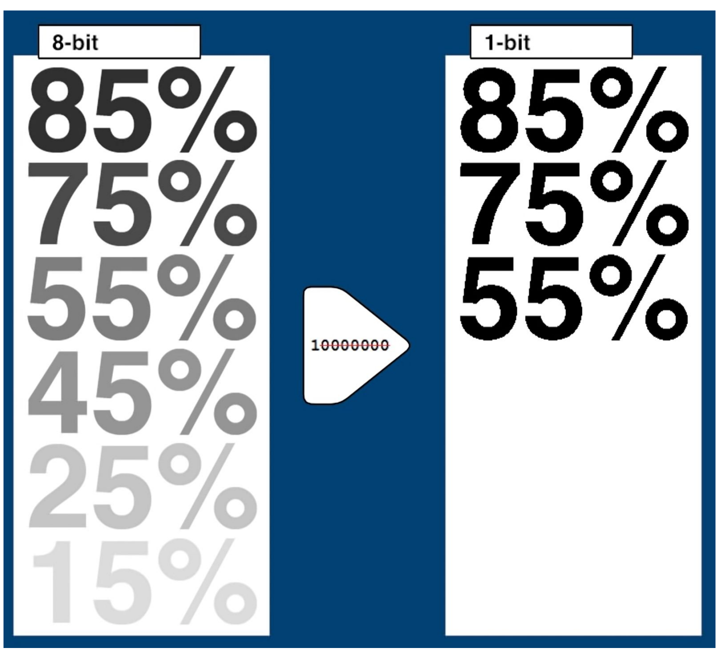

Figure 1.1

Next, we will apply dithering in the beginning. We apply noise at the 50% level, which means for each pixel we randomly increase or decrease pixel values from 0 to 50% of the full range before we truncate the image resolution like we did. As a result, we can see how we've retained the original information this time. Although there are some obvious noise in the background, the image's black level percentage stays the same from 15% to 85%.

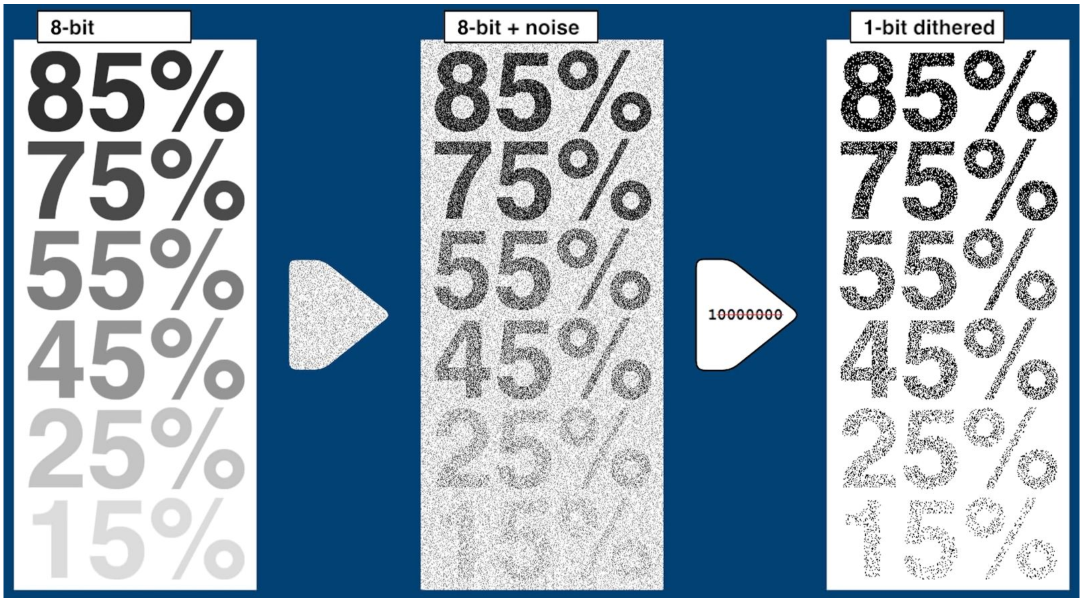

Figure 1.2

2. Problems and solutions

Problem 1:

Define the parameter vector $\boldsymbol{\theta}$ and the observation vector $\mathbf{y}$ for this problem. Write an explicit form for the PMF of $\mathbf{y}$ given $\boldsymbol{\theta}$, $p_{\mathbf{Y}}(\mathbf{y}|\boldsymbol{\theta})$.

Solution:

1.1

$$parameter\ \theta = A$$

$$observation = y(1),\ y(2), ..., y(n)$$

i.e.

$$y(1) = \begin{array}{l} +1\ with\ prob\ P(A+V \geq 0) = P(A) \\ -1\ with\ prob\ P(A+V < 0) = 1 - P(A) \end{array}$$

1.2

Note:

$$\text{Bernoulli: } \bar{y} = \begin{array}{l} 1\ w.p.\ P \\ 0\ w.p.\ 1-P \end{array}$$

$$\Rightarrow PMF\ :\ P(\bar{y}) = p^{\bar{y}}(1-p)^{(1-\bar{y})}$$

Discrete Distributions

Let $X \sim Bernoulli(p)$. The pmf is $p(x) = \begin{cases} p^x(1-p)^{1-x}, & x = 0, 1 \\ 0 & \text{elsewhere} \end{cases}$

1.3

Similarly,

$$\Rightarrow PMF\ : P(\bar{y}) = P(A)^{\frac{1+\bar{y}}{2}}(1 - P(A))^{\frac{1-\bar{y}}{2}}$$

Option 1 (Uniform distribution):

$$P(A+V \geq 0) = P(V \geq -A) = 1 - \frac{V-a}{b-a} = 1 - \frac{-A+\sqrt{3}}{2\sqrt{3}}$$

$$PMF\ : P(y_1, y_2, ..., y_n|A) = \prod_{i=1}^{n}(1 - \frac{-A+\sqrt{3}}{2\sqrt{3}})^{\frac{1+y_i}{2}}(\frac{-A+\sqrt{3}}{2\sqrt{3}})^{\frac{1-y_i}{2}}$$

$$= (1 - \frac{-A+\sqrt{3}}{2\sqrt{3}})^{\sum_{i=1}^{n}\frac{1+y_i}{2}}(\frac{-A+\sqrt{3}}{2\sqrt{3}})^{\sum_{i=1}^{n}\frac{1-y_i}{2}}$$

Option 2 (Gaussian distribution):

$$P(A+V\geq 0)=P(V\geq -A)=1-(\frac{-A}{})$$

$$PMF: P(y_1,y_2,...,y_n|A)=\prod_{i=1}^{n}(1-(\frac{-A}{}))^{\frac{1+y_i}{2}}((\frac{-A}{}))^{\frac{1-y_i}{2}}$$

$$=(1-(\frac{-A}{}))^{\sum_{i=1}^{n}\frac{1+y_i}{2}}((\frac{-A}{}))^{\sum_{i=1}^{n}\frac{1-y_i}{2}}$$

Problem 2:

Find the CRLB for A.

Solution:

2.1

Option 1 (Uniform distribution):

$$logP(y_1,y_2,...,y_n|A)=log(1-\frac{-A+\sqrt{3}}{2\sqrt{3}})^{\sum_{i=1}^{n}\frac{1+y_i}{2}}(\frac{-A+\sqrt{3}}{2\sqrt{3}})^{\sum_{i=1}^{n}\frac{1-y_i}{2}}$$

$$=\sum_{i=1}^{n}\frac{1+y_i}{2}log(1-\frac{-A+\sqrt{3}}{2\sqrt{3}})+\sum_{i=1}^{n}\frac{1-y_i}{2}log(\frac{-A+\sqrt{3}}{2\sqrt{3}})$$

2.2

$$First\ derivative: \frac{dlogP(y1,y2,...,yn|A)}{dA}=\frac{-\sum_{i=1}^{n}\frac{1-y_i}{2}}{-A+\sqrt{3}}+\frac{\sum_{i=1}^{n}\frac{1+y_i}{2}}{A+\sqrt{3}}$$

2.3

$$Second\ derivative: \frac{d^2logP(y1,y2,...,yn|A)}{dA^2}=\frac{-\sum_{i=1}^{n}\frac{1-y_i}{2}}{(-A+\sqrt{3})^2}+\frac{\sum_{i=1}^{n}\frac{1+y_i}{2}}{(A+\sqrt{3})^2}$$

2.4

$$E[y_i]=(+1)(1-\frac{-A+\sqrt{3}}{2\sqrt{3}})+(-1)(\frac{-A+\sqrt{3}}{2\sqrt{3}})$$

$$=1-2(\frac{-A+\sqrt{3}}{2\sqrt{3}})=1+\frac{A-\sqrt{3}}{3}$$

$$\sum_{i=1}^{n}\frac{1-E[y_i]}{2}=\frac{1-(1+\frac{A-\sqrt{3}}{3})}{2}=\frac{\sqrt{3}-A}{2\sqrt{3}}$$

$$\sum_{i=1}^{n}\frac{1+E[y_i]}{2}=\frac{1+(1+\frac{A-\sqrt{3}}{3})}{2}=1+\frac{A-\sqrt{3}}{2\sqrt{3}}=\frac{\sqrt{3}+A}{2\sqrt{3}}$$

Move the expectation inside:

$$FIM=E[-\frac{d^2logP(y1,y2,...,yn|A)}{dA^2}]=E[-(\frac{-\sum_{i=1}^{n}\frac{1-y_i}{2}}{(-A+\sqrt{3})^2}+\frac{\sum_{i=1}^{n}\frac{1+y_i}{2}}{(A+\sqrt{3})^2})]$$

$$= E[-(\frac{-\sum_{i=1}^{n}\frac{1-y_i}{2}}{(-A+\sqrt{3})^2} + \frac{\sum_{i=1}^{n}\frac{1+y_i}{2}}{(A+\sqrt{3})^2})]$$
$$= \sum_{i=1}^{n}\frac{1-E[y_i]}{2}\frac{1}{(\sqrt{3}-A)^2} - \sum_{i=1}^{n}\frac{1+E[y_i]}{2}\frac{1}{(\sqrt{3}-A)^2}$$
$$= \frac{1}{2\sqrt{3}}(\frac{1}{\sqrt{3}-A} - \frac{1}{\sqrt{3}+A})$$
$$= \frac{1}{2\sqrt{3}}(\frac{2\sqrt{3}}{3^2-A^2})$$
$$= \frac{1}{3^2-A^2}$$

$$\Rightarrow CRLB = \frac{1}{FIM} = 3^{\,2} - A^{\,2}$$

```
FIM = 1/(3*sigma^2 - A^2);
CRLB = 1/FIM;
CRLB = CRLB/n;
```

Figure 2.1 MATLAB code for CRLB (uniform)

2.5

Option 2 (Gaussian distribution):

$$logP(y_1, y_2, ..., y_n|A) = log(1-(\frac{-A}{}))^{\sum_{i=1}^{n}\frac{1+y_i}{2}}((\frac{-A}{}))^{\sum_{i=1}^{n}\frac{1-y_i}{2}}$$
$$= \sum_{i=1}^{n}\frac{1+y_i}{2}log(1-(\frac{-A}{})) + \sum_{i=1}^{n}\frac{1-y_i}{2}log((\frac{-A}{}))$$

2.6

Note: $(X)' = (X)$

$$First\ derivative: \frac{dlogP(y1,y2,...,yn|A)}{dA} = \sum_{i=1}^{n}\frac{1-y_i}{2}\frac{(\frac{-A}{})(\frac{-1}{})}{(\frac{-A}{})} - \sum_{i=1}^{n}\frac{1+y_i}{2}\frac{(\frac{-A}{})(\frac{-1}{})}{1-(\frac{-A}{})}$$

2.7

$$Second\ derivative:$$
$$\sum_{i=1}^{n}\frac{1-y_i}{2}\frac{(\frac{-A}{})(\frac{1}{2})'(\frac{-A}{})-(\frac{1}{2})^2(\frac{-A}{})}{(1-(\frac{-A}{}))^2} - \sum_{i=1}^{n}\frac{1+y_i}{2}\frac{(1-(\frac{-A}{}))(\frac{1}{2})'(\frac{-A}{})-(\frac{1}{2})^2(\frac{-A}{})}{(1-(\frac{-A}{}))^2}$$

2.8

$$E[y_i] = (+1)(1-(\frac{-A}{})) + (-1)((\frac{-A}{})) = 1-2((\frac{-A}{}))$$

2.9

Find the expectation of second derivative

$$E[\frac{d^2logP(y1,y2,...,yn|A)}{dA^2}]$$

$$= (\sum_{i=1}^{n} \frac{1-E[y_i]}{2})(\frac{((\frac{-A}{})(\frac{-1}{2})'(\frac{-A}{})-(\frac{1}{2})^{2}(\frac{-A}{})}{(\frac{-A}{})^{2}}) - (\sum_{i=1}^{n} \frac{1+E[y_i]}{2})(\frac{(1-(\frac{-A}{}))(\frac{-1}{2})'(\frac{-A}{})+(\frac{1}{2})^{2}(\frac{-A}{})}{(1-(\frac{-A}{}))^{2}})$$

$$= \frac{(\frac{-A}{})(\frac{-1}{2})'(\frac{-A}{})-(\frac{1}{2})^{2}(\frac{-A}{})}{(\frac{-A}{})} - \frac{(1-(\frac{-A}{}))(\frac{-1}{2})'(\frac{-A}{})+(\frac{1}{2})^{2}(\frac{-A}{})}{1-(\frac{-A}{})}$$

$$= \frac{(\frac{-1}{2})^{2}(\frac{-A}{})}{(\frac{-A}{})} = \frac{(\frac{-1}{2})^{2}(\frac{-A}{})}{1-(\frac{-A}{})}$$

$$= \frac{-1}{2}^{\ 2}(\frac{-A}{})(\frac{1}{(\frac{-A}{})(1-(\frac{-A}{}))})$$

$$\Rightarrow FIM =- E[\frac{d^2 logP(y1,y2,...,yn|A)}{dA^2}]$$

$$= \frac{1}{2}^{\ 2}(\frac{-A}{})(\frac{1}{(\frac{-A}{})(1-(\frac{-A}{}))}$$

$$\Rightarrow CRLB = \frac{1}{FIM}$$

```
FIM = (1/(sigma^2))*(phi(-A/sigma))^2 * (1/((PHI(-A/sigma)*(1-PHI(-A/sigma)))));
CRLB = 1/FIM;
CRLB = CRLB/n;
```

Figure 2.2 MATLAB code for CRLB (Gaussian)

Problem 3:

Derive the MLE of A.

Solution:

Uniform Case:

$$\log P(\underline{y}|A) = \left(\sum_{i=1}^{n} \frac{1+y_i}{2}\right) \log\left(\frac{1}{2} + \frac{A}{2\sqrt{3}\sigma}\right) + \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \log\left(\frac{1}{2} - \frac{A}{2\sqrt{3}\sigma}\right)$$

$$\frac{d \log p(\underline{y}|A)}{dA} = \left(\sum_{i=1}^{n} \frac{1+y_i}{2}\right)\left(\frac{\sqrt{3}}{\sqrt{3}A + 3\sigma}\right) + \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right)\left(\frac{\sqrt{3}}{\sqrt{3}A - 3\sigma}\right)$$

Setting the derivative to 0, we get

$$0 = \frac{-\sum_{i=1}^{n} \frac{1-y_i}{2}}{-A + \sqrt{(3)}\sigma} + \frac{\sum_{i=1}^{n} \frac{1+y_i}{2}}{A + \sqrt{(3)}\sigma}$$

$$\frac{\sum_{i=1}^{n} \frac{1-y_i}{2}}{-A + \sqrt{(3)}\sigma} = \frac{\sum_{i=1}^{n} \frac{1+y_i}{2}}{A + \sqrt{(3)}\sigma}$$

Simplifying, we obtain the MLE.

$$(-A + \sqrt{3}\sigma) \sum_{i=1}^{n} \frac{1+y_i}{2} = (A + \sqrt{3}\sigma \sum_{i=1}^{n} \frac{1-y_i}{2})$$

$$\sqrt{3}\left(\sigma \sum_{i=1}^{n} \frac{1+y_i}{2} - \sum_{i=1}^{n} \frac{1-y_i}{2}\right) = nA$$

$$\sqrt{3}\sigma \sum_{i=1}^{n} y_i = nA$$

$$A_{MLE} = \frac{\sqrt{3}\sigma \sum_{i=1}^{n} y_i}{n} = \sqrt{3}\sigma\bar{y}$$

Gaussian Case:

$$\log P(\underline{y}|A) = \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \log(1 - \Phi(\frac{-A}{\sigma})) + \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \log(\Phi(\frac{-A}{\sigma}))$$

$$\frac{d \log p(\underline{y}|A)}{dA} = \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \frac{\frac{d}{dA}\Phi(\frac{-A}{\sigma})}{\Phi(\frac{-A}{\sigma})} - \left(\sum_{i=1}^{n} \frac{1+y_i}{2}\right) \frac{\frac{d}{dA}\Phi(\frac{-A}{\sigma})}{1-\Phi(\frac{-A}{\sigma})}$$

Where

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}(\frac{x}{\sqrt{2}})$$

And

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

Note that

$$\frac{d}{dA}\Phi(\frac{-A}{\sigma}) = \frac{-1}{\sigma}\phi(\frac{-A}{\sigma})$$

Where

$$\phi(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$$

This can then be written as

$$\frac{d \log p(y_1, y_2, ..., y_n|A)}{dA} = \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \frac{\frac{-1}{\sigma}\phi(\frac{-A}{\sigma})}{\Phi(\frac{-A}{\sigma})} - \left(\sum_{i=1}^{n} \frac{1+y_i}{2}\right) \frac{\frac{-1}{\sigma}\phi(\frac{-A}{\sigma})}{1-\Phi(\frac{-A}{\sigma})}$$

Setting the derivative to 0, we get

$$0 = \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \frac{\frac{-1}{\sigma}\phi(\frac{-A}{\sigma})}{\Phi(\frac{-A}{\sigma})} - \left(\sum_{i=1}^{n} \frac{1+y_i}{2}\right) \frac{\frac{-1}{\sigma}\phi(\frac{-A}{\sigma})}{1-\Phi(\frac{-A}{\sigma})}$$

$$\left(\sum_{i=1}^{n} \frac{1+y_i}{2}\right) \frac{\frac{-1}{\sigma}\phi(\frac{-A}{\sigma})}{1-\Phi(\frac{-A}{\sigma})} = \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \frac{\frac{-1}{\sigma}\phi(\frac{-A}{\sigma})}{\Phi(\frac{-A}{\sigma})}$$

Cross multiplying and simplifying further, we get

$$\left(\sum_{i=1}^{n} \frac{1+y_i}{2}\right) \Phi(\frac{-A}{\sigma}) = \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \left(1 - \Phi(\frac{-A}{\sigma})\right)$$

$$\sum_{i=1}^{n} \frac{1-y_i}{2} = \Phi(\frac{-A}{\sigma}) \left(\sum_{i=1}^{n} \frac{1+y_i}{2} + \sum_{i=1}^{n} \frac{1-y_i}{2}\right)$$

$$\sum_{i=1}^{n} \frac{1-y_i}{2} = n\Phi(\frac{-A}{\sigma})$$

$$\Phi(\frac{-A}{\sigma}) = \frac{1}{n} \sum_{i=1}^{n} \frac{1-y_i}{2}$$

$$A_{MLE} = -\sigma \Phi^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1-y_i}{2}\right)$$

Where

$$\Phi^{-1}(x) = \sqrt{2}\,\mathrm{erfinv}(2x-1)$$

erfinv being the inverse of the error function.

Problem 4:

Assume that $A \sim \mathcal{N}(0, \sigma_A^2)$ and derive two Bayesian estimator for A.

Solution: (Question changed to only one Bayesian estimator)

Uniform Case:

We first find the conditional and marginal probabilities respectively

$$\log P(\underline{y}|A) = \left(\sum_{i=1}^{n} \frac{1+y_i}{2}\right) \log\left(\frac{1}{2} + \frac{A}{2\sqrt{3}\sigma}\right) + \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \log\left(\frac{1}{2} - \frac{A}{2\sqrt{3}\sigma}\right)$$

$$\log P(A) = \log(\frac{-A+\sqrt{3}\sigma}{2\sqrt{3}\sigma})$$

Following the definition of MAP,

$$A_{MAP} = \max_A P(A|\underline{y}) = \max_A \frac{P(\underline{y}|A)P(A)}{P(\underline{y})}$$

$$A_{MAP} = \max_A \log P(A|\underline{y}) = \max_A \left(\log P(\underline{y}|A) + \log P(A) - \log P(\underline{y})\right)$$

Which is the same as the equation below after removing the constant logP(y)

$$A_{MAP} = \max_A \left(\log P(\underline{y}|A) + \log P(A)\right)$$

We plug in the conditional and marginal probabilities to obtain the following optimization problem

$$A_{MAP} = \max_A \left(\left(\sum_{i=1}^{n} \frac{1+y_i}{2}\right) \log\left(\frac{1}{2} + \frac{A}{2\sqrt{3}\sigma}\right) + \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \log\left(\frac{1}{2} - \frac{A}{2\sqrt{3}\sigma}\right) + \log(\frac{-A+\sqrt{3}\sigma}{2\sqrt{3}\sigma})\right)$$

We then obtain the MAP estimator by solving this problem using MATLAB’s fminunc function, as no closed-form solution exists.

Gaussian Case:
We first find the conditional and marginal probabilities respectively

$$\log P(\underline{y}|A) = \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \log(1 - \Phi(\frac{-A}{\sigma})) + \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \log(\Phi(\frac{-A}{\sigma}))$$

$$\log P(A) = \frac{-A^2}{2\sigma_A^2}$$

Again, following the same definition of MAP and plugging in our conditional and marginal probabilities like in the uniform case into the following form:

$$A_{MAP} = \max_{A} \left(\log P(\underline{y}|A) + \log P(A)\right)$$

we obtain the following problem:

$$A_{MAP} = \max_{A} \left(\left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \log(1 - \Phi(\frac{-A}{\sigma})) + \left(\sum_{i=1}^{n} \frac{1-y_i}{2}\right) \log(\Phi(\frac{-A}{\sigma})) + \frac{-A^2}{2\sigma_A^2} \right)$$

Which we also solve using MATLAB's fminunc function to obtain the MAP estimator, as there is no closed form solution to this problem as well.

Problem 5:

Using 200 Monte-Carlo runs, compute the ML and the Bayesian estimators of A for the realization at each run. From the 200 estimates, find the empirical MSE for each estimator. Repeat this process while varying the value of A/σ from 0.1 to 10. Plot the empirical MSE of the estimators of A against the value of A/σ on a logarithmic scale (for both A/σ and the MSE). Include on the same graph the CRLB.

Solution:

Uniform Case:

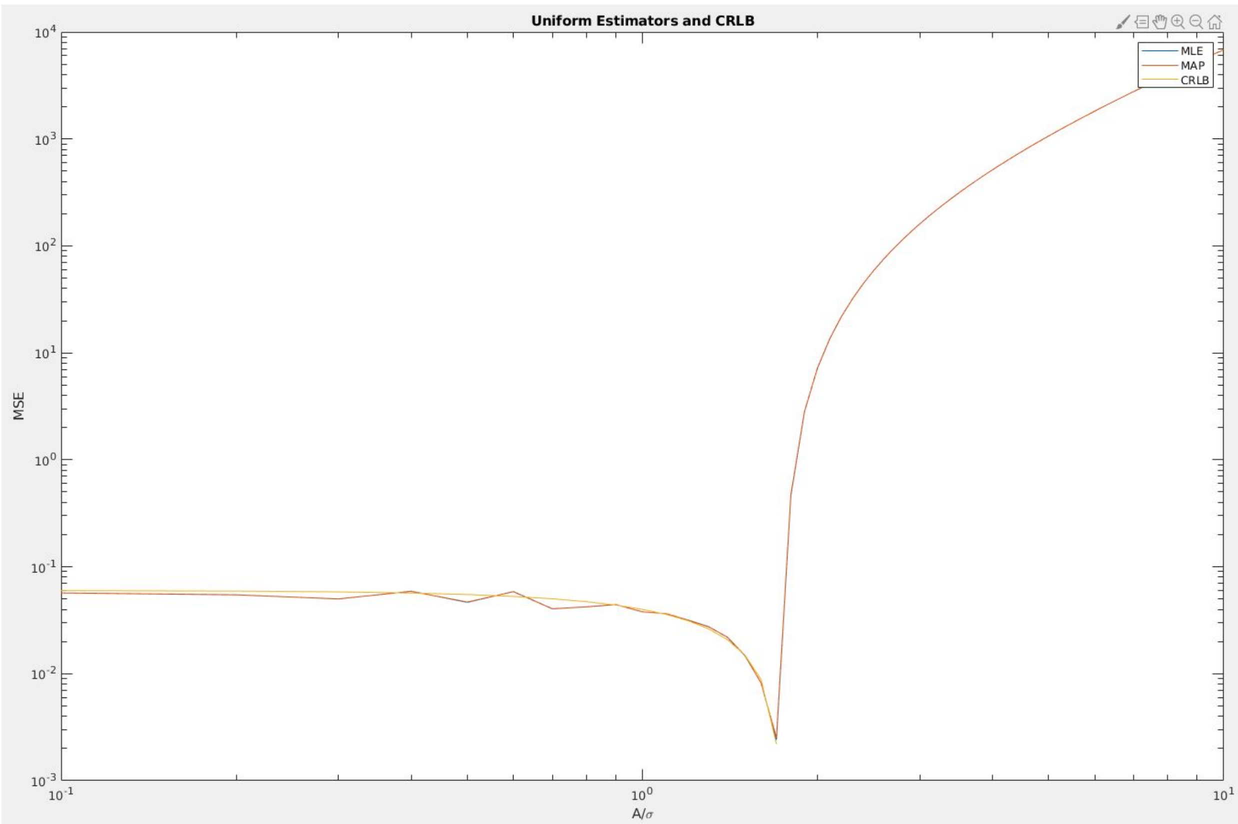

Figure 2.3 Note: Since it might be hard to read, x-axis is A/sigma, y-axis is MSE. The MLE and MAP overlap so the MLE is invisible on this plot, but it's still there. MLE is blue, MAP is orange, CRLB is yellow.

Gaussian Case:

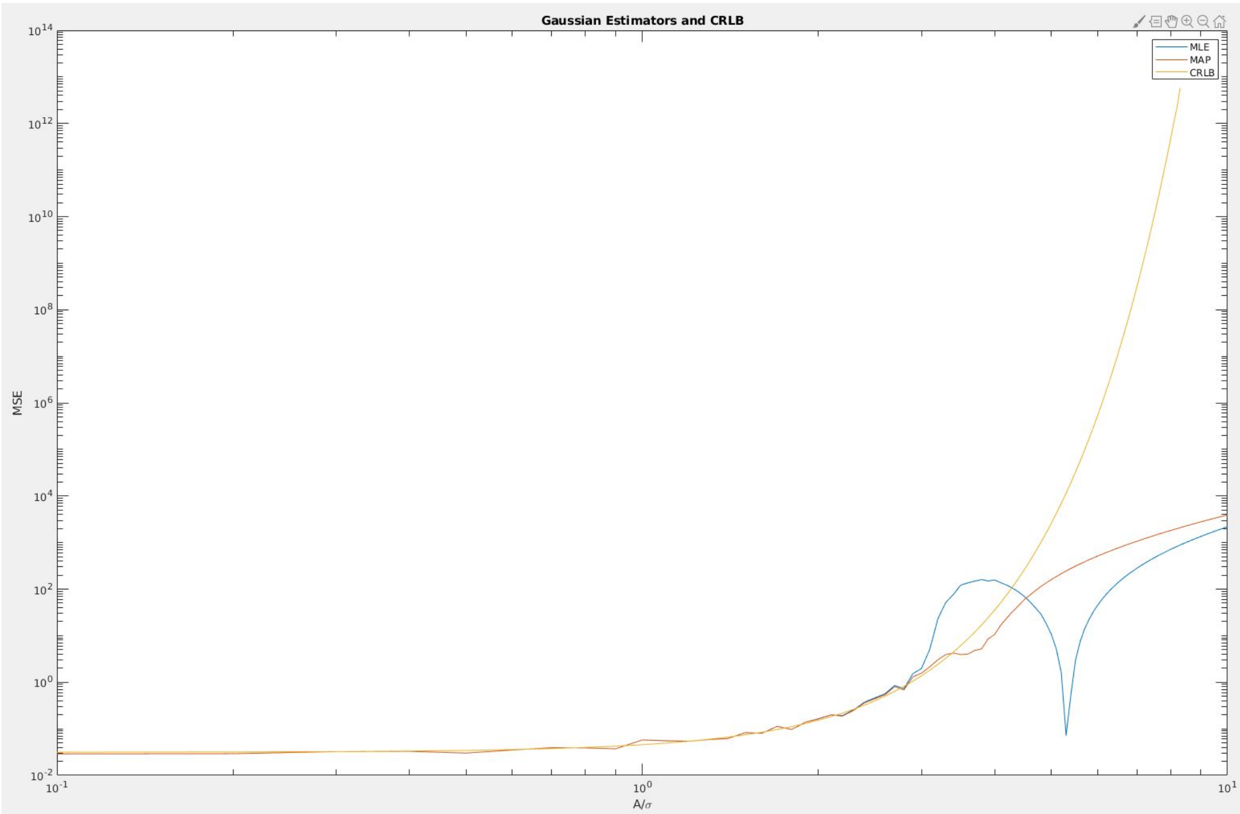

Figure 2.4 Note: Since it might be hard to read, x-axis is A/sigma, y-axis is MSE. MLE is blue, MAP is orange, CRLB is yellow.

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3. Conclusions and Results

We've considered dithering and its mathematical formulation as estimating a parameter given noise that translates to either +1 or -1 based on the sum of the parameter and the noise. In this problem formulation, we've looked at two estimators that try to find that parameter A: the maximum likelihood estimator (MLE) and the maximum a-posteriori estimator (MAP).

The problem seems very intuitive at first, when we only consider the value A to be such that A plus the noise would have a high probability of producing both positive and negative samples, as then both estimators can use these ratios to find an estimate for A. However, when all samples are either positive or negative, it becomes impossible to find an unbiased estimator for A, since the fundamental ratio that both estimators depend on (positive samples to negative samples) is useless.

In the uniform case, when the absolute value of A is less than $\sqrt{3}\sigma$ and with enough samples, we're guaranteed to be able to find an unbiased estimator for A because we will have both positive and negative values. However, when the absolute value of A is greater than $\sqrt{3}\sigma$, then A plus the uniform noise will always either be positive or negative, which translates to the estimator always being biased and either choosing $-\sqrt{3}\sigma$ if all values are negative or $\sqrt{3}\sigma$ if all values are positive. This applies both to MLE and MAP.

The Gaussian case is similar but there is no clear cut-off for when the estimator becomes biased. Naturally when there are enough positive and negative samples we can estimate A optimally, since both MLE and MAP estimators give the same MSE as the CRLB when the ratio between A and sigma is fairly small. However, the larger this ratio gets, the more likely we are to have strictly positive or strictly negative samples for each of the Monte Carlo runs, which will gradually bias the estimator.

The MSE in the Gaussian case becomes biased when all of the samples are the same value because we define the MLE differently in those cases in MATLAB. Since the MLE depends on the inverse of the error function which is either infinity or negative infinity when all of the samples are the same value, this makes it infeasible to evaluate normally, meaning we needed to define it differently in those cases, making it biased by evaluating at the same value whenever all values are the same.

The MAP estimator in the Gaussian case is also biased, but for less known reasons. Since the MAP has no closed form solution, we rely on MATLAB's fminunc function to solve the optimization problem. Since the objective function plateaus when the ratio of positives to negatives is either a 1 or 0, the optimizer iterates until its iteration limit, getting stuck and returning the same value. Therefore, it becomes biased.

4. Applications

One common application of dither technology is to more accurately display images or videos that contain a greater color range than the hardware's capability. For example, if we have a colored video that contains millions of colors while our display hardware is only able to show 256 colors at a time, dither technology might be utilized in order to avoid a significant distortion effect, which means the colors in the video might simply be "rounded off" to the closest available color. Without dithering, an image presented in our display hardware will be a poor representation of the original image. In fact, the reason that dithering would work for human's eyes is taking advantage of the human brain's tendency to combine two colors in close proximity to one another.

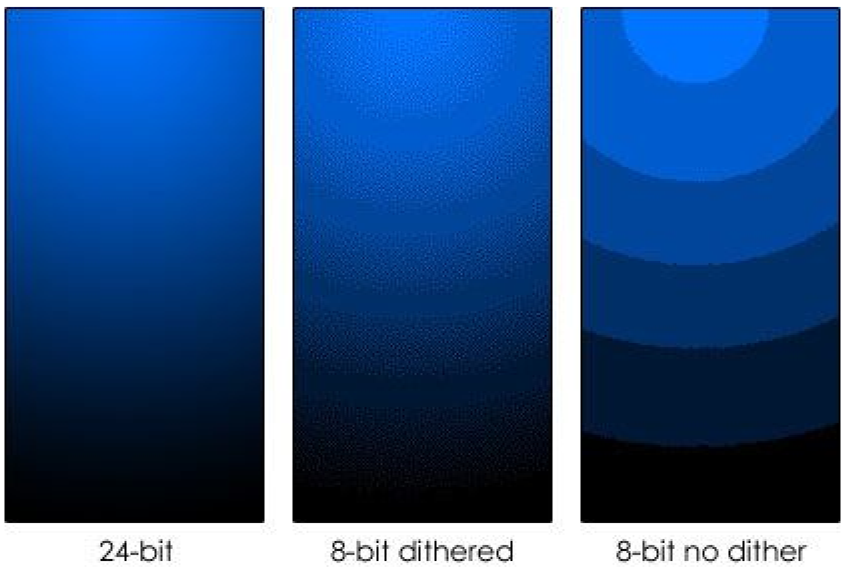

Figure 4.1 a comparison between truncation with dithering and without dithering

Another widely used application of dither is for solving image displaying situations when graphic file format has a limiting factor. For example, the commonly used GIF format is usually restricted to 256 or fewer colors in graphics design or editing softwares. Similar situations also happens in other image formats like PNG. For such situations, most graphic design or editing software will have the ability to apply dithering to the original images before saving them in formats such as GIF or PNG.

Figure 4.2 The image The original image on the left contains 16.7million colors. Both of the converted GIF images are 256-color.

5. References

- Earlevel - http://www.earlevel.com/main/category/digital-audio/dither-digital-audio/
- Normal distribution: (Cumulative) Distribution Function - http://www.matematicasvisuales.com/english/html/probability/varaleat/normaldistribution.html
- Jake Blanchard Monte Carlo Simulation in Matlab - https://www.youtube.com/watch?v=7ZkdDHHkVQ8&feature=emb_logo
- Dither - https://en.wikipedia.org/wiki/Dither#Applications
- Discrete Distributions - https://www.csus.edu/indiv/r/ramachandrang/formulasheet.pdf
- Dither in animation studio - https://www.coffeecup.com/help/articles/dither-modes-in-animation-studio/

6. Appendix

```
options = optimoptions(@fminunc, 'Display','off');


%Gaussian:
gaussian = 0;

if gaussian
    vals1 = []; %for MLE
    vals2 = []; %for MAP
    vals3 = []; %for CRLB

        for i=1:100
        MSE1=0; %for MLE
        MSE2=0; %for MAP
        A=i;
        R=200;
        for j=1:R
            n=5000;
            sigma=10;
            y = sign(repmat(A,n,1)+ normrnd(0,sigma,[n,1]));
            %MLE:
            if (1/n) * sum((y)) == 1
                A_mle = -sigma * sqrt(2)*erfinv(-.9999999);
            elseif (1/n) * sum((y)) == -1
                A_mle = -sigma * sqrt(2)*erfinv(.9999999);
            else
                A_mle = -sigma * sqrt(2)*erfinv(-(1/n) * sum((y)));
            end

            %MAP:
            obj = @(a) -((sum((1+y)/2))*log(1-PHI(-a/sigma)) + (sum((1-y)/2)) * log(PHI(-a/sigma))
-a^2/((2*sigma)^2));
            [A_map, fval] = fminunc(obj, 0, options);

            MSE1 = MSE1 + (A-A_mle)^2;
            MSE2 = MSE2 + (A-A_map)^2;
        end
        FIM = (1/(sigma^2))*(phi(-A/sigma))^2 * (1/((PHI(-A/sigma)*(1-PHI(-A/sigma)))));
        CRLB = 1/FIM;
        CRLB = CRLB/n;
        ratio = A/sigma
        mses = [mses; MSE1];
        MSE1 = MSE1/R;
        MSE2 = MSE2/R;
        vals1 = [vals1; MSE1];
        vals2 = [vals2; MSE2];
```

```
        vals3 = [vals3; CRLB];

    end
    figure(1)
    loglog([0.1:0.1:10], vals1)
    hold on;
    loglog([0.1:0.1:10], vals2)
    loglog([0.1:0.1:10], vals3)



end



%Uniform:
uniform = 1;

if uniform
    vals1 = []; %for MLE
    vals2 = []; %for MAP
    vals3 = []; %for CRLB

    for i=1:100
        MSE1=0; %for MLE
        MSE2=0; %for MAP
        for j=1:200
            A = i;
            n = 5000;
            sigma = 10;
            pd = makedist('Uniform', 'lower',-sqrt(3)*sigma, 'upper',sqrt(3)*sigma);

            y = sign(repmat(A,n,1)+ random(pd, [n, 1]));

            %MLE:
            A_mle = (sqrt(3)*sigma * sum(y))/n;

            %MAP:
            obj = @(a) -((sum((1+y)/2))*log(1-((-a+sqrt(3)*sigma)/(2*sqrt(3)*sigma))) + ...
                (sum((1-y)/2))* log(((-a+sqrt(3)*sigma)/(2*sqrt(3)*sigma))) + log(-a+sqrt(3)*sigma) -
log(2*sqrt(3)*sigma));

            [A_map, fval] = fminunc(obj, 0, options);

            MSE1 = MSE1 + (A-A_mle)^2;
            MSE2 = MSE2 + (A-A_map)^2;
        end
        %CRLB:
        FIM = 1/(3*sigma^2 - A^2);
```

```
        CRLB = 1/FIM;
        CRLB = CRLB/n;
        ratio = A/sigma
        MSE1 = MSE1/200;
        MSE2 = MSE2/200;
        vals1 = [vals1; MSE1];
        vals2 = [vals2; MSE2];
        vals3 = [vals3; CRLB];
    end
    figure(2)
    loglog([0.1:0.1:10], vals1)
    hold on;
    loglog([0.1:0.1:10], vals2)
    loglog([0.1:0.1:10], vals3)

end

function s = PHI(x)
    s = 1/2 + 1/2 * erf(x/sqrt(2));
end

function s = phi(x)
    s = (exp(-(x.^2)/2))/(sqrt(2*pi));
end

function s = phip(x)
    s = (-x).*phi(x);
end
```