Math 2605 Project Part 3

April 1, 2015

Introduction

The Leslie Matrix is a model used to forecast the growth of a populations over time. Though this model is often used in ecology, it is not always a reliable method for forecasting populations. It does not take into account various outside factors, like immigration, emigration, or long term changes in birthrate caused by social changes, such as the legalization of abortion, or economic industrialization. For now, we will assume that none of these outside factors apply.

Leslie Matrix

The Leslie Matrix takes the form

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{pmatrix} \tag{1}$$

where f represents fecundity (birthrate) and s represents survival rate (the number of individuals that advance from age class x to age class $x + 1$.

Each column x represents an age class (a group of individuals whose age is within a certain bound. In our example city, each age class represents 10 years, so age class $x = 1$ represents individuals who are $[0, 9]$ years old, age class $x = 2$ represents $[10, 19]$ year olds, etc.

A Leslie Matrix is always accompanied by a population vector

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \tag{2}$$

where n_x represents the number of individuals in population class x, and k represents the number of iterations used to reach reach this population estimate.

Therefore the equation

$$\overline{n}(k+1) = L\overline{n}(k) \tag{3}$$

where $\overline{n}(k)$ is the previous population vector, $\overline{n}(k+1)$ is the population after one iteration, and L is the Leslie Matrix.

Since each of our age classes represents 10 years, each iteration will produce a population vector that occurs 10 years after the previous vector.

Modeling a Certain City's population

In this city, the Leslie Matrix is given by

$$L = \begin{pmatrix} 0 & 1.2 & 1.1 & .9 & .1 & 0 & 0 & 0 & 0 \\ .7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .88 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .77 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .40 & 0 \end{pmatrix} \tag{4}$$

and the initial population in the year 2000 is given by the population vector

$$\overline{n}(0) = \begin{pmatrix} 210,000 \\ 190,000 \\ 180,000 \\ 210,000 \\ 200,000 \\ 170,000 \\ 120,000 \\ 90,000 \\ 50,000 \end{pmatrix} \tag{5}$$

Question 1

Analysis of fecundity Fecundity is the average rate at which individuals in a given population reproduce.

$f_1 = 0$, so the birthrate for children who are 0 - 9 years old is zero. It's physically impossible for children of this age to have children, because females become fertile only after onset of puberty, at around age 10. (*1*)

$f_2 = 1.2$ Teenagers have the highest birthrate out of all the population groups, which coincides with the onset of puberty, when humans are first able to reproduce. (*1*)

$f_3 = 1.1$, $f_4 = 0.9$ As humans age, fertility slowly drops.

$f_5 = 0.1$ The people in this city stop reproducing between the ages of 40 and 49 due to the start of menopause, when females are no longer able to bear children. (*2*)

The distribution of birth rates in this city is consistent with the distribution in developing countries. In developing countries, it is advantageous to have more children, as more children can provide more labor, and generate more money. Children often stay at home with their parents, and support them during the later stages of their lives. (*3*)

In developed countries such as the United States, this is not necessarily the case. Children generally leave home when they turn 18, in search of jobs or education. This, combined with the fact that Social Security supports parents when they enter retirement means having more children is not necessarily an advantage. Rearing more children is also extremely costly, and expenses like daycare, food, clothing, housing, and schooling all add up. The average cost to raise a child in the United States is $245,340, an expense that many simply cannot afford. (*4*)

Accessibility to birth control and sexually education in developed countries also greatly reduces teenage birth rates. For example, all students of American public schools are taught about safe sex, and condoms can be bought from neighborhood convenience stores. In developing countries, and especially in rural areas, education is lacking, and birth control is hard to obtain, leading to higher birth rates. (*5*)

Analysis of survival rate Survival rate of individuals in a population is primarily determined by outside factors, such as natural predators, malnutrition, disease, or aging. In a human population, the death rate is primarily determined by instances of disease.

$s_1 = 0.7$ The survival rate for children less than 10 years old is low because they are often not strong enough to fight disease or survive malnutrition. (*6*)

$s_2 = 0.85, s_3 = 0.9, s_4 = 0.9, s_5 = 0.88$ Humans who are 10 - 60 years old have the highest survival rates, because they have the strongest immune systems and bodies. They can withstand stress (like malnutrition and hunger) far better than the young or old.

$s_6 = 0.8, s_7 = 0.4$ As humans age and become seniors, their immune systems and bodies weaken. Unable to fight off illness, they begin dying in large numbers.

Question 2

Population distributions from 2000 to 2050

Age	2000	2010	2020	2030	2040	2050
0 - 9	210000.00	635000.00	518750.00	816240.00	965648.50	1341322.68
10 - 19	190000.00	147000.00	444500.00	363125.00	571368.00	675953.95
20 - 29	180000.00	161500.00	124950.00	377825.00	308656.25	485662.80
30 - 39	210000.00	162000.00	145350.00	112455.00	340042.50	277790.63
40 - 49	200000.00	189000.00	145800.00	130815.00	101209.50	306038.25
50 - 59	170000.00	176000.00	166320.00	128304.00	115117.20	89064.36
60 - 69	120000.00	136000.00	140800.00	133056.00	102643.20	92093.76
70 - 79	90000.00	92400.00	104720.00	108416.00	102453.12	79035.26
80 - 89	50000.00	36000.00	36960.00	41888.00	43366.40	40981.25
Sum	1420000.00	1734900.00	1828150.00	2212124.00	2650504.67	3387942.93

Total Population and change in population 2000 - 2050

Year	Population	Δ Population
2000	1420000.00	-
2010	1734900.00	22.17605634
2020	1828150.00	5.374949565
2030	2212124.00	21.00341876
2040	2650504.67	19.81718339
2050	3387942.93	27.82256037

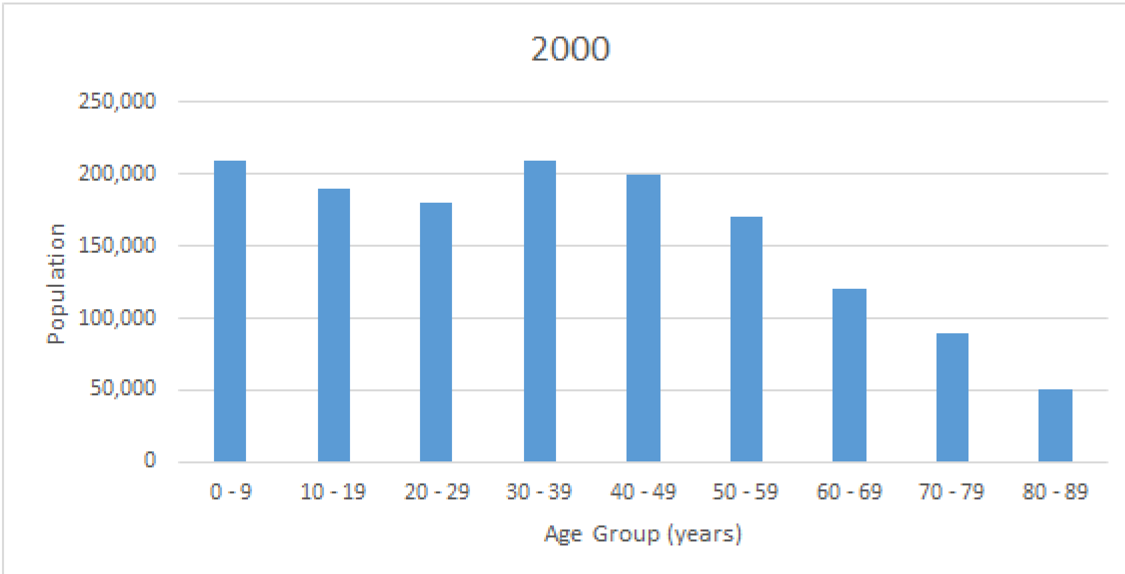

Figure 1: Population distribution in 2000

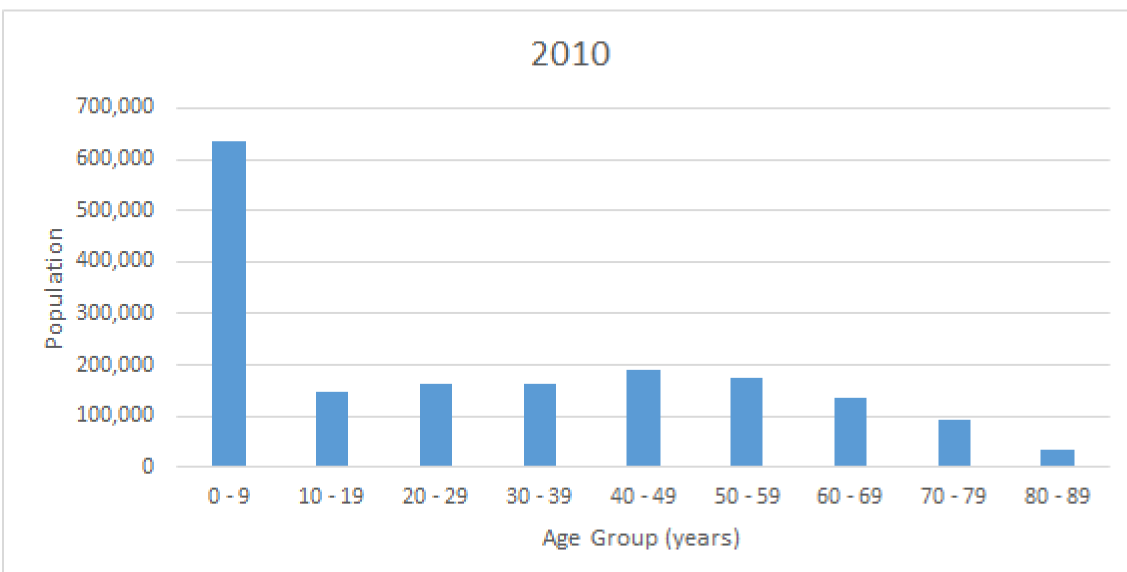

Figure 2: Population distribution in 2010

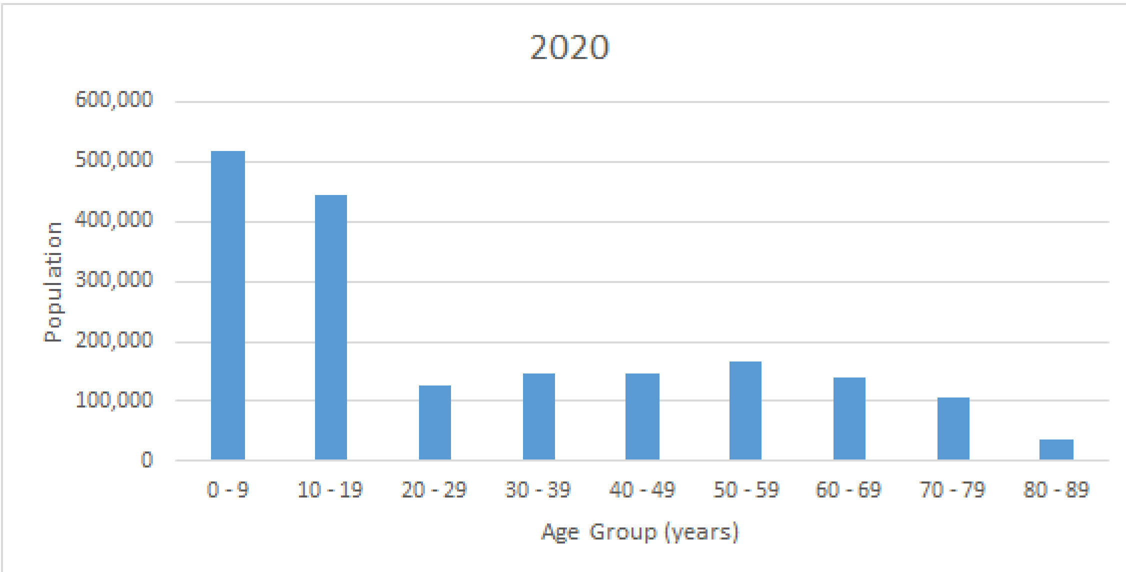

Figure 3: Population distribution in 2020

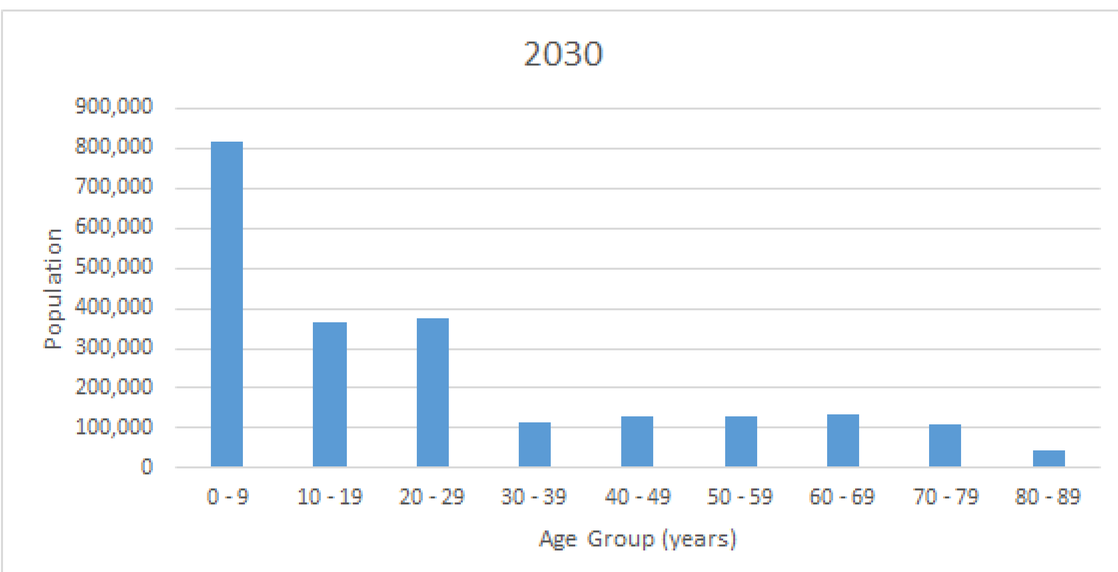

Figure 4: Population distribution in 2030

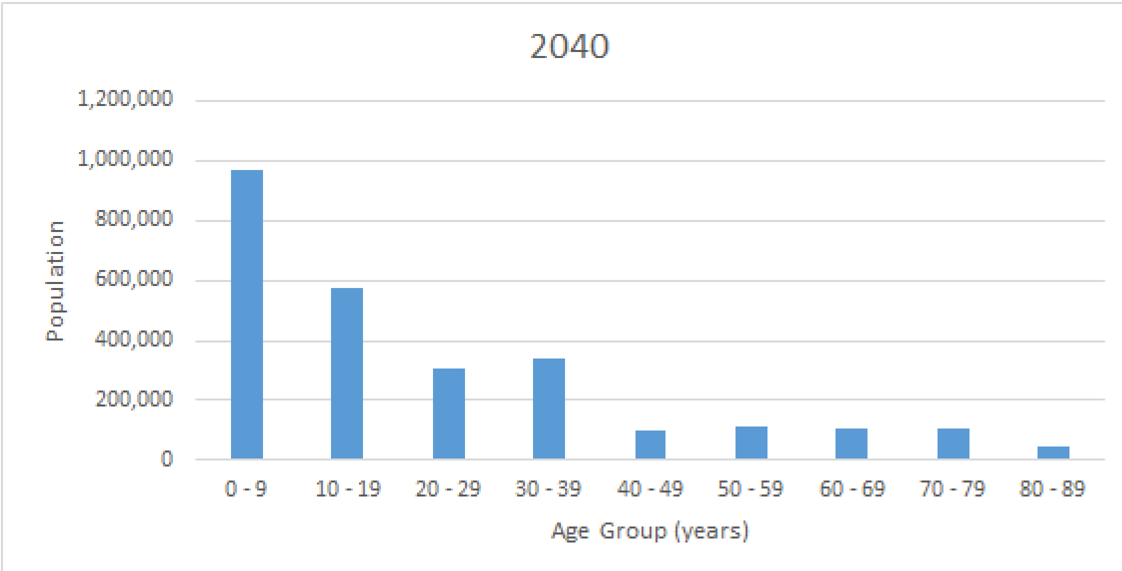

Figure 5: Population distribution in 2040

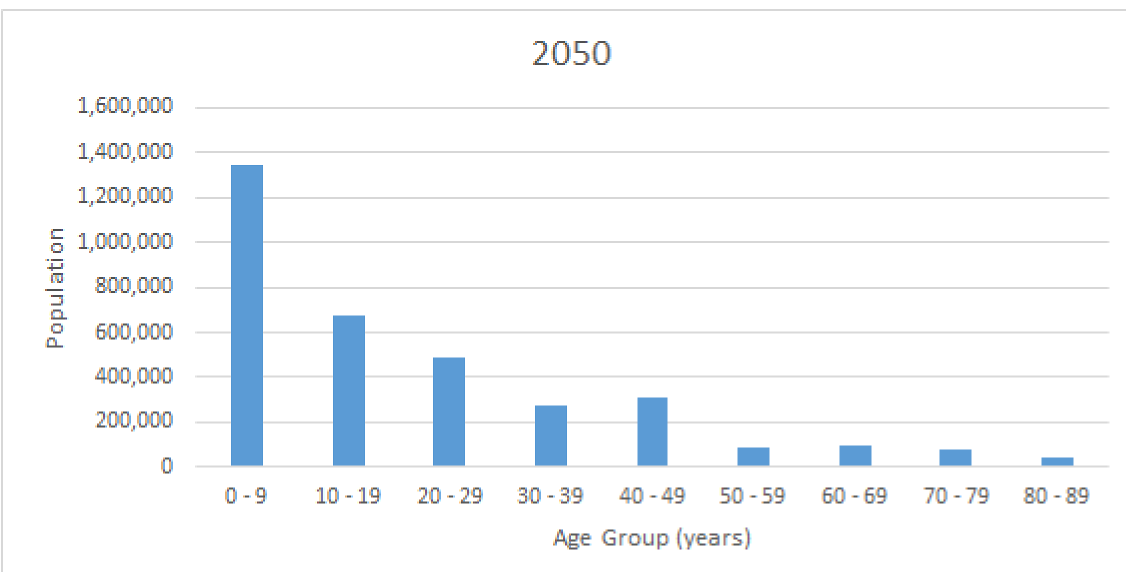

Figure 6: Population distribution in 2040

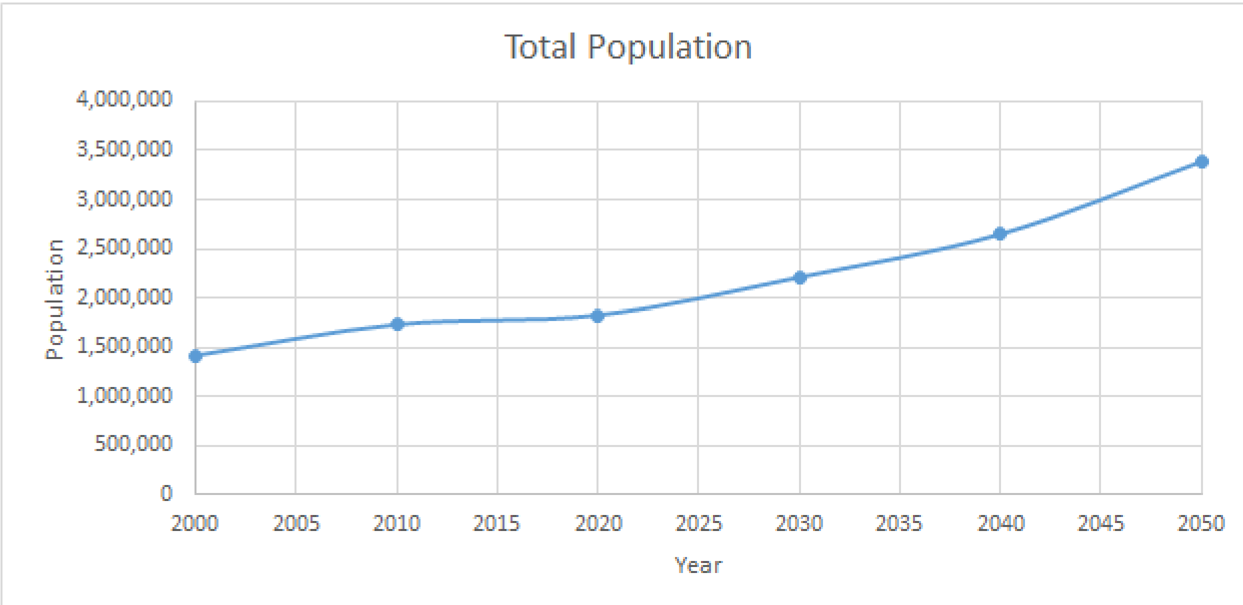

Figure 7: Total Population 2000 - 2050

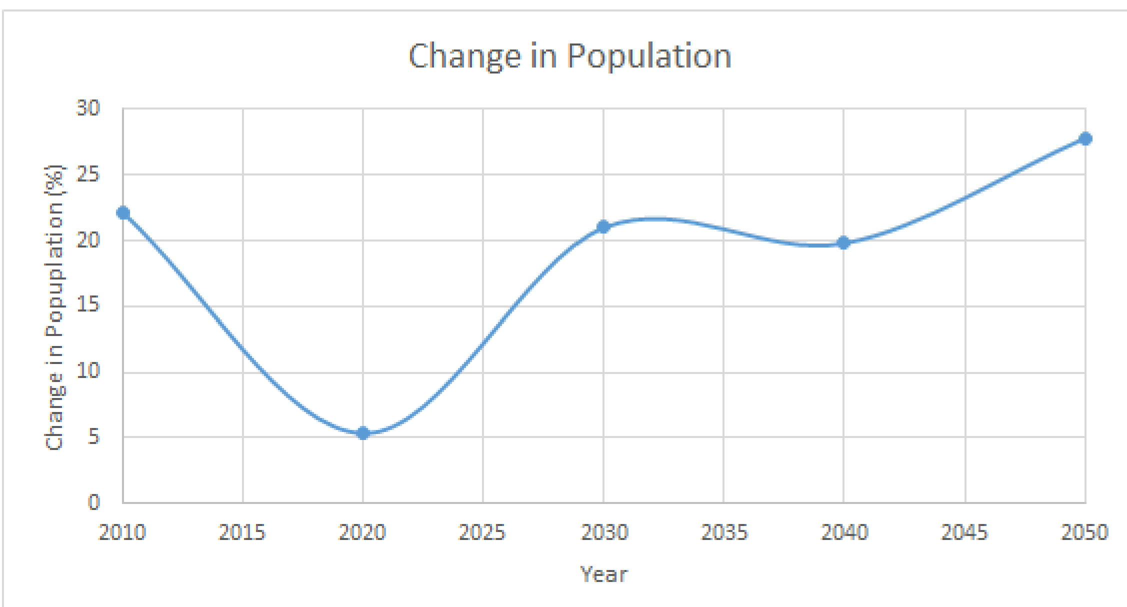

Figure 8: Change in Population 2000 - 2050

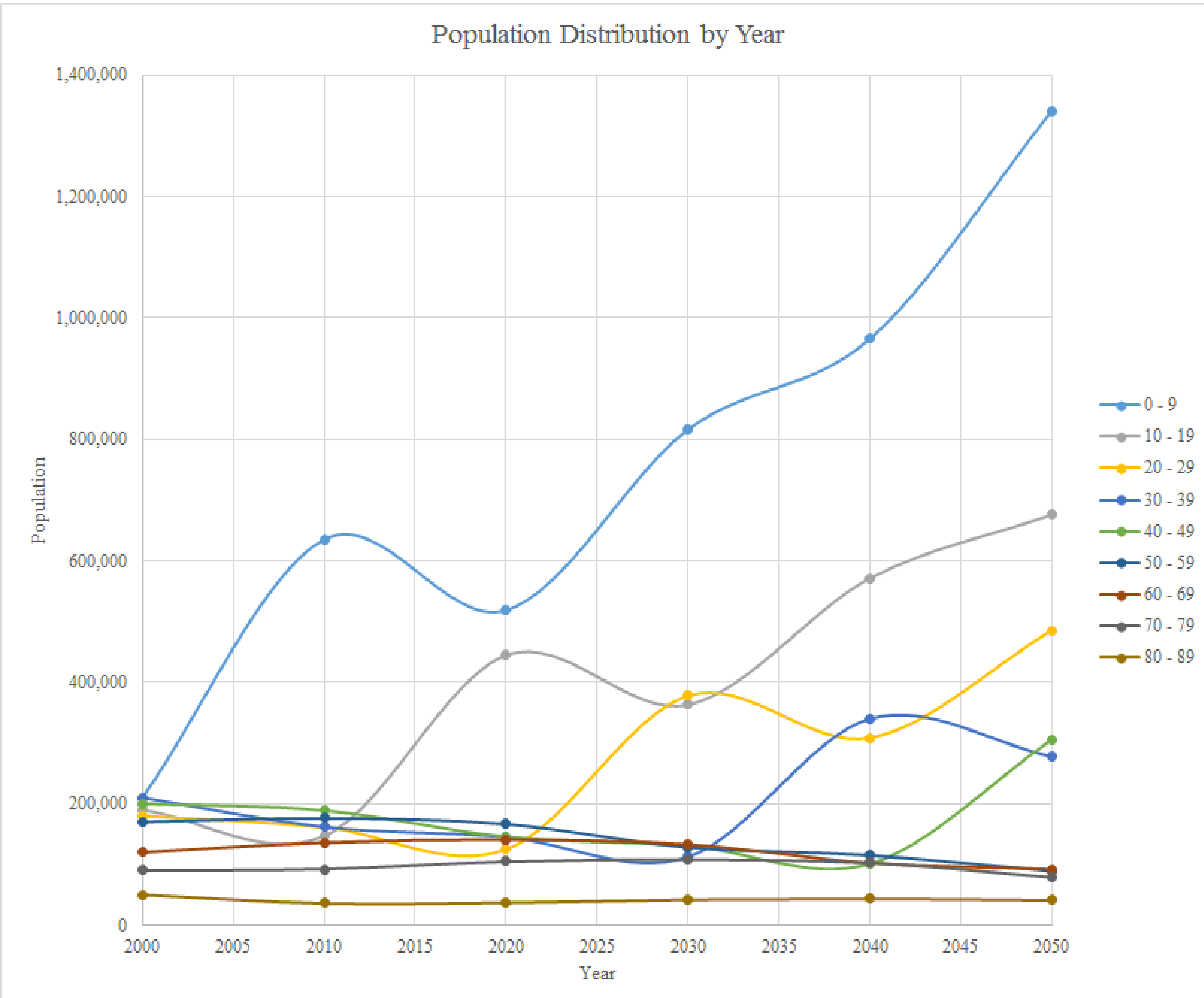

Figure 9: Population distribution 2000 - 2050

Question 3

Give the Euler-Lokta equation

$$1 = \sum_{a=1}^{\omega} \lambda^{-a} \ell(a) b(a) = \lambda^{-1} f_1 + \lambda^{-2} f_2 s_1 + \lambda^{-3} f_3 s_1 s_2 + \lambda^{-4} f_4 s_1 s_2 s_3 \tag{6}$$

where λ is the dominant eigenvalue of the Leslie Matrix (the growth rate), ω is the maximum age reached by the population, $\ell(a) = s_1 ... s_n$ is the number of individuals who survive to age a, and $b(a) = f_n s_1 ... s_n$ the number of individuals born at age a. (7)

This means that if $\lambda < 1$ population will decrease, $\lambda > 1$ population will increase, and if $\lambda = 1$ population will be stable

Since the maximum eigenvalue $\lambda = 1.28865623$, and $\lambda > 1$, the population will grow exponentially. $||A^k||$ will not converge, which is corroborated by running a Leslie model for $n = 100$ iterations. If the city's population keeps growing at the rate specified in the Leslie matrix, it will reach 9.686e16 (roughly 10 quadrillion) by the year 3000.

Question 4

If the 2nd age group's birth rate is halved in 2020, so $f_2 = 0.6$, the population distribution becomes

Year	Population 1	Population 2
2000	1420000.00	1420000.00
2010	1734900.00	1734900.00
2020	1828150.00	1828150.00
2030	1945424.00	1945424.00
2040	2245939.67	2245939.67
2050	2621909.13	2621909.13

and $\lambda = 1.16790273$

since $\lambda > 1$, the population will grow exponentially in the long run, but at a slower rate than if the pregnancy rate is age class 2 hadn't been cut in half.

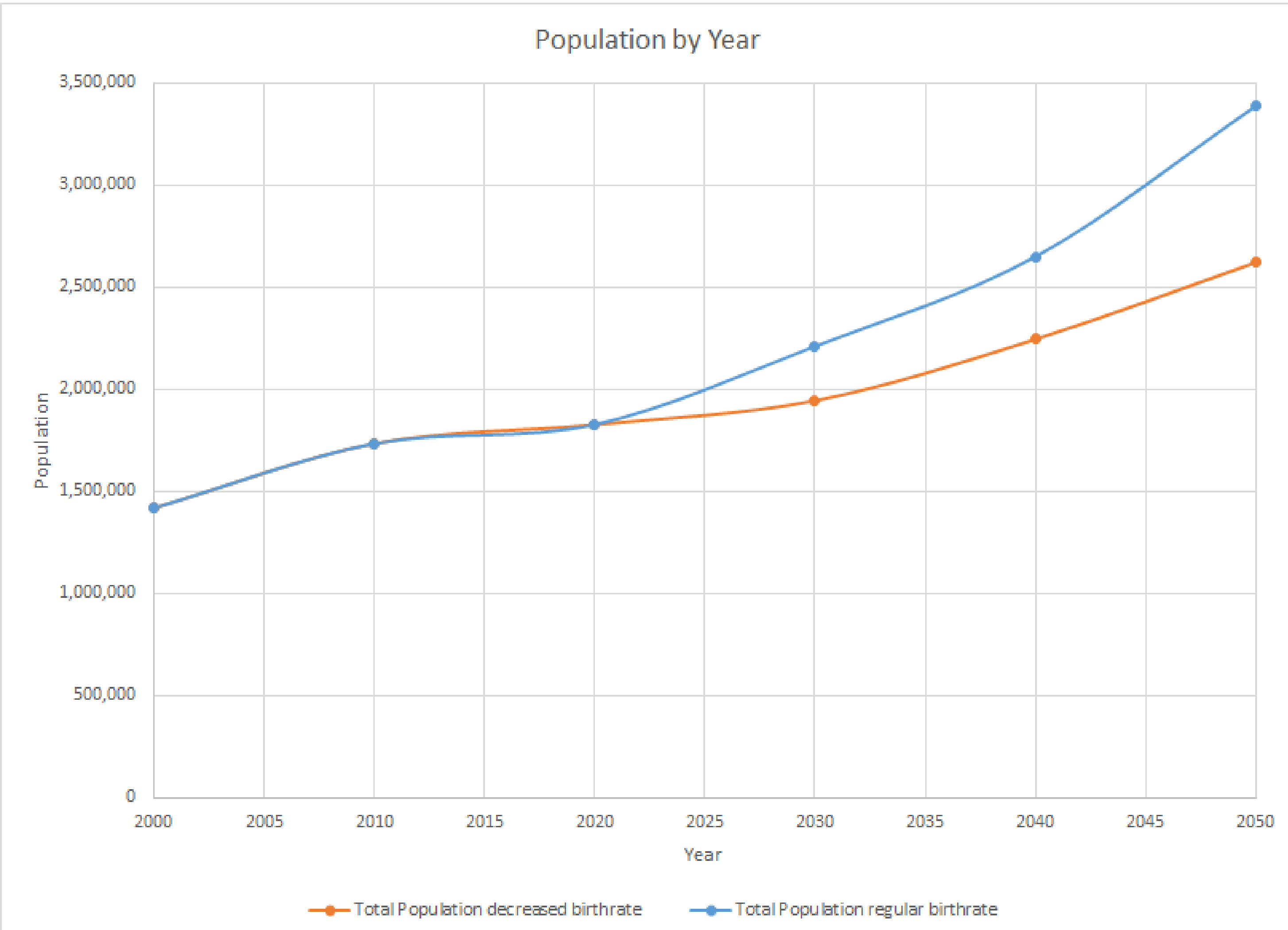

Figure 10: The total population of the city, with the original Leslie Matrix, and the total population of the city with the modified Leslie Matrix.

References and Notes

1. U. S. O. on Women's Health, Timing and stages of puberty (2014).

2. U. S. N. I. on Aging, Menopause (2013).

3. T. W. Bank, World population growth (2003).

4. U. S. D. of Agriculture, Parents projected to spend $245,340 to raise a child born in 2013, according to usda report (2014).

5. D. M. A. Julie DaVanzo, Family planning in developing countries (1998).

6. W. H. Organization, Children: reducing mortality, year = 2014, url = http://www.who.int/mediacentre/factsheets/fs178/en/, urldate = 2015-3-29.

7. R. Howard.

8. All code is original, except for the print methods in the Matrix class. That code was taken from NIST's JAMA library.