# Bethe phase variation due to a non-exponential nuclear amplitude and the possibility of using a t-dependent phase to determine the $\rho$ -parameter from elastic scattering data

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#### Abstract

We evaluate the possible deviation (from the conventional Cahn's result) of the phase between the one-photon-exchange and the 'nuclear' high energy pp scattering amplitudes in a small  $t \to 0$  region caused by a more complicated (not just exp(Bt)) behaviour of the nuclear amplitude. Furthermore we look at the possible role of the t-dependence of the  $\rho(t) \equiv \text{Real/Imaginary}$  amplitude ratio. It turns out that both effects are rather small - much smaller than to have any influence on the experimental accuracy of  $\rho(t=0)$  extracted from the elastic proton-proton scattering data.

## 1 Introduction

The real part of the high energy strong interaction (nuclear) pp elastic amplitude,  $F^N$ , was measured recently by TOTEM collaboration at  $\sqrt{s} = 13$  TeV with unprecedented accuracy of 0.01 for the  $\rho = \text{Re}F^N(t=0)/\text{Im}F^N(t=0)$  ratio [1].

The conventional way to measure the real part of the strong interaction (nuclear) forward amplitude is to consider its interference <sup>1</sup> with the pure real one-photon-exchange QED amplitude,  $F^C$ , at very small momentum transfer  $t \to 0$ . However this interference is affected by the possibility of multiphoton exchange processes which result in the additional phase difference  $\alpha\Phi$ . That is the total amplitude reads

$$F^{TOT} = F^N + e^{i\alpha\Phi}F^C . (1)$$

<sup>&</sup>lt;sup>1</sup>it is called the Coulomb Nuclear Interference (CNI).

Here  $\alpha = \alpha^{\rm QED} = 1/137$ . The phase  $\Phi$  (the so-called Bethe phase) was calculated first by Bethe [2] using the WKB approach, and then was re-examined by West and Yennie [3] in terms of Feynman diagrams. A more precise calculation was performed by R. Cahn [4] in 1982 which accounts for the details of the proton form factor. It gives

$$\Phi(t) = -[\ln(-Bt/2) + \gamma_E + C] , \qquad (2)$$

where B is the t-slope of the elastic cross section  $(d\sigma_{\rm el}/dt \propto e^{Bt})$ ,  $\gamma_E = 0.577...$  is Euler's constant and the constant  $C \sim 0.4 - 0.6$  depends on the precise form of the proton electromagnetic form factor and the t dependence of the nuclear amplitude. In the Cahn's paper [4] the usual dipole electromagnetic formfactor  $f(t) = 1/(1 - t/0.71 \text{GeV}^2)^2$  was used and the pure exponential t dependence of  $F^N \propto \exp(Bt/2)$  was assumed. In such a case the value of C = 0.60 for CERN-ISR energies and C = 0.45 for the LHC case when the slope  $B \simeq 20 \text{ GeV}^{-2}$ .

However the real t dependence of nuclear amplitude is more complicated even at a rather low |t|. In particular the deviation from the pure exponent was observed at 8 TeV in [5, 6]. This deviation should affect the constant C calculation and the main aim of the present paper is to evaluate how large this effect can be. We also evaluate in section 3 the effect of a t-dependence of the nuclear phase on the determination of the  $\rho$ -parameter from elastic scattering at low |t|.

#### 2 Estimate of the constant C alteration

Note that thanks to a small QED coupling  $\alpha = 1/137$  the absolute value of the phase  $\alpha \Phi$  is small. Within the relevant for Coulomb-nuclear interference interval  $|t| = 0.001 - 0.03 \text{ GeV}^2$  it does not exceed 0.03. That is actually the reason why we are looking for the contribution of the first diagram with one additional photon exchange (see Fig.1).

Next term in the  $e^{i\alpha\Phi}$  expansion,  $(\alpha\Phi)^2/2$ , is of the order of  $10^{-3}$  and the small possible change of this contribution is already negligible in comparison with the present experimental accuracy  $\sim 10^{-2}$ . Moreover, actually this second term interferes only with the real part of  $F^N$ . That is the expected effect should be of about  $\rho(\alpha\Phi)^2/2 \lesssim 10^{-4}$ 

Thus to estimate the C variation caused by a more complicate t dependence of the nuclear amplitude we have to compare the contributions of Fig.1 diagram with the complete t dependence of  $F_{exact}^N$  with that with  $F_{exp}^N(t) = F^N(0) \exp(Bt/2)$ .

$$\delta C(t) = -\int d^2 q_t \frac{f^2(q_t^2)}{q_t^2} \left( \frac{F_{exact}^N(t')}{F_{exact}^N(t)} - \frac{F_{exp}^N(t')}{F_{exp}^N(t)} \right) . \tag{3}$$

<sup>&</sup>lt;sup>2</sup>Here in the intermediate state we consider only the proton. The possibility of the  $p \to N^*$  excitation was studied in [7]. It was shown that the effect of these possible additional contributions on the value of  $\rho$  does not exceed  $10^{-3}$ .

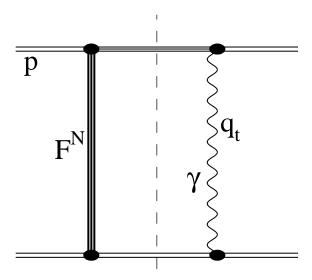


Figure 1: Diagrams responsible for the Bethe phase at first  $\alpha^{\rm QED}$  order. The nuclear amplitude is shown by the triple solid line and marked as  $F^N$ 

We denote the full transverse momentum transferred as  $Q_t$   $(t = -Q_t^2)$ . So the momentum transferred through the nuclear amplitude in Fig.1 is  $\mathbf{Q}'_t = \mathbf{Q}_t - \mathbf{q}_t$  and  $t' = -Q_t'^2$ .

Note that the integral (3) has no infrared divergence. The difference  $[F_{exact}(t')/F_{exact}(t) - F_{exp}(t')/F_{exp}(t)] \to 0$  at  $q_t \to 0$  since  $t' \to t$ .

The integral (3) was computed numerically for the case of 8 TeV pp scattering using the TOTEM parameterization of the nuclear amplitude  $F_{exact}^N(t) = F_{exact}^N(0) \exp(-\frac{1}{2} \sum_{1}^{3} b_k |t|^k)$ . The slope of  $F_{exp}^N$  amplitude was taken to be  $B = B(t=0) = b_1$  and the electromagnetic formfactor has the dipole form  $f(q_t^2) = 1/(1 + q_t^2/0.71 \text{GeV}^2)^2$ .

The results are shown in Fig.2. Continues curve corresponds to the parameterization of [6] which accounts for the Coulomb-nuclear interference while for the dashed curves the parameterization of [5], based on the description of a larger |t| interval (without the Coulomb term) was used.

It is seen from Fig.2 that the possible phase shift  $\alpha \delta C$  never exceed  $10^{-3}$  in the |t| < 0.03 GeV<sup>2</sup> region relevant for the Coulomb-nuclear interference.

This is consistent with the very naive estimate. Since the difference between the exact pp nuclear amplitude and its exponential approximation at low |t| is less than 10% we can expect  $\delta C < 0.1$ ; that is the phase shift  $\alpha \delta C < 10^{-3}$ . One can safely neglect this effect and use the Cahn's expression [4], written for the pure exponential  $F^N(t) \propto \exp(Bt/2)$  case (with B = B(t = 0)), bearing in mind the experimental accuracy of the order of  $10^{-2}$ .

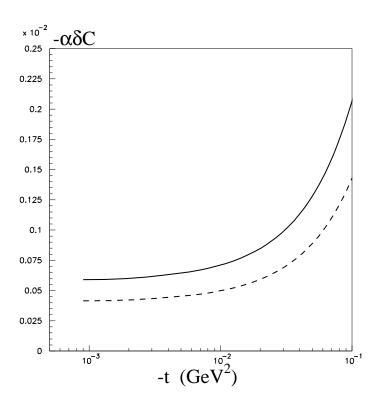


Figure 2: The deviation of the phase between the one photon exchange and the nuclear,  $F^N(t)$ , amplitudes caused by the more complicated t dependence of  $F^N_{exact}$  in comparison with the pure exponential behaviour used in Cahn's [4] calculations. Continues curve corresponds to the parameterization of [6], which accounts for the Coulomb-nuclear interference, while for the dashed curve the parameters of [5] (without the Coulomb term) was used.

## 3 t-dependence of $\rho$

Another point which should be considered is the following. Actually the value of  $\rho(0)$  is extracted from the elastic scattering differential cross section  $d\sigma_{el}/dt$  measured not at t=0 but in some interval of small but non-zero t. On the other hand we know that  $\rho(t) \neq const(t)$ . The real part of the elastic amplitude should vanish at some relatively low $|t| \sim 0.1 \text{ GeV}^2$  [8] (see e.g. Fig.1 of [9] as an example). The corresponding t dependence of the Re/Im ratio may also affect the value of  $\rho(0)$  obtained by fitting the  $d\sigma/dt$  data under the usual/simplified assumption that  $\rho(t) = const(t)$ .

The most straightforward way to take into account the fact that the real part should vanish at  $|t| \sim 0.1 \text{ GeV}^2$  is to use the following simplified formula for the t-dependence of the nuclear phase

$$\arg F^{N}(t) = \frac{\pi}{2} - \arctan(\rho(0)(1 + t/0.1)) \tag{4}$$

A similar t-dependence was also recently proposed by Durand-Ha [10] taking into account in addition the fact that the imaginary part has a zero around the dip region  $|t| \sim 0.45 \text{ GeV}^2$ .

$$\arg F^{N}(t) = \frac{\pi}{2} - \arctan\left(\rho(0)\frac{1 + t/t_{R}}{1 + t/t_{I}}\right) \tag{5}$$

with  $t_R=0.16~{\rm GeV^2}$  and  $t_I=0.42~{\rm GeV^2}$  at 13 TeV.

Naively one would expect a small effect here because looking at the corresponding t-dependence of  $\rho$  one sees a very weak dependence of  $\rho$  in the coulomb interference region. At the point of maximum sensitivity to the interference effect the deviation between the  $\rho$  value for a constant phase and the phase from (4,5) is less than  $10^{-3}$  ( remember that the best experimental uncertainty up to now is  $10^{-2}$ ).

To study this effect/question more quantitatively we have analyzed the published TOTEM 13 TeV data [1] using the t-dependent phase of (4,5). We also tried a couple of other versions of possible t-dependent phases published earlier: the so called standard parameterization [11]

$$\arg F^{N}(t) = \frac{\pi}{2} - \arctan \rho(0) + \arctan \left(\frac{|t| - |t_{0}|}{\tau}\right) - \arctan \left(\frac{-|t_{0}|}{\tau}\right)$$
 (6)

with  $t_0 = -0.5 \text{ GeV}^2$  and  $\tau = 0.1 \text{ GeV}^2$ ,

and the Bailly parameterization [12]

$$\arg F^{N}(t) = \frac{\pi}{2} - \arctan \frac{\rho(0)}{1 - t/t_0} \tag{7}$$

with  $t_0 = -0.53 \text{ GeV}^2$ 

Besides those we consider a more extreme hypotheses

the so called "peripheral" model [13] where

$$\arg F^{N}(t) = \frac{\pi}{2} - \arctan \rho(0) - \xi_1 \left( -\frac{t}{1 \text{GeV}^2} \right)^{\kappa} e^{\nu t}$$
(8)

with  $\xi_1 = 800$ ,  $\kappa = 2.311$  and  $\nu = 8.161~{\rm GeV^{-2}}$  and for comparison just the

$$\rho(t) = const \tag{9}$$

The corresponding t-dependence of all the mentioned phases are shown in Fig.3.

In all these six cases the data [1] where fitted using the conventional Cahn's phase (2). The nuclear amplitude was parametrized as

$$|F^N(t)|^2 = A \exp(Bt + Ct^2)$$
 (10)

Thus we have four free parameters: A, B, C and  $\rho(0)$ .

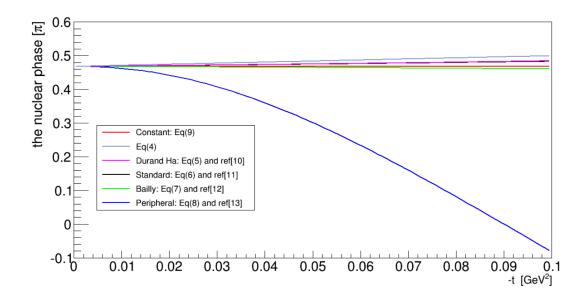


Figure 3: t-dependence of the nuclear phase for the six models ref [10, 11, 12, 13] and equation (4) and (9).

We have confirmed that, indeed, the difference in  $\rho$  for the case of a constant phase and the phase of (4-8) is less than  $10^{-3}$  by fitting the TOTEM data 13 TeV in the t-range 0.0008 GeV<sup>2</sup>- 0.12 GeV<sup>2</sup> and using the parameterization (10). The only exception is the "peripheral" model. In this model the value of  $\rho(0)$  differs from that in the  $\rho = const(t)$  case by about  $6 \cdot 10^{-3}$  but still smaller than the typical experimental uncertainty of  $10^{-2}$ . <sup>3</sup>

Note however that the peripheral model (8) is inconsistent with the dispersion relations for the C-even amplitude. In the simplified form the dispersion relation at fixed t reads  $^4$ 

$$\rho \simeq \frac{\pi}{2} \frac{\partial \ln(\operatorname{Im} F^{N}(t))}{\partial \ln s} . \tag{11}$$

As it follows from the experimental data this value should be positive in the  $|t| \sim 0.1 \; \mathrm{GeV^2}$  region while in the peripheral model it becomes negative (see Fig.3).

<sup>&</sup>lt;sup>3</sup>Since the parameters for the peripheral model/scenario at 13 TeV were not published we have used the numbers from 8 TeV [6]. Due to the weak/logarithmic behaviour of elastic pp-amplitude the difference should not be large while on another hand all this examples are just to demonstrate the expected order of the size of the effect. The same  $6 \cdot 10^{-3}$  difference was observed fitting with this t-dependence of  $\rho$  the 8 TeV data [6].

<sup>&</sup>lt;sup>4</sup>Here we use the  $2 \text{Im} F^N(0) = \sigma_{tot}$  normalization.

Thus we conclude that both effects - the possible t-dependence of the  $\rho = \text{Re}F^N(t)/\text{Im}F^N(t)$  and some deviation from the pure exponential t behaviour of the nuclear amplitude  $F^N(t)$  in the small |t| region relevant for extraction of  $\rho(t=0)$ , via the Coulomb-nuclear interference, do not exceed the  $10^{-3}$  level and can be neglected in comparison with the today experimental accuracy of about  $10^{-2}$ .

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