About soft photon resummation

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A contribution to Earle Lomon's 90th birthday celebration.

A DEDICATION

The first time one of us (G.P.) encountered Earle was in Summer 1966, when she was directed to study Earle's papers on radiative corrections to quasi-elastic electron scattering [1, 2]. The suggestion had come from Bruno Touschek [3], at the time head of the theoretical physics group at the Frascati National Laboratories near Rome. About the same time, Earle came from MIT to visit University of Rome and Frascati. G.P. was a young post-graduate, who had studied Earle's papers and was awed by his already impressive scientific figure. After almost 40 years had passed, Earle visited Italy with his wife Ruth, making Frascati their base for an extended visit of almost a month. They were housed in what was then the laboratory hostel for foreign visitors, a small villa higher up above the hill, toward the town of Frascati. Since then, we became close friends, a friendship which included both his family and ours, and which has been very important for us. In memory of that first visit and in gratitude for the many years of friendship, we will tell here a story of infrared radiative corrections to charged particle scattering, to which Earle's papers gave an important contribution.

I. THE INFRARED CATASTROPHE

In late 1940's, one of the problems waiting to be solved in the newly emerging discipline of Quantum Electro Dynamics was that of the infrared catastrophe, namely the apparent divergence of quasi-elastic electron scattering appearing when calculated order by order in perturbation theory. It had been known that this difficulty was arising from the neglect of more than one soft photon emission [4]. In 1949, Schwinger examined quasi-elastic electron scattering and showed that the divergence arising from the emission of a real photon in the limit of its energy going to zero, is cancelled in the cross-section by a similarly divergent term arising from virtual photon absorption and emission [5]. Schwinger calculated that emission up to a maximum resolution energy ΔE reduces the measured cross-section by a factor $\delta(\Delta E)$, namely

$$\sigma_{quasi-el} = \sigma_{el-theor}[1-\delta(\Delta E)] \tag{1}$$

This corresponds to the well known feature that the emission of radiation reduces the cross-section of any scattering process among charged particles.

The problem however was not resolved by Eq. (1), since $\lim_{\Delta E\to 0}\delta(\Delta E)=\infty$ and the cross-section would become negative. To avoid this highly unpleasant occurrence to show up in the calculated cross-section, it was found necessary to add the contribution of more and more soft photons. Schwinger put forward the ansatz that the single soft photon contribution should be exponentiated, i.e.

$$\sigma_{quasi-el} \to \sigma_{el-theor} e^{-\delta(\Delta E)} \tag{2}$$

The cancellation between real photons, for which $k_\mu k^\mu = 0$ and virtual photons for which $k_\mu k^\mu \neq 0$ points to the physical fact that the two types of photons are indistinguishable in the zero energy-momentum limit, when the apparent divergence in photon emission arises. Brown and Feynman [6] noticed that real and virtual emissions are physically related through the uncertainty principle: when a measurement is taken in a given small time interval, the uncertainty introduced in the photon energy would allow the virtual photon to be detected as a real one.

In [1, 2], based on [7], the measured cross-section had been written as

$$\sigma(\Delta E, E, \theta) = b(\Delta E, \varepsilon, C)\sigma_n(\varepsilon, E, \theta) + \mathscr{O}(\varepsilon/(E-m)) + \mathscr{O}[\alpha \ln(\varepsilon/E)]^{n+1}] \tag{3}$$

with the infrared correction factor $b(\Delta E)$, showing the explicit cancellation in the exponentiated photon spectrum, given as [7, 8]

$$b(\Delta E,\varepsilon,C) = \frac{1}{2\pi}\int_o^{\Delta E} d\omega' \int_{-\infty}^{+\infty} dt\; e^{-i\omega' t - h(\varepsilon;t)} \tag{4}$$

$$= \frac{1}{2\pi}\int_o^{\Delta E} d\omega' \int_{-\infty}^{+\infty} dt\; e^{-i\omega' t - \alpha C\int_o^{\varepsilon}\frac{d\omega}{\omega}[1-e^{i\omega t}]} \tag{5}$$

where the factor $C \equiv C(E,E')$ is a function of the incoming and outgoing electron momenta. In the exponential at the r.h.s. of Eq. (5), the cancellation between real and virtual soft photons is evident, with the first term in the square bracket coming from virtual photons, while the $e^{i\omega t}$ coming from the summation of real soft photons, each one of energy $\omega \le \varepsilon$. The problem of the upper integration limit in the exponential, at the right hand side of Eq. (5), was discussed by Lomon [1]. He was later inspired by Yennie and Suura's work [9] to write in closed form the expression for the b-factor which modifies the theoretical cross-section as [2]

$$b(\Delta E;\varepsilon;C) = (\frac{2}{\pi})\Big(cos\frac{\alpha C\pi}{2}\Big)\Big(\frac{\Delta E}{\varepsilon\gamma}\Big)^{\alpha C} \times I;$$

$$I = \int_o^{\infty}(\frac{d\sigma}{\sigma^{1+\alpha C}})\; sin\sigma\; exp\Big((\alpha C)\; Ci[\frac{\varepsilon}{\Delta E}\sigma]\Big);$$

γ is the Euler constant.

This equation shows the well known power law for the energy dependence of the correction factor. The complete formulation of the problem - in all four dimensions - was given in 1961 by Yennie, Frautschi and Suura (YFS) [10]. In the following, we shall show how to obtain Lomon's expression, with the explicit power law and normalization factor, as well as the subsequent 4-dimensional formulation by YFS, through the semi-classical method developed by Touschek to calculate infrared radiative corrections to electron positron experiments [11]. We shall also discuss Touschek's method in the context of a special kind of Abelian gauge theories, and under what conditions it could be extended to the presently still important problem of soft gluon re-summation in QCD.

II. TOUSCHEK RESUMMATION PROCEDURE

In November 1960, Bruno Touschek prepared a *memo* for his colleagues at the University of Rome and Frascati Laboratories, in which he proposed the construction of an electron-positron storage ring of c.m. energy $\sqrt{s} = 3.0$ GeV, the highest energy then under any sort of planning. It was a neat number, an energy chosen so as to include production of a pair of all the particles known to exist at the time, starting from two γ's up to a $p\bar{p}$ or $n\bar{n}$. Ten months before, the scientific staff of the Frascati Laboratories had approved the construction of a smaller lower energy collider, AdA, Anello di Accumulazione in Italian, storage ring in English. By November of the same year 1960, AdA was on the way to start functioning, the first of such type of machines in the world. Encouraged by this success, Touschek went on to propose building a real collider, one able to produce interesting physics, being of higher energy and higher luminosity, one which he called ADONE, a better AdA.

Both AdA and ADONE were electron (against positron) machines, but with a major difference: at ADONE's energy, electromagnetic radiation emission called for important radiative corrections. In 1963, he prepared an internal Laboratory note [12], reviewing the status of the contemporary scientific literature on the subject and laying the basis for the subsequent more complete work of [11]. In tribute to Lomon's work, the note says :'The theoretical background for the idea here discussed can be found in the works by Jauch, Rohlich and Lomon', and then again 'Applying the ideas of Lomon to the problem of administering the radiative corrections to the work with ADONE . . . ".

To calculate the correction factor to the measured $e^+e^- \to A\bar{A}$ cross-section, Touschek used the Bloch and Nordsieck result about soft photon emission from a classical source [4]. Bloch and Nordsieck showed that the distribution in the number of photons was given by a Poisson distribution, namely

$$P(\{n_{\mathbf{k}}\}) = \Pi_{\mathbf{k}}\frac{\bar{n}_{\mathbf{k}}^{n_{\mathbf{k}}}}{n_{\mathbf{k}}!}\exp[-\bar{\mathrm{n}}_{\mathbf{k}}] \tag{6}$$

with $n_{\mathbf{k}}$ and $\bar{n}_{\mathbf{k}}$ the number of photons emitted with momentum $\mathbf{k}$ and their average value. We notice that Eq. (6) describes a discrete momentum spectrum of the emitted photons, corresponding to quantization of the electromagnetic field in a finite box. In the following, we shall first assume that a smooth continuum limit exists. Later, in Sect. III, we shall discuss possible subtleties with the continuum limit.

Letting K^μ be the overall four-momentum loss due to photon emission, one can sum on all values of all the number of emitted photons $n_{\mathbf{k}}$, and write the probability $P(K)$ as

$$d^4P(K) = \sum P(\{n_{\mathbf{k}}\})\delta_4(\sum_{\mathbf{k}} k n_{\mathbf{k}} - K)d^4K \tag{7}$$

where the four-dimensional δ-function imposes overall energy momentum conservation and allows to exchange the sum with the product in Eq. (7). One can then write

$$d^4P(K) = \frac{d^4K}{(2\pi)^4}\int d^4x \exp[\mathrm{iK}\cdot\mathrm{x} - \sum_{\mathbf{k}} \bar{\mathrm{n}}_{\mathbf{k}}(1-\mathrm{e}^{-\mathrm{ik}\cdot\mathrm{x}})] \tag{8}$$

In this formulation, an important property of the integrand in Eq. (8) is that by its definition $d^4P(K) \neq 0$ only for $K_0 = \omega \geq 0$.

If one takes the continuum limit, integrating over the three momentum $\mathbf{K}$ leads to the probability of finding an energy loss in the interval $d\omega$ as

$$dP(\omega) = \frac{d\omega}{2\pi}\int_{-\infty}^{+\infty} dt \exp[\mathrm{i}\omega\,\mathrm{t} - \beta\int_{\mathrm{o}}^{\varepsilon}\frac{\mathrm{dk}}{\mathrm{k}}(1-\mathrm{e}^{-\mathrm{ikt}})] = \frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{+\infty}\mathrm{dt}\exp[\mathrm{i}\omega\,\mathrm{t} - \mathrm{h(t)}] \tag{9}$$

where β is a function of the incoming and outgoing particle momenta, and ε an energy scale valid for single photon emission, to be determined to the order of precision in the perturbation treatment of the process under examination. In [11] the function β was shown to be a relativistic invariant, and its expression in terms of the Mandelstam variables s,t,u can be found in [13].

We postpone the implication of taking this continuum limit to Sect.III, and pass to evaluate Eq. (9) in closed form for $\varepsilon/\omega > 1$. Following the steps taken from Eq. [11] through Eq. [17] of [11], the analiticity properties of $h(t)$ in the lower half of the t-plane lead to

$$N(\beta)dP(\omega) = \beta\frac{d\omega}{\omega}(\frac{\omega}{\varepsilon})^{\beta} \quad for\ \omega < \varepsilon \tag{10}$$

with the normalization factor given by

$$N(\beta) = \frac{\int_o^\infty dP(\omega)}{\int_o^\varepsilon dP(\omega)} = \gamma^\beta\Gamma(1+\beta) \tag{11}$$

which one obtains following the procedure outlined in Appendix III of [11] and which corresponds to the results in [1, 2, 14]. This approach is based on a separation of soft from hard processes in the observed cross-section. The application of Touschek's method to the scattering amplitudes led to the coherent state approach to infrared effects proposed by Greco and Rossi [15].

The question of the scale ε in resummation procedures acquires a particular relevance when the matrix element of the hard scattering process has a strong dependence on energy. In such case a straightforward separation of the soft photon factor from the cross-section cannot be done. Lomon considered this possibility in [1], when he proposed the use of an expression such as

$$M = \int \mathscr{B}(K)Q(p-K)dK \tag{12}$$

where $\mathscr{B}(K)$ is the matrix element for the soft radiation component with total momentum K. Not so relevant phenomenologically in 1955 (when Lomon submitted his paper, from the Institut of Theoretish Fysik in Copenhagen), the extension of the radiative correction calculation with Eq. (12) to processes in which a narrow resonance is produced, was considered in [12, 16] and became essential in 1974 with the discovery [17–19] of a very narrow resonance, to be called J/Ψ, a bound state of a new type of quarks, the charm [20]. In [21] the application of the methods inspired by Lomon's expression of Eq. (12) led to what was, at the time, the most precise determination of the J/Ψ width. In this paper, it was found that for very narrow resonances the scale which controls radiative effects is not the experimental resolution ΔE [22], but, most importantly, the resonance width $\Gamma \leq \Delta E$.

III. THE ZERO MOMENTUM MODE OF ABELIAN GAUGE FIELDS

In this section we return to Eq. (8) and discuss the separation of the zero momentum mode from the continuum in Abelian gauge theories in the presence of different boundary conditions [23].

Up to Eq. (8), the method developed to obtain the energy-momentum distribution K_μ is a classical statistical mechanics exercise. Going further requires to input an expression for the average number of photons of momentum $\mathbf{k}$ and the choice of the boundary conditions imposed upon the field. To take the continuum limit, let the quantization volume be $V = L^3$, and introduce μ, a fictitious photon mass. We must eventually take the limit $L \to \infty$ and $\mu \to 0$. Separating the zero momentum mode of energy ω_0 from all the other modes, we write

$$h(t) = n_0(t)[1-e^{-i\omega_0 t}] + \bar{h}(t) = n_0(t)[1-e^{-i\omega_0 t}] + \beta\int_0^E \frac{dk}{k}[1-e^{-ikt}] \tag{13}$$

with the photon mass now safely taken to be zero in the integral and

$$\beta = \frac{\alpha}{(2\pi)^2}\int d^2\mathbf{n}\sum_{\hat{e}}|\sum_i \frac{(p_i\cdot\hat{e})\varepsilon_i}{(\mathbf{p}_i\cdot\hat{n}-p_{0i})}|^2 \tag{14}$$

where p_i and $\hat{e}$ are the 4-momenta and polarization of the incoming and outgoing particles, $\varepsilon_i = \pm 1$, for incoming particles or antiparticles; E the maximum single photon energy characterizing the process, to be determined through the inclusion of higher order corrections and the theoretical precision required for the calculation.

For the zero mode, the $\mu \to 0$ limit is more delicate. One has

$$n_0(t)[1-e^{-i\omega_0 t}] \approx iW_0 t \tag{15}$$

with $W_0 = \frac{2\pi e^2}{L^3\mu^2}|\sum_i \varepsilon_i \mathbf{v_i}|^2$. In general, one can then write

$$d\mathscr{P}(K_0) = \frac{1}{N(\beta)}\beta\frac{dK_0}{2\pi}\int dt e^{i(K_0-W_0)t-\bar{h}(t)} = \frac{1}{N(\beta)}\beta\frac{dK_0}{(K_0-W_0)}(\frac{K_0-W_0}{E})^{\beta}\Theta(K_0-W_0) \tag{16}$$

If one takes first the limit $L\to\infty$, $W_0 = 0$ and the zero mode gives zero contribution, as in the case of vanishing boundary conditions. The case $W_0 \neq 0$ on the other hand might be present in theories with a different infrared regularization scheme. One cannot exclude the zero mode to be relevant in the discussion of the still unknown infrared behaviour of QCD, or in cosmology.

IV. CONCLUSIONS AND ACKNOWLEGMENTS

We have shown how Earle Lomon's work of the 1950's about infrared radiative corrections [1, 2] took the way of Frascati, where ADONE, a 3 GeV c.m. electron-positron collider, was being built. The need to 'administer' such corrections in order to extract meaningful physics from future experimental measurements, was keenly felt by Bruno Touschek, who had proposed and built the first electron-positron collider, AdA [24]. Through the 1960's, Earle's work influenced Bruno Touschek to develop a method for infrared photon resummation, a legacy which Touschek passed on to the young theorists of the Frascati theory group, and which, along the years, can still be found in many extensions to QCD [25, 26].

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