The magical file to quickly get a solution to a problem encountered before without searching the Internet.

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Aliasing

Example :

$$x_a(t) = 3\cos(600\pi t) + 2\cos(1800\pi t)$$

- Sampling Frequency : 1000 Hz
- Aliasing : Sampling frequency $\leq 2F$

Frequencies in Resulting Discrete-time signal : - $f_1 = \frac{F_1}{F_s} = \frac{300}{1000} = 0.3$ cycles/sample - $f_2 = \frac{F_2}{F_s} = \frac{900}{1000} = 0.9$ cycles/sample

f_2 appearing as $f_2^{\text{alias}} = 0.9 - 1 = -0.1$ cycles/sample

- $f_2 = \frac{F_2}{F_s} = \frac{900}{1000} = 0.9$ cycles/sample

Power and Energy of Signal

Energy

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Power

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

For $x(n) = A\cos(2\pi f n + \phi)$:

$$P = \frac{A^2}{2}$$

Impulse Response

1. System Equation
2. Impulse Response
3. Characteristic Equation from Poles
4. General Form
5. Impulse Response from General Form

Example :

$$y(n) = 0.25y(n-1) + \frac{1}{8}y(n-2) + x(n) - x(n-1)$$

$$\begin{aligned} \text{Impulse Response : } & h(n) = 0.25h(n-1) + \frac{1}{8}h(n-2) + \delta(n) - \delta(n-1) \\ \text{Characteristic Equation : } & \lambda^2 - 0.25\lambda - 0.125 = 0 \quad \lambda_1 = 0.5, \lambda_2 = -0.25 \\ \text{General Form : } & h(n) = c_1(0.5)^n + c_2(-0.25)^n \text{ for n } \geq 1 \end{aligned}$$

From Impulse Response :

$$h(0) = 1, h(1) = -0.75, h(2) = -0.0625$$

From General Form :

$n = 1 : 0.5c_1 - 0.25c_2 = -0.75$

$n = 2 : 0.25c_1 - 0.0625c_2 = -0.0625$

$c_1 = -\frac{2}{3}, c_2 = \frac{5}{3}$

Impulse Response from General Form :

$$h(n) = (-2/3)(0.5)^n + (5/3)(-0.25)^n \text{ for } n \geq 0$$

Steady-state response

Example :

$$x(n) = 2\cos(0.1\pi n)u(n)$$
$$H(z) = \frac{1 - 0.5z^{-1}}{1 - \frac{1}{12}z^{-1} + \frac{1}{12}z^{-2}}$$

1. Frequency Response. $H(jw) = H(z)|_{z \equiv e^{jw}} = 1.0332 \times e^{0.358j}$
2. Fundamental Frequency. $w_0 = 0.1\pi$
3. Substitute Fundamental Frequency into Frequency Response.
4. Steady-state Response. $y(n) = 2 \times 1.0332 \times \cos(0.1\pi n + 0.358)u(n)$

Zero-state Step Response

The step response is the system's output when the input is $u(n)$.

Zero-input Response

- Poles of the system.

$$y_{zi} = [A(p_1)^n + B(p_2)^n]u(n)$$

Stability of a System

Determine if the poles of the system are within the unit circle.

Initials and Final Values of Causal Signal

- Signal is causal

Initial Value

$$\lim_{z \to \infty} z^{-1} = 0$$

Final Value

$$\lim_{z \to 1} z^{-1} = 1$$

Statistical Calculations

Mean

$$\mu = E[X] = \int_{-\infty}^{\infty} xp(x)dx$$

Mean Square

$$\mu = E[X^2] = \int_{-\infty}^{\infty} x^2p(x)dx$$

Variance

$$Var[X] = E[X^2] - \mu^2$$

Standard Deviation

$$\sigma = \sqrt{Var[X]}$$

Quantisation Noise

$$\Delta = \frac{x_{max} - x_{min}}{L}$$

Variance of Quantisation Noise Signal Power is also equal to Quantisation Noise variance.

$$P_{signal} = \delta^2 = \Delta/12$$

Signal to Quantisation-Noise Ratio

$$SQNR = \frac{P_{signal}}{P_{noise}} = L^2$$

Autocorrelation Sequence

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

With number sequences :

`x(n) = {1, 2, 1, 1}`

- Do multiplication with

```
      1 2 1 1
      1 1 2 1 \\ Inverted x(n)
-------------
      1 2 1 1
    2 4 2 2 0
  1 2 1 1 0 0
1 2 1 1 0 0 0
-------------
1 3 5 7 5 3 1
```

Communication System

Information Source -> Input Transducer -> Transmitter -> Channel -> Receiver -> Output Transducer -> Sink

Source Encoder -> Channel Encoder -> Modulator

Demodulator -> Channel Decoder -> Source Decoder

Instantaneous Frequency

Example :

$$x(t) = 2\cos(2\pi 10t + 10\pi t^2)$$

Instantaneous Frequency :

$$f(t) = \frac{1}{2\pi}\frac{d(2\pi 10t + 10\pi t^2)}{dt} = 10 + 10t$$

VCO Instantaneous Frequency :

$$f(t) = f_c + k_f m(t)$$

AM Modulation

Amplitude of AM Signal :

$$A_{max} = A_C(1+\mu)$$
$$A_{min} = A_C(1-\mu)$$
$$A_m = A_C \times \mu$$

Example :

$$x_c(t) = A_c[1 + \mu x(t)]\cos(w_c t)$$

Assuming that $\mu < 1$.

- Transmitted Power $= P_t = \langle x_c^2(t) \rangle$
- Power in Message Signal $= P_x = \langle x^2(t) \rangle$

Find an expression for transmitted power in terms of message power and modulation index.

Substitute expression for transmitted power :

$$P_t = \frac{1}{2}A_c^2\langle 1 + 2\mu x(t) + \mu^2 x^2(t)\rangle + \frac{1}{2}A_c^2\langle[1+\mu x(t)]^2\cos(2w_c t)$$

- Time average of a constant is the same.
- Twice the carrier frequency component was removed
- Message Signal has no DC Component $\langle x(t) \rangle = 0$

$$P_t = \frac{1}{2}A_c^2(1 + \mu^2 P_x) = \frac{1}{2}A_c^2 + \frac{1}{2}A_c^2\mu^2 P_x$$

- First term is unmodulated carrier power $\mu = 0$
- Second term is power in carrier plus the signal
- Significant percentage of power resides in the carrier and conveys no information.

FM Modulation

FM Modulation Index

$$\mu_f = \frac{A_m k_f}{f_m}$$

- $A_m k_f \leftarrow$ Maximum Frequency Deviation , f_{max}
- Narrowband FM, $\mu_f << 1$

Example :

VCO has sensitivity constant of $3kHz/V$ and free running frequency of $1MHz$. Input to VCO is tone signal with peak amplitude equal to $2V$ and frequency $4kHz$. Output signal is FM. Calculate modulation index of the output FM Signal.

$$
\begin{aligned}
f_c &= 1 \times 10^6 \quad k_f = 3 \times 10^3 \quad f(t) = f_c + k_f m(t) \\
m(t) &= A_m \cos(2\pi f_m t) = 2\cos(2\pi \times 4000t) \\
f_{max} &= k_f max\{|m(t)|\} = 3 \times 10^3 \times 2 = 6 \times 10^3 Hz \\
\mu_f &= \frac{6kHz}{4KHz} = 1.5
\end{aligned}
$$

Narrowband FM Magnitude Spectrum :

$$e(t) \approx A_c \cos(w_c t) - A_c \mu_f \sin(w_m t)\sin(w_c t)$$

$$e(t) \approx A_c \cos(w_c t) - \frac{A_c \mu_f}{2}\cos(w_c - w_m)t + \frac{A_c \mu_f}{2}\cos(w_c + w_m)t$$

Example :

Draw a double-sided magnitude spectrum for the Narrowband FM signal.

$$e(t) = 10\cos[2\pi 10^7 t + 0.2\cos(2\pi 10^4 t)]$$

- $\mu_f = 0.2$

$$e(t) = A_c \cos(2\pi 10^7 t+) - A_c \mu_f \cos(2\pi 10^4 t)\sin(2\pi 10^7 t+)$$

- Carrier Signal has amplitude of 5 and two side-bands which are message signals with amplitude of 0.5

Wideband FM

$$
\begin{aligned}
e(t) = & A_c J_n(\mu_f)\cos(w_c t) \\
& + \sum_{n\text{even}}^{\infty} A_c J_n(\mu_f)[\cos(w_c + nw_m)t + \cos(w_c - nw_m)t] \\
& + \sum_{n\text{odd}}^{\infty} A_c J_n(\mu_f)[\cos(w_c + nw_m)t - \cos(w_c - nw_m)t]
\end{aligned}
$$

- FM spectrum has carrier frequency line and **infinite** number of side-band lines at $f_c \pm nf_m$.
- Lines equally spaced by f_m.
- Odd-order lower side-band lines reversed in phase.
- $J_n(\mu_f)$ obtained from the Bessel Function Values table.

Modulated Signal Example :

Carrier given by $c(t) = 10\cos(2pif_c t)$. Message Signal $\cos(20\pi t)$. The message is used to frequency modulate the carrier with $k_f = 50$. Find expression for modulated signal, determine how many harmonics selected to contain 99% of modulated power.

From message signal :

$$f_m = 10Hz$$

Modulated Signal :

$$
\begin{aligned}
e(t) &= A\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} \cos(20\pi\tau)d\tau \\
&= 10\cos(2\pi f_c t + \frac{100\pi}{20\pi}\sin(20\pi t)) \\
&= 10\cos(2\pi f_c t + 5\sin(20\pi t)
\end{aligned}
$$

Modulation index :

$$\mu_f = 5$$

FM Modulated Signal :

$$e(t) = 10\cos(2\pi f_c t + 5\sin(20\pi t)$$

Total Power :

$$P = \frac{A^2}{2} = 100/2 = 50$$

Spectral lines are at $f_c \pm 10n$ - How many of the lines contain 99% of the total power.

General FM Signal Spectrum :

$$
\begin{aligned}
e(t) = & A_c J_n(\mu_f)\cos(w_c t) \\
& + \sum_{n\text{even}}^{\infty} A_c J_n(\mu_f)[\cos(w_c + nw_m)t + \cos(w_c - nw_m)t] \\
& + \sum_{n\text{odd}}^{\infty} A_c J_n(\mu_f)[\cos(w_c + nw_m)t - \cos(w_c - nw_m)t]
\end{aligned}
$$

- $A_c = 10$ and $\mu_f = 5$
- Power in carrier line $\frac{A_c J_0(\mu_f))^2}{2} = \frac{100 J_0^2(5)}{2}$

Value of k such that at least 99% of power is covered :

$$\sum_{n=-k}^{k} \frac{100 J_0^2(5)}{2} \geq 0.99P$$

- $k = 6$
- Significant spectral lines at $f_c \pm 10k$, $k = 1, ..., 6$
- Effective bandwidth $= 120Hz$ ($60Hz$ on both sides of carrier)

Bandwidth of FM Signals

- Bandwidth of FM signal depends on f_m and μ_f

Carson's Rule : 98% of total power is contained in bandwidth.

$$B_c = 2(\mu_f + 1)W$$

- Modulation Index, μ_f
- Bandwidth of message signal, W

Example :

$$m(t) = 10\text{sinc}\ (10^4 t)$$

Determine transition bandwidth of FM modulated signal with $k_f = 4000$.

$$m(t) = \frac{10}{10^4 t}\sin(10^4 \pi t)$$

- Sinc function can be converted to sine function.
- Bandwidth , $W = 5000Hz$

$$f_{max} = k_f max\{|m(t)|\} = 4000 \times 10 = 40kHz$$

$$\mu_f = \frac{f_{max}}{W} = \frac{40k}{5k} = 8$$

$$B_c = 2(8+1)5k = 90kHz$$

Information Measure

Entropy Example

```
Source = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

P(1)=P(6)=P(7)=P(10)=1/16

P(2)=P(3)=P(4)=P(5)=P(8)=P(9)=2/16
```

$$\begin{aligned} Entropy &= 4 \times -\frac{1}{16}\log_2\frac{1}{16} + 6 \times -\frac{2}{16}\log_2\frac{2}{16} \\ &= 3.25 bits \end{aligned}$$

Source Coding Example

```
% Source has alphabet = {A B C D}

P(A)=0.25,  P(B)=0.5,  P(C)=0.125,  P(D)= 0.125
```

Encoder 1 : - Encode 4 elements into 2 bits

Element	Bit
A	00
B	01
C	10
D	11

Encoder 2 : - Account Probabilities

Element	Bit
A	10
B	0
C	110

Element	Bit
D	111

Source Rate

$$R = \frac{H}{T}$$

- H, Entropy
- T, Time

Noiseless Channel Capacity

$$C = 2B \log_2 M$$

$$C = B \log_2 (1 + \frac{S}{N})$$

- B, Bandwidth
- M, Level Signalling
- C, Channel Capacity

Signal-to-Noise Ratio (SNR)

$$SNR = \frac{P_x}{P_n}$$

$$SNR_{db} = 10 \log_{10} SNR$$

Quantisation

Uniform Quantisation Step Size :

$$\Delta = \frac{(x_{max} - x_{min})}{L}$$

- $L = 2^n$, Number of discrete levels

$$Eq(l) = |x - q_l| \qquad q_l = x_{min} + \Delta/2 + l \times \Delta$$

Accuracy :

$$\text{Accuracy} = \frac{E}{D}$$

Maximum Quantisation Error :

$$E = \frac{\Delta}{2}$$

Bit Rate :

$$R = f_s \times n$$

μ-Law Warping/Unwarping Sample Compressing (Warping) + Uniform Quantisation + Sample Expanding (Unwarping)

μ-Law Warping :

$$F(x) = x_{max} \frac{ln(1 + \mu[\frac{|x|}{x_{max}}])}{ln(1 + \mu)} \cdot sgn(x)$$

μ-Law Unwarping :

$$\hat{x} = F^{-1}(F(x)) = \frac{x_{max}}{\mu}[(1 + \mu)^{\frac{|F(x)|}{x_{max}}} - 1] \cdot sgn(F(x))$$

Encoding

Type of Encoding	Characteristics
Unipolar	Only 1s and 0s
NRZ-L	Positive voltage mean 0 Negative voltage mean 1
NRZ-I	Transition when next bit is 1
RZ	Positive, Negative and 0 Change to 0 and Positive for 1 and Negative for 0
Manchester	Low -> High represents 1 High -> Low represents 0
Differential Manchester	Transition at beginning means 0 No transition means 1

Raised Cosine Pulse Shaping

Bandwidth :

$$B_{RC} = \frac{1 - \alpha}{2T}$$

Efficiency :

$$E_{RC} = \frac{2}{1 - \alpha}$$

Pulse Shaping

Type of Pulse Shaping	Characteristics
Nyquist	Problems :- Time duration long- Ideal Synchronisation Required
Raised Cosine	Reduce Intersymbol Interference (ISI)

Matched Filtering

Matched Filter :

$$H(f) = K \frac{S^*(f)}{P_n(f)} e^{-j2\pi f t_0}$$

Impulse Response :

$$h(t) = Cs(t_0 - t)$$

- Known signal played backward and translated by t_0
- Autocorrelation function maximum when perfectly aligned

Power Spectral Density of White Noise :

$$P_n(f) = \frac{N_0}{2}$$

Probability of bit error :

$$P_e < \frac{e^{-E_b/N_0}}{2\sqrt{\pi E_b/N_0}}$$

- E_b, Transmitted signal energy per bit
- N_0, Noise spectral density
- $\frac{E_b}{N_0}$ proportional to SNR

Equalisation

- Equalising Filter to minimise ISI
- Two types :
  - Zero Forcing Equaliser
  - MMSE Equaliser

Digital Modulation

$$a\cos(2\pi f_c t + \theta)$$

Binary Shift Keyring	Characteristics
Frequency Shift Keyring	Carrier Frequency of signal changed to represent 1,0
Amplitude Shift Keyring	Amplitude of signal changed to represent 1,0
Phase Shift Keyring	Phase of signal changed to represent 1,0

M-ary Digital Modulation	Characteristics
Quadrature PSK	Four phased used, represented by dibit
M-ary Quadrature Amplitude Modulation	Combines ASK and PSK
Differential PSK	Encode information in phase difference