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Fuzzy Control Systems: Past, Present and Future

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Abstract

More than 40 years after fuzzy logic control appeared as an effective tool to deal with complex processes, the research on fuzzy control systems has constantly evolved. Mamdani fuzzy control was originally introduced as a model-free control approach based on expert's experience and knowledge. Due to the lack of a systematic framework to study Mamdani fuzzy systems, we have witnessed growing interest in fuzzy model-based approaches with Takagi-Sugeno fuzzy systems and singleton-type fuzzy systems (also called piecewise multiaffine systems) over the past decades. This paper reviews the key features of the three above types of fuzzy systems. Through these features, we point out the historical rationale for each type of fuzzy systems and its current research mainstreams. However, the focus is put on fuzzy model-based approaches developed via Lyapunov stability theorem and linear matrix inequality (LMI) formulations. Finally, our personal viewpoint on the perspectives and challenges of the future fuzzy control research is discussed.

I. Introduction

Fuzzy control was initiated by Mamdani [1] in 1974 stimulated by the Zadeh's two seminal papers on fuzzy algorithms [2] in 1968 and linguistic analysis [3] in 1973. In these papers, Zadeh presented a method of system modeling based on fuzzy IF-THEN rules with linguistic variables. The first application of fuzzy logic control was performed by Mamdani and Assilian on a laboratory steam engine [4] which led to a great impact on the fuzzy control research. Indeed, many fuzzy control systems have been proposed since the publication of the original paper in 1975. Generally speaking, there are three types of fuzzy systems as classified by Sugeno in [5] according to the consequent parts of IF-THEN rules. First, Mamdani-type fuzzy systems [4] are defined by IF-THEN rules associated with linguistic variables as

$$\begin{aligned} \text{Rule } R_i: \quad & \text{IF } \; x_1 \text{ is } \mathcal{G}_1^i, \; x_2 \text{ is } \mathcal{G}_2^i, \ldots, x_m \text{ is } \mathcal{G}_m^i \\ & \text{THEN } \; y \text{ is } \mathcal{H}^i, \; i = 1, 2, \ldots, n \end{aligned}$$

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where $\mathcal{H}^i$ and G^i_j, $j = 1, 2, \ldots, m$, are fuzzy sets, n is the number of fuzzy rules. Second, Takagi-Sugeno (T-S) fuzzy systems [6] are with functional consequents

$$\begin{aligned} \text{Rule} \quad R_i: \quad & \text{IF} \quad x_1 \text{ is } \mathcal{G}^i_1, \; x_2 \text{ is } \mathcal{G}^i_2, \ldots, x_m \text{ is } \mathcal{G}^i_m \\ & \text{THEN} \quad y \text{ is } f^i(x_1, x_2, \ldots, x_m), \; i = 1, 2, \ldots, n \end{aligned}$$

The function $f^i(\cdot)$ is usually linear as follows:

$$f^i(x_1, x_2, \ldots, x_m) = b^i + a^i_1 x_1 + a^i_2 x_2 + \cdots + a^i_m x_m,$$

where the coefficients b^i, a^i_j, with $j = 1, \ldots, m$, are constants. Third, singleton-type fuzzy systems [5] are with fuzzy rules of the following form:

$$\begin{aligned} \text{Rule} \quad R_i: \quad & \text{IF} \quad x_1 \text{ is } \mathcal{G}^i_1, \; x_2 \text{ is } \mathcal{G}^i_2, \ldots, x_m \text{ is } \mathcal{G}^i_m \\ & \text{THEN} \quad y \text{ is } b^i, \; i = 1, 2, \ldots, n \end{aligned}$$

where b^i is a singleton, *i.e.*, a real number. A singleton-type fuzzy system is recently called piecewise multiaffine (PMA) system in [7] since its input-output relation with respect to an affine-in-control system is found to be a multiaffine function. This type of fuzzy systems is used to be called PB (piecewise bilinear, more precisely *biaffine*) systems because for a two-dimensional case, the output can be expressed as $y = ax_1 + bx_2 + cx_1x_2 + d$, for some scalars a, b, c and d, see [5], [8].

Mamdani-type fuzzy systems are linguistically understandable since fuzzy variables are used in both the premises and the consequents. However, T-S fuzzy systems do not have linguistic variables since only functional membership functions (MFs) are used without labels. PMA systems are a simplified special case of both previous types of fuzzy systems. Indeed, PMA systems are Mamdani fuzzy systems with singleton consequents and they can be obtained from T-S fuzzy systems when only the constant terms in the consequents are present. While Mamdani fuzzy systems are deeply concerned with fuzzy set and logic, T-S fuzzy systems are only concerned with fuzzy set in their premises. PMA systems stand independently of fuzzy set and logic since there is no need to use neither linguistic variables nor MFs. In the case of PMA systems, these functions in the premises are of a triangular shape and only play roles as parameters for interpolation which are not necessarily interpreted as membership functions in the conventional "fuzzy" sense. However, PMA systems derived from Mamdani may keep their status in fuzzy control and linguistic labels could be assigned to singletons if necessary. Note that all three types of fuzzy systems are known to have general approximation capability for any nonlinear functions [9]. However, compared to two other types, T-S fuzzy modeling can drastically reduce the number of fuzzy rules, especially for high dimensional complex systems [5].

During the first ten years since Mamdani's successful application of fuzzy logic control, researchers were faced with a lot of criticisms from the conventional control theorists as reported in [5], [10], [11]. The main reason is that there was no stability analysis available for fuzzy control at that time [12]. To answer these criticisms, T-S fuzzy systems were newly introduced in 1985. Since T-S fuzzy systems are associated with linear models in their consequents, model-based stability analysis and control design can be performed for fuzzy systems using conventional Lyapunov-based approaches as firstly shown in [13]. T-S fuzzy modeling can be used to represent exactly a nonlinear system in a compact set of the state space, where the nonlinearities are embedded in membership functions [14]. Therefore, it is generally impossible to make a non-conservative stability analysis of a given nonlinear system with T-S fuzzy model-based approaches. More clearly, T-S fuzzy systems belong to a class of polytopic uncertain systems whose uncertainties are caused by the nonlinearity of the MFs. As such, T-S fuzzy model-based stability analysis always remains conservative [15]. PMA models were then presented to overcome this drawback of T-S fuzzy models. As shown in [5], PMA models are fully parametric like linear systems, though they are approximate models of nonlinear systems in their nature. Because of this advantage, it is theoretically possible to derive *necessary and sufficient* stability conditions for PMA systems just as in the case of linear systems [16]. This cannot be the case of other types of fuzzy systems. Since T-S fuzzy systems and PMA systems include linear systems as a special case, it is expected that any theoretical framework developed for these both types of fuzzy systems would be more general than its linear counterpart.

The intention of this paper is to provide a concise overview on the fuzzy control research since the pioneering works performed by Mamdani's group at Queen Mary College. Here, no attempt is made to comprehensively review the literature which includes several excellent books [14], [17]–[20] and thousands of technical articles, see for instance [5], [10], [12], [21]–[23] and references therein. Instead, only a selective and exemplary list of some breakthrough results is given to tracing back the evolution of fuzzy control systems. Our primary aim is to provide a historical rationale for each type of fuzzy systems and its current research status without excessive mathematical complexity. The emphasis is put more on the fuzzy model-based stability analysis than model-free fuzzy one. In particular, for the sake of simplicity and illustration, only approaches based on Lyapunov stability theory for continuous-time dynamical systems without any specific performance issue are focused in the paper. However, it is stressed that theoretical results on stability analysis and control design with or without various performance specifications, for instance $\mathcal{H}_\infty$ and $\mathcal{H}_2$ with respect to external disturbances, robustness with respect to time delay or modeling uncertainty, etc., and their discrete-time counterparts have also been widely reported in the fuzzy control literature [14], [22]. Finally, we present our viewpoint on the perspectives and challenges of the future fuzzy control research.

Notation: Ω_N denotes the set $\{1, 2, \dots, N\}$. $\mathbb{R}$ is the field of real numbers. For a vector $x \in \mathbb{R}^n$ and $i \in \Omega_n$, x_i denotes the ith entry of x. I denotes the identity matrix of appropriate dimension. For a matrix X, $X^\top$ indicates its transpose and $X_{(ij)}$ denotes its element of the ith row and jth column. For any square matrix X, $X > 0$ indicates a symmetric positive definite matrix, and $\mathrm{He}X = X + X^\top$. The symbol $\star$ stands for matrix blocks that can be deduced by symmetry. The time dependency of the variables is omitted when convenient.

II. Mamdani Fuzzy Systems

The basic idea of Mamdani fuzzy control (MFC) is to represent the process states by means of *linguistic* variables and to exploit these variables as inputs to control rules [18]. Hence, MFC enables to incorporate the expert's skills and experience through a set of fuzzy If-Then rules [17], [19]. Thanks to this particular feature, the effectiveness of MFC has been clearly proved in the following situations [12]. First, no acceptable mathematical model is available for the controlled plant. Second, human operators play a crucial role in the control process and can provide qualitative control rules in terms of fuzzy logic sentences. Moreover, as pointed out by Zadeh, MFC is *task-oriented* control which is in contrast to *set-point-oriented* feature of conventional control approaches [24]. This allows MFC to achieve easily multi-objective goals by simply setting some fuzzy control rules under one criterion and others under a different performance criterion. The coordination between different control objectives can be performed by fuzzy reasoning. For these reasons, until now MFC has been successfully applied to a large number of industrial processes, see [12], [23], [25], [26] for constructive surveys on prominent applications of MFC.

Unfortunately, except for a few exceptions, the design of MFC remains model-free and essentially heuristic. As a direct consequence, we still lack at present a systematic framework as well as analytical tools to study rigorously the stability of Mamdani fuzzy systems [22]. Establishing such a theoretical stability framework for Mamdani fuzzy systems is expected to be particularly challenging due to the novelty of the fuzzy mathematics and language [7]. The mainstream idea for stability analysis has been to consider the Mamdani fuzzy controller as a nonlinear controller, then the fuzzy control design is recast as a nonlinear control approach using absolute stability theory, sliding mode control, adaptive fuzzy control, etc. Excellent reviews on the stability issues of MFC systems can be found, for instance, in [5], [12], [22].

To overcome the "model-free" major drawback of MFC, fuzzy model-based control approaches were originally proposed by Sugeno's research group at Tokyo Institute of Technology [6], [13]. Such control approaches enable a systematic framework to deal with the stability analysis and control design of nonlinear dynamical systems in the following general form:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control input. Without loss of generality, the nonlinear system (1) satisfies the following assumption.

Assumption 1. The origin $x_0 = 0 \in \mathbb{R}^n$ is an equilibrium of system (1) such that $f(0) = 0$. Moreover, the vector fields $f(\cdot)$ and $g(\cdot)$ are sufficiently smooth, *i.e.*, $f \in \mathcal{C}^2$ and $g \in \mathcal{C}^2$.

Fuzzy model-based approches to study theoretically the nonlinear system (1) are discussed in the subsequent sections.

III. Takagi-Sugeno Fuzzy Systems

After a brief description of T-S fuzzy models, this section reviews some key points on the stability analysis and control design of this type of fuzzy systems.

A. System Description

The T-S fuzzy model of the continuous-time nonlinear system (1) is described in the following form [14]:

$$\begin{aligned} \text{Rule } R_i: \quad & \text{IF} \;\; z_1(t) \text{ is } \mathcal{M}_1^i \text{ and } \cdots \text{ and } z_p(t) \text{ is } \mathcal{M}_p^i \\ & \text{THEN} \;\; \dot{x}(t) = A_i x(t) + B_i u(t) \end{aligned} \tag{2}$$

where R_i denotes the ith fuzzy inference rule, r is the number of inference rules, $\mathcal{M}_j^i$, with $i \in \Omega_r$ and $j \in \Omega_p$, are the fuzzy sets, and (A_i, B_i) the state-space matrices of appropriate dimensions of the ith local model. The vector of premise variables is defined as $z(t) = [z_1(t) \cdots z_p(t)]$. Using the center-of-gravity method for defuzzification, the T-S fuzzy model (2) can be represented in the following compact form:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z)(A_i x(t) + B_i u(t)) \tag{3}$$

where the normalized MF $h_i(z)$ is defined as

$$h_i(z) = \frac{\omega_i(z)}{\sum_{i=1}^{r} \omega_i(z)}, \quad \omega_i(z) = \prod_{j=1}^{p} \mu_j^i(z_j), \quad i \in \Omega_r.$$

The grades of membership of the premise variables in the respective fuzzy sets $\mathcal{M}_j^i$ are given as $\mu_j^i(z_j)$. Note that the normalized MFs satisfy the following convex sum property:

$$0 \le h_i(z) \le 1, \quad \sum_{i=1}^{r} h_i(z) = 1, \quad \sum_{i=1}^{r} \dot{h}_i(z) = 0 \tag{4}$$

Remark 1. Although the premise variables can represent any type of variables in nonlinear systems, most of the works in the T-S fuzzy literature deal with the case for which the premise variables are composed of a subset of the system state $x(t)$. This assumption is also adopted along this paper.

Remark 2. Using the sector nonlinearity approach in [14], the T-S fuzzy model (3) can be directly derived from (1). In this case, both representations of the affine nonlinear system are strictly *equivalent* in a compact set of the state space.

B. Lyapunov-Based Stability Analysis

Consider the T-S fuzzy system (3) with $u(t) \equiv 0$ as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z) A_i x(t) \tag{5}$$

The first results on the stability analysis were proposed in [27] for the T-S fuzzy system (5), and in [13] for its discrete-time counterpart. Consider a quadratic Lyapunov function (QLF) of the form

$$V(x) = x^\top P x, \quad P > 0 \tag{6}$$

The following result is readily obtained.

Theorem 1. [27] The equilibrium of the T-S fuzzy system (5) is globally asymptotically stable if there exists a common positive definite matrix P such that

$$A_i^\top P + P A_i < 0, \quad i \in \Omega_r \tag{7}$$

The three following remarks deserve particular attention.

- Conditions (7) are expressed in terms of linear matrix inequalities (LMIs). Hence, the stability analysis can be easily checked with available numerical solvers [28].
- A common Lyapunov matrix variable P has to exist for all local linear subsystems.
- The MFs of the T-S fuzzy system (5), *i.e.*, its nonlinearities, are considered as *uncertainty*. Then, the stability analysis is embedded in the conventional robust control theory.

Concerning the first remark, the possibility to reformulate the stability analysis of T-S fuzzy systems as a convex optimization problem has undeniably sparked the growing interest in this type of fuzzy systems [14]. This enables systematic and effective frameworks for stability analysis and control design of general nonlinear systems, see detailed discussions on prominent results in the 90s and in the early of 2010s in [11], [22]. Another very recent survey on this research topic can be found in [29].

The two other remarks are concerned with the *conservativeness* issue of T-S fuzzy control theory [15]. Currently, most of the research effort has been focused on this issue. There are two mainstreams to reduce the conservatism of T-S fuzzy model-based approaches, which are briefly discussed hereafter. The first one is based on the choice of different families of Lyapunov function candidates eventually combined with slack variables introduced via robust control tools such as Finsler lemma, $\mathcal{S}-$procedure, etc. [28]. The second mainstream consists in finding ways to exploit more efficiently the information on the MFs for stability analysis.

1) Relaxations with Different Choices of Lyapunov Functions: The conservatism of the results due to the use of a single quadratic Lyapunov function (6) was emphasized in [30], [31]. To overcome this drawback, more general classes of Lyapunov functions have been suggested in the T-S fuzzy literature, including piecewise Lyapunov functions (PLFs) [30], [32], [33], fuzzy Lyapunov functions (FLFs) depending on the MFs [31], [34], [35], line integral Lyapunov functions (LILFs) [34], [36], [37], polynomial Lyapunov functions depending on the MFs with arbitrary degree [38]–[40], multidimensional fuzzy Lyapunov functions [41], [42], and so on.

An interesting remark when using FLFs for stability analysis of the continuous-time T-S fuzzy system (5) is that the stability conditions depend on the time-derivatives of the MFs, thus the derivatives of the system state. This implies more numerical and theoretical complexities and requires a much larger effort to obtain convex formulations than a quadratic Lyapunov based framework. To overcome this difficulty, different alternatives have been proposed to consider the upper bounds of the time-derivatives of the MFs. However, this usually leads to a local analysis setting [41]–[44]. Another alternative is to make use of PLFs or LILFs to avoid the presence of the time-derivatives of the MFs in the stability conditions. However, in these cases some special structures should be imposed on the Lyapunov matrices which introduce some conservatism. In addition, using PLFs is only suitable for T-S fuzzy systems with triangular or trapezoidal MFs inducing particular state-space partitions [22], [30], [32]. Moreover, LILFs-based approaches usually require that the premise variables are the system state, *i.e.*, $z \equiv x$. Nevertheless, this assumption can be recently avoided in [37]. Applying PLFs for the stability analysis of T-S fuzzy systems was already discussed in detail in [22]. Hence, only the two other cases, namely FLFs and LILFs, are given below to illustrate the above discussion.

Consider a simple example of FLFs depending explicitly on MFs as follows [31]:

$$V(x) = x^\top \left(\sum_{i=1}^{r} h_i(z) P_i \right) x, \quad P_i > 0 \tag{8}$$

whose time-derivative is given by

$$\dot{V}(x) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}^\top \begin{bmatrix} \sum_{i=1}^{r} \dot{h}_i(z) P_i & \star \\ \sum_{i=1}^{r} h_i(z) P_i & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

Remark 3. When the positive definite matrices are imposed as $P_1 = \cdots = P_r = P$, then the QLF (6) is straightforwardly recovered from (8). Therefore, the QLFs are only a special case of the FLFs. Observe also that the terms $\dot{h}_i(z)$, $i \in \Omega_r$, appear *explicitly* in the expression of $\dot{V}(x)$. This makes the stability analysis much more involved with this choice. It should be stressed that although the fuzzy Lyapunov function (8) belongs to a more general class of Lyapunov function candidates compared to the quadratic one, there is no guarantee that FLFs-based results are less conservative than those derived from QLFs, see [45] for a counterexample. This is primarily due to the presence of *a priori* unknown time-derivatives of the MFs in the theoretical developments when using FLFs [35]. Hence, the stability conservatism strongly depends on the way how such *unknown* time-derivatives are handled.

For simplicity, one could exploit the ideas related to properties of the MFs to deal with their time-derivatives. For instance, from the convex sum property (4), it follows that $\sum_{i=1}^{r} \dot{h}_i(z)X = 0$, for any matrix X. This implies that

$$\sum_{i=1}^{r} \dot{h}_i(z)P_i = \sum_{i=1}^{r} \dot{h}_i(z)(X + P_i).$$

Assume there exist upper bounds of the time-derivatives of the MFs such that $|\dot{h}_i(z)| \leq \phi_i$, for some positive scalars ϕ_i, $i \in \Omega_r$. Then, it follows that

$$x^\top \left(\sum_{i=1}^{r} \dot{h}_i(z)(P_i + X) \right) x \leq x^\top P_\phi x \tag{9}$$

where $P_\phi = \sum_{i=1}^{r} \phi_i(P_i + X)$. Using a null-term approach or Finsler lemma [28], inequality (9) can be exploited for stability analysis as summarized in the following theorem.

Theorem 2. [34] Given a T-S fuzzy system (5), and upper bounds on the time-derivatives of the MFs as $|\dot{h}_i(z)| \leq \phi_i$, $i \in \Omega_r$. If there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, with $i \in \Omega_r$, and matrices $X \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$, satisfying the following linear matrix inequalities:

$$P_i + X > 0, \quad i \in \Omega_r \tag{10}$$

$$\begin{bmatrix} P_\phi - MA_i - A_i^\top M^\top & \star \\ P_i + M^\top - NA_i & N + N^\top \end{bmatrix} < 0, \quad i \in \Omega_r \tag{11}$$

with P_ϕ given by (9). Then, the T-S fuzzy system (5) is asymptotically stable.

Remark 4. Differently from Theorem 1, Theorem 2 only allows for *local* stability analysis of system (5) since the system state has to satisfy $|\dot{h}_i(z)| \leq \phi_i$, $i \in \Omega_r$. Hence, an implicit goal is also to maximize the domain of attraction included inside the state region defined by the above assumption [41]. Note that exploiting different types of properties of fuzzy MFs and relaxing tools from robust control theory, several local stability conditions have been proposed in the FLFs literature [44]. It is important to note also that unlike the continuous-time case, the stability analysis for discrete-time T-S fuzzy systems does not suffer from this kind of difficulty in dealing with the time-derivatives of MFs when using FLFs [22], [46].

Remark 5. In Theorem 2, matrices X, M, N, are considered as slack variables, used for relaxation purposes. By imposing $X = 0$, $P_i = P > 0$, for $\forall i \in \Omega_r$, and $M = -P$, it is easy to prove that the result of Theorem 2 includes that of Theorem 1. This theoretically confirms that compared to quadratic Lyapunov functions, FLFs enable less conservative stability analysis, see also Remark 3.

For T-S fuzzy systems with $z_k = x_k$, $k \in \Omega_n$, *i.e.*, the premise variables are *explicitly* the state variables, line integral Lyapunov functions can be exploited to avoid dealing with the time-derivatives of the MFs [36]. Consider now a Lyapunov function of the following form:

$$V(x) = 2\int_{\Gamma[0,x]} \left\langle \sum_{i=1}^{r} h_i(\zeta)P_i\,\zeta, d\zeta \right\rangle \tag{12}$$

where $\Gamma[0, x]$ is a path from the origin to the present state, $d\zeta$ is an infinitesimal displacement, $\langle\cdot,\cdot\rangle$ denotes the inner product. As shown in [36], if the matrices P_i are written as

$$P_i = D_0 + D_i \tag{13}$$

where

$$D_0 = \begin{bmatrix} 0 & d_{12} & \cdots & d_{1n} \\ d_{12} & 0 & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1n} & d_{2n} & \cdots & 0 \end{bmatrix},$$

$$D_i = \begin{bmatrix} d_{11}^{\alpha_{i1}} & 0 & \cdots & 0 \\ 0 & d_{22}^{\alpha_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^{\alpha_{in}} \end{bmatrix},$$

and α_{ik} indicating which $\mathcal{M}_k$ fuzzy set is used for the ith rule. Then, it follows from (12) and (13) that

$$\nabla V(x) = 2\sum_{i=1}^{r} h_i(x) P_i x.$$

Exploiting this special structure of the Lyapunov function, the following theorem can be stated.

Theorem 3. [34] Given a T-S fuzzy system (5) with $z \equiv x$. If there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, with $i \in \Omega_r$, and matrices $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$ such that

$$P_i > 0, \quad i \in \Omega_r \tag{14}$$

$$\begin{bmatrix} -MA_i - A_i^\top M^\top & \star \\ P_i + M^\top - NA_i & N + N^\top \end{bmatrix} < 0, \quad i \in \Omega_r \tag{15}$$

with P_i having the structure in (13). Then, the T-S fuzzy system (5) is globally asymptotically stable.

Remark 6. A great advantage of Theorem 3 compared to Theorem 2 is that the assumption on the upper bounds of the time-derivatives of the MFs is not required anymore for LMI-based stability analysis. Such an assumption is not always verified, especially in T-S fuzzy control context [35]. However, the special structure of the matrices P_i as shown in (13) may induce some conservatism. In fact, up to now it is still hard to get a definitive answer on which approach leads to less conservative stability conditions.

2) Exploiting the Knowledge of the Membership Functions for Stability Relaxations: The MFs used to "blend" the local linear submodels of the T-S fuzzy system (3) represent the *nonlinearity* of system (1). However, these MFs have been widely considered as system *uncertainty*, and only their convex sum property (4) has been exploited in most of the existing works based on quadratic, fuzzy and polynomial Lyapunov functions [15]. Whenever the shape of the MFs and its intrinsic time-varying characteristic are not *explicitly* taken into account in the stability analysis, the conservativeness issue still remains, see further details in [29], [47]–[49].

Several approaches have been proposed to consider *explicitly* the shape of the MFs in the stability analysis. These approaches can be classified into two following categories [50]. First, the membership-function-approximation approaches exploit the MF information via alternative similar functions such as staircase MFs [51], piecewise linear MFs [52]. Second, the membership-bound-dependent approaches exploit the bound information of MFs for stability analysis [48]. In addition, the MFs image space and the order relations among the MFs have been also exploited, see for instance [50], [53], [54]. Note that membership-function-approximation and image-space approaches generally lead to a higher number of convex stability constraints. Concerning membership-bound-dependent approaches, slack variables are usually introduced into the stability conditions for relaxation purposes. As an illustrative example, the following theorem presents the idea in [48] on using the MFs shape information to reduce further the stability conservatism of Theorem 3 by introducing new relaxation variables. More discussions on the membership-function-dependent stability analysis can be found in the recent survey [29].

Theorem 4. (adapted from [48] and [34]) Given a T-S fuzzy system (5) with $z \equiv x$. Consider the MFs vector $h(x) = \begin{bmatrix} h_1(x) & h_2(x) & \dots & h_r(x) \end{bmatrix}^\top$ such that

$$h(x)^\top S h(x) + h(x)^\top w + v \leq 0 \tag{16}$$

where $S \in \mathbb{R}^{r\times r}$, $w \in \mathbb{R}^r$ and $v \in \mathbb{R}$ are given. If there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, $i \in \Omega_r$, and matrices $Z \in \mathbb{R}^{2n\times 2n}$, $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$ such that

$$Z > 0, \quad P_i > 0, \quad i \in \Omega_r \tag{17}$$

$$Q_i - \beta_{ii} Z < 0, \quad i \in \Omega_r \tag{18}$$

$$Q_i + Q_j - (\beta_{ij} + \beta_{ji})Z < 0, \quad i,j \in \Omega_r, \quad j > i \tag{19}$$

with P_i having the structure in (13), $\beta_{ij} = S_{(ij)} + w_i + v$ and

$$Q_i = \begin{bmatrix} -MA_i - A_i^\top M^\top & \star \\ P_i + M^\top - NA_i & N + N^\top \end{bmatrix}.$$

Then, the T-S fuzzy system (5) is globally asymptotically stable.

Remark 7. The key difference between Theorems 3 and 4 lies on the extra decision variable Z. Via this slack variable, the shape information of the MFs (16), also represented by the scalars β_{ij}, can be exploited to reduce the stability conservatism in Theorem 4. This shape information can be used to deal with relatively complex MF shapes, *i.e.*, with some minor modifications, multiple constraints on the MF shapes are easily handled. However, it usually requires a preliminary optimization step to find the tightest constraint verifying (16) that could be applied to the MFs of the studied system.

Besides the two above mainstreams, research efforts have been also devoted to reduce the stability conservatism caused by the *sufficiency* of fuzzy summations, *i.e.*, the ways how MFs are dropped out to obtain a *finite* set of LMI conditions. Although numerous results have been proposed to deal with this source of conservativeness [55]–[61], the most prominent approach relies on Pólya's theorem. Based on checking the positiveness of multidimensional matrices, the authors in [58] applied Pólya's theorem to derive *asymptotically necessary and sufficient* LMI-based conditions for the stability and performance of T-S fuzzy systems. It should be stressed that as the homogeneous degree of a multiple summation fuzzy Lyapunov function increases, the conservatism of the stability conditions decreases thanks to the introduction of more degrees of freedom. However, Pólya's theorem based approaches are conceptual rather than implementable since the computational burden swiftly increases in a way that most numerical solvers crash.

C. Takagi-Sugeno Fuzzy Control Design

An important application of stability theorems is to design stabilizing fuzzy controllers based on T-S fuzzy models of nonlinear plants. The control design of T-S fuzzy models are usually addressed with the following steps: (1) choose a specific form of the control law, (2) find the respective closed-loop T-S fuzzy representation, (3) apply a set of stability analysis conditions to the closed-loop representation, and (4) transform these conditions into LMI-based formulations. The challenge usually lies on the last step since without any transformation at this step, the design conditions are expressed in terms of bilinear matrix inequalities (BMIs) instead of LMIs, leading to numerical difficulties.

Different (state or output) feedback control schemes can be applied to T-S fuzzy models [14], [22], [46]. The most commonly used control law is based on the so-called parallel distributed compensation (PDC) concept, for which the fuzzy controller shares the same fuzzy rules and sets as the T-S fuzzy model. As a result, a PDC controller is obtained from a convex blending of the linear local feedback gains and the MFs of the T-S fuzzy model. Within LMI-based control framework, this control approach was originally proposed in [62] and was named "PDC" in [55].

For illustration, consider a PDC control law of the form

$$u(t) = \sum_{j=1}^{r} h_j(z) K_j x(t) \tag{20}$$

Then, the closed-loop T-S fuzzy system can be obtained from (3) and (20) as follows:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\left(A_i + B_i K_j\right)x(t) \tag{21}$$

The basic result is based on the quadratic Lyapunov function (6). For the closed-loop stability analysis, its time-derivative along the trajectory of system (21) is required to be negative, *i.e.*, $\dot{V}(x) = \dot{x}^\top P x + x^\top P \dot{x} < 0$, which results in

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\mathrm{He}\left[(A_i + B_i K_j)^\top P\right] < 0 \tag{22}$$

Note that (22) is expressed in terms of BMI due to the coupling between the Lyapunov matrix P and the feedback gains K_j, $j \in \Omega_r$. However, by applying a simple congruence transformation [28] to (22), with $X = P^{-1} > 0$, it is easy to show that if the following LMI constraint holds:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\mathrm{He}[A_i X + B_i Y_j] < 0 \tag{23}$$

Then, the PDC control law (20) asymptotically stabilizes the T-S fuzzy system (3) with $K_j = Y_j X^{-1}$, for $j \in \Omega_r$.

Remark 8. A direct consequence of sharing the same MFs for the PDC control and the T-S fuzzy system is that double convex sums appear, see for example (23). Similar to the stability analysis discussed above, several approaches are available to reduce the design conservatism at the expense of a higher computational cost. The key difference between stability analysis and control design is that the latter usually requires additional matrix transformations and/or matrix inversions of decision variables to derive LMI-based formulations. Apart from some special cases, this makes the use of some classes of Lyapunov functions with special structures on their parameter matrices, for instance LILFs and PLFs, very challenging for control design purposes.

Remark 9. Compared to LILFs and PLFs, FLFs as given in (8) can be applied more easily to the synthesis problem [35], [42], [63]. However, in this case, for T-S fuzzy systems (5) with $z \equiv x$, the time-derivatives of the MFs involve the system dynamics, *i.e.*, $\dot{h}_i(x) = \left(\frac{\partial h_i}{\partial x}\right)^\top \dot{x}$, which can include the control action $u(t)$ to be designed. Hence, it is not obvious to check *a posteriori* the validity region of the designed controller [35]. This is the biggest gap between stability analysis and control design when using fuzzy Lyapunov functions.

Remark 10. It should be stressed that various alternative control schemes with different degrees of design conservatism have been proposed in the T-S fuzzy literature, including non-PDC control [35], [46], [64], [65], fuzzy observer-based control with measurable premises [57], [66]–[68], with unmeasurable premises [69]–[72], fuzzy static output feedback control [73]–[77], etc., and their numerous extensions. Since full state-information is generally not available in practice, observers are usually designed to reconstruct the system state for state-feedback control. T-S fuzzy observer designs with or without performance specifications have been largely developed using Lyapunov stability arguments and LMI techniques, see for instance [22], [78]–[84] and related references. Note that in the framework of T-S fuzzy systems whose premises are state variables, two following cases are distinguished. If the premise variables are measurable, then the approaches in [57], [66], [67], [78], [79] can be directly applied. The second case, in which the premise variables are not measurable, is much more challenging. This is due to a mismatch between the T-S fuzzy systems and the corresponding T-S fuzzy observers. In this situation, it is necessary to estimate the premise variables as shown in [69]–[72], [83].

IV. Piecewise Multiaffine Systems

This section discusses the use of PMA modeling for theoretical studies of the nonlinear system (1).

A. Description of PMA Systems

To introduce the PMA modeling, let us first consider the nonlinear system (1) with $u \equiv 0$ for simplicity, which results in the following autonomous system:

$$\dot{x} = f(x) \tag{24}$$

Remark 11. It is important to stress that T-S fuzzy model-based approaches require rewriting system (24) in the form $\dot{x} = A(z)x$, where the premise vector z regroups all *nonlinear* terms involved in the matrix $A(\cdot)$. Up to now, no *systematic* procedure is available to select, among the *infinite* possibilities, the "best" parameterization $f(x) = A(z)x$ guaranteeing to achieve stability and other performance specifications. Therefore, the issue on the non-uniqueness of T-S fuzzy representation still remains open [15]. As shown below, we do not have this source of conservatism with PMA model-based approaches since the form (24) is directly dealt with. However, PMA modeling leads to appropriation errors. At the present stage of progress, taking into account such errors for stability analysis of the nonlinear system (24) still remains open in PMA model-based framework.

Without loss of generality, we assume that $\underline{x}_i \le x_i \le \overline{x}_i$, $i \in \Omega_n$, where $\underline{x}_i$ and $\overline{x}_i$ denote respectively the upper and lower bounds of the ith entry of x. As a consequence, the state x belongs to the set $\mathcal{R} = [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$. Let us partition the state-space of system (24) as follows:

$$\underline{x}_j = \chi_j^{[1]} < \chi_j^{[2]} < \ldots < \chi_j^{[N_j+1]} = \overline{x}_j, \quad j \in \Omega_n \tag{25}$$

Let $\mathcal{K}_v = \Omega_{N_1+1} \times \ldots \times \Omega_{N_n+1}$ be the set of multi-indexes corresponding to all the vertices induced by the partition (25) and $\mathcal{K}_r = \Omega_{N_1} \times \ldots \times \Omega_{N_n}$ the set of multi-indexes corresponding to the regions. For $i = (i_1, \ldots, i_n) \in \mathcal{K}_r$, the region $\left[\chi_1^{[i_1]}, \chi_1^{[i_1+1]}\right] \times \ldots \times \left[\chi_n^{[i_n]}, \chi_n^{[i_n+1]}\right]$ is denoted by $\mathcal{R}_i$ and $\mathcal{K}_i = \{i_1, \; i_1+1\} \times \ldots \times \{i_n, \; i_n+1\}$ is the set of multi-indexes corresponding to all vertices of $\mathcal{R}_i$. For $k \in \mathcal{K}_i$, χ_k is the vertex of $\mathcal{R}_i$ whose jth component is defined as $\chi_j^{[k_j]}$, for $j \in \Omega_n$. For a hyper-rectangle region $\mathcal{R}_i$, for $i \in \mathcal{K}_r$, we consider the following set of fuzzy rules:

$$\begin{aligned} &\text{IF} \quad x_1(t) \text{ is } \eta_1^{[k_1]}(x_1) \text{ and } \cdots \text{ and } x_n(t) \text{ is } \eta_n^{[k_n]}(x_n), \\ &\text{THEN} \quad \dot{x}(t) \text{ is } \mathcal{F}_k, \quad k \in \mathcal{K}_i, \end{aligned}$$

where $\mathcal{F}_k = f(\chi_k)$ is the singleton vector, and the triangular membership function $\eta_j^{[k_j]}(x_j)$ with respect to $x_j(t)$, for $j \in \Omega_n$ and $k_j \in \Omega_{N_j+1}$, is defined as

$$\eta_j^{[k_j]}(x_j) = \begin{cases} \dfrac{x_j - \chi_j^{[k_j-1]}}{\chi_j^{[k_j]} - \chi_j^{[k_j-1]}}, & \text{if } x_j \in \left[\chi_j^{[k_j-1]}, \chi_j^{[k_j]}\right] \\ & \text{and } j \ge 2 \\ \dfrac{\chi_j^{[k_j+1]} - x_j}{\chi_j^{[k_j+1]} - \chi_j^{[k_j]}}, & \text{if } x_j \in \left[\chi_j^{[k_j]}, \chi_j^{[k_j+1]}\right] \\ & \text{and } j \le N_j \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Note that the MFs (26) satisfy the following properties [5]:

$$\eta_j^{[k_j]}(x_j) \ge 0, \quad \sum_{k_j=i_j}^{i_j+1} \eta_j^{[k_j]}(x_j) = 1, \quad j \in \Omega_n.$$

Given $x(t)$ with these membership functions, $\dot{x}(t)$ can be inferred by taking the weighted average of $\mathcal{F}_k$ as follows:

$$\dot{x} = \sum_{k \in \mathcal{K}_i} \eta_k(x)\mathcal{F}_k, \quad \eta_k(x) = \prod_{j=1}^{n} \eta_j^{[k_j]}(x_j) \tag{27}$$

The following expression of $x(t)$ can be derived from that of the triangular MFs [5]:

$$x = \sum_{k \in \mathcal{K}_i} \eta_k(x)\chi_k \tag{28}$$

Based on (27) and (28), the parametric expression of PMA system on $\mathcal{R} = \bigcup_{i\in\mathcal{K}_r} \mathcal{R}_i$ can be expressed as follows [7]:

$$\dot{x} = \sum_{k\in\mathcal{K}_v} \eta_k(x)\mathcal{F}_k, \quad x = \sum_{k\in\mathcal{K}_v} \eta_k(x)\chi_k \tag{29}$$

For stability analysis, we assume that the equilibrium $x \equiv 0$ of system (29) corresponds to the vertex χ_{k_0} of the state-space partition, for a given $k_0 \in \mathcal{K}_v$. Let $\mathcal{K}_Z$ be the set of multi-indexes for regions containing the origin which is called *zero-regions*, and $\mathcal{K}_{NZ} = \mathcal{K}_r \backslash \mathcal{K}_Z$ (called *non-zero regions*). We denote also $\mathcal{K}_i^* = \mathcal{K}_i \backslash \{k_0\}$, for $i \in \mathcal{K}_Z$.

Remark 12. Observe in (29) that the weights with respect to x_j, $j \in \Omega_n$, in the premises are computed by the multiplication of $\eta_j^{[k_j]}(x_j)$, $k_j \in \Omega_{N_j+1}$. Therefore, the fuzzy reasoning used here is characterized by normalized membership functions, multiplicative weights calculation, and weighted average aggregation [5]. Note also that system (24) can be approximated by PMA model (29) with arbitrary accuracy on $\mathcal{R}$ by increasing the number of piecewise regions with (25).

Remark 13. PMA systems are characterized by the triangular MFs defined in (26). Other types of MFs for PMA systems, for instance Gaussian MFs or trapezoidal MFs, could be considered. However, the parametric expression (28) of $x(t)$ is crucial for the stability analysis of PMA systems, which is directly derived from triangular MFs. Moreover, note also that triangular MFs are largely employed in fuzzy control systems and in practice [85].

Consider a particular case with $n = 2$. Then, the resulting second-order PMA model (29) with the MFs (26) corresponds to the parametric expressions of the singleton-type fuzzy systems studied in [5], [16]. In this case, by eliminating one of the triangular MFs from the parametric expression (29), the following state-space representation of PMA systems can be readily obtained:

$$\begin{cases} \dot{x} = \mathcal{A}_i(x)x + \mu_i, \quad x \in \mathcal{R}_i, \ i \in \mathcal{K}_r \\ \mu_i = \displaystyle\sum_{k\in\mathcal{K}_i} \eta_k(0)\mathcal{F}_k \end{cases} \tag{30}$$

where $\mu_i = 0$, for $i \in \mathcal{K}_Z$, and $\eta_k(0)$ denotes the value of $\eta_k(x)$ for $x = 0$. The state-dependent matrix is of the form

$$\mathcal{A}_i(x) = \eta_1^{[k_1]}(x_1)\mathbf{S}(k_1,\cdot) + \eta_1^{[k_1+1]}(x_1)\mathbf{S}(k_1+1,\cdot) \tag{31}$$

for $k \in \mathcal{K}_i$ and $i \in \mathcal{K}_r$. The explicit expressions of $\mathbf{S}(k_1,\cdot)$ and $\mathbf{S}(k_1+1,\cdot)$, their derivations, and alternative equivalent state-space representations of (30)-(31) can be found in [5].

Remark 14. For second-order PMA systems, both representations (29) and (30) are strictly equivalent. Note also that system (30) is found to be a *biaffine* system. Moreover, with expression (31), the PMA system (30) can be viewed as a *piecewise polytopic affine system* with two vertex systems: $\dot{x} = \mathbf{S}(k_1,\cdot)x + \mu_i$ and $\dot{x} = \mathbf{S}(k_1+1,\cdot)x + \mu_i$.

B. Literature Review on PMA Systems

Many advantages of using PMA systems for control purposes were highlighted in [5]. First, a PMA model (29) is easily obtained from the mathematical expression (24) of a nonlinear system as shown above. Second, PMA models have a general approximation capacity of any smooth nonlinear systems. Third, PMA models are simply implemented by means of *look-up tables* (LUT) which are one of the most widely used practical tools in the industry for model approximation and control implementation, especially in automotive and aerospace engineering. Finally, PMA systems are fully parametric, *i.e.*, both the state vector and its rate can be expressed by parametric expressions in (29). As shown later, this enables a systematic framework to study PMA systems.

Despite these practical and theoretical advantages, up to 1999 there was no rigorous stability framework on PMA systems. Almost all papers had been focused on the practical applications of PMA systems rather than their theoretical studies [5]. By *fully* taking into account the information of the triangular MFs as functions of state variables, Sugeno first set the theoretical foundations on a quadratic Lyapunov stability framework for both continuous-time and discrete-time PMA systems [5]. These results were then improved in [16] to achieve *necessary and sufficient* stability conditions with respect to a common quadratic

Lyapunov function (6). The basic idea of the stability analysis in [5], [16] consists in using both parametric expressions (29) and the state-space representation (30) to analytically set $V(x) > 0$, $\dot{V}(x) < 0$, for $x \neq 0$, as the stability conditions for each piecewise region. Following the same line, Taniguchi and Sugeno proposed in [86] *necessary and sufficient* stability conditions with respect to a piecewise Lyapunov function of the form

$$V(x) = \frac{1}{2}x^\top P_i x + q_i^\top x + r_i, \quad i \in \mathcal{K}_r \tag{32}$$

where the explicit expressions of P_i, q_i and r_i, for $i \in \mathcal{K}_r$, can be found in [86]. In addition, the conditions guaranteeing that function (32) is a proper Lyapunov function candidate are also discussed therein. Note that the Lyapunov function (32) can be considered as a piecewise approximation of an *arbitrary* Lyapunov function by second-order functions. These preliminary results of Sugeno's group clearly show the great potential of PMA model-based approaches to get rid of the conservativeness issue of T-S fuzzy theory, for which MFs are usually regarded as *unknown* parameters [15].

In parallel, without considering the knowledge on the MFs, numerical approaches were developed for stability analysis and control design of PMA control systems in [8], [87]. Also based on Lyapunov stability arguments, there is however a *fundamental* difference compared to the results in [5], [16], [86]. That is, these works essentially focused on the characteristics of PMA controllers performed on T-S fuzzy objective systems. As such, the stability analysis of PMA control systems is embedded into the conventional T-S fuzzy theory, which is not compatible with the original motivation of PMA control systems. Indeed, the final goal in PMA control theory is to provide a stability analysis framework of nonlinear systems by embedding them in PMA systems to avoid the inherent drawbacks of T-S fuzzy control systems [5].

It should be stressed that all the above-mentioned results on PMA control systems suffer from two major drawbacks [7]. First, the proposed stability conditions are expressed in terms of nonlinear matrix inequalities, which obviously induce numerical difficulties. Second and more importantly, these conditions are only applicable to second-order PMA systems. Note that state-space representation (30) of PMA systems was *exclusively* employed to study PMA systems in [5], [8], [16], [86], [87]. However, such a representation is very challenging for theoretical study due to its *piecewise polytopic affine* feature [5], see also Remark 14. As a consequence, the extensions of these results to high-dimensional PMA systems are not obvious. Due to these restrictive drawbacks, PMA control systems have not yet received much attention from the (fuzzy) control community. Only very recently, it is demonstrated in [7] that the *specific* representation of PMA via parametric expressions (29) enables a simple and effective stability framework. The key idea relies on the fact that within a piecewise region, each of these expressions is *uniquely* determined by its vertex values and its restriction to the region is a *convex combination* of these values. Moreover, the piecewise feature of PMA systems is fully exploited via a piecewise Lyapunov function. This led to promising results on stability analysis of PMA systems of any order, and has opened new research perspectives as discussed in the subsequent sections.

C. LMI-Based Stability Analysis for General PMA Systems

Consider a piecewise quadratic Lyapunov function candidate parameterized as follows [30]:

$$V(x) = \begin{cases} x^\top P_i x, \text{ for } x \in \mathcal{R}_i,\ i \in \mathcal{K}_Z \\ \hat{x}^\top \hat{P}_j \hat{x}, \text{ for } x \in \mathcal{R}_j,\ j \in \mathcal{K}_{NZ} \end{cases} \tag{33}$$

where $\hat{x} = \begin{bmatrix} x^\top & 1 \end{bmatrix}^\top$ and

$$\begin{cases} P_i = M_i^\top T M_i, \text{ for } i \in \mathcal{K}_Z \\ \hat{P}_j = \hat{M}_j^\top T \hat{M}_j, \text{ for } j \in \mathcal{K}_{NZ} \end{cases} \tag{34}$$

The constraint matrices are constructed as follows [30]:

$$\hat{L}_i = \begin{bmatrix} L_i & l_i \end{bmatrix}, \quad \hat{M}_i = \begin{bmatrix} M_i & m_i \end{bmatrix} \tag{35}$$

such that

$$\hat{L}_i \hat{x} \geq 0, \quad x \in \mathcal{R}_i, \quad i \in \mathcal{K}_r,$$
$$\hat{M}_i \hat{x} = \hat{M}_j \hat{x}, \quad x \in \mathcal{R}_i \cap \mathcal{R}_j, \quad i,j \in \mathcal{K}_r,$$

where $l_i = 0$, $m_i = 0$, $i \in \mathcal{K}_Z$.

Remark 15. Since the Lyapunov function (33) combines the power of quadratic functions near an equilibrium point with the flexibility of piecewise functions in the large, it can lead to less conservative results compared to those based on a common quadratic Lyapunov function [7], [30]. Indeed, the latter can be regarded as a special case of (33) by considering $M_i = I$ and $L_i = 0$.

Note that all free parameters of the Lyapunov function $V(x)$ defined in (33) are collected in the symmetric matrix T and the expression of P_i is linear in T. This feature allows for an LMI-based framework to study the stability of PMA systems as shown in the following theorem.

Theorem 5. [7, Theorem 1] Given an PMA system in (29). If there exist a symmetric matrix $T \in \mathbb{R}^{N\times N}$, symmetric matrices with nonnegative entries $U_q \in \mathbb{R}^{2n\times 2n}$ and $W_q \in \mathbb{R}^{2n\times 2n}$ (for $q \in \mathcal{K}_r$), matrices $Y_{1i} \in \mathbb{R}^{n\times n}$, $Y_{2i} \in \mathbb{R}^{1\times n}$ (for $i \in \mathcal{K}_Z$), $\hat{Y}_{1j} \in \mathbb{R}^{(n+1)\times(n+1)}$ and $\hat{Y}_{2j} \in \mathbb{R}^{1\times(n+1)}$ (for $j \in \mathcal{K}_{NZ}$), satisfying the LMI constraints (36), (37), (38) and (39). Then, the PMA system (29) is asymptotically stable. Moreover, the piecewise quadratic function $V(x)$ defined in (33) is a Lyapunov function of this system.

$$P_i - L_i^\top U_i L_i > 0, \quad i \in \mathcal{K}_Z \tag{36}$$

$$\mathrm{He}\begin{bmatrix} Y_{1i} + L_i^\top W_i L_i/2 & -Y_{1i}\chi_k + P_i\mathcal{F}_k \\ Y_{2i} & -Y_{2i}\chi_k \end{bmatrix} < 0, \quad i \in \mathcal{K}_Z,\ k \in \mathcal{K}_i^* \tag{37}$$

$$\hat{P}_j - \hat{L}_j^\top U_j \hat{L}_j > 0, \quad j \in \mathcal{K}_{NZ} \tag{38}$$

$$\mathrm{He}\begin{bmatrix} \hat{Y}_{1j} + \hat{L}_j^\top W_j \hat{L}_j/2 & -\hat{Y}_{1j}\hat{\chi}_k + \hat{P}_j\hat{\mathcal{F}}_k \\ \hat{Y}_{2j} & -\hat{Y}_{2j}\hat{\chi}_k \end{bmatrix} < 0, \quad j \in \mathcal{K}_{NZ},\ k \in \mathcal{K}_j \tag{39}$$

with P_i and $\hat{P}_j$ defined in (34) and

$$\hat{\mathcal{F}}_k = \begin{bmatrix} \mathcal{F}_k \\ 0 \end{bmatrix}, \quad \hat{\chi}_k = \begin{bmatrix} \chi_k \\ 1 \end{bmatrix}, \quad k \in \mathcal{K}_j, \quad j \in \mathcal{K}_{NZ}.$$

Remark 16. LMI conditions (36) and (38) are imposed for each region to guarantee that (33) is a proper Lyapunov function candidate. The set of LMIs (37) and (39) ensures that the value of the Lyapunov function decreases along any trajectory of the PMA system (29).

Remark 17. For the stability results in Theorem 5, the parametric expressions of both x and $\dot{x}$ in (28) can be fully exploited via Finsler lemma [28], see [7] for details. Since expression (28) is *equivalent* to that of the triangular MFs (26), the information on the MFs can be thus easily taken into account in the theoretical studies of PMA systems to reduce the conservatism. Note that this is not the case of T-S fuzzy systems whose MFs are often considered as *uncertainty* [15].

D. Lyapunov-Based LUT Control Design

Following the same modeling procedure as in Section IV-A, the following PMA model of system (1) is straightforwardly obtained on $\mathcal{R}$:

$$\dot{x} = \sum_{k\in\mathcal{K}_v} \eta_k(x)(\mathcal{F}_k + \mathcal{G}_k u), \quad x = \sum_{k\in\mathcal{K}_v} \eta_k(x)\chi_k \tag{40}$$

where $\mathcal{G}_k = g(\chi_k)$. Assume that the linearized system (A, B) of (1) around $x = 0$ is stabilizable. This means that there exists a linear feedback gain H such that $(A + BH)$ is Hurwitz with

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0}, \quad B = \left[\frac{\partial g}{\partial x}\right]_{x=0} \tag{41}$$

This control gain H can be designed in advance by any linear control technique to guarantee some local closed-loop properties of (1). To stabilize the PMA system (40), let us consider the state-feedback control law of the form

$$u(x) = \sum_{k \in \mathcal{K}_v} \eta_k(x) \upsilon_k + Hx \tag{42}$$

where the control vertex at the origin must be assigned as $\upsilon_0 = 0$ and other input vertices υ_k, with $k \in \mathcal{K}_v \setminus \{0\}$, have to be designed.

Remark 18. The control law (42) is composed of two parts. The *parametric expression* part $\sum_{k \in \mathcal{K}_v} \eta_k(x) \upsilon_k$ is defined in the same way as x and $\dot{x}$, see (40). The incorporation of the linear feedback part Hx in (42) is obviously crucial to guarantee the closed-loop stability in the case where system (1) is open-loop unstable around $x = 0$. Note that (42) can be easily implemented with look-up-tables which is an outstanding feature for real-time control applications [5].

Using similar arguments as for Theorem 5, the following theorem provides sufficient conditions to design LUT controller (42) for nonlinear systems.

Theorem 6. Given an PMA system in (40) and a stabilizing feedback gain H of the linearized system (A, B) defined in (41). Assume there exist a symmetric matrix $T \in \mathbb{R}^{N \times N}$, symmetric matrices with nonnegative entries $U_q \in \mathbb{R}^{2n \times 2n}$ and $W_q \in \mathbb{R}^{2n \times 2n}$ (for $q \in \mathcal{K}_r$), matrices $Y_{1i} \in \mathbb{R}^{n \times n}$, $Y_{2i} \in \mathbb{R}^{m \times n}$, $Y_{3i} \in \mathbb{R}^{1 \times n}$ (for $i \in \mathcal{K}_Z$), $\hat{Y}_{1j} \in \mathbb{R}^{(n+1) \times (n+1)}$, $\hat{Y}_{2j} \in \mathbb{R}^{m \times (n+1)}$, $\hat{Y}_{3j} \in \mathbb{R}^{1 \times (n+1)}$ (for $j \in \mathcal{K}_{NZ}$), and the control input vertices $\upsilon_k \in \mathbb{R}^{m \times 1}$ (for $k \in \mathcal{K}_v \setminus \{0\}$), satisfying the following matrix inequalities (43), (44), (45) and (46). Then, the control law (42) asymptotically stabilizes the PMA system (40). Furthermore, the piecewise quadratic function (33) is a Lyapunov function of the closed-loop system.

$$P_i - L_i^\top U_i L_i > 0, \quad i \in \mathcal{K}_Z \tag{43}$$

$$\mathrm{He} \begin{bmatrix} \Gamma_i & P_i \mathcal{G}_q & -Y_{1i}\chi_k + P_i \mathcal{F}_k + H\upsilon_k \\ Y_{2i} + H & -I/2 & \upsilon_k - Y_{2i}\chi_k \\ Y_{3i} & 0 & -\upsilon_k^\top \upsilon_l/2 - Y_{3i}\chi_k \end{bmatrix} < 0, \quad i \in \mathcal{K}_Z,\ q \in \mathcal{K}_i,\ k \in \mathcal{K}_i^*,\ l \in \mathcal{K}_i^* \tag{44}$$

$$\hat{P}_j - \hat{L}_j^\top U_j \hat{L}_j > 0, \quad j \in \mathcal{K}_{NZ} \tag{45}$$

$$\mathrm{He} \begin{bmatrix} \hat{\Gamma}_j & \hat{P}_j \hat{\mathcal{G}}_k & -\hat{Y}_{1j}\hat{\chi}_k + \hat{P}_j \hat{\mathcal{F}}_k + \hat{H}\upsilon_k \\ \hat{Y}_{2j} + \hat{H} & -I/2 & \upsilon_k - \hat{Y}_{2j}\hat{\chi}_k \\ \hat{Y}_{3j} & 0 & -\upsilon_k^\top \upsilon_l/2 - \hat{Y}_{3j}\hat{\chi}_k \end{bmatrix} < 0, \quad j \in \mathcal{K}_{NZ},\ k \in \mathcal{K}_j,\ l \in \mathcal{K}_j \tag{46}$$

where $\hat{H} = \begin{bmatrix} H & 0 \end{bmatrix}$ and

$$\hat{\mathcal{G}}_k = \begin{bmatrix} \mathcal{G}_k \\ 0 \end{bmatrix}, \quad k \in \mathcal{K}_j, \quad j \in \mathcal{K}_{NZ},$$
$$\Gamma_i = Y_{1i} + \left(L_i^\top W_i L_i + H^\top H \right)/2, \quad i \in \mathcal{K}_Z,$$
$$\hat{\Gamma}_j = \hat{Y}_{1j} + \left(\hat{L}_j^\top W_j \hat{L}_j + \hat{H}^\top \hat{H} \right)/2, \quad j \in \mathcal{K}_{NZ}.$$

Remark 19. The slack matrices U_q and W_q, for $q \in \mathcal{K}_r$, are introduced into the conditions of Theorems 5 and 6 through the $\mathcal{S}-$procedure [28]. This contributes to reduce the conservatism since the piecewise-region feature of PMA systems can be fully exploited via the constraint matrices defined in (35).

Remark 20. The design conditions in Theorem 6 are expressed in terms of BMIs due to the product $\upsilon_k^\top \upsilon_l$ involved in (44) and (46). Note that for each vertex, there are 2^n associated BMI-based conditions to be verified. Hence, the value of υ_k, for $k \in \mathcal{K}_v \setminus \{k_0\}$, should be imposed identical for 2^n conditions concerning the same vertex to avoid sliding modes and chattering phenomena.

V. Future Perspectives of Fuzzy Control Systems

This paper provides a concise discussion on the evolution of fuzzy control systems. Through a selective list of references, we present the historical motivations and the current research progress of three types of fuzzy systems: Mamdani-type fuzzy systems, T-S fuzzy systems and PMA (or singleton-type fuzzy) systems. Great advances on both fundamental and application aspects of fuzzy control have been made with a huge number of available publications on the topic. However, many interesting and important issues still remain challenging, which provide fantastic opportunities for the research on fuzzy control systems in the future. Since we believe that within fuzzy control context, only fuzzy model-based approaches enable systematic frameworks for stability analysis and control design of nonlinear systems, below our personal perspectives are focused on T-S fuzzy systems and PMA systems.

A. Open Issues on Takagi-Sugeno Fuzzy Systems

A large research effort has been focused on reducing the conservativeness of stability analysis and control design. As discussed above, this has been done by searching for more general Lyapunov function candidates and clever manipulations to bring the problem to an LMI framework.

Although there are conditions considering the information on the shape of the MFs, see for instance [29], [47], we believe that more powerful methodologies could arise with the use of interval type-2 T-S fuzzy modeling techniques [88]–[91]. These fuzzy models allow to deal with uncertain grades of membership. Hence, type-2 T-S fuzzy systems are very useful in the cases where exact MFs are difficult to be chosen and/or there is a need to cope with large amounts of uncertainties [89], [90]. However, for stability analysis and control design, the lower and upper MFs have to be simultaneously considered leading to more elaborated manipulations and computational burden [92], [93]. Possible contributions go toward building less conservative LMI-based conditions for interval type-2 T-S fuzzy models depending on MFs, reducing the gap between type-1 and type-2 T-S fuzzy modeling methodologies. For instance, membership-dependent stability conditions for both type-1 and type-2 T-S fuzzy systems were recently proposed in [50]. To this end, it is considered that the MFs belong to a unified space. Then, an extrema-based method is proposed to construct a polyhedron convex hull to enclose the membership distribution in this space. Despite the interests of the proposed results, much research effort should be still devoted to the stabilization problem, and even to the stability analysis to reduce further the gap between both types of fuzzy systems.

In our opinion, the design of T-S fuzzy controllers/observers that do not share the same MFs or premise variables with T-S fuzzy models is another important topic. This topic should deserve more thorough consideration since it can greatly improve the flexibility of the control/observer design in many cases [69]. Also, such a design scheme is involved in current research mainstreams on T-S fuzzy model-based approaches as shown in the two following illustrative examples.

- The first example is concerned with the design of nonlinear networked control systems (NCSs) in which network-induced imperfections, for instance network-induced delay and sampling issues, are intrinsic [70], [94]. One of the major difficulties in this case is the fact that, when considering the network-induced delay, the information transmitted over the network to the controller is usually delayed, causing a mismatch between the premise variables. This mismatch can be seen as extra constraints in the design of NCSs [95]–[99]. Moreover, it can be very restrictive to assume that the premise variables of the fuzzy systems and the fuzzy controllers/observers are synchronous if bilateral networks (controller-to-actuator and sensor-to-controller) have to be designed [99]. Then, the asynchronous transmission issue should be considered.
- The second example is related to the fact that in any control system, fault detection, diagnosis and recovery is decisive to have a robust and resilient system operation. In this context, control reconfiguration plays a crucial role to obtain a fault tolerant control (FTC) that can effectively handle severe actuator/sensor faults while still guaranteeing a desired closed-loop performance [100]. To achieve this goal, it is crucial to have effective fault detection and isolation (FDI) schemes to support the control reconfiguration in a proper way [101]. One idea behind the control reconfiguration is to redesign the problem for the faulty system by choosing possibly a new structure and adapting the control law to this new scenario. A possible solution for this problem relies on the use of two-step design procedures for which FDI and FTC

schemes are separately dealt with. To this end, virtual actuators/sensors can be used to mask the actuator/sensor faults [102]. More specifically, an FDI scheme via T-S fuzzy virtual actuator/sensor models have to be designed to achieve the reconfiguration goal. This requires the design of a T-S fuzzy observer-based controller, in which the T-S fuzzy controller and the T-S fuzzy observer possibly do not share the same MFs or it is necessary to estimate their unmeasurable premise variables.

B. Open Issues on Piecewise Multiaffine Systems

Lyapunov-based studies of PMA systems are much more limited compared to those of T-S fuzzy systems. As mentioned previously, this is due to theoretical challenges when dealing with this type of fuzzy systems, especially for high-order systems. Although the outstanding contributions in [5], [7], [16] throw new light on the research of PMA control systems, several important issues need to be solved in the future.

- How to *fully* take into account the knowledge of triangular MFs to derive tractable *necessary and sufficient* stability conditions for high-order PMA systems? Two of the authors have provided a discussion on this issue in [16], [86]. However, the stability conditions are only applicable to second-order PMA systems and expressed in terms of nonlinear matrix inequalities.
- How to develop an effective framework to design LUT controllers from the PMA stability results? BMI-based design conditions in Theorem 6 induce numerical difficulties and may be conservative, especially for high-order systems. PMA systems can be also used for LUT-based control implementation of *feedback error learning* (FEL) scheme [103]. In FEL control, it is crucial to build an online pseudo-inverse model of the general nonlinear plant for *feedforward* control purposes. Up to now, neural networks have been intensively used for this aim. However, due to computational and overfitting reasons, PMA systems are considered as a powerful alternative to neural networks. Despite the practical effectiveness of LUT-based FEL control, more rigorous stability analysis is required in the future.
- How to extend the theoretical results concerning the "classical" PMA system (40) to a wider class of systems such as time-delay PMA systems, descriptor PMA systems, switching PMA systems and so on? The research on this issue depends obviously on the progress of the two previous ones.
- Whether does the stability of PMA system (40) imply that of the nonlinear system (1)? To solve this issue, the approximation errors should be characterized, and *explicitly* considered in the stability analysis as highlighted in [7].
- For a given nonlinear system, how to obtain its "best" PMA model in terms of facilitating both accurate modeling and effective stability/performance analysis, and reducing the numerical complexity (namely the number of piecewise regions)? The results in [104] may bring some interesting ideas to this open issue.

A part from the above open issues, many other challenges and perspectives concerning fuzzy control systems have been raised in numerous publications, for instance [10], [11], [22], [23] and references therein. To conclude the paper, we strongly believe that PMA model-based approaches could provide a promising future for fuzzy control, which would have great impacts on the control community. From the theoretical viewpoint, they allow overcoming the conservativeness issue of T-S fuzzy model-based approaches [7], [16]. From the application viewpoint, a systematic LUT-based control approach is of special interest for real-time applications [5]. Therefore, PMA based approaches could provide a viable answer to the issue: "Only *straightforward in theory* plus *straightforward in practice* are *great* solutions to change a discipline", recently raised in [11].

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