P232-4

求 $f(x) = \sin x$ 在 $[0, \frac{\pi}{2}]$ 上的一次最佳一致逼近多项式

解 设 $f(x)$ 的一次最佳一致逼近多项式为 $P_1(x) = a_1x + a_0$，$f(x)$ 在 $[0, \frac{\pi}{2}]$ 上有 *3* 个交错点 $0, x_2, \frac{\pi}{2}$，满足

$$\begin{aligned} f'(x_2) - P_1(x_2) &= 0 \\ f(0) - P_1(0) &= f(\frac{\pi}{2}) - P_1(\frac{\pi}{2}) \\ f(0) - P_1(0) &= f(x_2) - P_1(x_2) \end{aligned}$$

解得

$$a_0 = \frac{1}{2}\sqrt{1 - \frac{4}{\pi^2}} - \frac{1}{\pi}\arccos\frac{2}{\pi} \qquad a_1 = \frac{2}{\pi} \qquad x_2 = \arccos\frac{2}{\pi}$$

于是

$$\begin{aligned} P_1(x) &= \frac{2}{\pi}x + \frac{1}{2}\sqrt{1 - \frac{4}{\pi^2}} - \frac{1}{\pi}\arccos\frac{2}{\pi} \\ &= 0.6366x + 0.1053 \end{aligned}$$

P233-6

求 $f(x) = 2x^4 + 3x^3 - x^2 + 1$ 在 $[-1, 1]$ 上的三次最佳一致逼近多项式

解 设 $f(x)$ 的三次最佳一致逼近多项式为 $P_3(x)$，则

$$\begin{aligned} \frac{f(x) - P_3(x)}{2} &= \tilde{T}_4(x) = x^4 - x^2 + \frac{1}{8} \\ P_3(x) &= f(x) - 2\tilde{T}_4(x) \\ &= (2x^4 + 3x^3 - x^2 + 1) - 2(x^4 - x^2 + \frac{1}{8}) \\ &= 3x^3 + x^2 + \frac{3}{4} \end{aligned}$$

P233-9

求函数 $f(x)$ 在指定区间上关于 $\Phi(x) = \mathrm{span}\{1, x\}$ 的最佳平方逼近多项式

(3) $f(x) = \cos\pi x, x \in [0, 1]$

解 设 $f(x)$ 的最佳平方逼近多项式为 $P_1(x)$，取 $\rho(x) \equiv 1$，有

$$\begin{aligned} b_0 &= (1, f) = \int_0^1 1 \cdot \cos\pi x dx = 0 & b_1 &= (x, f) = \int_0^1 x \cdot \cos\pi x dx = -\frac{2}{\pi^2} \\ g_{0,0} &= (1, 1) = \int_0^1 1 \cdot 1 dx = 1 & g_{0,1} &= (1, x) = \int_0^1 1 \cdot x dx = \frac{1}{2} \\ g_{1,0} &= (x, 1) = \int_0^1 x \cdot 1 dx = \frac{1}{2} & g_{1,1} &= (x, x) = \int_0^1 x \cdot x dx = \frac{1}{3} \end{aligned}$$

由

$$\begin{pmatrix} g_{0,0} & g_{0,1} \\ g_{1,0} & g_{1,1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$$

解得

$$a_0 = \frac{12}{\pi^2} \qquad a_1 = -\frac{24}{\pi^2}$$

$$\begin{aligned} P_1(x) &= (-\frac{24}{\pi^2})x + (\frac{12}{\pi^2}) \\ &= -2.4317x + 1.2159 \end{aligned}$$

(4) $f(x) = \ln x, x \in [1, 2]$

解 设 $f(x)$ 的最佳平方逼近多项式为 $P_1(x)$，取 $\rho(x) \equiv 1$，有

$$\begin{aligned} b_0 &= (1, f) = \int_1^2 1 \cdot \ln x dx = \ln 4 - 1 & b_1 &= (x, f) = \int_1^2 x \cdot \ln x dx = \ln 4 - \frac{3}{4} \\ g_{0,0} &= (1, 1) = \int_1^2 1 \cdot 1 dx = 1 & g_{0,1} &= (1, x) = \int_1^2 1 \cdot x dx = \frac{3}{2} \\ g_{1,0} &= (x, 1) = \int_1^2 x \cdot 1 dx = \frac{3}{2} & g_{1,1} &= (x, x) = \int_1^2 x \cdot x dx = \frac{7}{3} \end{aligned}$$

由

$$\begin{pmatrix} g_{0,0} & g_{0,1} \\ g_{1,0} & g_{1,1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$$

解得

$$a_0 = 10 \ln 4 - \frac{29}{2} \qquad a_1 = 9 - 6 \ln 4$$

$$\begin{aligned} P_1(x) &= (9 - 6 \ln 4)x + (10 \ln 4 - \frac{29}{2}) \\ &= 0.6822x - 0.6371 \end{aligned}$$

P233-11

$f(x) = \sin \frac{\pi}{2} x$ 在 $[-1, 1]$ 上按 Legendre 多项式展开，求三次最佳平方逼近多项式

解 令 $\Phi(x) = \mathrm{span}\{P_0, P_1, P_2, P_3\}$，$\rho(x) \equiv 1$，其中

$$\begin{aligned} P_0 &= 1 & P_1 &= x \\ P_2 &= \frac{3x^2 - 1}{2} & P_3 &= \frac{5x^3 - 3x}{2} \end{aligned}$$

计算可得

$$\begin{aligned} (P_0, P_0) &= \int_{-1}^1 1 \cdot 1 dx = 2 & (P_1, P_1) &= \int_{-1}^1 x \cdot x dx = \frac{2}{3} \\ (P_2, P_2) &= \int_{-1}^1 \frac{3x^2 - 1}{2} \cdot \frac{3x^2 - 1}{2} dx = \frac{2}{5} & (P_3, P_3) &= \int_{-1}^1 \frac{5x^3 - 3x}{2} \cdot \frac{5x^3 - 3x}{2} dx = \frac{2}{7} \\ (f, P_0) &= \int_{-1}^1 \sin \frac{\pi}{2} x \cdot 1 dx = 0 & (f, P_1) &= \int_{-1}^1 \sin \frac{\pi}{2} x \cdot x dx = \frac{8}{\pi^2} \\ (f, P_2) &= \int_{-1}^1 \sin \frac{\pi}{2} x \cdot \frac{3x^2 - 1}{2} dx = 0 & (f, P_3) &= \int_{-1}^1 \sin \frac{\pi}{2} x \cdot \frac{5x^3 - 3x}{2} dx = \frac{48(\pi^2 - 10)}{\pi^4} \end{aligned}$$

于是 $f(x)$ 的三次最佳平方逼近多项式为

$$\begin{aligned}P_3(x) &= \frac{(f,P_0)}{(P_0,P_0)}P_0 + \frac{(f,P_1)}{(P_1,P_1)}P_1 + \frac{(f,P_2)}{(P_2,P_2)}P_2 + \frac{(f,P_3)}{(P_3,P_3)}P_3 \\ &= (\frac{420}{\pi^2} - \frac{4200}{\pi^4})x^3 - (\frac{240}{\pi^2} - \frac{2520}{\pi^4})x \\ &= -0.5622x^3 + 1.5532x\end{aligned}$$

P233-13

求 $f(x) = \arctan x$ 在 $[-1,1]$ 上的三次 Chebyshev 插值多项式

解 选取 *Chebyshev* 插值点

$$\begin{aligned}x_0 &= \cos\frac{\pi}{8} = 0.9239 & f(x_0) &= \arctan\cos\frac{\pi}{8} = 0.7459 \\ x_1 &= \cos\frac{3\pi}{8} = 0.3827 & f(x_1) &= \arctan\cos\frac{3\pi}{8} = 0.3655 \\ x_2 &= \cos\frac{5\pi}{8} = -0.3827 & f(x_2) &= \arctan\cos\frac{5\pi}{8} = -0.3655 \\ x_3 &= \cos\frac{7\pi}{8} = -0.9239 & f(x_3) &= \arctan\cos\frac{7\pi}{8} = -0.7459\end{aligned}$$

做 *Lagrange* 插值

$$\begin{aligned}P_3(x) &= f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x) + f(x_3)l_3(x) \\ &= 0.4268x^3 - 0.2433x^2 + 0.4430x + 0.2077\end{aligned}$$