COMP H2026

Information Technology Mathematics

Matrices

Semester 3

Today's lecture

- Matrices (Part 1)
  - What are Matrices
  - Uses in computing and Information Technology
  - Indexing a matrix
  - Addition of matrices
  - Multiplication of matrices
  - The transpose of a matrix

- Matrices (Part 2)
  - Determinants
  - Inverses
  - transformations

Matrices …

- what is a Matrix?
  - a matrix is a rectangular array of numbers which are set out in rows and columns. A matrix with *m* rows and *n* columns is called an *m x n* matrix.

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -1 & 7 \\ 9 & 5 & 1 \\ 4 & 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 7 & 1 & 3 \\ 0 & 2 & -1 & 2 \\ 5 & 9 & 3 & 2 \end{pmatrix}$$

Matrices

- Matrices have a wide variety of uses in Information Technology and Computing
  - e.g. There are a wide variety of problems that come up in computer graphics that require the numerical solution of matrix equations (simulation of materials such as water , creating a surface that drapes over a given collection of points…)
  - They provide a means of representing data and it is important to be able to manipulate matrices for computing tasks

Application Images & Image Compression: The basic idea behind this method of compression is to treat a digital image as an array of numbers i.e., a matrix. Each image consists of a fairly large number of little squares called **pixels** (picture elements). The matrix corresponding to a digital image assigns a whole number to each pixel. For example, in the case of a 256x256 pixel gray scale image, the image is stored as a 256x256 matrix, with each element of the matrix being a whole number ranging from 0 (for black) to 225 (for white). The JPEG compression technique divides an image into 8x8 blocks and assigns a matrix to each block. One can use some linear algebra techniques to maximize compression of the image and maintain a suitable level of detail

Images are comprised of pixels represented by numbers

$$\begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{bmatrix}$$

Ref: http://aix1.uottawa.ca/~jkhoury/haar.htm See website for further details on image compression

Problem…

- Write a computer program to:
  - Add two matrices **A** + **B** (both *m x n*)
  - Multiply *m x n* matrix **A** by the scalar value 2
  - Given an *m x k* matrix **A** and a *k x n* matrix *B*, Find **AB**

  - where do you start?
  - Lets look at the Maths of it first… and we'll come back to the program

Below are two more matrices, C is a 4 by 5 matrix, and D is a 5 by 4 matrix:

$$C = \begin{pmatrix} 1 & 2 & 9 & -3 & 0 \\ 0 & -3 & 6 & 10 & 2 \\ -1 & 4 & 5 & 3 & -1 \\ 7 & 5 & -2 & 1 & 4 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 4 & 6 & 3 \\ 0 & 5 & 9 & 5 \\ 7 & -1 & 6 & -5 \\ 12 & 6 & 8 & 4 \\ -2 & 3 & 4 & 0 \end{pmatrix}$$

These numbers which describe the size of the matrix, are known as its **rank**, while the numbers in the array are called **elements**. These elements may be denoted by,

$$a_{ij}$$

where i is the row and j is the column containing that element. The a refers to the name of the matrix.

For example, the element a_{32} of the matrix A below is 5 and the Element b_{23} of the matrix B below is -1.

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -1 & 7 \\ 9 & 5 & 1 \\ 4 & 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 7 & 1 & 3 \\ 0 & 2 & -1 & 2 \\ 5 & 9 & 3 & 2 \end{pmatrix}$$

Note that an 'ordinary' number, a 1 by 1 matrix, is now referred to as a **scalar**. Also, a row matrix has a single row and a column matrix has a single column, as shown below.

$$A = \begin{pmatrix} 1 & 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 \\ -7 \\ 8 \end{pmatrix}$$

Matrices have a well-defined algebra.

They can always be multiplied by scalars.

They can be added, subtracted and multiplied by other matrices, given certain conditions on their rank.

Given stricter conditions, they can be inverted, and so one matrix can be divided by another.

Matrix Addition, Subtraction and Scalar Multiplication

As might be expected, two matrices are added by adding each of the corresponding elements. But this means that two matrices can only be added if they have **identical rank**. Thus none of the matrices seen so far could be added, as they are all of different rank.

Can the following two matrices be added?

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix}$$

They can, since both are 2 by 2 matrices. Each element of A can be added to the corresponding element of B.

The result is,

$$A + B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Each of the corresponding elements has been added.

Subtraction is done in an analogous way.

$$A - B = \begin{pmatrix} 2 & 5 \\ -9 & 6 \end{pmatrix}$$

Example Add and subtract the following two matrices:

$$A = \begin{pmatrix} 10 & -6 \\ 13 & -5 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & -4 \\ 7 & -3 \end{pmatrix}$$

Solution:

A matrix is multiplied by a scalar by multiplying each element of the matrix by the scalar. For example, if A is given by

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}$$

then multiplying A by the number 5 gives:

$$5A = \begin{pmatrix} 5 & 15 \\ -25 & 15 \end{pmatrix}$$

Dividing this matrix by 2 is the same as multiplying by 1/2, giving:

$$\frac{A}{2} = \frac{1}{2}A = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix}$$

Example Multiply the following matrix by 7 and -1/3.

$$A = \begin{pmatrix} 10 & -6 \\ 13 & -5 \end{pmatrix}$$

Solution:

$$7A = \begin{pmatrix} 70 & -42 \\ 91 & -35 \end{pmatrix}$$

$$\left(-\frac{1}{3}\right)A = \begin{pmatrix} -\frac{10}{3} & 2 \\ -\frac{13}{3} & \frac{5}{3} \end{pmatrix}$$

Exercises:

- What is the rank of the following Matrices?

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -5 \\ 1 & 1 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ -2 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix}$$

- Using the matrices above write down a_{12}, b_{31}, c_{13}

- Using the matrices below evaluate the following: $(-2\mathbf{A} + \mathbf{B})$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & -2 \\ 1 & 3 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 3 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

Matrix by Matrix Multiplication

This idea is slightly more involved. Although the form of multiplication shown here may seem unduly complicated, it is precisely this definition which makes matrices so useful.

They originally arose from situations where this form of multiplication arrived from studying scientific, computing and engineering systems.

Consider these two matrices:

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

The number of columns in A (2) is the same as the number of rows in B (2). This means that each row in A has the same number of elements as each column of B.

Consider each element of the first row in A and the corresponding element in the first column in B. Multiply these corresponding elements, and add the results:

$$1\times 2 \quad + \quad 3\times(-1) \quad = \quad 2-3 \quad = \quad -1$$

This will be an element in the resulting product matrix. Its position must be decided.

It came from row 1 in A and column 1 in B, suggesting it should be in row 1, column 1 of the resulting product matrix.

When this process is repeated with row 2 in A and column 1 in B, the result is –5. Again, this number will be the element in row 2, column 1.

Repeating this procedure with the first row of A and the second column of B, gives 3. This will then occupy row 1, column 2.

Taking the second row of A and the second column of B, gives 3. This will then occupy row 2, column 2 of the product matrix. Putting these results together gives the product:

$$A \times B = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -5 & 3 \end{pmatrix}$$

Notice that the result is a 2 by 2 matrix.

For this method of multiplication to be possible, the number of columns in the first matrix must be the same as the number of rows in the second.

The same procedure is applied for all matrices. Consider the following example:

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}$$

The product here will be calculated exactly as before. Each row of A is taken with each column of B, and a number produced. There are 3 columns in B, and so 3 numbers are produced by the first row in A,

$$(1\times 1)+(3\times(-1)) = -2, \quad (1\times 0)+(3\times 2) = 6, \quad (1\times(-1))+(3\times(-2)) = -7$$

These form the top row in the product matrix.

Taking the second row, another 3 numbers are produced, –8, 6 and –1. These form the second row in the product matrix.

The overall result is a 2 by 3 matrix as expected,

$$A = 2\times 2, \quad B = 2\times 3, \quad Answer = (2\times 2)\times(2\times 3) = 2\times 3$$

In general, if two matrices, call them A and B, are to be multiplied and A is an m by n matrix, the number of columns in A must be the same as the number of rows in B, in order that they can be multiplied. Therefore, B must be an n by k matrix, where k is some number.

The number of rows in the product matrix will be the same as those of the first matrix, and the number of columns the same as those of the second matrix. The result will be an m by k matrix.

Matrix Multiplication

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} + b_{13}a_{31} & b_{11}a_{12} + b_{12}a_{22} + b_{13}a_{32} \\ b_{21}a_{11} + b_{22}a_{21} + b_{23}a_{31} & b_{21}a_{12} + b_{22}a_{22} + b_{23}a_{32} \end{pmatrix}$$

Matrix Multiplication

$$\begin{array}{cc}
 & \begin{array}{ccc} \text{col 1} & \text{col 2} & \text{col 3} \end{array} \\
 & \begin{pmatrix} 0 & -2 & 1 \\ 1 & 1 & 2 \\ -3 & 2 & 0 \end{pmatrix} \\
\begin{array}{c} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{array} \begin{pmatrix} 1 & -1 & 2 \\ 4 & 0 & -3 \\ -1 & 2 & 1 \end{pmatrix} & \begin{pmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{pmatrix}
\end{array}$$

Question: Multiply the following two matrices in the order they are given.

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The result is:

$$AB = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\times(-1)+3\times 1 \\ -5\times(-1)+3\times 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

A 2 by 2 matrix by a 2 by 1 matrix is a 2 by 1 matrix.

Exercise:

- Multiply the following matrices (on board):

Recall the two matrices multiplied before:

$$A \times B = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -5 & 3 \end{pmatrix}$$

The product $B \times A$ is also well defined, but will give a different answer:

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$$B \times A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ -2 & 0 \end{pmatrix}$$

Thus in this case

$$\boxed{A \times B \neq B \times A}$$

In this way, matrix multiplication is different from ordinary multiplication of scalars. The order in which two matrices are multiplied affects the answer.

This means that multiplication between matrices is **not commutative**.

It can be seen in advance that the same will be true of the following two matrices:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix}$$

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The result of the multiplication *AB* will be a 2 by 2 matrix, whereas the product *BA* will be a 3 by 3 matrix. The results of the calculations are as follows.

Firstly,
$$AB = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 6 & -9 \end{pmatrix}$$

Then
$$BA = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -4 \\ 7 & -8 & 5 \\ -1 & 0 & 1 \end{pmatrix}$$

Square and Identity Matrices

We have seen that the suitability of matrices for multiplication and other arithmetical operations depends on their rank.

The matrices with the most interesting algebra are those of the form *n*x*n*, in other words, the number of rows is the same as the number of columns.

These are called **square matrices.**

There are certain matrices which come up often in calculations because of their useful properties.

The most important example is the **identity** matrix, so-called because it is the equivalent for matrices as the number 1 in ordinary numbers. The 2 by 2 identity matrix is:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If this matrix is pre-multiplied by any 2 by n matrix, the 2 by n matrix is unchanged. If it is post-multiplied by any n by 2 matrix, again, the n by 2 matrix is unchanged.

For 3 by 3 matrices, the identity is

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In the same way, this matrix leaves 3 by n or n by 3 matrices unchanged.

Consider the following two matrices, both 3 by 3 matrices.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & -2 \\ 3 & -4 & -1 \\ -1 & 0 & 3 \end{pmatrix}$$

We can calculate AB and BA, the results are:

$$AB = \begin{pmatrix} 3 & 1 & -5 \\ 6 & -9 & -6 \\ -1 & 11 & -1 \end{pmatrix} \qquad BA = \begin{pmatrix} -5 & 6 & -6 \\ 4 & -6 & 4 \\ 8 & -6 & 4 \end{pmatrix}$$

Because the matrices A and B are square, the results of these calculations can be directly compared, and in particular, they can be subtracted:

$$AB - BA = \begin{pmatrix} 8 & -5 & 1 \\ 2 & -3 & -10 \\ -9 & 17 & -5 \end{pmatrix}$$

This resultant matrix is called the **commutator** of A and B.

The Transpose of a Matrix

Consider the following 2 by 3 matrix:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}$$

Consider the second, related matrix formed by switching this matrix around on its diagonal. The rows of this matrix become columns, the columns become rows. For A the matrix found by doing this is:

$$A^T = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ -1 & -2 \end{pmatrix}$$

This matrix is known as the **transpose** of A, and is written as A^T.

Questions?

Inverses

With square matrices, it is possible to consider the inverse of a matrix. First we will define exactly what we mean by this. As will be seen, the use of the inverse is analogous to division with scalar numbers.

Let A be a square matrix. Let B be a square matrix of the same rank, so that the products BA and AB are well defined. Let I denote the identity matrix for square matrices of this rank. If B has the property that

$$BA = I \text{ and } AB = I$$

then B is said to be the **inverse** of A.

As in ordinary arithmetic, this can be denoted using negative powers:

$$A^{-1} \text{ is the inverse of } A.$$

Let us first look at the relatively simple case of 2 by 2 matrices. A rule is required for finding the inverse of the general matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Consider the second 2 by 2 matrix

$$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and consider what happens when the two are pre- and post-multiplied. The results are:

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$BA = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

Both of these results are simply the 2 by 2 identity matrix multiplied by the factor $ad - bc$. Thus if I is the 2 by 2 identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then $BA = AB = (ad - bc)I$.

From this it can be seen that if the matrix B is divided by the factor $ad - bc$, it will satisfy the requirements of an inverse. In summary, the inverse of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example Find the inverse of the matrix below.

$$A = \begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}$$

Solution:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{(3)(1) - (2)(3)}\begin{pmatrix} 1 & -2 \\ -3 & 3 \end{pmatrix}$$

$$= \frac{1}{3 - 6}\begin{pmatrix} 1 & -2 \\ -3 & 3 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & -1 \end{pmatrix}$$

The Determinant of the 2 by 2 Matrix

Clearly the quantity $ad - bc$ is of paramount importance in this definition. If it is zero, as was the case in the last example above, then the inverse is not defined, and so does not exist for A. This quantity is known as the **determinant.**

The determinant of the 2 by 2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is given by $ad - bc$.

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There are several ways of denoting the determinant of a matrix A. It can be denoted by det A, |A| or by a delta symbol: D(A)

Thus

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Notice that with the straight lines notation, the curly brackets are dropped.

The inverse can then be calculated by

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinants of Higher Rank Matrices

Before we can calculate the inverses of matrices of a higher rank than 2, we need to discuss determinants of matrices other than the two by two case. We saw that the determinant of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

was $ad - bc$. More formally, the notation is, in terms of A,

$$\det A = |A| = \mathrm{D}(A).$$

The determinants of higher order matrices are defined in terms of the 2 by 2 case.

Here is an example of a 3 by 3 matrix A,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$$

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Firstly, let aij be the element of matrix A in row i and column j. Thus, for example, in this case of A,

$$a13 = 3,\ a23 = 5 \text{ and } a31 = 6$$

For a given element of A, consider the **sub-matrix** of A formed by leaving out the row and column containing this element.

For example, for the element $a_{12} = 2$, in the 1st row and 2nd column, the corresponding sub-matrix is found by leaving out the first row and the second column.

The elements left are

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 5 \\ 6 & 2 \end{pmatrix}$$

Note that the three by three has gone to a two by two. We know how to get the determinant of this matrix,

$$(4\times 2)-(6\times 5)=-22$$

For each row *i* and column *j* of the matrix *A*, let *Mij* be the value of the determinant formed in this way.

These are called ***minors***. Thus *Mij* is the determinant of the matrix formed by leaving out row *i* and column *j*.

We next define the ***cofactors*** of the matrix A.

For each minor Mij, we define the cofactor Cij as follows. If the value of $i + j$ is even, then the cofactor is

$$C_{ij} = M_{ij}$$

and if the value of $i + j$ is odd, then the cofactor is

$$C_{ij} = -M_{ij}$$

We could put this more generally as,

$$C_{ij} = (-1)^{i+j} M_{ij}$$

We now have a cofactor for each value of i and j, in other words, for each element of the matrix.

The determinant for the 3 by 3 matrix A is found by expanding one row of the matrix:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This equation means that we follow the top row across, multiply each element in the row by it’s corresponding cofactor, and then take the total.

The result is the value of the determinant.

Here are the calculations….

For $a_{11} = 1$, the sub-matrix is found by omitting the 1st row and 1st column, giving sub-matrix

$$\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$$

The determinant of this matrix is $1 \times 2 - 0 \times 5 = 2$. Thus

$$M_{11} = 2$$

The value of $i + j$ is 2, which is even, so the cofactor is

$$C_{11} = 2$$

For the 2 in the 1st row and 2nd column, the sub-matrix is

$$\begin{pmatrix} 4 & 5 \\ 6 & 2 \end{pmatrix}$$

The determinant of this sub-matrix is –22, so $M12 = -22$

The value of $i + j$ is 3, which is odd, so the cofactor is

$$C_{12} = -(-22) = 22$$

Finally, for the 3 in the 1st row and 3rd column, the sub-matrix is

$$\begin{pmatrix} 4 & 1 \\ 6 & 0 \end{pmatrix}$$

The determinant here is –6, so $M_{13} = -6$.

The value of $i + j$ is 4, which is even, so the cofactor is $C_{13} = -6$.

These values can now all be put in the equation

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This gives

$$\det A = 1 \times 2 + 2 \times 22 + 3 \times (-6) = 28$$

In fact, we could use any row, or column, as long as we find the appropriate cofactors.

It will always give us the same value for the determinant.

Using Matrices for Computer Graphics

As you work through this section you may wonder why the relatively laborious mathematical techniques of matrices are used for the relatively simple calculations for computer graphics.

For example, to translate (move) a point to the right by 3 steps on the x-axis can be performed as follows:

$$P = (x, y)$$
$$P' = (x+3, y)$$

The same calculation performed using matrices can be written as follows:

$$P' = TP = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (1 \times x) + (0 \times y) + (3 \times 1) \\ (0 \times x) + (1 \times y) + (0 \times 1) \\ (0 \times x) + (0 \times y) + (1 \times 1) \end{bmatrix} = \begin{bmatrix} x + 0 + 3 \\ 0 + y + 0 \\ 0 + 0 + 1 \end{bmatrix} = \begin{bmatrix} x + 3 \\ y \\ 1 \end{bmatrix}$$

However, computers are extremely good at performing regular calculations fast and without error.

Matrices offer a way to work with points and a wide range of graphical transformations in a single representation.

It is possible to represent any sequence of graphical transformations as a single matrix, so that with a few simple computer routines for creating identity matrices, and multiplying matrices, we can implement complex graphical scenes and animations.

Representing a 2D point as a 2 By 1 Column Matrix

A point can be represented as either a row or column matrix.

In this section, we shall follow the convention of representing **points as column matrices**. So a point (x, y) in a two-dimensional space could be represented as a matrix as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

Representing a 2D Point as a 3 By 1 (homogeneous) Column Matrix

While 2×1 column matrices may seem the most natural way to represent points in two-dimensional spaces, to enable many 2D graphical transformations to be represented (and executed) in a single way, we will choose to represent points in 2D space as 3×1 column matrices as follows:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This 3×1 column matrix representation is called a **homogeneous coordinate representation**.

Representing a 3D point as a 4 By 1 (homogeneous) Column Matrix

Just as we can represent a 2D (***x, y***) point as a 3 × 1 homogeneous column matrix, we can represent a point in 3D space (***x, y, z***) as a 4 × 1 column matrix as follows:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Representing 2D Translation as 3 By 3 Matrices

An example of a two-dimensional graphical transformation is to **translate** all x-coordinate values by a given factor, such as 5.

This would result in all points having some factor added to their x-coordinate value, moving the points left or right on the screen. We can represent any x-axis translation factor $\boldsymbol{t_x}$ in a 3 × 3 matrix as follows:

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice how this is essentially an identity matrix, except that the element at location (1, 3) has been replaced with the x-axis scaling factor $\boldsymbol{t_x}$.

To see the result of transforming a point with a 3 × 3 matrix like this we perform a matrix multiplication with a 1 × 3 homogenous column matrix for any (***x, y***) point:

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = TP = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & t_x \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (1 \times x) + (0 \times y) + (t_x \times 1) \\ (0 \times x) + (1 \times y) + (0 \times 1) \\ (0 \times x) + (0 \times y) + (1 \times 1) \end{bmatrix} = \begin{bmatrix} x + 0 + t_x \\ 0 + y + 0 \\ 0 + 0 + 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y \\ 1 \end{bmatrix}$$

Let us assume we have a point P (10, 5) and wish to apply an x-axis translation factor of 2. The calculation using matrices to find the new point P' would be:

$$T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}$$

$$P' = TP = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10+2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ 1 \end{bmatrix}$$

Thus the new point P' has the Cartesian coordinates (12, 5)

Representing 2D Scaling as 3 By 3 Matrices

An example of a two-dimensional graphical transformation is to **scale** all x-coordinate values by a given factor, such as 2 or 3. We can represent any x-axis scaling factor $\boldsymbol{s_x}$ in a 3 × 3 matrix as follows:

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice how this is essentially an identity matrix, except that the element at location (1, 1) has been replaced with the x-axis scaling factor $\boldsymbol{s_x}$.

To see the result of transforming a point with a 3 × 3 matrix like this we perform a matrix multiplication with a 1 × 3 homogenous column matrix for any (***x, y***) point:

$$S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = SP = \begin{bmatrix} s_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} s_x & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (s_x \times x) + (0 \times y) + (0 \times 1) \\ (0 \times x) + (1 \times y) + (0 \times 1) \\ (0 \times x) + (0 \times y) + (1 \times 1) \end{bmatrix} = \begin{bmatrix} (s_x \times x) + 0 + 0 \\ 0 + y + 0 \\ 0 + 0 + 1 \end{bmatrix} = \begin{bmatrix} s_x \times x \\ y \\ 1 \end{bmatrix}$$

Let us assume we have a point P (10, 5) and wish to apply an x-axis scaling factor of 2. The calculation using matrices to find the new point P' would be:

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}$$

$$P' = SP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 1 \end{bmatrix}$$

Thus the new point P' has the Cartesian coordinates (20, 5).

General Properties of Matrices

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules are of matrix arithmetic are valid:

$A + B = B + A$ — Commutative law for addition

$A + (B + C) = (A + B) + C$ — Associative law for addition

$A(BC) = (AB)C$ — Associative law for multiplication

$A(B + C) = AB + AC$ — Left distributive law

$(B + C)A = BA + CA$ — Right distributive law

$$A(B - C) = AB - AC$$

$$(B - C)A = BA - CA$$

$$a(B + C) = aB + aC$$

$$a(B - C) = aB - aC$$

$$(a + b)C = aC + bC$$

$$(a - b)C = aC - bC$$

$$a(bC) = (ab)C$$

$$a(BC) = (aB)C = B(aC)$$

Summary

- Matrices
  - What is a Matrix
  - Addition/Subtraction
  - Matrix multiplication
  - Determinants
  - Inverses
  - Transformations

  …

Questions?