Probability – Contents

2 Probability

2.1 Permutations and combinations

Most small club lotteries are based around picking 4 correct numbers out of 20, 24 or 28. The national lottery picks 6 numbers from 42, and the British takes 6 from 49. To calculate the probability of winning these lotteries, we have to see how many ways 4 numbers can be taken out of 24, or 6 out of 42, and so on.

Calculations like this fall into the category of permutations and combinations, and are generally very important for probability. Before going any further, take note of the following notation.

2.1.1 Definition: *n* Factorial

For a number n, the number $n!$ is defined as

$$n! = n(n-1)(n-2)\ldots 2.1.$$

This number is called n factorial.

2.1.2 Examples of Factorial Calculations

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$

$6! = 6 \times 5 \times \ldots \times 1 = 720.$

$7! = 7 \times 6 \times 5 \times \ldots \times 1 = 5{,}040.$

The values increase rapidly:

10! = 10 x 9 x … x 1 = 3,628,800.

It can be seen that when calculating the value of 6!, it was not necessary to find the value of 5x4x3x2x1, since this had already been done for 5!.

In other words, 6! = 6 x 5!.

Generally,

$$n! = n \times (n - 1)!.$$

2.1.3 Permutations

A tourist is visiting the south-west of Ireland, and has time to visit four towns out of Cork, Killarney, Ennis, Limerick, Tralee, Bantry.

If the order in which the towns are visited is taken into account, how many possible trips are there?

There are 6 towns, and 4 choices are made, so the number of ways in which the tourist could choose the towns is

- 6 choices for the first town, then
- 5 for the second,
- 4 for the third and
- 3 for the fourth.

The result is

$$6 \text{ by } 5 \text{ by } 4 \text{ by } 3, \text{ which is } 360.$$

Each particular choice of towns, in a particular order, is a *permutation* of four of the six names in the list.

2.1.4 Definition: Permutation

A permutation of r items from a list of n is a choice of r items in a particular order.

When calculating how many permutations there are of r items from a list of n, the calculation always goes the same way. The result is

$$n(n-1)(n-2)(n-3)(n-4)\ldots$$

with r numbers being multiplied in this list. This number is defined as follows.

2.1.5 Definition: ^nP_r

The number of ways of selecting r objects from a list of n objects, where the order does count, is given by ^nP_r, where

$$^nP_r = n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)\ldots$$

where there are r numbers in this list.

This number is referred to as ‘n P r’.

This number ^nP_r can be written in terms of the factorial notation:

$$^nP_r = \frac{n!}{(n-r)!}.$$

2.1.6 Example – Permutation of Towns

For the example of the tourist, the number of ways of choosing 4 towns out of 6, in a particular order was

$$^6P_4 = 6.5.4.3 = 360.$$

In terms of the factorial notation:

$$^{6}P_{4} = \frac{6!}{2!} = \frac{6.5.4.3.2.1}{2.1} = 6.5.4.3\,.$$

The 2 by 1 term cancels from above and below the line.

2.1.7 An Example from Cards

From a pack of 52 cards, 5 cards are to be dealt. How many possible permutations of the cards are possible? In other words, how many lists of cards could be seen as they are dealt?

The answer, in the notation we have adopted, is:

$$^{52}P_5 = 52 \text{x} 51 \text{x} 50 \text{x} 49 \text{x} 48 = 311{,}875{,}200.$$

This calculation can be looked at in terms of the factorial notation:

$$^{52}P_5 = \frac{52!}{47!} =$$

$$= \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47 \times 46 \times ... \times 3 \times 2 \times 1}{47 \times ... \times 3 \times 2 \times 1} =$$

$$= 52 \times 51 \times 50 \times 49 \times 48$$

2.1.8 Combinations

What if the order the towns were visited in did not matter?

If the towns are to be selected and the order does not matter, the selection involved is called a *combination*. In this case, there will be far fewer separate lists.

2.1.9 Definition: Combination

A combination of r items from a list of n is a choice of r items, where the order does not matter.

To find how many ways the towns can be visited, the number ${}^6P_4$ must be divided by the number of ways the 4 towns can be ordered, to ignore the different orders the towns could be listed in.

For a particular choice of 4 towns, how many ways can they be reordered? The answer is 4x3x2x1, in other words 4!.

This means the number of ways the 4 towns can be visited, where the order does not matter, is

$$\frac{6\times 5\times 4\times 3}{4\times 3\times 2\times 1}=15\,.$$

This shows that the number of ways of taking out r objects (names, numbers, etc.) from n is given by the number ${}^nP_r$, divided by $r!$.

2.1.10 Definition: ${}^nC_r$

The number of ways of selecting r objects from a list of n objects, where the order does not count, is given by ${}^nC_r$, where

$${}^nC_r = n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)\ldots \div r!$$

In factorial notation, this is equal to:

$$ {}^{n}C_{r} = \frac{n!}{r! \times (n-r)!} . $$

To actually calculate the value of ${}^{n}C_{r}$, note that the n-r numbers in (n-r)! cancel with the last n-r numbers in n!, leaving r numbers.

This means there are r numbers above and below the line. So this number is the same as:

$$\frac{n\times(n-1)\times(n-2)\times\ldots}{r\times(r-1)\times\ldots\times 1}.$$

This makes it very easy to calculate the value.

2.1.11 Another Example from Cards

From a pack of 52 cards, 5 cards are to be dealt. How many possible combinations of the cards are possible? In other words, how many hands will be seen as they are dealt?

The answer, in the notation we have adopted, in terms of the factorial notation, is:

$$^{52}C_5 = \frac{52!}{5! \times 47!}.$$

When the 47! is cancelled with the 52!, only the numbers 52, 51, 50, 49, 48 are left multiplying above the line:

$$^{52}C_5 = \frac{52\times 51\times 50\times 49\times 48}{5\times 4\times 3\times 2\times 1}$$

This leaves 5 numbers above the line and below the line.

From here, there will always be more cancellations that can happen:

$$^{52}C_5 = \frac{52\times 51\times 10\times 49\times 2}{1} = 52\times 51\times 10\times 49\times 2 .$$

The result is 2,598,960. This is a large number, but a lot less than before.

2.1.12 Example – lists of people

Four candidates' names are to be short-listed from a panel of ten for an interview. Assuming the top 4 are to be interviewed:

- How many possible orders of candidates for interview?

- How many possible lists of candidates are there?

The first question is asking for the number of possible interview lists, if the order is taken into account. This is then:

$$^{10}P_4 = 10.9.8.7 = 5{,}040.$$

The answer to the second part is simply the number of ways of taking 4 from 10, without the ordering, which is given by

$$^{10}C_4 = \frac{10\times9\times8\times7}{4\times3\times2\times1} = \frac{10\times3\times7}{1} = 210$$

2.1.13 Example – Panels for Teams

A five-a-side soccer team is to be selected from a panel of 8 players. How many possible teams are there?

Firstly, treat each possible team as a list of names, in other words, a combination.

The number of ways of selecting 5 players out of 8 is given by:

$$^{8}C_{5} = \frac{8\times7\times6\times5\times4}{5\times4\times3\times2\times1} =$$

$$= \frac{8\times7\times6}{3\times2\times1} = 56$$

It is worth noting that in this case, where 5 players must be chosen from 8, any choice of 5 players is the same as a choice of 3 to leave out.

It follows then that the number of ways of choosing the 5 is the same as the number of ways of leaving 3 players on the bench.

Looking back to the definition of ${}^{n}C_{r}$, shows why this might be. The expression for ${}^{8}C_{5}$ is

$$ {}^{8}C_{5} = \frac{8\times7\times6\times5\times4}{5\times4\times3\times2\times1} \,. $$

The last two numbers on the top cancel with the first two in the bottom line, giving

$$^{8}C_{5} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1},$$

which is identical to $^{8}C_{3}$.

When calculating a value of ${}^nC_r$, whichever of the two numbers of r or n-r is the lower, any more than these numbers above or below the line will cancel. So it can be seen that

$${}^nC_r = {}^nC_{n-r}.$$

That this is the case can be seen from the definition:

$${}^nC_r = \frac{n!}{r! \times (n-r)!}.$$

Since n and $n - r$ are multiplied in this equation, it doesn't matter which order they appear in.

2.1.14 Example – Teams as a Combination

The example above with the 5-a-side soccer team assumes that the team is just a list of five names. What if the same five players, but in different positions, counts as a different team?

The number of possible teams is then

$$^{8}P_{5} = 8 \times 7 \times 6 \times 5 \times 4 = 6{,}720.$$

2.2 Lotteries

The definitions of permutations and combinations are used to calculate the odds for the lotteries mentioned above. The probability will be the number of combinations which satisfy the criteria for the event, which of course is 1, divided by the number of possible combinations.

Thus once it is known how many possible combinations there are when selecting *r* numbers from *n*, then the probability of getting a particular combination in a lottery is just the inverse of this.

Example – Club Lotteries

Most small club lotteries are based around picking 4 correct numbers out of 20, 24 or 28. What is the probability of winning in each of these cases?

The probability of winning with a particular choice of 4 numbers is 1 in *N*, where *N* is the number or ways of taking 4 numbers from 20, 24 or 28.

For 20, the number of ways of selecting 4 numbers out of 20 is given by

$$^{20}C_4 = \frac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1} = 4845$$

Thus the probability of winning such a lottery is 1/4845.

For 24 numbers, the number of combinations is

$$^{24}C_4 = \frac{24\times 23\times 22\times 21}{4\times 3\times 2\times 1} = 10{,}626 .$$

The probability of winning this lottery is 1/10,626.

For 28, the number of combinations is

$$^{28}C_4 = \frac{28 \times 27 \times 26 \times 25}{4 \times 3 \times 2 \times 1} = 20{,}475 .$$

The probability of winning this a lottery is 1/20,475.

The Irish National Lottery picks 6 numbers from 42, so the number of combinations is

$$^{42}C_6 = \frac{42 \times 41 \times 40 \times 39 \times 38 \times 37}{6 \times 5 \times 4 \times 3 \times 2 \times 1} .$$

Some of these numbers can be cancelled to give:

$$41 \times 13 \times 38 \times 37 = 5{,}245{,}786.$$

The probability of winning this a lottery is one in 5 million, or 1.9×10^{-7}.

The British lottery picks 6 numbers from 49, so the number of combinations is

$$^{49}C_6 = \frac{49\times 48\times 47\times 46\times 45\times 44}{6\times 5\times 4\times 3\times 2\times 1} =$$

$$= 49\text{x}47\text{x}46\text{x}44\text{x}3 = 13{,}983{,}816.$$

The probability of winning this lottery is then one in 14 million, or $7.15\text{x}10^{-8}$.

2.3 Further Examples of Combinations

Many of the problems in probability can be solved using the same type of calculations as in the Birthday Enigma, or others like it.

These are all cases where an experiment gives rise to a number of equally probable outcomes, and the specified event is a well-defined subset of these outcomes. Calculating the probability is then a matter of counting; the probability of an event happening is then the ratio of the number of ways it can happen, which is counted from the list of outcomes, to the total number of outcomes, in other words, the number of ways the experiment produces outcomes, whether or not they satisfy the conditions of the event. This is often phrased as ‘without restriction’.

2.3.1 Example – A class

A class consists of 10 men and 15 women. Calculate the probability that if four names are chosen at random, all are women.

To answer this question, firstly look at the number of ways of picking the four names without any restrictions. There are 25 people in the class, so this is $^{25}C_4$.

How many ways can four women's names be picked? There are 15 women, so the answer here is ${}^{15}C_4$.

The probability is then ${}^{15}C_4 \div {}^{25}C_4 = 1{,}365/12{,}650 = 0.108$.

The probability that all four names would be men can be calculated as:

$$ {}^{10}C_4 \div {}^{25}C_4 = 210/12{,}650 = 0.0166 $$

2.3.2 Another Example

In the same situation, calculate the probability that if six names are chosen, half are men and half women.

As in the first case, the number of ways of picking the six names is

$$^{25}C_6.$$

The event here is that 3 women and 3 men are picked. The number of ways of doing this is the number of ways of picking 3 women's names multiplied by the corresponding figure for men.

The number of ways of picking 3 women's names is ${}^{15}C_3$.
The number of ways of picking 3 men's names is ${}^{10}C_3$.

These figures are multiplied to give the number of ways of finding 3 women's names and 3 men's names – this is $^{15}C_3$ by $^{10}C_3$. The probability of the event is then

$$\frac{^{15}C_3 \times {}^{10}C_3}{^{25}C_6} = \frac{455 \times 120}{177{,}100} = 0.31\,.$$

2.4 The Laws of Probability

Recall from the example of two dice being thrown, that if A is an event and B is the exact opposite event, then

$$P[A] + P[B] = 1.$$

In other words, the probabilities of two distinct events, which cover all possibilities, add up to 1.

A generalisation of the above rule concerns n distinct possible events:

$$E_1, E_2, E_3, \ldots, E_n,$$

and no others. Then

$$P[E_1] + P[E_2] + \ldots + P[E_n] = 1.$$

In other words, when all distinct possibilities have been taken care of, the sum of the probabilities of the events is 1.

These laws will now be phrased more rigorously, with some definitions of events and how they are related, and laws which govern their probabilities.

2.4.1 The Addition Law of Probabilities

The ‘addition law’ concerns how we calculate the probability of the event ‘A or B’. before we look at this, we have to define the type of events we are looking at.

2.4.2 Definition – Mutually exclusive events

Two events A and B, the possible results of the same experiment, are said to be *mutually exclusive* if it is impossible for them to happen together. If two events A and B are not mutually exclusive, that is, A and B can occur together, they are said to be mutually *non-exclusive* events.

With this definition, we can now write down two laws governing how the probabilities of the event A or B.

2.4.3 The addition law for mutually exclusive events

If two events A and B are mutually exclusive, then the following law holds:

$$P[A \text{ or } B] = P[A] + P[B].$$

This means the probability of one of event A or event B occurring is given by the sum of their two probabilities.

2.4.4 The addition law for mutually non-exclusive events

If the two events are not mutually exclusive, the law above is broadened so that the probability of A or B occurring is given by:

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B].$$

This means the probability of one of event A or event B occurring is given by the sum of their two probabilities, less the probability of them both occurring.

2.4.5 Example of the addition law

If a single dice is thrown, determine the probability of getting a multiple of 3 or a multiple of 2, and then the probability of one or the other.

To start, the events should be defined. Let A be the event of getting a multiple of 3, and B be the event of getting a multiple of 2.

The probability of scoring a multiple of 3 is that of getting a 3 or a 6. Then

$$P[A] = 2/6 = 1/3.$$

The probability of scoring a multiple of 2 is therefore:

$$P[B] = 3/6 = ½.$$

Now both of these events could happen at the same time; a number can be a multiple of 2 and 3, and one of these is one the die: 6. It then follows that A and B are non-exclusive events, so

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B].$$

The event 'A and B' means that a number is a multiple of 2 and 3, so this is equivalent to getting a 6. Thus

$$P[A \text{ and } B] = 1/6.$$

The final probability is then

$$P[A \text{ or } B] = 2/6 + 3/6 - 1/6 = 4/6 = 2/3.$$

In this case, finding the probability $P[A \text{ or } B]$ from scratch would not require too much calculation. It would simply be a matter of counting the number of ways a number on the dice could be a multiple of 3 *or* a multiple of 2. The ways of getting this are 3 and 6 for the first, and 2, 4 and 6 for the second.

This is a list of 4 distinct numbers, so the answer is 4/6 = 2/3. This confirms the answer already found.

Example – the addition law

With a full deck of cards, determine the probability of drawing an ace or a red card.

Let A be the event of drawing an ace and B the event of drawing a red card. The question is then to determine $P[A$ or $B]$.

To solve this problem, first find the probabilities of drawing an ace and of drawing a red card.

There are 4 aces in a pack, so $P[A] = 1/13$.
There are 26 red cards in a pack, so $P[B] = ½$.

These are mutually non-exclusive events (ace of hearts or aces of diamonds both are successes), so the law

$$P[A \text{ or } B] = P[A] + P[B] – P[A \text{ and } B]$$

must be applied again.

Finding the probability: there are 2 red aces in a pack, so

$$P[A \text{ and } B] = 2/52 = 1/26.$$

Substituting the values found;

$$P[A \text{ or } B] = P[A] + P[B] – P[A \text{ and } B] =$$

$$= 1/13 + ½ - 1/26 = 7/13.$$

As with the last case, this probability could have been calculated by simply counting out the number of ways the event could happen. The number of ways of taking out an ace or a red card is clearly the 26 red cards plus the remaining two aces, giving 28 in total. Therefore the probability is:

$$P[A \text{ or } B] = 14/52 = 7/13.$$

2.4.6 Independent and Dependent Events

The next concept in probability concerns which events do or don't affect the occurrence of other events. An example would be the rolling of a dice on two occasions. The outcome of the first throw will not affect the probabilities for the second throw; the dice is picked up and thrown again with no link to the last throw.

Now consider the event of drawing a king from a full deck of cards without replacing it, and the second event of drawing a king after this. Clearly whether the first event *does* or *does not* happen, this will alter the probability of the second event.

2.4.7 Definition – Independent events

Two events are *independent* when the occurrence of one event **does not** affect the probability of the occurrence of the second event. If the outcome of one event **does** affect the probability of the second event, they are said to be *dependent*.

To go back to the case of two dependent events, consider the event F defined as drawing a king from a full deck of cards, and event N, drawing a king after this without replacement.

The probability of F is:

$$P[F] = 4/52 = 1/13.$$

The card is not replaced, and the same experiment is carried out again.

If the *first* card was a king, the probability is:

$$P[N] = 3/51.$$

If the *first* card was not a king, the probability is:

$$P[N] = 4/51.$$

Thus the outcome of the first experiment, drawing a king, has affected the probabilities for the second.

The two events are not independent, i.e. they are dependent.

2.4.8 The Multiplication Law of Probability

For two *independent events* A and B, the probability of the occurrence of both events A and B, is given by

$$P[A \text{ and } B] = P[A].P[B]$$

The probability of the occurrence of both events is the product of the two individual probabilities.

2.4.9 Example of the Multiplication Law

A single fair dice is thrown 4 times. Calculate the probability of getting 4 5's in a row.

The event of rolling the die each time is an independent event.

Each time, the probability of getting as 5 is 1/6.

Thus using the multiplicative law, the probability is

$$(1/6)^4 = 1/6^4.$$

2.4.10 Dependant events – Conditional Probability

To deal with dependent events, some notation will be needed to indicate when the probability of one event depends on whether another event has happened.

This idea comes up a lot in calculations of failure probabilities in complex systems of components, where the failure of the system depends on different structures or ‘pathways’ in the system.

Consider the situation of two events A and B, where the occurrence or not of event A does effect the probability of event B. In other words, B is dependent on A. we will set up a notation for this.

2.4.11 Notation – conditional probability

The probability of event B, providing that event A has already occurred is denoted by the notation:

$$P[B \mid A].$$

This is 'the probability of B, given A.'

For two independent events A and B, by definition, the fact that A has already occurred does not affect the probability of event B. It then follows that

$$P[A \mid B] = P[A].$$

If the events are dependent, these probabilities are not the same.

Now consider two events A and B; with event A dependent on B. The probability of the occurrence of both events is going to be

$$P[A \text{ and } B] = P[A].\ P[B \mid A]$$

The probability of both events happening is the probability of A times the probability of B, given that A has occurred.

2.4.12 Example of a conditional probability

Consider two events A and B, where

- A is throwing a six with a fair die, and
- B is drawing a king from a full deck of cards.

Determine the probability of the occurrence of both events. In other words, find $P[A \text{ and } B]$.

Since these are independent events, we use

$$P[A \text{ and } B] = P[A].P[B]$$

Thus

$$P[A \text{ and } B] = (1/6) \text{ x } (4/52) =$$

$$= (1/6) \text{ x } (1/13) = 1/78.$$

Example – conditional probabilities

A box contains five 10 kΩ resistors and twelve 20 kΩ resistors. Determine

- The probability of randomly picking a 10 kΩ resistor from the box.
- The probability of randomly picking a 10 kΩ resistor from the box and then a 20 kΩ resistor.

Solution

Let event A denote the event of picking a 10 kΩ resistor, and let B denote the event of picking a 20 kΩ resistor.

The first probability is just $P[A]$, and since the total number of resistors is 17, it is

$$P[A] = 5/17.$$

To find the probability of both, that is, $P[A \text{ and } B]$, observe that B depends on A, since A is the event that is happening first. The probability law for dependent events must be used:

$$P[A \text{ and } B] = P[B \mid A].P[A].$$

To find $P[B \mid A]$, we need the probability that a second resistor picked from the box will be a 20 kΩ resistor, providing that the first one was a 10 kΩ resistor.

For this case, $P[B|A] = 12/16 = ¾$.

So the probability of both events, picking a 20Ω k resistor after getting a 10 kΩ resistor is:

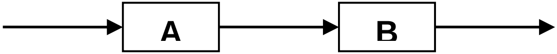
$$P[A \text{ and } B] = P[B|A].P[A] = ¾ \text{ x } 5/17 = 15/68.$$

2.4.13 Some Examples of complex systems

In each of the following systems, the probability that each individual component of type A will fail is 0.03, and the probability that each individual component of type B will fail is 0.05. All components A and components B are independent.

For each system, we will calculate the overall probability that each system will work.

Example 1

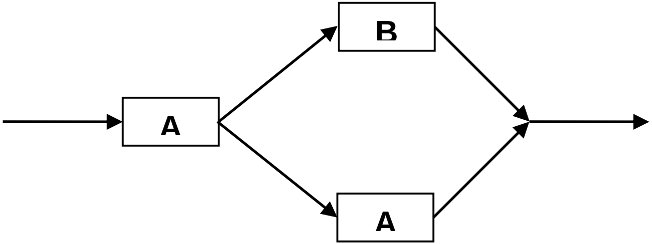


For this example, clearly both components have to work so that the system overall works. The two events, *A* and *B*, are independent, so the probability they both work is:

$$0.97 \times 0.95 = 0.9215.$$

This then is the probability the system works.

Example 2



In this case, going from left to right, the first component A has to work, followed by the second stage, which is *either* the B or the second A.

For the second stage to work, A or B must work.

For the system overall, both stages must work.

$$P[\text{system works}] = P[\text{stage 1 works}] \times P[\text{stage 2 works}].$$

Firstly,

$$P[\text{stage 1 works}] = P[\text{A}].$$

In the second stage, *A* working and *B* working are not mutually exclusive. Then

$$P[\text{stage 2 works}] = P[\text{A or B works}]$$

So to find the probability this part of the system works, we have to use the second version of the addition law, this is just

$$P[\text{A or B}] = P[\text{A}] + P[\text{B}] - P[\text{A}]P[\text{B}].$$

The overall picture is now: P[system works] =

$= P[\mathrm{A}] \times P[\mathrm{A\ or\ B}] = P[\mathrm{A}]\,(\,P[\mathrm{A}] + P[\mathrm{B}] - P[\mathrm{A}]P[\mathrm{B}]\,).$

In this case, going from left to right, the first component A has to

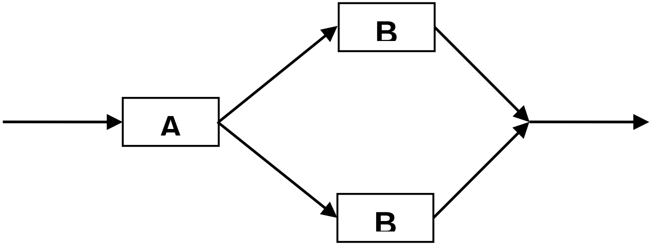
$$P[\mathrm{A\ or\ B}] = P[\mathrm{A}] + P[\mathrm{B}] - P[\mathrm{A}]P[\mathrm{B}] =$$

$$= 0.97 + 0.95 - 0.97 \mathrm{x} 0.95 = 0.9985.$$

The laws then mean that the probability the whole system works is

P[system works] = P[A] x 0.9985 = 0.97x0.9985 = 0.968,545.

Example 2 (a)



In this case, going from left to right, the first component A has to work, followed by the second stage, which is *either* one B or the other.

For the system overall, both stages must work.

Therefore:

$$P[\text{system works}] = P[\text{stage 1 works and stage 2 works}] =$$

$$= P[\text{stage 1 works}] \times P[\text{stage 2 works}].$$

Firstly, stage 1 is equivalent to component A:

$$P[\text{stage 1 works}] = P[\text{A}].$$

For stage 2:

$$P[\text{stage 2 works}] = P[\text{B or B}] = P[\text{B}] + P[\text{B}] - P[\text{B and B}]$$

In the second stage, *B* working and *B* working are not mutually exclusive. So to find the probability this part of the system works, we have to use the second version of the addition law, this is just

$$P[\text{B or B}] = P[\text{B}] + P[\text{B}] - P[\text{B}]P[\text{B}].$$

Putting in the known probabilities:

$$P[\text{B or B}] = 0.95 + 0.95 - 0.95\text{x}0.95 = 0.9975.$$

The laws then mean that the probability the whole system works is

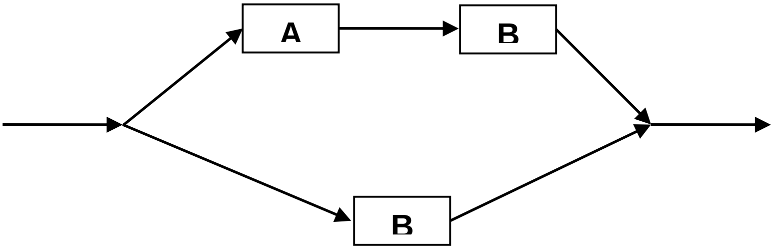
$$P[\text{system works}] = P[\text{A}] \; (\; P[\text{B}] + P[\text{B}] - P[\text{B}]P[\text{B}] \;) =$$

$$= 0.97\text{x}0.9975 = 0.9676.$$

Example 3

In this example, for the system to work;

The lower B has to work, *or* the upper A and the B together, *or* both scenarios.



For the probabilities, this is then:

$$P[\text{A and B}] + P[\text{B}] - P[\,[\text{A and B}] \text{ and B}].$$

The part P[A and B] is given by

$$P[\text{A and B}] = P[\text{A}]\text{x}P[\text{B}].$$

The last term can be broken down as:

$$P[\,[\text{A and B}] \text{ and B}] = P[\text{A and B}]\text{x}P[\text{B}] = P[\text{A}]\text{x}P[\text{B}]\text{x}P[\text{B}].$$

The probability of the system working is then:

$$P[\mathrm{A}] \mathrm{x} P[\mathrm{B}] + P[\mathrm{B}] – P[\mathrm{A}] \mathrm{x} P[\mathrm{B}] \mathrm{x} P[\mathrm{B}].$$

Then the probability the system works is

$$0.97 \mathrm{x} 0.95 + 0.95 – 0.97 \mathrm{x} 0.95 \mathrm{x} 0.95 =$$

$$= (0.97 + 1 – 0.97 \mathrm{x} 0.95) \mathrm{x} 0.95 =$$

$$= 0.996075.$$

3 Random Variables

Say an experiment is being done, the outcomes of which are quantifiable; that is, they are measurable numbers. Such quantities are called Random Variables.

The range of possible outcomes may be any real number, such as weight, height etc., or it may be an integer, if items are being counted.

With a Random Variable

- The experiment is producing and then measuring the variable
- The event is the variable taking on a particular value, or range of values.

3.1.1 Example – revisit

In the case of rolling two dice, what is the probability that the difference is 2?

The possible outcomes which make up this event are:

$$(1,3), (2,4), (3,1), (3,5), (4,2), (4,6), (5,3), (6,4).$$

In this case:

- The experiment is the rolling of the dice, and seeing which numbers come up, and then finding the difference.

- The outcomes are whatever numbers show up.
- The Random variable is the difference of the two numbers, call it D.
- The event is the random variable taking on particular values or ranges of values, in this case, $D = 2$.

In this case, the variable D has been assigned to the difference between the two numbers. So the event in this case is

$$D = 2,$$

and from counting the possible ways in which this comes about,

$$P[D = 2] = 8/36 = 2/9.$$

The key thing here is that we are treating the numbers that come up on the dice as such, that is, adding and subtracting with them.

3.1.2 Discrete and continuous

A random variable is called *discrete* if it takes on distinct values, such as integers or integer multiples of a real number. Such random variables usually arise as the result of a counting process.

An example might be 'the number of students who turn up for a lecture.'

A random variable which can take on *any* numerical value is called a continuous random variable. These are usually physical quantities, such as height, weight, distance, resistance, capacitance and times. These are phrased here as definitions:

3.1.3 Definition – Discrete Random Variable

A Random Variable is said to be *discrete* if it can only take on values from a set of distinct numbers, such as integers or a finite subset of numbers.

3.1.4 Definition – Continuous Random Variable

A Random Variable is said to be *continuous* if it can take on any real numbers as a value.

3.2 Distributions

Random variables often have a distribution; this is a law governing the probability of particular values of the variable coming up. In the case of a discrete random variable, this could simply be a list of probabilities for each value. It can also be an equation giving the probability in terms of the value.

3.2.1 A simple example of a distribution

In the example of rolling two dice and calculating the difference, we drew up the event space, and from this we can write down the probability of all events:

$$P[D = 0] = 6/36 = 0.1667.$$

$$P[D = 1] = 10/36 = 0.2778.$$

$$P[D = 2] = 8/36 = 0.2222.$$

$$P[D = 3] = 6/36 = 0.\ 1667.$$

$$P[D = 4] = 4/36 = 0.1111.$$

$$P[D = 5] = 2/36 = 0.0555.$$

3.3 The Binomial Distribution

Consider the following problem – it is known that 17% of the Irish population is left-handed. In a randomly selected group of 12 people, what is the probability of finding 3 people who are left-handed?

To answer this question, look at one example – in the 12 people, what is the probability of the first 3 of the 12 being left-handed?

Stating at the start that the group have been randomly selected means that we can assume there is no connection between them, so each person being left or right-handed is an independent event.

This means that the probabilities of these events for each person are multiplied to find the overall probability.

Thus the probability of the first 3 being left-handed, and the remaining 9 being right-handed, is

$$(0.17)^3.(0.83)^9.$$

But the first 3 being left-handed, and the next 9 being right-handed, is just one way of having the result in question.

The last 3 could be left-handed, or the middle 3, or indeed any selection of 3 from the 12 could be the order.

But for each one, the probability of that selection coming up is

$$(0.17)^3.(0.83)^9.$$

These possible outcomes are all mutually exclusive, so to find the overall probability, add on

$$(0.17)^3.(0.83)^9.$$

for each combination. In other words, multiply by the number of combinations of 3 from 12 – this is our friend ${}^{12}C_3$.

The probability of finding 5 left-handed people in a group of 30 is then

$$^{30}C_5(0.17)^5(0.83)^{25}.$$

This is an example of what is called the Binomial distribution.

3.3.1 Definition – The Binomial Distribution

A trial is being repeated, with a possible result *A*. The following is known:

- Each time the trial is done, the probability of result A turning up is p.
- The trial is repeated n times.

Let X be the random variable of the number of times event A comes up. The probability of getting r results from n trials is:

$$\mathrm{P}[X = r] = {}^{n}C_{r}\, p^{r}(1 - p)^{n-r}.$$

The case of the number of left-handed people fits into this pattern.

- The trial being repeated is checking whether or not a person is left-handed.
- In the example we studied, this is being repeated 12 times, so this means $n = 12$.
- The probability of this occurring for each person ‘tested’ is $p = 0.17$. This number comes from the proportion of 0.17 of the population being left-handed,

and from our original definition of what a probability actually is.

Let L be the random variable of the number of left-handers in the group. The probability we are looking at is then: $P[L = 3]$.

Using the binomial distribution,

$$P[L = r] = {}^{n}C_{r} p^{r} (1 - p)^{n-r}.$$

$$P[L = 3] = {}^{12}C_{3}\,(0.17)^{3}(0.83)^{9} =$$

$$= 220 \times 0.004913 \times 0.18694 =$$

$$= 0.20206.$$

3.3.2 Example – defective components

A factory is producing components, of which 1.5% are defective. They are packed in boxes, each containing 20 components. Calculate the probability that a box has 2 defective components.

This is a case of the binomial distribution.

- There are 20 components in each box, so the value of n is 20.
- Each component has a probability of 0.015 of being defective so $p = 0.015$.

To apply the binomial distribution, let X be the random variable of the number of defective components in a given box. The event we are looking at is $X = 2$. So $r = 2$ in the equation.

Then

$$P[X = r] = {}^{n}C_{r}\, p^{r}(1 - p)^{n-r}:$$

For this case:

$$P[X = 2] = {}^{20}C_{2}(0.015)^{2}(1 - 0.015)^{18} =$$

$$= 190 \text{x} 0.015^{2} \text{x} 0.985^{18} = 0.033.$$

Thus of every 1000 boxes coming out of the factory, 33 would have two defectives.

3.3.3 More on this example

In this case, find the probability of getting

1. no defectives,
2. one defective,

and so find the probability of

3. less than 3 defectives.
4. 2 or more are defective

Part 1:

The probability of getting no defectives is

$$P[X = 0] = {}^{20}C_0\,(0.015)^0(1 - 0.015)^{20} =$$

$$= 1 \times 1 \times 0.985^{20} = 0.739.$$

Bear in mind that ${}^{20}C_{0} = 1$, this is because there is only one way of having no defectives in the box.

Part 2:

The probability of getting one defective is:

$$P[X = 1] = {}^{20}C_1(0.015)^1(1 - 0.015)^{19} =$$

$$= 20 \text{x} 0.015 \text{x} 0.985^{29} = 0.225.$$

Part 3:

To find the probability of getting less than three defectives, first look at exactly what this means. It means that

$$X = 0 \text{ or } X = 1 \text{ or } X = 2.$$

The probability of getting less than three defectives is then

$$P[X < 3] = P[X = 0 \text{ or } X = 1 \text{ or } X = 2].$$

Since these are mutually exclusive events, the probabilities can be added:

$$P[X < 3] = P[X = 0] + P[X = 1] + P[X = 2].$$

These are all probabilities we have worked out. So this means

$$P[X < 3] = 0.739 + 0.225 + 0.033 = 0.997.$$

Part 4:

To calculate the probability that in a box of 20, two or more are defective, we could add the probabilities for $X = 2$, $X = 3$, $X = 4$ and so on. However, this would be simpler to use the fact that the event of getting two or more defectives is the converse of getting none or one. Thus

$$P[X \geq 2] = 1 - P[X = 0 \text{ or } X = 1].$$

Since these are mutually exclusive events, the probabilities can be added:

$$P[X = 0 \text{ or } X = 1] = P[X = 0] + P[X = 1].$$

So this means

$$P[X > 2] = 1 - \big(P[X = 0] + P[X = 1]\big) =$$

$$= 1 - (0.739 + 0.225) = 1 - 0.964 = 0.036.$$

3.3.4 Multiple Choice Exams

A classic example of a binomial distribution is the question of a student, lacking in knowledge, trying to pass a multiple choice examination by answering the questions at random. Consider the following case.

A multiple choice exam has 20 questions, each one with a choice of five answers. A student chooses their answers at random. What is the probability they pass the exam, if 40% is the pass mark?

- A student is attempting 20 questions, and each time has a chance of getting it right. This is repeated 20 times, so $n = 20$.
- Each time a student answers a question, they have a chance of 0.2 of getting it right, so $p = 0.2$.

Let X be the random variable of the number of questions the student gets right. Then;

$$P[X = r] = {}^{20}C_r 0.2^r\ 0.8^{20-r}.$$

To pass the exam, a student needs 8 questions or more right. The event is:

$$X \geq 8 \text{ which is } X = 8 \text{ or } X = 9 \text{ or} \ldots \text{ or } X = 20.$$

Since these are all mutually exclusive events, the probabilities are

$$P[X \geq 8] = P[X = 8] + P[X = 9] + \ldots + P[X = 20].$$

Each of these probabilities is calculated and the results summed up.

In practice, when calculating these probabilities, they will be quite low, and if a level of accuracy is chosen, say three decimal places, then the individual probabilities will quickly fall below the next level, say four decimal places. It is therefore quite easy to calculate. The results are;

$$P[X = 8] = {}^{20}C_8 0.2^8\ 0.8^{12} =$$

$$= 125{,}970 \text{x} 0.00000256 \text{x} 0.0687194 =$$

$$= 0.022161.$$

$$P[X = 9] = {}^{20}C_9 0.2^9\ 0.8^{11} =$$

$$= 167{,}960 \text{x} 0.00000512 \text{x}\ 0.08589934592$$

$$= 0.0073866.$$

Etc.

$$P[X = 12] = {}^{20}C_{12} 0.2^{12}\, 0.8^{8} =$$

$$125{,}970 \times 0.000000004096 \times 0.16777216$$

$$= 0.000086.$$

To three decimal places,

$$P[X \geq 8] = P[X = 8] + P[X = 9] + \ldots + P[X = 20] = 0.0321.$$

Thus the probability of passing is 0.0321, so the probability of failing is 0.9679.

3.4 The Normal Distribution

In the case of the binomial distribution, the random variable was a number of outcomes. This number, as a count, will always be an integer.

The next distribution we will look at is for continuous variables. These are variables which can take on any possible value, such as 1, 1.2, or 1.2345...

Most quantities in nature are like this, and very many of them follow the normal distribution. This includes measurable variables such as height, or rainfall averages. When dealing with a continuous variable, we can no longer talk about the chances of the variable being equal to particular values; since there are an infinite number of possible values, the chances of one particular value coming up are zero.

Instead we talk about the probability of the variable being in a particular range. So if a variable r has a binomial distribution, the events were

$$r = 0; r = 1 \text{ or } r > 4.$$

With a continuous variable, say height H, the events are

$$H > 160\text{cm or } H < 170\text{cm}.$$

The Normal distribution for a particular variable is characterised by two numbers. This is similar to the binomial distribution, where the value of the probabilities for the variable X is determined by the numbers n and p.

In the case of the normal distribution, the two numbers are

- The mean, or average, which is denoted μ, and
- The standard deviation, denoted σ.

Recall that for many cases of the use of the binomial distribution, the probability p of the event being counted often came from proportions of the population the sample came from.

In a similar way, the mean and the standard deviation may come from the analysis of a population. Thus the ideas of the average and standard deviation are the same as those from the study of data presentation.

If a variable X is normally distributed, this means that if a large number of values are generated of the variable, then they will be more likely to be close to the mean μ, and unlikely to be far from it.

Just how 'likely' or 'unlikely' is determined by the standard deviation σ.

The probabilities for the Normal distribution are given by the following equation. Let X be a normally distributed variable, with mean μ and standard deviation σ. Then the probability that it is less than a value a is

$$P[X < a] = \tfrac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^a e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x.$$

Now this is clearly a very complex integral, and in fact it is impossible to work out this integral using normal integration. Instead it must be calculated numerically using a computer.

This cannot be done for every possible choice of mean μ, and standard deviation σ. Instead it is done for one choice of μ and σ, and these are related to all others.

The table gives probabilities for a normally distributed random variable, with mean $\mu = 0$ and standard deviation $\sigma = 1$.

This is called the *standard* normal distribution. The probabilities given in the tables are that Z takes on a value greater than a given value a, in other words:

$$P[\, Z > a].$$

Its probabilities have been calculated by computer from the equation above.

3.4.1 Standardising

The Standard Normal Variable is a normal variable *Z* with mean 0 and standard deviation 1. These probabilities can then be used to find the probabilities of any normal variable. This is possible because of the following property of the normal distribution.

If the variable X is normally distributed with of mean μ, and standard deviation σ, then the variable Z, given by the relation

$$Z = \frac{X - \mu}{\sigma},$$

has mean 0 and standard deviation 1, in other words, it is the standard normal variable. This comes from the defining integral equation for the distribution. It can be shown that the event X > a is the same as the event

$$Z > \frac{a - \mu}{\sigma}.$$

This then means that the probabilities are the same:

$$P[X > a] = P\left[Z > \frac{a-\mu}{\sigma}\right].$$

An event involving *X* has been shown to be equivalent to an event involving *Z*, so if the probabilities for *Z* are known, then those for *X* are known too.

3.4.2 Summary of standardisation

If X is a normally distributed variable with mean μ and standard deviation σ, and Z is the normal variable with mean 0 and standard deviation 1, then

$$P[X > a] = P\left[Z > \frac{a-\mu}{\sigma}\right].$$

The normal variable can be reduced to one involving the standard normal variable. What this means in practice is that to find $P[X > a]$, firstly work out the number:

$$\frac{a-\mu}{\sigma}$$

and then look it up in the table which has been calculated for the standard normal distribution. It is then not necessary to recalculate the probabilities once they have been done for the standard variable.

3.4.3 Example of a standard normal distribution

The diameter of the ball bearings being produced in a factory is a normally distributed random variable, D with mean 4mm and standard deviation 0.01mm. What is the probability that a ball bearing chosen at random has a diameter greater than 4.015mm?

The question means we are calculating the probability

$$P[\, D > 4.015].$$

To find this probability, firstly work out the standardised number:

$$\frac{4.015-\mu}{\sigma}=\frac{4.015-4.0}{0.01}=1.5.$$

Use the fact that

$$P[\,D > 4.015] = P[\,Z > 1.5]$$

and then look it up in the table. It gives 0.0668.

So the result is that we now know that:

$$P[\,D > 4.015] = 0.0668.$$

The procedure here meant that we translated an event involving D to one involving Z.

Example

In the same situation, what is the probability that a ball bearing chosen at random has a diameter less than 4.0185 mm?

We are calculating $P[\ D < 4.0185]$. Standardise:

$$\frac{4.0185-\mu}{\sigma}=\frac{4.0185-4.0}{0.01}=1.85\,.$$

This means that

$$P[\,D < 4.0185] = P[\,Z < 1.85].$$

However, if we look this number 1.85 up in the table, the value given is the probability that Z is greater than 1.85:

$$P[\,Z > 1.85] = 0.0322.$$

To deal with this, use the fact that

$$P[\ Z < 1.85] = 1 - P[\ Z > 1.85].$$

Then

$$P[\ Z < 1.85] = 1 - 0.0322 = 0.9678.$$

So we have found that

$$P[\ D < 4.0185] = 0.9678.$$

To summarise these steps,

$$P[\ D < 4.0185] = P[\ Z < 1.85],$$

from standardising, and then

$$P[\,Z < 1.85] = 1 - P[\,Z > 1.85] = 1 - 0.0322 = 0.9678.$$

3.4.4 The Symmetry of the Standard Normal Distribution

The values in the log tables are probabilities $P[Z > a]$, for positive numbers a.

We must be able to extend this to probabilities like $P[Z > a]$ for negative values of a, or $P[Z < a]$ and for ranges, $P[a < Z < b]$.

The standard normal distribution has some important properties arising from its definition as an integral which make it possible to calculate these probabilities.

To handle problems such as $P[Z < a]$, for some number a, use the fact that $Z < a$ and $Z > a$ cover all eventualities, and so their probabilities add up to 1. It then follows that

$$P[\ Z < a] = 1 - P[\ Z > \mathrm{a}].$$

To handle negative values of a, we use a property of the standard normal variable called *symmetry*. It means that

$$P[Z < -a] = P[Z > a],$$

$$P[Z > -a] = P[Z < a].$$

These two rules can be summarised by saying that to change the direction of the inequality, the sign must be changed also.

This also applies in reverse – to change the sign, the direction of the inequality must be changed too.

Example

In the case of ball bearing production, what is the probability that a ball bearing chosen at random has a diameter greater than 3.985mm?

Calculate:

$$P[\ D > 3.985].$$

The problem here is that when we standardise, we are left with a negative number:

$$\frac{3.985 - 4.0}{0.01} = -1.5 .$$

Applying symmetry to this example:

$$P[Z > -1.5] = P[Z < 1.5].$$

Now use the fact that

$$P[Z < 1.5] = 1 - P[Z > 1.5].$$

The last probability can be found in the tables, so:

$$P[D > 3.985] = 1 - 0.0668 = 0.9332.$$

To summarise, first standardise to get:

$$P[D > 3.985] = P[Z > -1.5].$$

Then apply symmetry:

$$P[Z > -1.5] = P[Z < 1.5],$$

and now the basic law:

$$P[Z < 1.5] = 1 - P[Z > 1.5] = 1 - 0.0668 = 0.9332.$$

Example

The height of men is normally distributed, with mean 1.71 metres and standard deviation 0.11 metres. Find the probability that the height of a man chosen at random is:

- Greater than 1.76 metres.
- Less than 1.74 metres.
- Less than 1.64 metres.
- Greater than 1.54 metres.

Let H be the random variable of height in men. The mean and standard deviation are $\mu = 1.71$ $\sigma = 0.11$.

The first question here is straightforward. Standardise the value given, 1.76:

$$\frac{1.76-\mu}{\sigma} = \frac{1.76-1.71}{0.11} = 0.45\ .$$

This means that

$$P[H > 1.76] = P[Z > 0.45],$$

and this is then a matter of looking up the tables, to get 0.3264.

In summary

$$P[H > 1.76] = P[Z > 0.45] = 0.3264.$$

For the second part, standardising gives

$$\frac{1.74 - 1.71}{0.11} = 0.27.$$

It then follows that

$$P[\, H < 1.74] = P[\, Z < 0.27] =$$

$$= 1 - P[\, Z > 0.27] =$$

$$= 1 - 0.3936 = 0.6064.$$

For the last part, first standardise to get:

$$\frac{1.64 - 1.71}{0.11} = -0.64,$$

so that $P[H < 1.64] = P[Z < -0.64]$.

Then apply symmetry:

$$P[Z < -0.64] = P[Z > 0.64].$$

This is a straightforward case from the tables, giving a final answer of 0.2611.

$$P[Z < 1.85] = 1 - P[Z > 1.85].$$

In summary:

$$P[H < 1.64] = P[Z < -0.64] = P[Z > 0.64] = 0.2611.$$

3.4.5 Ranges

Recall the example of ball-bearings being produced in a factory – the diameter of the ball bearings is a normally distributed random variable, D with μ = 4mm, and σ = 0.01mm.

Consider the following question – what is the probability that a ball bearing chosen at random has a diameter between 4.015mm and 4.02mm?

To address this problem, consider the following three possible events; the diameter is greater than 4.015mm, greater than 4.02mm and between the two.

These probabilities are

$$P[D > 4.015],\ P[D > 4.02],\ \text{and}\ P[4.015 < D < 4.02].$$

Since 4.015 is the lower number, the event ‘*D* between 4.015 and 4.02’ and ‘*D* greater than 4.02’, can be combined to be ‘*D* greater than 4.015’. The three probabilities are connected by the following relation:

$$P[4.015 < D < 4.02] + P[D > 4.02] = P[D > 4.015].$$

Bringing one of the probabilities across the equals sign gives:

$$P[4.015 < D < 4.02] = P[D > 4.015] - P[D > 4.02].$$

Both of the probabilities on the RHS can be calculated in the usual way. Standardising gives

$$P[D > 4.015] = P[Z > 1.5] = 0.0668,$$

and

$$P[D > 4.02] = P[Z > 2.0] = 0.0228.$$

The result of the calculation is:

$$P[4.015 < D < 4.02] = 0.0668 - 0.0228 = 0.033.$$

The key step here was writing the probability of the range as

$$P[4.015 < D < 4.02] = P[D > 4.015] - P[D > 4.02].$$

3.4.6 Example of ranges – Heights

The height of men is normally distributed, with mean 1.71 metres and standard deviation 0.11 metres. Find the probability that the height of a man chosen at random is:

- Between 1.74 metres and 1.76 metres.

- Between 1.64 metres and 1.76 metres.

Let H be the random variable of height in men, with

$$\mu = 1.71 \text{ and } \sigma = 0.11.$$

For the first question, the probability is broken up as

$$P[1.74 < H < 1.76] = P[H > 1.74] - P[H > 1.76].$$

Standardise the values 1.76 and 1.74 gives:

$$\frac{1.74-1.71}{0.11}=0.27 \text{ and } \frac{1.76-1.71}{0.11}=0.45.$$

This means that

$$P[H > 1.74] = P[Z > 0.27] = 0.3936.$$

and

$$P[H > 1.76] = P[Z > 0.45] = 0.3264,$$

Then the final step is:

$$P[1.74 < H < 1.76] = 0.3936 - 0.3264 = 0.0672.$$

For the second part, the probability is

$$P[1.64 < H < 1.76] = P[H > 1.64] - P[H > 1.76].$$

Standardise 1.64, in the same way as above, to get:

$$P[H > 1.64] = P[Z > -0.64].$$

Then apply symmetry:

$$P[Z > -0.64] = P[Z < 0.64],$$

and the laws of probability

$$P[Z < 0.64] = 1 – P[Z > 0.64] = 1 – 0.2611 = 0.7389.$$

From the first part of the question, we already know that

$$P[H > 1.76] = P[Z > 0.45] = 0.3264.$$

So the overall result is

$$\mathrm{P}[1.64 < H < 1.76] = P[H > 1.64] – P[H > 1.76] =$$

$$= 0.7389 – 0.3264 = 0.4125.$$

3.4.7 Example – Sales from a Petrol Station

The manager of a petrol station finds that the number of litres of petrol sold in a week, S, is a normally distributed random variable with mean 2,500L and standard deviation 240L.

Find the probability that they sell between 2,400L and 2,600L of petrol in a week.

The probability is a range, and is split up as

$$P[2{,}400 < S < 2{,}600] = P[S > 2{,}400] - P[S > 2{,}600].$$

Standardising the first, 2,400, gives –100/240 = –0.42. From this it follows that

$$P[S > 2{,}400] = P[Z > -0.42],$$

and applying symmetry gives

$$P[S > 2{,}400] = P[Z > -0.42] = P[Z < 0.42].$$

This is one step away from a probability in the tables:

$$P[Z < 0.42] = 1 - P[Z > 0.42] = 1 - 0.3372 = 0.6628.$$

The second is a straightforward case of standardising:

$$P[S > 2{,}600] = P[Z > 0.42] = 0.3372.$$

The overall result of this calculation is

$$P[2{,}400 < S < 2{,}600] = P[S > 2{,}400] - P[S > 2{,}600] =$$

$$= 0.6628 - 0.3372 = 0.3256.$$

3.5 Working in Reverse

Consider the following example: A petrol station finds that the number of litres of petrol sold in a week, S, is a normally distributed random variable with mean 2,500L and standard deviation 200L. Calculate how much petrol they should stock so the probability of running out is 0.02.

This is a new type of question – we have the probability, we need the figure such that the probability that the sales exceed it is 0.02.

The sales of petrol is a normally distributed random variable, S, with mean and standard deviation

$$\mu = 2{,}500\text{L, and } \sigma = 200\text{L.}$$

Now find a value a such that $P[S > a] = 0.02$.

If we find the probability 0.02, within the body of the tables, we see that it came from the value 2.05. Mathematically,

$$P[Z > 2.05] = 0.02.$$

Thus we need the value of petrol sales, which was standardised to give 2.05, and then we will have our figure.

To standardise, we subtract the mean μ (= 2,500L), and divide by the standard deviation σ (= 200L).

To reverse this, we multiply by σ and add μ.

To look at this mathematically, return to the equation that is the basis for standardising, and apply our information to it.

We have, firstly,

$$P[S > a] = 0.02,$$

and then from the tables,

$$P[Z > 2.05] = 0.02.$$

The defining equation is

$$P[S > a] = P\left[Z > \frac{a-\mu}{\sigma}\right].$$

We know that the left-hand side is 0.02, and so comparing the right-hand side with the observation

$$P[Z > 2.05] = 0.02,$$

we can say that

$$\frac{a - \mu}{\sigma} = 2.05.$$

This can be looked on as an equation to solve for a, yielding

$$a = 2.05\sigma + \mu.$$

Using the values we have,

$$a = 2.05\text{x}200 + 2{,}500 = 2{,}910.$$

When we had the sales figures, we standardised to get the Z values, and now that we have the Z values, we reverse to get the sales figures.

3.5.1 Example – Ball Bearing Production

Recall the diameter of the ball bearings produced in a factory is a normally distributed random variable, D with mean and standard deviation μ = 4mm, and σ = 0.01mm. Find a diameter such that there is only a 5% chance a ball bearing exceeds it. Repeat for a 2.5% chance.

For the first question, we are looking for a number a, a diameter, such that

$$P[D > a] = 0.05.$$

If we find the probability 0.05 within the tables, we see it came from the number 1.65:

$$P[Z > 1.65] = 0.05.$$

We can look on this value 1.65 as the result of standardising a.

$$P[D > \mathrm{a}] = P[Z > 1.65] = 0.05.$$

So we must 'reverse standardise' this figure.

This is simply:

$$a = 1.65\mathrm{x}0.01\mathrm{mm} + 4\mathrm{mm} = 4.0165\mathrm{mm}.$$

For the second part, we are looking for a number a, a diameter, such that

$$P[D > \mathrm{a}] = 0.025.$$

If we find the probability 0.025 within the tables, we see it came from the number 1.96:

$$P[Z > 1.96] = 0.025.$$

So we must ‘reverse standardise’ this figure. This is simply:

$$a = 1.96\mathrm{x}0.01\mathrm{mm} + 4\mathrm{mm} = 4.0196\mathrm{mm}.$$

3.5.2 Example – Heights again

The height of men is normally distributed, with $\mu = 1.71$ $\sigma = 0.11$. Find heights such that the probability that a man chosen at random is taller than:

- 0.025.
- 0.1.

For the first question, look for a number a, a height, such that

$$P[H > a] = 0.025.$$

From the tables, the probability 0.025 came from the number 1.96:

$$P[Z > 1.96] = 0.025.$$

Reverse standardise this figure:

$$a = 1.96\text{x}0.11\text{m} + 1.71\text{m} = 1.9256\text{m}.$$

For the second part, we are looking for a height a, such that:

$$P[H > a] = 0.1.$$

The probability 0.1 comes from the number 1.28:

$$P[Z > 1.28] = 0.1.$$

Reverse standardise this figure:

$$a = 1.28\text{x}0.11\text{m} + 1.71\text{mm} = 1.8508\text{m}.$$

So we can say that

$$P[H > 1.8508] = 0.1.$$

Consider the following, slightly different case:

3.5.3 Example – Probabilities above 0.5

For the example of height, find heights such that the probability that a man chosen at random is not as tall is (i) 0.9, (ii) 0.1.

Find a height such that the probability that a man chosen at random is taller is 0.95.

Let H be the random variable of height in men, with

$$\mu = 1.71 \text{ and } \sigma = 0.11.$$

For the first question, we are looking for a height a such that

$$P[H < a] = 0.9.$$

If we try and find the probability 0.9 in the tables, it is not there – the way to deal with this is simply change the question. Equivalently, we are looking for a height a such that

$$P[H > a] = 0.1.$$

This is a straightforward case:

$$P[Z > 1.28] = 0.1.$$

Reverse standardise this figure:

$$a = 1.28\text{x}0.11\text{m} + 1.71\text{mm} = 1.8508\text{m}.$$

For the second part, we are looking for a height *a*, such that:

$$\text{P}[H < \text{a}] = 0.1.$$

The probability 0.1 came from the number 1.28:

$$P[Z > 1.28] = 0.1.$$

The inequality signs go the wrong way. Use symmetry: if

$$P[Z > 1.28] = 0.1, \text{ then } P[Z < –1.28] = 0.1.$$

We can now just reverse standardise this figure:

$$a = –1.28 \text{x} 0.11\text{m} + 1.71\text{mm} = 1.5692\text{m}.$$

Thus

$$P[\ H < 1.5692\text{m}] = 0.1.$$

For the third question, we are looking for a height a such that

$$P[\, H > a] = 0.95.$$

We must switch around to get probabilities less than 0.5:

$$P[\, H < a] = 0.05.$$

If we find the probability 0.05 within the tables, we see it came from the number 1.65. But the inequality goes the wrong way:

$$P[\, Z > 1.65] = 0.05.$$

Using symmetry

$$P[\ Z < -1.65] = 0.05.$$

‘Reverse standardise’ this figure:

$$a = -1.65\text{x}0.11\text{m} + 1.71\text{m} = 1.5285\text{m}.$$

3.5.4 Example – Middle Ranges

For the case of height in men, find the middle 90% of heights.

This means two values a and b such that $P[a < H < b] = 0.9$, but because it is the *middle* 90%, the probabilities of H being above or below the range are the same.

This gives us a way of tackling the calculation, since we have two probabilities:

$$P[H < a] = 0.05 \text{ and } P[H > b] = 0.05.$$

The probabilities of these three possibilities add up to 1.

(The 0.05 comes from $0.05 = ½(1 – 0.9)$.)

These cases are just like the problems above, and both involve the same z-value from the tables, except for a difference in sign.

We have done the second question already: $P[H > b] = 0.05$. The value from the tables is 1.65, and this gives

$$b = 1.65 \text{x} 0.11\text{m} + 1.71\text{m} = 1.8915\text{m}.$$

The second part is done above, and involved the value –1.65; the calculation was

$$a = -1.65 \times 0.11\text{m} + 1.71\text{m} = 1.5285\text{m}.$$

So once we have the *z*-value for one part, we have it for the other, with a change of sign.