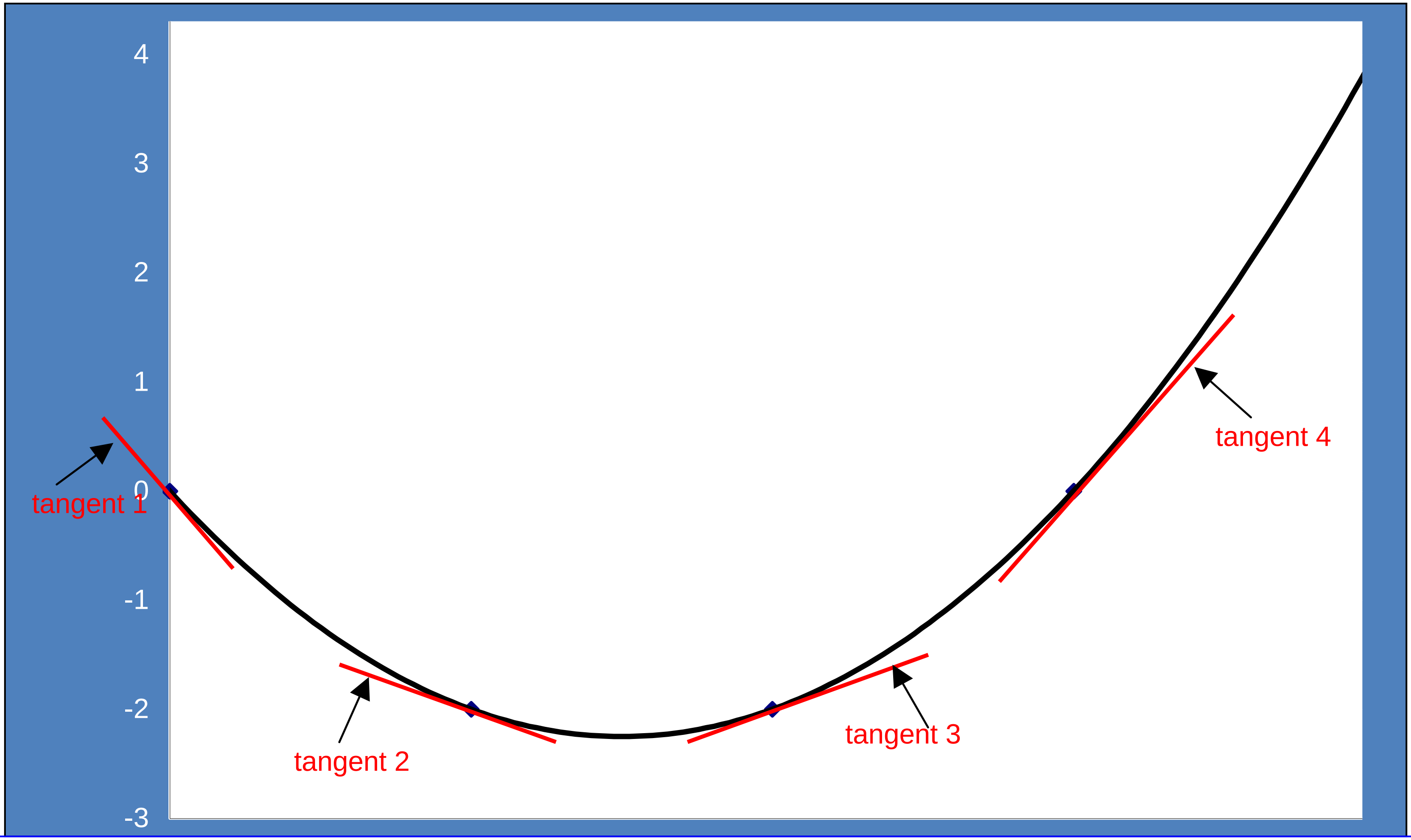
DIFFERENTIAL CALCULUS

Contents

Introduction

Consider the curve below. It has an infinite number of tangents, the slopes of which change with respect to the *x*-value. There is no longer a constant slope value to show how the variables change with respect to each other. The value of *y* with respect to *x* varies.



Introduction (*cont.*)

For every tangent point or (x, f(x)) or (x, y(x)), there is a tangent line with a different slope.

There is an infinite number of tangent points on a curve. If we plotted the tangents at each point we would be able to see how the shape of the curve changes.

If we think of the curve as being a series of y-values dependent on x-values, then we want to find out how y changes with respect to x at any instantaneous point on the curve.

This is the basis of *differentiation*.

Aside: The following is of interest and good to have as a reference but will not be on the exam

A Quick Lesson on Limits

Take the function $f(x) = 2x + 1$

$$f(4) = 2(4) + 1 = 9$$

$$f(4.8) = 2(4.8) + 1 = 10.6$$

$$f(4.9) = 2(4.9) + 1 = 10.8$$

$$f(4.999) = 2(4.999) + 1 = 10.998$$

It is clear that as x gets closer and closer to 5, $f(x)$ gets closer to 11. This statement can be written mathematically as,

Scan me!!

$$\lim_{x \to 5} (2x + 1) = 11 \quad or \quad \lim_{x \to 5} f(x) = 11$$

A Quick Lesson on Limits (*cont.*)

Now look at the function $g(x) = \dfrac{x-2}{x^2-4}$

What is $\lim_{x \to 2} \dfrac{x-2}{x^2-4}$? In other words, as x gets closer and closer to 2, what will $g(x)$ get closer to ?

The problem in this case is that $g(2) = \dfrac{0}{0}$, which is indeterminate.

To find the limit we will first use a rough and ready approach and then a proper mathematical approach.

A Quick Lesson on Limits (*cont.*)

Rough and Ready Approach:

Here is a table of values of x and g(x) correct to 4 decimal places. The values are getting closer and closer to 2.

x	g(x)
1	0.3333
1.5	0.2857
1.8	0.2632
1.9	0.2564
1.99	0.2506
1.999	0.2501

It would APPEAR that as x approaches 2, g(x) approaches .25.

This can be proven in the following mathematical method.

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Mathematical Approach:

$$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{(x-2)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}$$

The Laws of Limits

$$\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a} f(x) \times g(x) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

The Laws of Limits (*cont.*)

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Example: Evaluate $\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt{x+3}-2}$

$$f(x)=\frac{x-1}{\sqrt{x+3}-2} \qquad f(1)=\frac{1-1}{2-2}=\frac{0}{0}$$ which is indeterminate

But

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}$$

$$=\lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x+3-2)} = \lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x-1)}$$

$$=\lim_{x\to 1}(\sqrt{x+3}+2) = (\sqrt{4}+2)=2+2=4$$

Check: let x = 0.999, $f(x)$ = 3.99975, which is very near 4.

End Aside

Back to Differentiation

Mathematically:

Differentiate a function $y = f(x) = y(x)$

means that we get an expression for the rate of change of y with respect to x. There is a shorthand way of expressing this, as follows:

$$\text{Rate of change of y with respect to x} = \frac{dy}{dx} = f'(x)$$

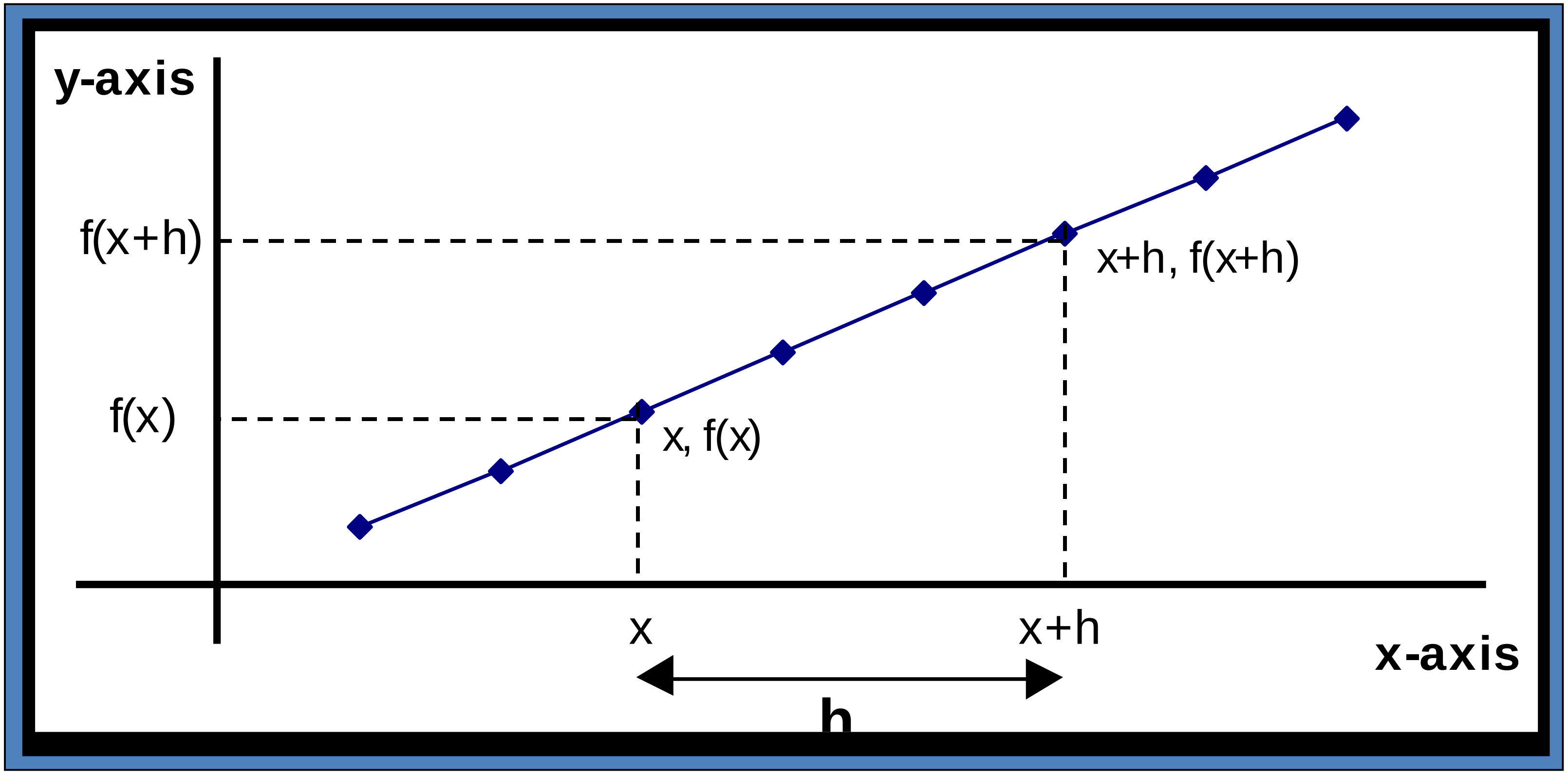
Definition:

Calculus is the area of mathematics which covers situations in which one physical quantity changes with respect to another.

Approximating the Slope

Consider the straight line below as a chord on a curve. The slope, m, is given by:

$$m = \frac{f(x+h) - f(x)}{h}$$



Approximating the Slope (*cont.*)

As h approaches zero, the chord approaches a tangent to the curve at a single point. The distance between the 2 points above is h.

The mathematical notation for letting h approach zero is:

$$\lim_{h \to 0}$$

Therefore the above expression becomes:

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This gives us an expression for the slope of a tangent to the curve in terms of x, since h is a distance along the x-axis.

Approximating the Slope (*cont.*)

The rate of change of one quantity with respect to another is called its *derivative*.

Therefore if we have a function with 2 variables, then we can get the derivative of the function using the formula above.

This is the basis of *differentiation*.

Hence, for a function $f(x) = y(x)$

The rate of change of y with respect to x is:

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Approximating the Slope (*cont.*)

$f'(a)$ is the gradient of the tangent to the curve at the point a.

The gradient of the chord gives the average rate of change of a function over an interval. The gradient of the tangent or the derivative gives the instantaneous rate of change of the function at a point.

Differentiation is the rate of change of one quantity with another.

Therefore, it is the

$$\frac{\textit{Change in dependent variable}}{\textit{Change in independent variable}}$$

as the bottom interval approaches zero. There are many physical quantities related in this manner.

Approximating the Slope (*cont.*)

Example: The average rate of change is that taken over an interval. For example the average speed is given by

$$\frac{\textit{Change in distance}}{\textit{Change in time}}$$

The smaller the time interval gets the closer we get to a value for the speed at an instantaneous moment. In other words this is an approximation to the instantaneous rate of change of distance with respect to time.

Differentiating from First Principles

To differentiate from first principles we use the formula,

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: Find an expression for the rate of change with respect to x of the simple quadratic function

$$y = x^2 \quad or \quad f(x) = x^2$$

$$f(x) = x^2$$

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$$

Differentiating from First Principles (*cont.*)

Therefore,
$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} 2x + h$$

$$= 2x$$

Therefore the rate of change of the function with respect to x is $2x$.

Differentiating from First Principles (*cont.*)

What does this mean graphically?

From our earlier theory, the above solution of $2x$ must allow us to calculate the slope of the function at any point.

Take the point (2, 4). What is the slope if the tangent at this point?

At this point we have x = 2, therefore the slope of the tangent at this point is $2x = 2$ by $2 = 4$.

In summary we have found an expression that allows us to calculate the slope of the curve for any value of x.

Remember that the rate of change of the slope gives us the rate of change of the function with respect to x or dy/dx.

Differentiating from First Principles (*cont.*)

Example: Using the equation $y = f(x) = x^2$, find

i) 5 points on the curve

ii) the slope at each of the 5 points

$x = -2, \quad y = x^2 = (-2)^2 = 4$ (-2, 4) is on the curve

$x = -1, \quad y = x^2 = (-1)^2 = 1$ (-1, 1) is on the curve

$x = 0, \quad y = x^2 = (0)^2 = 0$ (0, 0) is on the curve

$x = 1, \quad y = x^2 = (1)^2 = 1$ (1, 1) is on the curve

$x = 2, \quad y = x^2 = (2)^2 = 4$ (2, 4) is on the curve

Differentiating from First Principles (*cont.*)

$$x = -2, \quad slope = 2x = 2(-2) = 4$$

$$x = -1, \quad slope = 2x = 2(-1) = -2$$

$$x = 0, \quad slope = 2x = 2(0) = 0$$

$$x = 1, \quad slope = 2x = 2(1) = 2$$

$$x = 2, \quad slope = 2x = 2(2) = 4$$

Differentiating from First Principles (*cont.*)

How do these results substantiate our earlier theories?

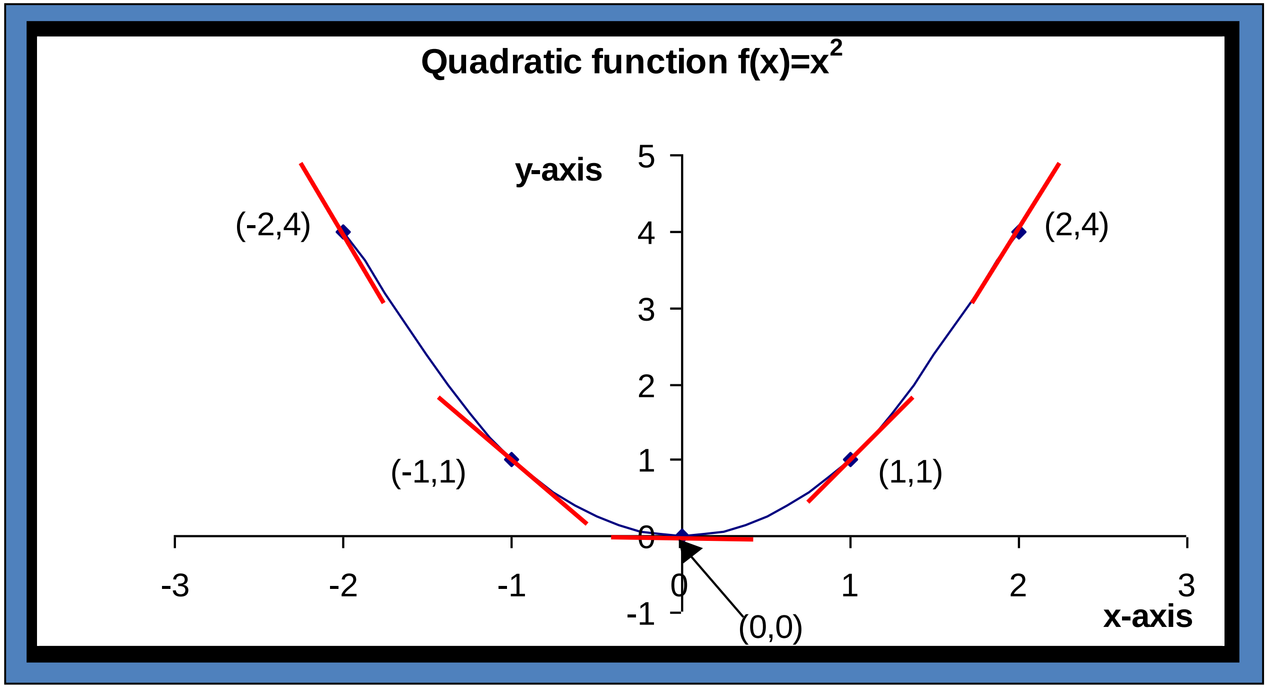
1. there is different slope at each different value of x

2. the slope of the lowest point (minima) is zero

3. the slope of the points where x is less than the minima all have a different sign to those after the minima. Recall that a negative slope means the slope is decreasing from left to right and vice versa for a positive slope.

Differentiating from First Principles (*cont.*)

If we plot the 5 points that we found for the quadratic function $f(x) = x^2$ we get the following graph:



Points to note on graph:

- graph is symmetrical about the y-axis
- the slope at the lowest point (0, 0) is zero (horizontal line)
- because the graph is symmetrical, the slope at point (-1, 1) is the same in magnitude but opposite in direction to that at point (1, 1). The same holds for other symmetrical pairs of points.

Differentiating from First Principles (*cont.*)

Example: Differentiate the following function from first principles:

$$y = f(x) = x^2 - 1$$

$$f(x) = x^2 - 1 \qquad f(x+h) = (x+h)^2 - 1 = x^2 + 2xh + h^2 - 1$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 1 - (x^2 - 1)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 1 - x^2 + 1)}{h} \qquad = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h \qquad = 2x$$

Therefore the rate of change of the function with respect to x is $2x$.

Six Important Differential Rules

Rule 1: The Sum Rule

If $u(x)$ and $v(x)$ are two functions, and if $f(x) = u(x) + v(x)$

then,
$$\boxed{\frac{df}{dx} = \frac{du}{dx} + \frac{dv}{dx}}$$

Rule 2: The Difference Rule

If $u(x)$ and $v(x)$ are two functions, and if $f(x) = u(x) - v(x)$

then,
$$\boxed{\frac{df}{dx} = \frac{du}{dx} - \frac{dv}{dx}}$$

Six Important Differential Rules (*cont.*)

Rule 3: The Product Rule

If $u(x)$ and $v(x)$ are two functions, and if $f(x) = u(x)v(x)$

then, $$\boxed{\frac{df}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$ (Will be given in exam)

Rule 4: The Quotient Rule

If $u(x)$ and $v(x)$ are two functions, and if $f(x) = \dfrac{u(x)}{v(x)}$

then, $$\boxed{\frac{df}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}}$$ (Will be given in exam)

Six Important Differential Rules (*cont.*)

Rule 5:

If $g(x)$ is a function and if $f(x) = kg(x)$

then,

$$\boxed{\frac{df}{dx} = k\frac{dg}{dx}}$$

Rule 6:

If $f(x) = x^n$, then $\boxed{\frac{df}{dx} = nx^{n-1}}$ for $n \in R$ (any real number)

Six Important Differential Rules (*cont.*)

Example: Differentiate the following:

$$5x^3+8x^2+11x+7, \quad \sqrt{x}, \quad \frac{5x-1}{x^2+3}$$

Solutions:

$$y=5x^3+8x^2+11x+7$$

$$\frac{dy}{dx}=15x^2+16x+11$$

$$y=\sqrt{x}=x^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2}\frac{1}{\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

$$y=\frac{5x-1}{x^2+3}=\frac{u}{v}$$

$$\frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}=\frac{(x^2+3)(5)-(5x-1)(2x)}{(x^2+3)^2}$$

$$=\frac{15+2x-5x^2}{(x^2+3)^2}$$

Six Important Differential Rules (*cont.*)

Example: If $s(t) = \dfrac{10}{t^2}$, find $\dfrac{ds}{dt}$

Do this yourselves now !!

Solution:

$$\frac{ds}{dt} = -20t^{-3} = -20\frac{1}{t^3} = \frac{-20}{t^3}$$

This is called the differentiation of *s* with respect to *t*.

Differentiating Trigonometric Functions

We can differentiate trigonometric functions by simply using the formulae on the right.

These formulae may be found on Page 41 of the standard Mathematics Tables

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

Differentiating Trigonometric Functions (*cont.*)

Example: Differentiate $x \sin x$

Do this yourselves now !!

Solution:

$$y = x \sin x = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x(\cos x) + \sin x(1)$$

$$= x \cos x + \sin x$$

The Chain Rule for Composite Functions

Let us consider these three functions:

$$f(x) = \sin x, \quad g(x) = x^2, \quad h(x) = 7x + 1$$

Here is a list of composite functions made up from pairs of these:

$$fg(x) = \sin(x^2)$$

$$gf(x) = (\sin x)^2 = \sin^2 x$$

$$fh(x) = \sin(7x + 1)$$

$$hf(x) = 7\sin x + 1$$

$$gh(x) = (7x + 1)^2$$

$$hg(x) = What\ do\ you\ think? \quad 7x^2 + 1$$

The Chain Rule for Composite Functions (*cont.*)

The Chain Rule is the rule used to differentiate composite functions:

If $y = fg(x)$, let $u = g(x)$

Then,
$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example: Differentiate $(3x+2)^5$

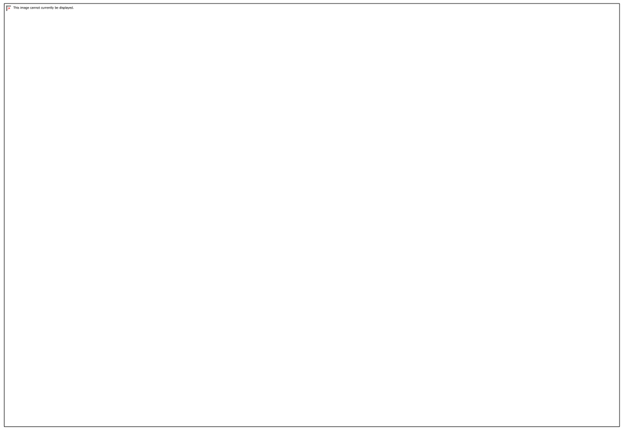
Solution: Let $u = (3x+2)$, hence $y = u^5$

The Chain Rule for Composite Functions (*cont.*)

Example: Differentiate $\sin(7x^2+1)$

Do this yourselves now !!

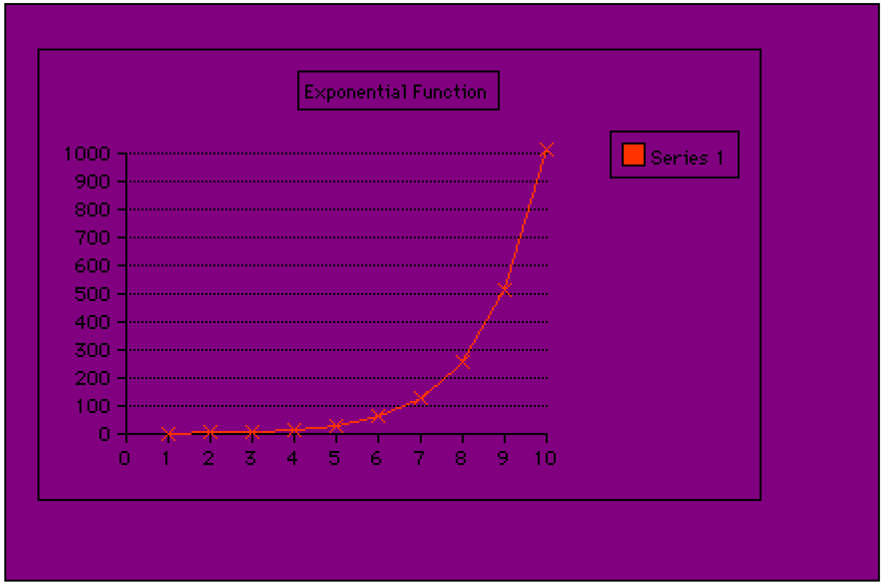
Solution: Let $u=(7x^2+1)$, hence $y=\sin u$



The Exponential Function, e^x

The irrational number e, which is equal to 2.718 correct to 3 decimal Places has a special property…

If you draw the graph of $y = e^x$, you will find that the slope of the tangent at any point is equal to the y-coordinate at that point.



We can therefore write,
$$\frac{d}{dx}(e^x) = e^x$$

The Natural Logarithmic Function, $\log_e x$

This function is sometimes written in the form $\ln(x)$; l for log and *n* for natural.

The interesting thing about a graph of this function is that the slope of the tangent at $x = 1$ is 1, at $x = 2$ it is 1/2, at $x = 3$ it is 1/3 and so on.

In fact, the slope at any point $(x, \ln x)$, is given by $1/x$.

We write

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Second Derivatives

If $y = f(x)$, then $\dfrac{dy}{dx}$ is known as the first derivative of y with respect to x.

The second derivative is the differential of $\dfrac{dy}{dx}$, which is written as, $\dfrac{d^2y}{dx^2}$

For example, if

$$y = x^3, \quad \frac{dy}{dx} = 3x^2, \quad \frac{d^2y}{dx^2} = 6x$$

Second Derivatives (*cont.*)

Example: Write down $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of these:

$$y = x^5, \quad y = 4x^3$$

Solutions:

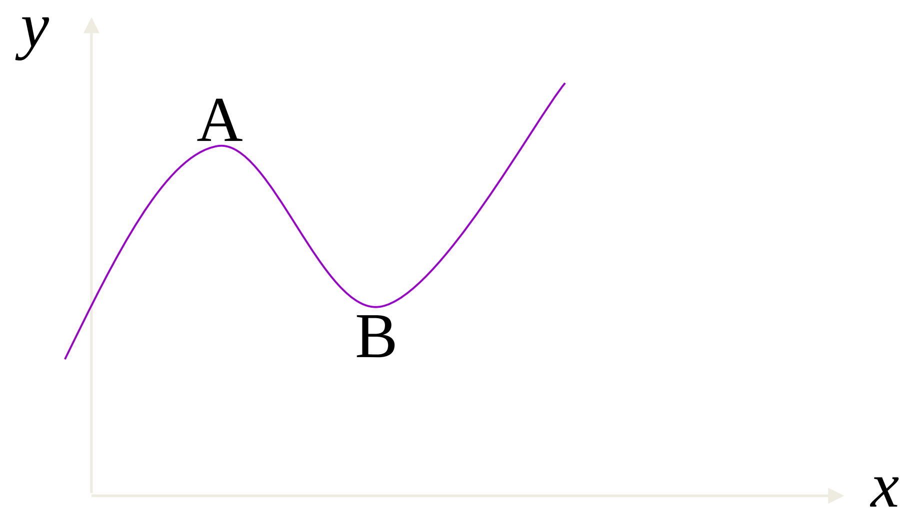
$$y = x^5 \quad \Rightarrow \quad \frac{dy}{dx} = 5x^4 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 20x^3$$

$$y = 4x^3 \quad \Rightarrow \quad \frac{dy}{dx} = 12x^2 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 24x$$

Maximum and Minimum Values

In the graph of the function below, A is known as a local maximum and B is known as a local minimum.

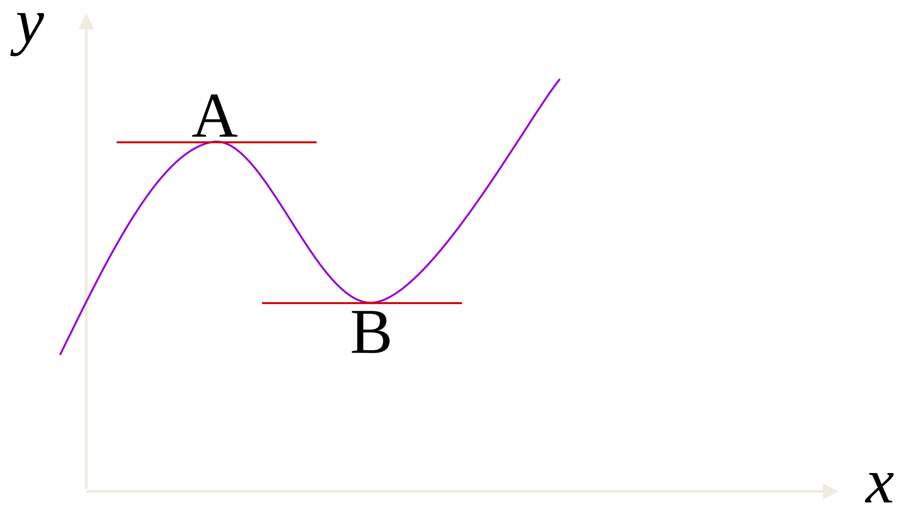
It is important to note that A is not the highest point on the graph, however, in the locality of A, A is the highest. This can be clearly seen since if we move away from A in either direction, the graph decreases. The same is true for B. As we move away from B in either direction, the graph increases.



Maximum and Minimum Values (*cont.*)

At the points A and B, the tangents are parallel to the x-axis. Therefore, we can write

$$\frac{dy}{dx} = 0$$



We can now say that at maximum and minimum turning points,

$$\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} \text{ does not exist.}$$

Maximum and Minimum Values (*cont.*)

To distinguish if a point is a maximum or minimum, we use what is known as the first derivative test. Basically,

To the left of a maximum point, $\dfrac{dy}{dx}$ is positive.

To the right of a maximum point, $\dfrac{dy}{dx}$ is negative.

To the left of a minimum point, $\dfrac{dy}{dx}$ is negative.

To the right of a minimum point, $\dfrac{dy}{dx}$ is positive.

Maximum and Minimum Values (*cont.*)

Example: Determine the position of any maximum or minimum points on the curve $y = 2x - x^2$

$$\frac{dy}{dx} = 2 - 2x$$

At max or min points, $\frac{dy}{dx} = 0$

$$2 - 2x = 0$$

$$2x = 2$$

$$x = 1$$

Let x = 0.9:

Since the derivative is positive to the left and negative to the right, the point is a maximum.

Let x = 1.1: $\frac{dy}{dx} = 2 - 2x = -0.2$

Maximum and Minimum Values (*cont.*)

Another method we can use is called the second derivative test. We use the following simple rules:

If $y' = 0$ and $y'' < 0$ at a point, then the point is a maximum point.

If $y' = 0$ and $y'' > 0$ at a point, then the point is a minimum point.

If $y' = 0$ and $y'' = 0$ at a point, then the second derivative test fails and we must return to the first derivative test.

Maximum and Minimum Values (*cont.*)

Example: Use the second derivative test to find all maximum and minimum turning points of the function

$$y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + 2$$

$$\frac{dy}{dx} = x^2 - x - 6 = (x+2)(x-3) = 0 \quad \therefore \quad x = -2, \quad x = 3$$

$$\frac{d^2y}{dx^2} = 2x - 1$$

$$x = -2: \quad \frac{d^2y}{dx^2} = -5 \quad \text{therefore max}$$

$$x = 3: \quad \frac{d^2y}{dx^2} = 5 \quad \text{therefore min}$$

$$x = -2: \quad y = 28/3$$

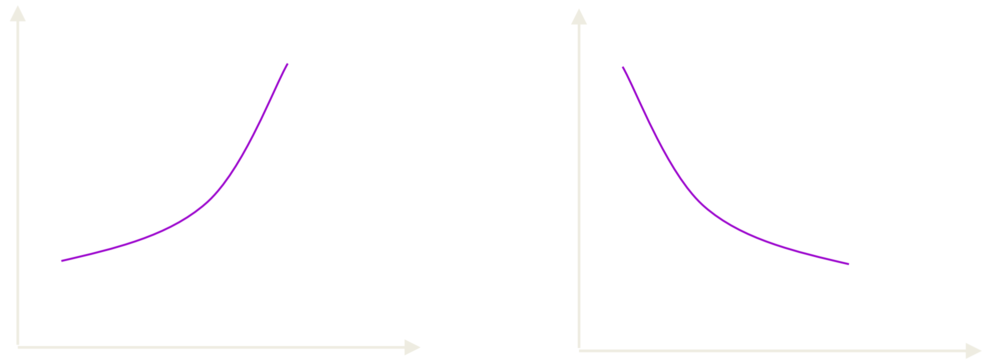
$$x = 3: \quad y = -23/2$$

maximum point is (-2, 28/3)
minimum point is (3, -23/2)

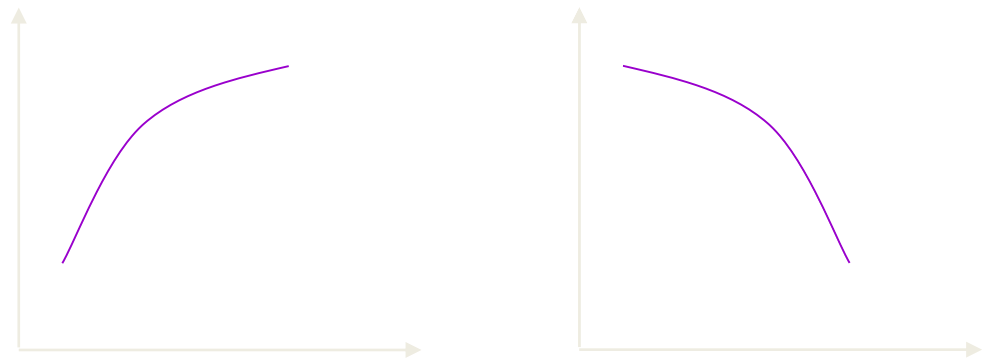
Points of Inflection

Look at the curves shown below. The top two are said to be **concave up**, while the bottom two are said to be **concave down**.

Concave up:

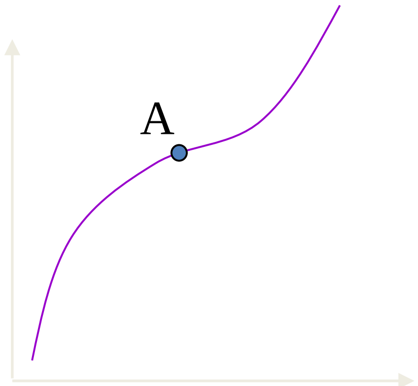


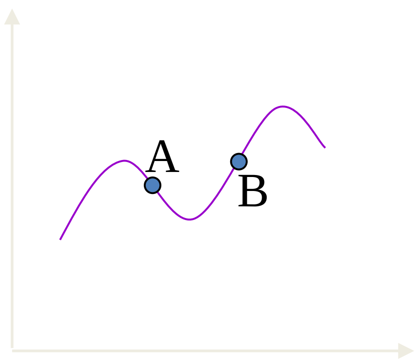
Concave down:



Points of Inflection (*cont.*)

A point at which concavity changes from concave up to concave down, or vice versa, is called a **point of inflection**.





A and B are points of inflection.

A curve is concave up when y”> 0, and concave down when y”< 0.