PROBABILITY

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Introduction

Probability is the area of mathematics and statistics which deals with chance. The two ideas central to the topic are <u>events</u> and the <u>probability of their occurrence</u>.

An *event* is a well-defined occurrence that may or may not happen.

To be discussed formally in mathematics, an event must be exactly described. If a fair dice is thrown, there are six possible outcomes, in other words, the number left facing up. Any events can be defined in terms of this number. Taking the number itself, there are six well-defined possible events – each number.

Alternatively, an event could be the occurrence of an even number, or a number less than 4. In each of these cases, when a number comes up on the dice, it is immediately clear whether or not the criteria defining the event have been satisfied.

For a less clear example, consider the event of a “White Christmas.”

Although our image of a White Christmas may involve fields of white and rows of snowmen, the bookies have an exact description, in order that they can decide if it has occurred, and whether or not they must pay out any money.

In this case, it is a certain amount of snow at certain weather stations throughout Ireland.

The *probability* of an event is a measure of how likely it is to occur. It is a number between 0 and 1. A probability of 0 for an event means it will not occur, while a probability of 1 means it definitely will occur.

Notation

Mathematically, an event can be denoted by any symbol. Thus the event E could be the event of a particular number showing up on a dice. The probability of event E is written mathematically as $P[E]$.

The probability of an event can be defined more rigorously if it may occur many times.

For example, in the case of the dice, we are concerned with the result of a particular action; the throwing of the dice. Such an action is referred to as an *experiment*.

The event is now the result of this experiment, that is, it is defined in terms of one or more of the outcomes of the experiment. Quantifying how likely an event is to occur can now be understood as the proportion of times it occurs if the experiment is repeated many times.

In the example of the dice, when the experiment is done, that is, the dice has been thrown, we are interested in the number left facing up. The set of possible outcomes is

$$\{1, 2, 3, 4, 5, 6\}.$$

The various possible events can be defined in terms of these outcomes. To see what the probabilities are, the dice can be thrown many times, and the outcomes, and so events, counted.

Even without doing this, it would be expected that after a large number of throws, each side would come up a roughly equal number of times. Thus the probabilities of various events may be derived from this consideration.

We will now frame a definition of probability and, with this in mind, find the probability of a given number coming up.

Definition:

Let E denote an event, which is the possible outcome of an experiment. In order to discuss it meaningfully, it must be in some sense repeatable.

Recall that the probability of event E is written mathematically as $P[E]$. The probability $P[E]$ is the proportion of times event E occurs, as the experiment is repeated a very large number of times.

Mathematically, if the experiment is repeated N times, and E turns up N_E times, then the probability of E occurring, is

$$P[E] = \lim_{N \to \infty} \frac{N_E}{N}$$

Consider the case of the dice: what are the probabilities of getting 1, 2, 3 and so on?

Let us denote these probabilities as $P[1]$, $P[2]$, etc. Let the dice be thrown N times, where N is a large number. Each face will come up approximately the same number of times. Call this number be n. Then we would expect that $n = N/6$.

From the definition of probability

$$P[1] = \frac{n}{N} = \frac{N/6}{N} = \frac{1}{6}$$

and the calculation for all the other numbers is the same.

Example

Consider the throw of two dice. What is the probability of the result adding up to 5?

With each dice having 6 possible outcomes, there are 36 possible outcomes in all. For a pair of numbers, with either number going from 1 to 6, the possibilities are

(1,1), (1,2), …, (1,6), (2,1), (2,2),…, (2,6),… etc…

(6,1), (6,2),…, (6,6).

All of these are equally likely to come up. Only 4 have the correct sum of 5:

(1, 4), (2, 3), (3, 2) and (4, 1).

If the dice are thrown a large number of times, then each pair comes up 1/36 of the time, and pairs adding up to 5 come up 4/36 of the time.

For example, if the dice were thrown 36 million times, then the right pairs would come up roughly 4 million times. Thus the probability is
$4/36 = 1/9$.

Note the distinction between the outcome of the experiment; the numbers which turn up, and the event, which is they add up to 5.

As this example shows, the probability of a particular event can often be found by considering all the possible outcomes of the experiment being run, and seeing how they compare.

This is particularly useful if there is a basic list of outcomes, all of which are equally likely.

Example

In the same experiment, rolling two dice, what is the probability that the two numbers, when multiplied, give 6?

The possibilities are:

$$(1,6), (2,3), (3,2), (6,1).$$

There are four possible outcomes that give 6, so the probability is $4/36 = 1/9$.

Example

For the case of two dice, what is the probability that the sum is below 6?

The most methodical way of counting the number of outcomes which satisfy the definition of the event is to calculate the sum for each possible outcome in a table.

Thus every possible outcome, in terms of the numbers coming up and their relevant sum, will be listed. Such a table is called an event space, since every possible event is a subset of this table.

If the probability required was that of a product or difference, a similar event space could be drawn up, but with the appropriate figures calculated

In this case, we make up the table and count those below 6.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are 10 results below 6, so the probability is 10/36 = 0.277778.

The Complement

What is the probability that the sum is equal to or greater than 6?

It is known that there are 10 outcomes with the sum less than 6. Therefore the number of outcomes equal to or greater than 6 is 36 – 10 = 26.

The probability of the sum being equal to or greater than 6 is 26/36. This is an example of a particularly important rule.

In an experiment with a number of possible outcomes, let A be an event and let B be the exactly opposite event. The event B is called the complement of A.

Also, we can say that $P[A] + P[B] = 1.$

In other words, when two distinct possibilities, which cover all possibilities, have been taken care of, the sum is 1.

Example

What is the probability that the sum adds up to less than 7, equal to 7, and greater than 7?

Counting the outcomes which give the correct respective sum,there are 15, 6, and 15 possibilities respectively. The probabilities are then 15/36, 6/36 15/36.

Note that these all add up to 1, as they should from the counting process.

A generalisation of the above rule can now be surmised from this example of the dice. Say an experiment has n distinct possible events:

$$E1, E2, E3, \ldots, En,$$

and no others. Then

$$P[E1] + P[E2] + \ldots + P[En] = 1.$$

In other words, when all distinct possibilities have been taken care of, the sum of the probabilities is 1.

Example

20 people are asked to choose a number from 1 to 100. What is the probability that two or more people pick the same number?

To work on this question, it is simpler to look at the equivalent question – find the probability that no two people pick the same number.

This is answered by seeing how many ways there are in which the 20 people can pick their numbers, with no restrictions. This is the total number of outcomes.

Then find the number of outcomes which satisfy the criteria for the event, in other words, the number of ways they can pick the numbers *with no two the same*.

The probability is then the second figure divided by the first.

How many ways can 20 people pick a number between 1 and 100?

Each of the 20 people can pick any of 100 numbers. So the number of possible combinations of the 20 numbers is 100 multiplied by itself 20 times:

$$100 \times 100 \times \ldots \times 100 = 100^{20}.$$

With the laws of indices, this is

$$100^{20} = (10^2)^{20} = 10^{40}.$$

This number is 1 with 40 zeros.

Now find how many ways the numbers can be chosen so none are the same. Let the first person pick their number. They have a choice of the full 100.

The second person now has the choice of the other 99 numbers. The number of ways the two numbers can be chosen is:

$$100 \times 99.$$

The third person will have a choice of 98 numbers. The total possible combinations are now: $100 \times 99 \times 98.$

Continuing like this, the last person will have a choice of 81 numbers, so the full number of combinations will be

$$100 \times 99 \times 98 \times \ldots \times 81.$$

This number works out to be roughly $100 \times 99 \times 98 \times \ldots \times 81 = 1.3 \text{x} 10^{39}$.

If the 20 numbers are chosen at random, the probability that none are the same is this number

$$100 \times 99 \times 98 \times \ldots \times 81,$$

divided by the first, which was 10^{40}.

Call S the event that one or more of the numbers are the same. Let N be the opposite event, that none are the same. Then we have just worked out that

$$\boxed{P[N] = \frac{100 \times 99 \times 98 \times \ldots \times 81}{100 \times 100 \times 100 \times \ldots \times 100} = 0.13}$$

and so $P[S] = 0.87$.

To make it more feasible to calculate the overall fraction, split it into fractions:

$$P[N] = \frac{100}{100} \times \frac{99}{100} \times \frac{98}{100} \times \ldots \times \frac{81}{100}$$

This is:

$$P[N] = 1 \times 0.99 \times 0.98 \times \ldots\ 0.81.$$

This works out as $P[N] = 0.13$.

In summary, if 20 people pick a number between 1 and 100, there is an 87% chance that 2 or more will pick the same number.

Permutations and Combinations

Most small club lotteries are based around picking 4 correct numbers out of 20, 24 or 28. The national lottery picks 6 numbers from 42, and the British Lottery takes 6 from 49.

To calculate the probability of winning these lotteries, we have to see how many ways 4 numbers can be taken out of 24, or 6 out of 42, etc.

Firstly, take note of the following notation. For a number n, the number $n!$ is defined as

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

Examples

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$6! = 6 \times 5 \times \ldots \times 1 = 720$

$7! = 7 \times 6 \times 5 \times \ldots \times 1 = 5{,}040$

The values increase rapidly:

$10! = 10 \times 9 \times \ldots \times 1 = 3{,}628{,}800.$

Permutations

A tourist is visiting the south-west of Ireland, and has time to visit four towns out of Cork, Killarney, Ennis, Limerick, Tralee, Bantry. If the order in which the towns are visited is taken into account, how many possible trips are there?

There are 6 towns, and 4 choices are made, so the number of ways in which the tourist could choose the towns is:

6 by 5 by 4 by 3.

This is 360.

Each particular choice of towns, in a particular order, is a *permutation* of four of the six names in the list. It is called this because the order in which the names are chosen matters.

Calculations of the number of ways these permutations can be done usually involve multiplications like the one above, so the following notation is introduced.

When r numbers are multiplied, starting at n, the result is denoted ${}^nP_r$:

$${}^nP_r = n(n-1)(n-2)(n-3)(n-4)\ldots$$

where there are r numbers in this list.

In general, the number of ways of selecting r objects from n, where the order counts, is given by ${}^nP_r$.

In the example of the tourist, the number of ways of choosing 4 towns out of 6, in a particular order was

$${}^6P_4 = 6\times5 \times 4 \times 3 = 360$$

This number ^nP_r can be written in terms of the factorial notation:

$$^nP_r = \frac{n!}{(n-r)!}$$

The *n-r* numbers below the line cancel with the last *n-r* above the line, leaving *r* numbers. This means that this definition is the same as

$$n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)\ldots$$

where *r* numbers are being multiplied.

To look at this for the example of the figure $^6P_4$ in terms of the factorial notation,

$$^6P_4 = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2\times1}{2\times1} = 6\times5\times4\times3$$

The 2 by 1 term cancels from above and below the line.
What if the order the towns were visited in did not matter?

Combinations

In the case where the tourist was visiting the towns in any order, there will be far fewer separate lists. If the order does not matter, we are dealing with a *combination*.

For a particular choice of 4 towns, how many ways can they be reordered? The answer is $4\times3\times2\times1$, in other words 4!. So the original answer must be divided by 4!, to ignore the different orders the towns could be listed in.

Thus the answer for this case is

$$\boxed{\frac{6\times5\times4\times3}{4\times3\times2\times1}=15}$$

This shows that the number of ways of taking out r objects (names, numbers, etc.) from n is given by ${}^nP_r$, divided by $r!$, which is

$$\boxed{\frac{n!}{r!(n-r)!}}$$

This very important number is denoted by:

$$ {}^{n}C_{r} = \frac{n!}{r!(n-r)!} $$

The *n*-*r* numbers in (*n*-*r*)! cancel with the last *n*-*r* numbers in *n*!, giving *r* numbers. This means there are *r* numbers above and below the line.

So this number is the same as:

$$ \frac{n \times (n-1) \times (n-2) \times \ldots}{r \times (r-1) \times \ldots \times 1} $$

This makes it very easy to calculate the value.

Example

Four candidates names are to be short listed from a panel of ten for an interview. How many possible ways could the candidates be interviewed? How many possible lists of candidates are there?

If the order is taken into account, then the number of possible interview lists is:

$$^{10}P_4 = 10 \times 9 \times 8 \times 7 = 5{,}040$$

The answer to the second part is simply the number of ways of taking 4 from 10, which is given by

$$^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

These numbers allow us to calculate the odds for the lotteries mentioned above.

Example

A five-a-side soccer team is to be selected from a panel of 8 players. How many possible teams are there?

The number of ways of selecting 5 players out of 8 is given by

$$ {}^{8}C_{5} = \frac{8\times7\times6\times5\times4}{5\times4\times3\times2\times1} = 56 $$

It is worth noting that in this case, where 5 players must be chosen from 8, any choice of 5 players is the same as a choice of 3 to leave out.

It follows then that the number of ways of choosing the 5 is the same as the number of ways of leaving 3 players on the bench.

If we look back to the definition of ^nC_r, we can see why this might be the case. Looking at the expression for of $^8C_5$

$$^8C_5 = \frac{8\times7\times6\times5\times4}{5\times4\times3\times2\times1}$$

Notice that the last to numbers on the top cancel with the first two in the bottom line, giving

$$^8C_5 = \frac{8\times7\times6}{3\times2\times1}$$

which is identical to $^8C_3$.

When calculating a value of ^nC_r, whichever of the two numbers of r or n-r is the lower, any more than these numbers above or below the line will cancel. So it can be seen that

$$^nC_r = {}^nC_{n-r}$$

We can see that this is the case from the definition:

$$ {}^{n}C_{r} = \frac{n!}{r!(n-r)!} $$

Since n and $n - r$ are multiplied in this equation, it doesn't matter which order they appear in.

Example

The example above with the 5-a-side soccer team assumes that the team is just a list of five names. What if the same five players, but in different 0positions, counts as a different team?

The number of possible teams is then

$$^{8}P_{5} = 8 \times 7 \times 6 \times 5 \times 4 = 6{,}720.$$

This figure is the same as

$$\boxed{\frac{8!}{3!}}$$

since the 3! cancels the last 3 numbers in 8!

Example

Most small club lotteries are based around picking 4 correct numbers out of 20, 24 or 28. What is the probability of winning?

The probability of winning with a particular choice of 4 numbers is 1 in N, where N is the number or ways of taking 4 numbers from 20, 24 or 28.

For 20, the number of ways of selecting 4 numbers out of 20 is given by

$$^{20}C_4 = \frac{20!}{4! \times 16!} = \frac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1}$$

The 16 numbers in 16! cancel with the last 16 numbers in 20!, giving just 20 to 17. So then

$$^{20}C_4 = \frac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1} = 4845$$

Thus the probability of winning such a lottery is 1/4845.

For 24 numbers, the number of combinations is

$$^{24}C_4 = \frac{24\times 23\times 22\times 21}{4\times 3\times 2\times 1} = 10{,}626$$

The probability of winning this lottery is 1/10,626.

For 28, the number of combinations is

$$^{28}C_4 = \frac{28\times 27\times 26\times 25}{4\times 3\times 2\times 1} = 20{,}475$$

The probability of winning this a lottery is 1/20,475.

Example

Calculate Irish National Lottery allows the player to choose 6 numbers from a possible 42. Calculate the probability of winniny this lottery.

$$^{42}C_6 = \frac{42\times 41\times 40\times 39\times 38\times 37}{6\times 5\times 4\times 3\times 2\times 1} =$$

$$= 41\times 13 \times 38 \times 37 = 5{,}245{,}786.$$

The probability of winning this lottery is about one in 5 million, or 1.9×10^{-7}.

The Laws of Probability

We saw already that if A is an event ant B be the exactly opposite event, then $P[A] + P[B] = 1$.

In other words, when two distinct events, which cover all possibilities, have been taken care of, the sum is 1. A generalisation of the above rule concerns n distinct possible events:

$$E1, E2, E3, \ldots, En,$$

and no others. Then

$$P[E1] + P[E2] + \ldots + P[En] = 1$$

In other words, when all distinct possibilities have been taken care of, the sum of the probabilities is 1. We will now phrase these laws more rigorously, and introduce some definitions of events and how they are related, and laws which govern their probabilities.

The Addition Law of Probability

Two events A and B, the possible results of the same experiment, are said to be *mutually exclusive* if it is impossible for them to happen together.

If two events A and B are mutually exclusive, then the following law holds:

$$P[A \text{ or } B] = P[A] + P[B]$$

This means the probability of one of event A or event B occurring is given by the sum of their two probabilities.

If two events A and B are not mutually exclusive, that is, A and B can occur together, they are said to be mutually *non-exclusive* events. In this case, the law above is broadened so that the probability of A or B occurring is given by:

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B]$$

This means the probability of one of event A or event B occurring is given by the sum of their two probabilities, less the probability of them both occurring.

Example

If a single dice is thrown, let A be the event of scoring a multiple of 3, and B be the event of scoring a multiple of 2. Determine the probability of getting a multiple of 2 or a multiple of 3, and then the probability of both together.

The probability of scoring a multiple of 3 is that of getting a 3 or a 6. Thus

$$P[A] = 2/6 = 1/3$$

The probability of scoring a multiple of 2 is:

$$P[B] = 3/6 = ½.$$

These are non-exclusive events so

$$P[A \text{ or } B] = P[A] + P[B] – P[A \text{ and } B]$$

The event 'A and B' means that a number is a multiple of 2 and 3, so this is equivalent to getting a 6. Thus

$$P[A \text{ and } B] = 1/6$$

The final probability is then

$$P[A \text{ or } B] = 2/6 + 3/6 - 1/6 = 4/6 = 2/3$$

In this case, finding the probability $P[A \text{ or } B]$ from scratch would not require too much calculation. It would simply be a matter of counting the number of ways a number on the dice could be a multiple of 3 *or* a multiple of 2.

The ways of getting this are 3 and 6 for the first, and 2, 4 and 6 for the second. This is a list of 4 distinct numbers, so the answer is 4/6 = 2/3.

This confirms the answer we already found.

Independent and Dependent Events

Two events are *independent* when the occurrence of one event **does not** affect the probability of the occurrence of the second event.

An example would be the rolling of a die on two occasions. The outcome of the first throw **will not** affect the probabilities for the second throw.

If the outcome of one event **does** affect the probability of the second event, they are said to be *dependent*.

Consider the event E defined as drawing a king from a full deck of cards.

We know the probability is:

$$P[E] = 4/52 = 1/13 = 0.077$$

If the card is not replaced, and the same experiment carried out again, the probability becomes:

$$P[E] = 3/51 = 0.059$$

Thus the outcome of the first experiment, drawing a king, has affected the probabilities for the second.

The two events are not independent.

The Multiplication Law of Probability

Considering two *independent events* A and B, the probability of the occurrence of both events A and B, is given by

$$P[A \text{ and } B] = P[A] \times P[B]$$

Thus the probability is the product of the two individual probabilities.

To deal with dependent events, some notation will be needed to indicate when the probability of one event depends on whether another has happened.

The probability of an event B, providing that an event A has already occurred is denoted by the notation:

$$P[B \mid A]$$

This is 'the probability of B, given A.'

For two independent events A and B, it is obviously the case that

$$P[B \mid A] = P[B] \text{ and } P[A \mid B] = P[A]$$

This is because the fact that A has already occurred does not affect the probability of event B, by definition.

Now consider two events A and B with event A dependent on B. The probability of the occurrence of both events is going to be

$$P[A \text{ and } B] = P[A \mid B] \times P[B]$$

The probability of both events happening is the probability of A, given B, times the probability of B.

Example

Consider two events A and B, where A is throwing a six with a fair dice, and B is drawing a king from a full deck of cards. Determine the probability of the occurrence of both events.

Since these are independent events, we use

$$P[A \text{ and } B] = P[A] \times P[B]$$

Thus

$$P[A \text{ and } B] = 1/6 \times 4/52 = 4/312 = 1/78 = 0.013$$

Example

A box contains five 10 kΩ resistors and twelve 20 kΩ resistors. Determine the probability of randomly picking a 10 kΩ resistor from the box. Also, determine the probability of randomly picking a 10 kΩ resistor from the box and then a 20 kΩ resistor.

Solution

Let event A denote the event of picking a 10 kΩ resistor, and let B denote the event of picking a 20 kΩ resistor.

The first probability is just $P[A]$, and since the total number of resistors is 17, it is

$$P[A] = 5/17.$$

To find the probability of both, that is, $P[A \text{ and } B]$, we need to use the probability law

$$P[A \text{ and } B] = P[B \mid A] \times P[A]$$

Here B depends on A, since A is the event that is happening first.

To find $P[B \mid A]$, we need the probability that a second resistor picked randomly from the box will be a 20 kΩ resistor, providing that the first one picked from the box was a 10 kΩ resistor.

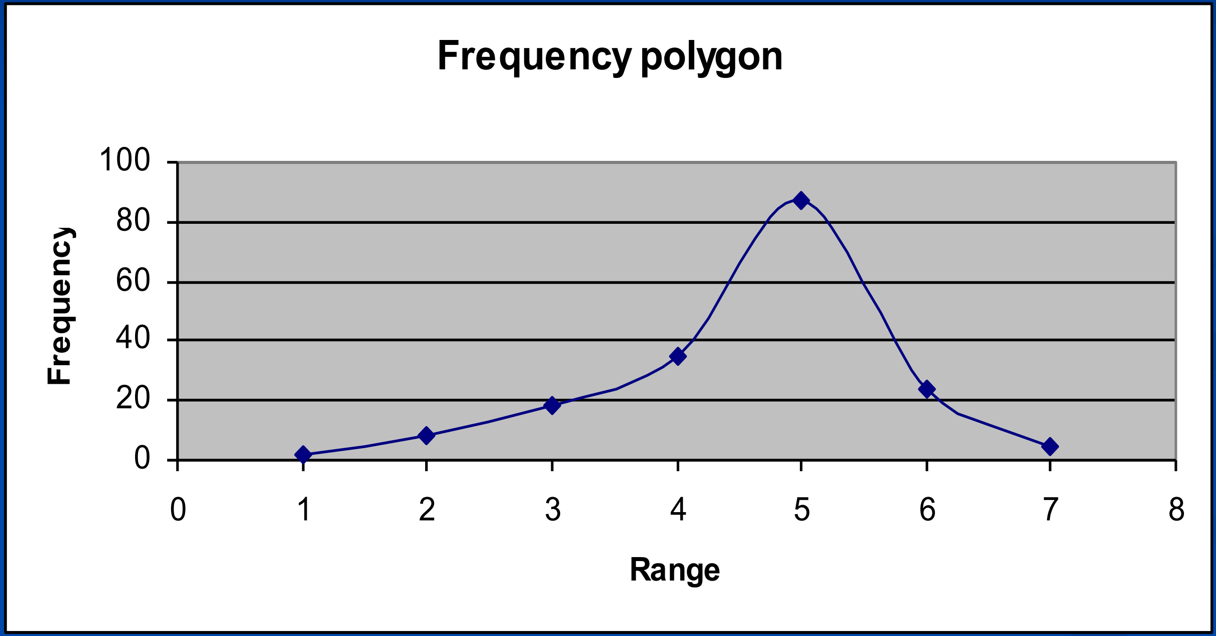
For this case, $P[B \mid A] = 12/16 = 3/4$.

So the probability of both events, picking a 20kΩ resistor after getting a 10 kΩ resistor is:

$$P[A \text{ and } B] = P[B \mid A] \times P[A] = 5/17 \times 3/4 = 15/68 = 0.22$$

Probability Distributions

The distribution pattern of the frequencies with which all the events occur in a trial can be represented graphically. Typical graphs which can be used to show the type of frequency distribution include the histogram and the frequency polygon.



Using mathematical tools we are able to determine important information about the distribution such as mean and standard deviation.
The Frequency Polygon above is typical of a Normal Distribution.

There are 3 main Probability Distributions:

- Binomial Distribution
- Poisson Distribution
- Normal Distribution

The Binomial and Poisson Distributions are **Discrete Probability Distributions**.

The Normal Distribution is a **Continuous Probability Distribution**.

We will see what this means later.

The Binomial Distribution

Consider an experiment that has two possible outcomes. The outcome can either be a success or a failure. We will use 'p' to denote the probability of a success and 'q' to denote the probability of a failure.

From our general probability laws it follows that $p + q = 1$.

Example

The percentage of people who use the internet in a sample of 10 people is 32%. What is 'p' and 'q'?

The probability that one of the 10 people use the internet is

$$p = \frac{32}{100} = 0.32$$

The probability that one of the 10 people do not use the internet is

$$q = 1 - p$$

Therefore $q = 1 - 0.32 = 0.68$

Probability of >1 Success Occurring

What if we want to know the probability that 3 people in the sample of 10 will use the internet.

To calculate this we use the Binomial Distribution

$$P(r \text{ successes in } n \text{ trials}) = {}^{n}C_{r}\ p^{r}\ q^{n-r} = \frac{n!}{(n-r)!\,r!}\ p^{r}\ q^{n-r}$$

Example

In the above example what is the probability that if 10 people are sampled, that 3 of them will use the internet.

$$P(r \text{ successes in } n \text{ trials}) = {}^{n}C_{r}\ p^{r}\ q^{n-r} = \frac{n!}{(n-r)!\,r!}\ p^{r}\ q^{n-r}$$

$$P(3\ successes\ in\ 10\ trials) = {}^{10}C_3 \ (0.32)^3\ (0.68)^{10-3}$$

$$= \frac{10!}{(10-3)!\,3!}(0.32)^3\ (0.68)^{10-3}$$

$$P\ (R = r) = 120 \times (0.32)^3 \times (0.68)^7$$

$$P\ (R = 3) = 120 \times 0.03277 \times 0.067$$

$$P\ (R = 3) = 0.26$$

Example

If a dice is rolled 7 times, what is the probability of getting 3 sixes?

$$p = \frac{1}{6} \;,\quad q = \frac{5}{6}$$
$$n = 7 \;,\quad r = 3$$

Therefore the probability of getting 3 sixes is:

$$P(3\ successes\ in\ 7\ trials) = {}^{7}C_{3}\ (\frac{1}{6})^{3}\ (\frac{5}{6})^{7-3} = \frac{7!}{(7-3)!\,3!}(\frac{1}{6})^{3}\ (\frac{5}{6})^{7-3}$$
$$P\ (R = r) = 35 \times (0.0046) \times (0.48)$$
$$P\ (R = 3) = 0.08$$

Example

2.5% of components in a factory are found to be defective. In a random sample of 40 components what is the probability that 4 of them will be defective?

$$p = \frac{2.5}{100} = 0.025 \quad , \quad q = (1 - p) = 0.975$$

$$n = 40 \quad , \quad r = 4$$

$$P(4 \text{ successes in } 40 \text{ trials}) = {}^{40}C_4 \ (0.025)^4 \ (0.975)^{40-4}$$

$$= \frac{40!}{(40-4)!\,4!} (0.025)^4 \ (0.975)^{40-4}$$

$$P\ (R = r) = 91390 \times (3.91 \times 10^{-7}) \times (0.402)$$

$$P\ (R = 4) = 0.01$$

Probability of an Event Occurring < or > a Particular Count

Example

35% of components are faulty. What is the probability that in a random sample of 40 components, less than 3 of them are faulty?

In this case all of the following satisfy this criteria:

$$P[R=2]\ or\ P[R=1]\ or\ P[R=0]$$

To solve this problem, we calculate the probability of each of these outcomes and use the Laws of Probability for 'OR', i.e.,

$$P[R<3]=P[R=2]+P[R=1]+P[R=0]$$

where

$$p=0.35 \quad , \quad q=0.65$$

$$n=40 \quad , \quad r=0\ or\ 1\ or\ 2$$

Finish this yourselves

The Poisson Distribution

The Poisson Distribution is given by the function:

$$P[R = r] = \frac{e^{-\mu}\ \mu^{r}}{r!}$$

where

$\mu = \text{the distribution mean} = np$

$r = \text{number of times that an event occurs}$

$p = \text{probability of a success in any one trial}$

$n = \text{total number of trials}$

When is the Poisson Distribution Used?

The Poisson Distribution is used when p is small ($p \leq 0.1$) and n is large ($n \geq 50$). The value of $P[R = r]$ computed using

$$\boxed{\frac{e^{-\mu}\ \mu^{r}}{r!}}$$

approaches the value computed using

$$\boxed{{}^{n}C_{r}\ p^{r}\ q^{n-r}}$$

for such values of p and n

Example

The percentage of defective components produced on an assembly line is 1%. In a random sample of 80 components, what is the probability of 5 of the components being defective?

$$p = \frac{1}{100} = 0.01 \quad (p \le 0.1)$$

$$n = 80 \quad (n > 50)$$

$$\mu = np = 80 \times 0.01 = 0.8$$

$$r = 5$$

Therefore:

$$P[R = r] = \frac{e^{-\mu} \ \mu^{r}}{r!}$$

$$P[R = 5] = \frac{e^{-0.8} \ 0.8^{5}}{5!}$$

$$P[R = 5] = \frac{e^{-0.8} \ 0.3278}{120}$$

$$P[R = 5] = 0.001$$

The Normal Distribution

In the case of the binomial distribution, the random variable was a number of outcomes. This number, as a count, will always be an integer.

The next distribution we will look at is for continuous variables. These are variables which can take on any possible value, such as 1, 1.2, or 1.2345...

Most quantities in nature are like this, and very many of them follow the normal distribution. This includes measurable variables such as height, or rainfall averages.

When dealing with a continuous variable, we can no longer talk about the chances of the variable being equal to particular values. Since there are an infinite number of possible values, the chances of one particular value coming up are zero.

Instead we talk about the probability of the variable taking on a value in a particular range.

So if a variable R has a binomial distribution, the events were

$$r = 0;\ r = 1 \text{ or } r > 4.$$

With a continuous variable, say height X, the events are

$$X > 160\text{cm or } X < 170\text{cm}.$$

The Normal distribution for a particular variable is characterised by two numbers.

This is similar to the binomial distribution, where the value of the probabilities for the variable X is determined by the numbers n and p.

In the case of the normal distribution, the two numbers are

1) the mean, or average, which is denoted μ, and
2) the standard deviation, denoted σ.

Recall that for many cases of the use of the binomial distribution, the probability p of the event being counted often came from proportions of the population the sample came from. In a similar way, the mean and the standard deviation may come from the analysis of a population. Thus the ideas of the average and standard deviation are the same as those from the study of data presentation.

If a variable X is normally distributed, this means that the values are likely to be close to the mean μ, and unlikely to be far from it. Just how likely or unlikely is determined by the standard deviation σ.

The table at the back of this handout gives probabilities for a normally distributed random variable, with mean $\mu = 0$ and standard deviation $\sigma = 1$. This is called the *standard* normal distribution.

The probabilities for the Normal distribution are given by the following equation. Let X be a normally distributed variable, with mean μ and standard deviation σ. Then the probability that it is less than a value a is

$$P[X < a] = \int_{-\infty}^{a} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,\mathrm{d}x.$$

This is clearly a very complex integral, and in fact it is impossible to work out this integral using normal integration. Instead it must be calculated numerically using a computer.

This can not be done for every possible choice of mean μ, and standard deviation s. Instead it is done for one choice of μ and σ, and these are related to all others.

The probabilities given in the tables are that Z takes on a value *greater* than a given value a, in other words:

$$P[Z > \mathrm{a}]$$

Standardising

The Standard Normal Variable is a normal variable Z with mean 0 and standard deviation 1. It's probabilities have been calculated by computer from the equation above.

These probabilities are available in statistical tables such as the one in mathematics log tables. It can then be used to find the probabilities of any normal variable. This is possible because of the following property of the normal distribution.

If the variable X is normally distributed with of mean μ, and standard deviation σ, then the variable Z, given by the relation

$$Z = \frac{X - \mu}{\sigma}$$

has mean 0 and standard deviation 1, in other words, it is the standard normal variable.

Consider then the event $X > a$. To relate this to the variable Z, subtract the mean μ from both sides, and then divide by the standard deviation σ to give:

$$\boxed{\frac{X-\mu}{\sigma} > \frac{a-\mu}{\sigma}}$$

In other words, the event $X > a$ is the same as the event

$$\boxed{Z > \frac{a-\mu}{\sigma}}$$

Thus the probabilities are the same:

$$\boxed{P[X > a] = P\left[Z > \frac{a-\mu}{\sigma}\right]}$$

An event involving X has been shown to be equivalent to an event involving Z, so if the probabilities for Z are known, then those for X are known too.

In summary, if X is a normally distributed variable with mean μ and standard deviation σ, and Z is the normal variable with mean 0 and standard deviation 1, then

$$\boxed{P(X > a) = P\left[Z > \frac{a-\mu}{\sigma}\right]}$$

The normal variable can be reduced to one involving the standard normal variable. The probabilities for the standard normal variable are available in the log tables, and probabilities for any normal variable can be calculated from them.

What this means in practice is that to find the probability $P[X > a]$, firstly work out the number:

$$\boxed{\frac{a-\mu}{\sigma}}$$

and then look it up in the table which has been calculated for the standard normal distribution.

These simple steps, which allow any probability involving a normal distribution to be transformed to one involving the standard normal distribution, mean it is not necessary to recalculate the probabilities once they have been done for the standard variable.

Example

The diameter of the ball bearings being produced in a factory is a normally distributed random variable, D with mean of 4mm and standard deviation 0.01mm. What is the probability that a ball bearing chosen at random has a diameter greater than 4.015mm?

The question means we are calculating the probability $P[\ D > 4.015]$.
To find this probability, firstly work out the standardised number:

$$\boxed{\frac{4.015-\mu}{\sigma} = \frac{4.015-4.0}{0.01} = 1.5}$$

We use the fact that

$$P(D > 4.015) = P(Z > 1.5)$$

and then look it up in the table. It gives 0.0668. So:

$$P(D > 4.015) = 0.0668$$

The procedure here meant that we translated an event involving D to one involving Z.

Example

What is the probability that a ball bearing chosen at random has a diameter less than 4.0185 mm?

We are calculating $P[\ D < 4.0185]$. Standardise:

$$\frac{4.0185-\mu}{\sigma}=\frac{4.0185-4.0}{0.01}=1.85$$

This means that $P(D < 4.0185) = P(\ Z < 1.85)$

However, if we look this number 1.85 up in the table, the value given is the probability that Z is greater than 1.85: $P(Z > 1.85) = 0.0322$

To deal with this, use the fact that

$$P(Z < 1.85) = 1 - P(Z > 1.85)$$

Then

$$P(Z < 1.85) = 1 - 0.0322 = 0.9678.$$

So we have found that

$$P(D < 4.0185) = 0.9678.$$

Properties of the Standard Normal Distribution

The values in the log tables are probabilities $P[Z > a]$, for positive numbers a.

We must be able to extend this to probabilities like $P[Z > a]$ for negative values of a, or $P[Z < a]$ and for ranges, $P[a < Z < b]$.

The standard normal distribution has some important properties arising from its definition as an integral which make it possible to calculate these probabilities.

To handle problems such as $P[Z < a]$, for some number a, use the fact that $Z < a$ and $Z > a$ cover all eventualities, and so their probabilities add up to 1. It then follows that

$$P[Z < a] = 1 - P[Z > a].$$

To handle negative values of *a*, we use a property of the standard normal variable called *symmetry*. It means that

$$P[Z < -a] = P[Z > a]$$

$$P[Z > -a] = P[Z < a]$$

These two rules can be summarised by saying that to change the direction of the inequality, the sign must be changed also.

This also applies in reverse – to change the sign, the direction of the inequality must be changed too.

Example

In the case of ball bearing production, what is the probability that a ball bearing chosen at random has a diameter greater than 3.985mm?

We are calculating: $P[D > 3.985]$

The problem here is that when we standardise, we are left with a negative number:

$$\frac{3.985 - 4.0}{0.01} = -1.5$$

Applying symmetry to this example:

$$P[Z > -1.5] = P[Z < 1.5]$$

Now use the fact that

$$P[Z < 1.5] = 1 - P[Z > 1.5]$$

The last probability can be found in the tables, so:

$$P[D > 3.985] = 1 - 0.0668 = 0.9332$$

To summarise, first standardise to get:

$$P[D > 3.985] = P[Z > -1.5]$$

Then apply symmetry:

$$P[Z > -1.5] = P[Z < 1.5]$$

and now the basic law:

$$P[Z < 1.5] = 1 - P[Z > 1.5] = 1 - 0.0668 = 0.9332$$

Ranges

Recall the example of ball-bearings being produced in a factory – the diameter of the ball bearings is a normally distributed random variable, D with mean and standard deviation given as $\mu = 4$mm, and $\sigma = 0.01$mm.

Consider the following question – what is the probability that a ball bearing chosen at random has a diameter between 4.015mm and 4.02mm?

To address this problem, consider the following three possible events; the diameter is greater than 4.015mm, greater than 4.02mm and between the two. These probabilities are

$$P[D > 4.015],\ P[D > 4.02],\ \text{and}\ P[4.015 < D < 4.02].$$

Since 4.015 is the lower number, the event ‘D between 4.015 and 4.02’ and ‘D greater than 4.02’, can be combined to be ‘D greater than 4.015’.

The three probabilities are connected by the following relation:

$$P[4.015 < D < 4.02] + P[D > 4.02] = P[D > 4.015]$$

This can be rearranged by bringing one of the probabilities across the equals sign:

$$P[4.015 < D < 4.02] = P[D > 4.015] - P[D > 4.02]$$

Both of the probabilities on the RHS can be calculated in the usual way, which means the problem can be solved. Standardising gives

$$P[D > 4.015] = P[Z > 1.5] = 0.0668$$

and

$$P[D > 4.02] = P[Z > 2.0] = 0.0228$$

The result of the calculation is:

$$P[4.015 < D < 4.02] = 0.0668 - 0.0228 = 0.033$$

Example

The height of men is normally distributed, with mean 1.71 metres and standard deviation 0.11 metres. Find the probability that the height of a man chosen at random is:

1. Between 1.74 metres and 1.76 metres,
2. Between 1.64 metres and 1.76 metres.

Let H be the random variable of height in men. We have $\mu = 1.71$ and $\sigma = 0.11$.

For the first question, the probability is broken up as

$$P[1.74 < H < 1.76] = P[H > 1.74] - P[H > 1.76]$$

Standardise the values 1.76 and 1.74 gives:

$$\boxed{\frac{1.74 - 1.71}{0.11} = 0.27}$$ and $$\boxed{\frac{1.76 - 1.71}{0.11} = 0.36}$$

This means that

$$P[H > 1.74] = P[Z > 0.27] = 0.3936$$

and

$$P[H > 1.76] = P[Z > 0.36] = 0.3594$$

Thus

$$P[1.74 < H < 1.76] = 0.3936 - 0.3594 = 0.0342$$

For the last part, the probability is

$$P[1.64 < H < 1.76] = P[H > 1.64] - P[H > 1.76]$$

Standardise 1.64, in the same way as above, to get:

$$P[H > 1.64] = P[Z > -0.64]$$

Then apply symmetry:

$$P[Z > -0.64] = P[Z < 0.64]$$

and the laws of probability

$$P[Z < 0.64] = 1 - P[Z > 0.64] = 1 - 0.2611 = 0.7389$$

We have already seen that

$$P[H > 1.76] = P[\ Z > 0.36] = 0.3594$$

Thus the overall result is

$$P[1.64 < H < 1.76] = P[H > 1.64] - P[H > 1.76] =$$
$$= 0.7389 - 0.3594 = 0.3795.$$