# DIFFERENTIAL CALCULUS

## What is differerential calculus?

This is an area of mathematics which is concerned with the **rate of change** of one physical quantity with respect to another…

…provides mathematical techniques for modelling and analysing dynamic systems.

## Uses in Computing…

Employed in such diverse areas as computer animation; the modelling of performance and costs of computer projects; the analysis of software reliability…

## This is what you are familiar with!

Power Rule of differentiation

$$y = x^n$$

$$\frac{dy}{dx} = nx^{n-1}$$

$$y = x^2$$

$$\frac{dy}{dx} = 2x^{2-1} = 2x$$

**Today's lecture:**

- What is differential calculus
- Recap of plotting a straight line by finding two points on it
- Finding the slope of a straight line using two points on line
- General equation for the slope of a straight line
- Tangent lines and curves
- Using formula for straight line slope to approximate the slope of the tangent line at a point on a curve
- Differentiation from first principles

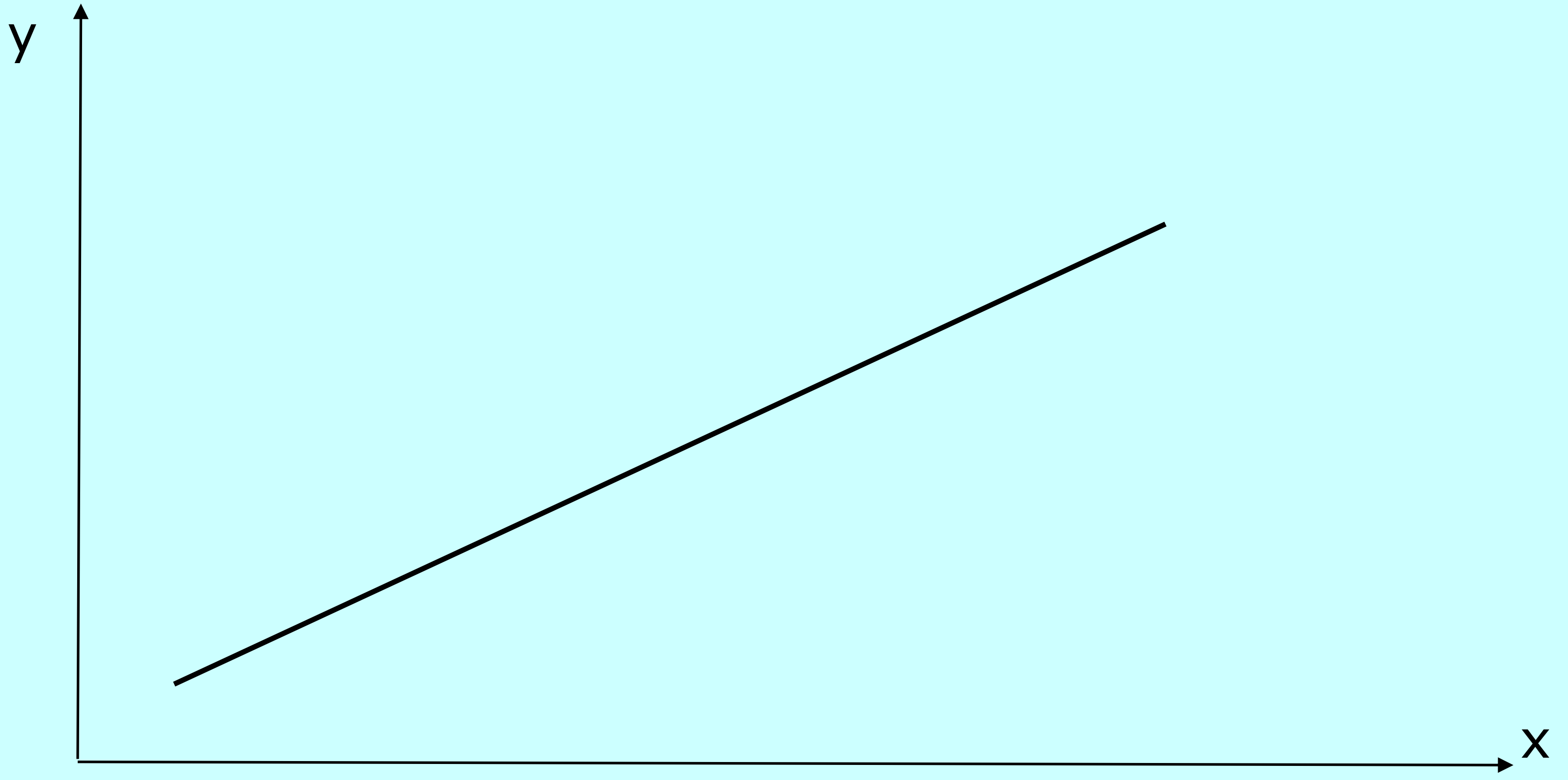
# Differerential calculus

What does **rate of change** of one physical quantity with respect to another mean in a physical sense?

Consider straight line graph below with variables x and y.

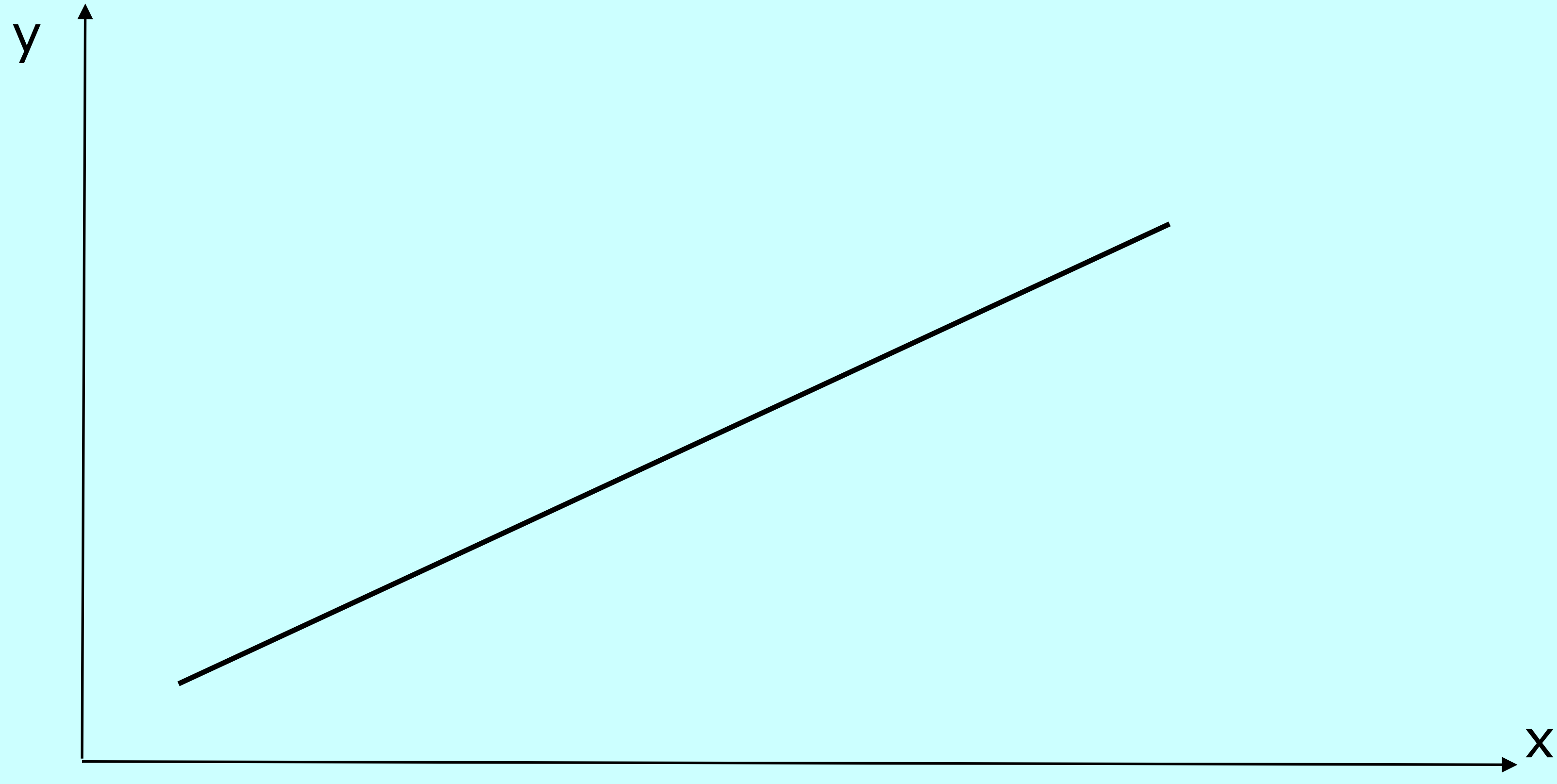
Each variable represents a physical quantity.

The value of y output depends on the value of x input.



**New question:**

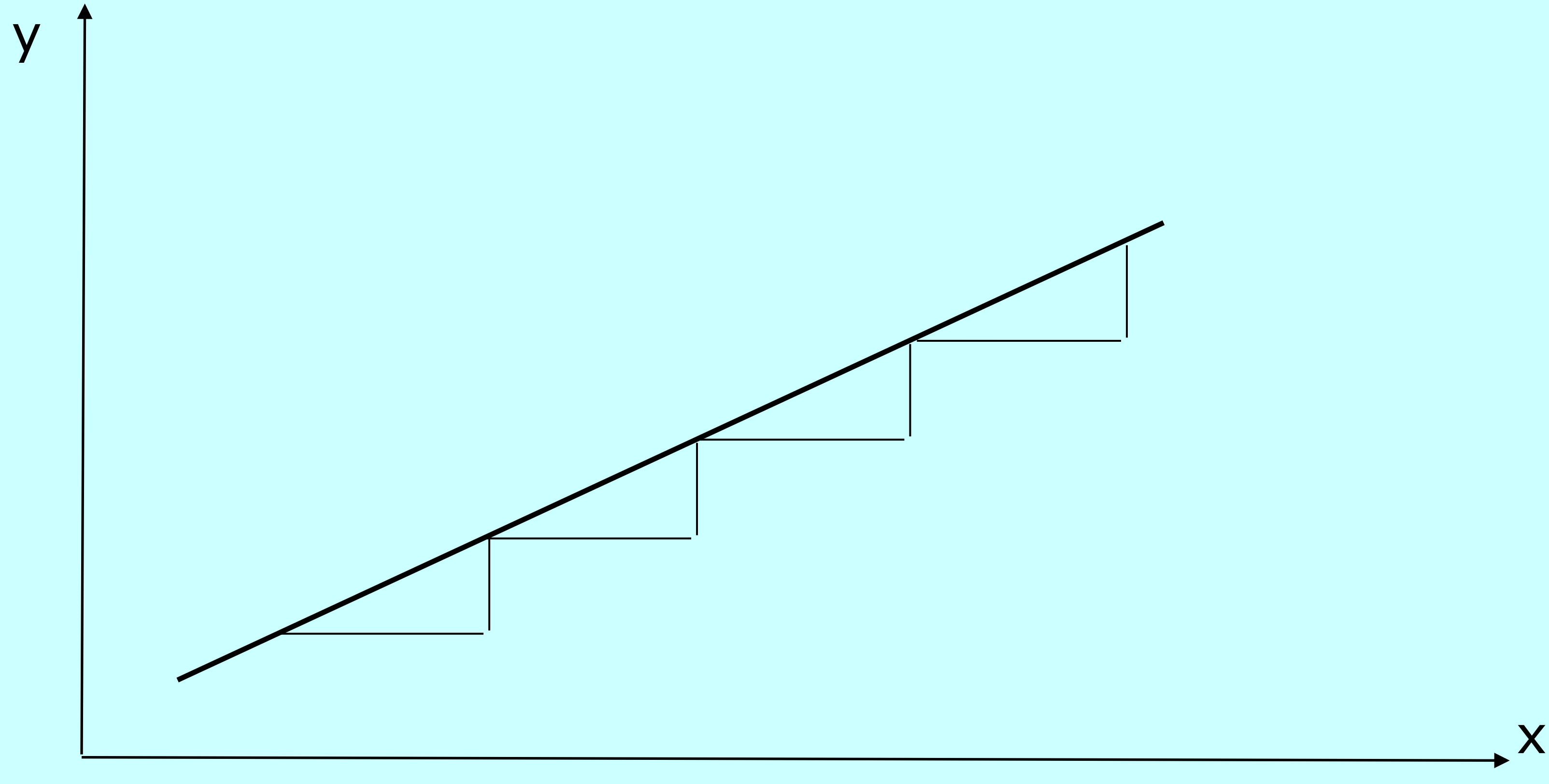
What is the **rate of change** of the variable y in terms of how the variable x changes?



**Rate of change of one variable with respect to another:**

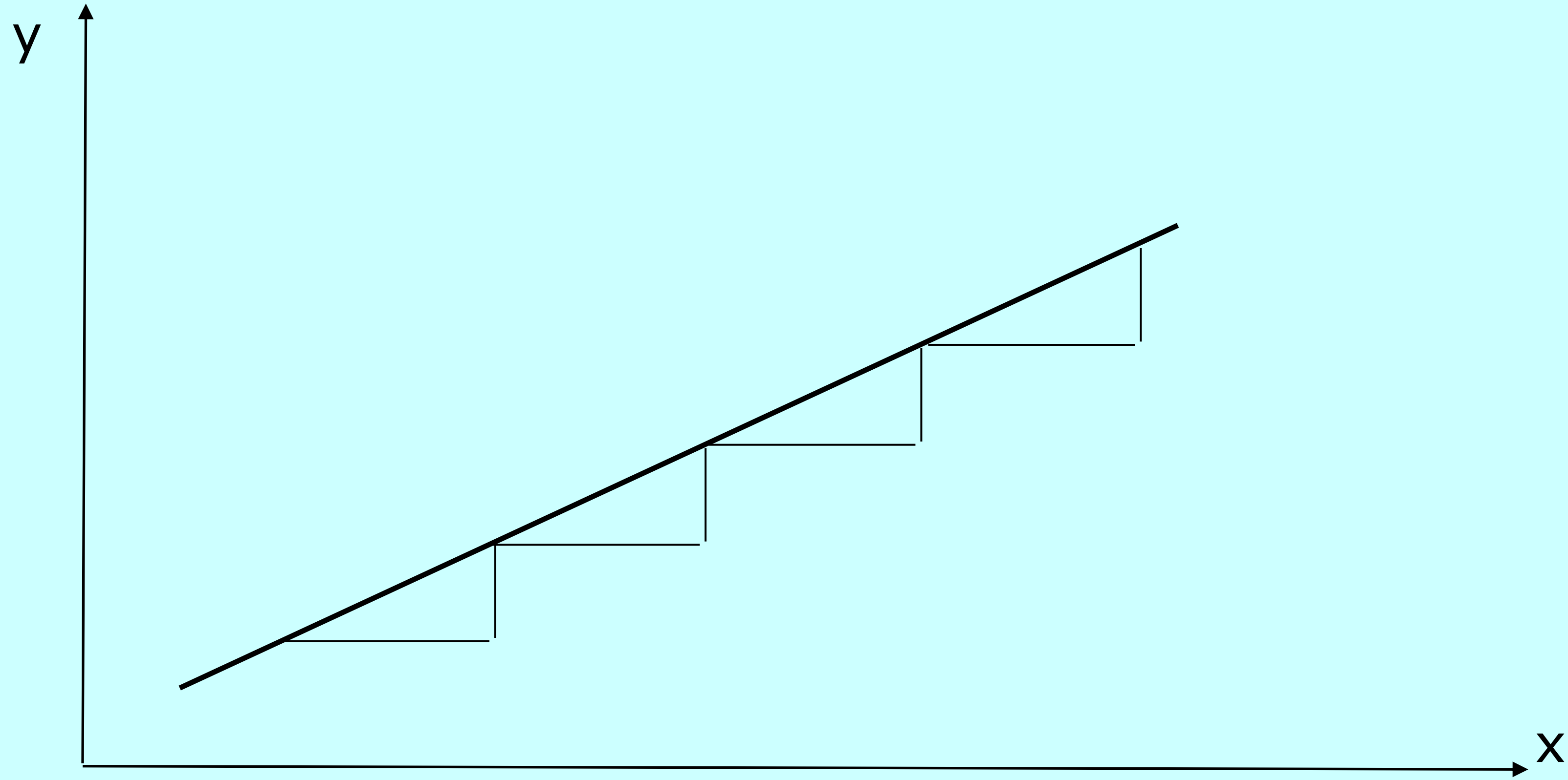
As the horizontal variable increases by equal amounts…

… then there is a constant increase in the vertical variable.



For any straight line such as this, the **rate of change** of the vertical variable with respect to the horizontal variable is a

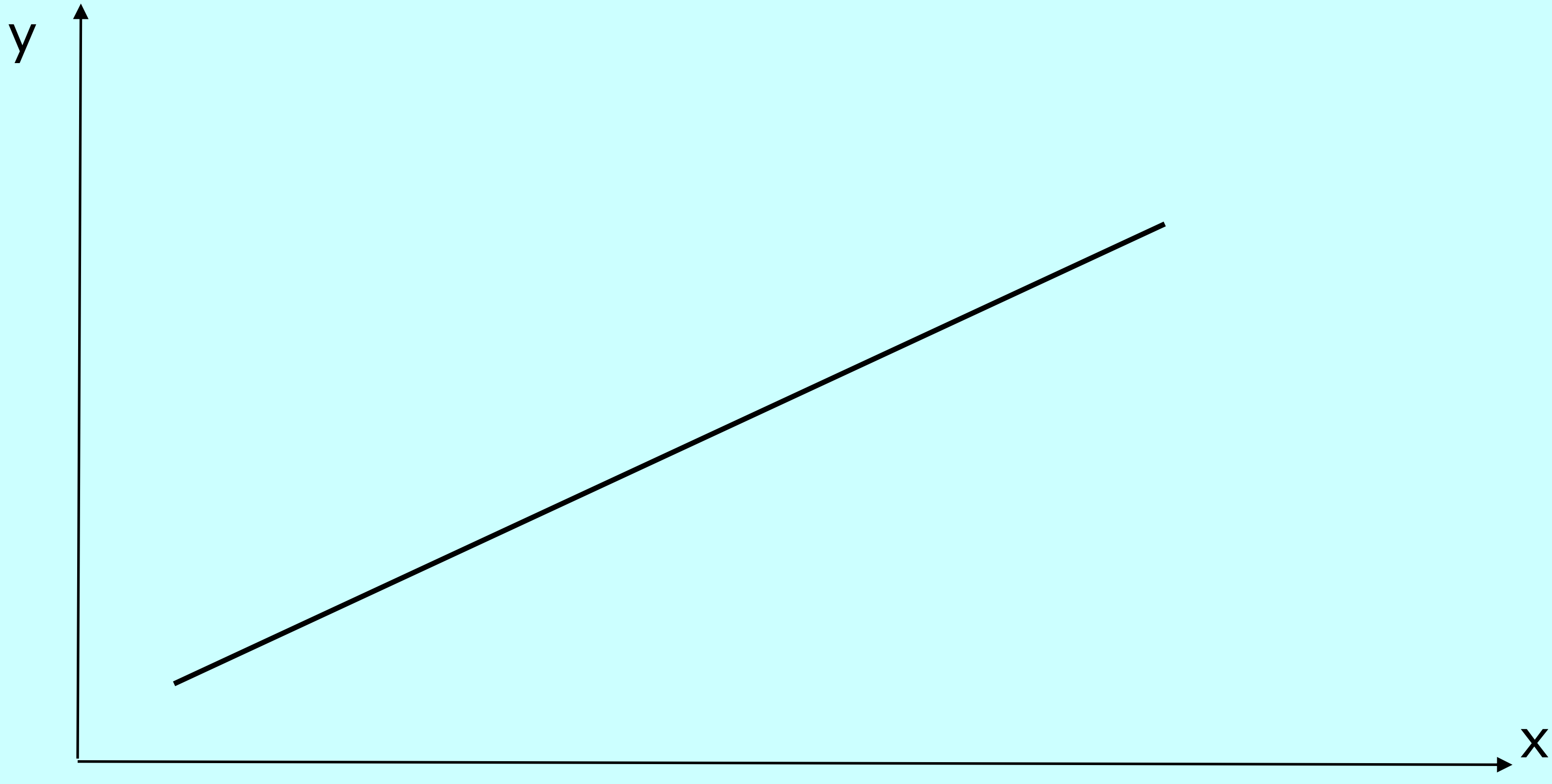
C O N S T A N T



If there is a linear relationship between the physical quantities…

…then rate of change of the dependent physical quantity with respect to the independent is a

C O N S T A N T

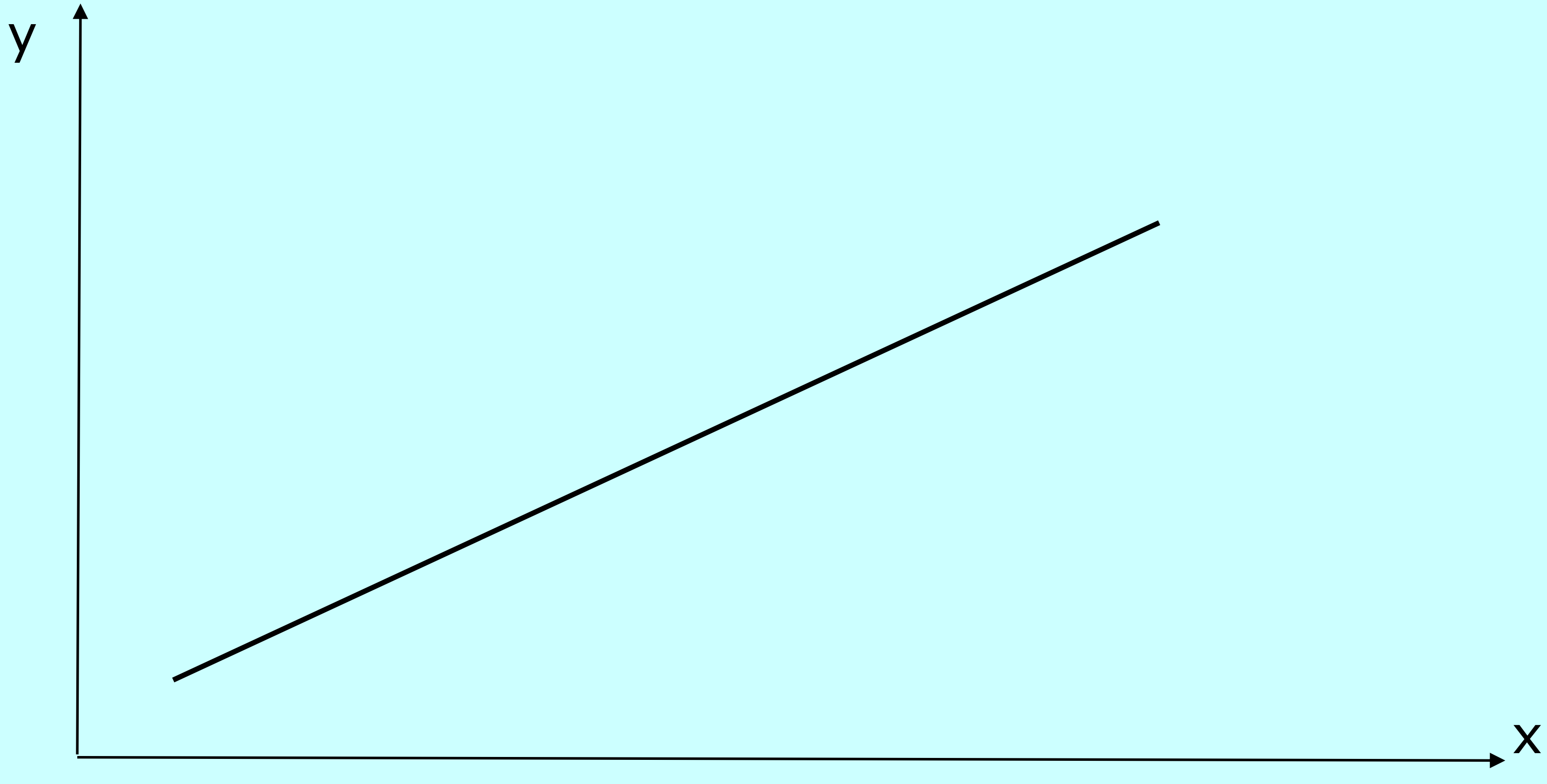


Is there a way of putting a **value** on this linear relationship between the two physical quantities?

The constant value which is **rate of change** of one physical quantity with respect to the other is got by finding the

S L O P E

of the line.

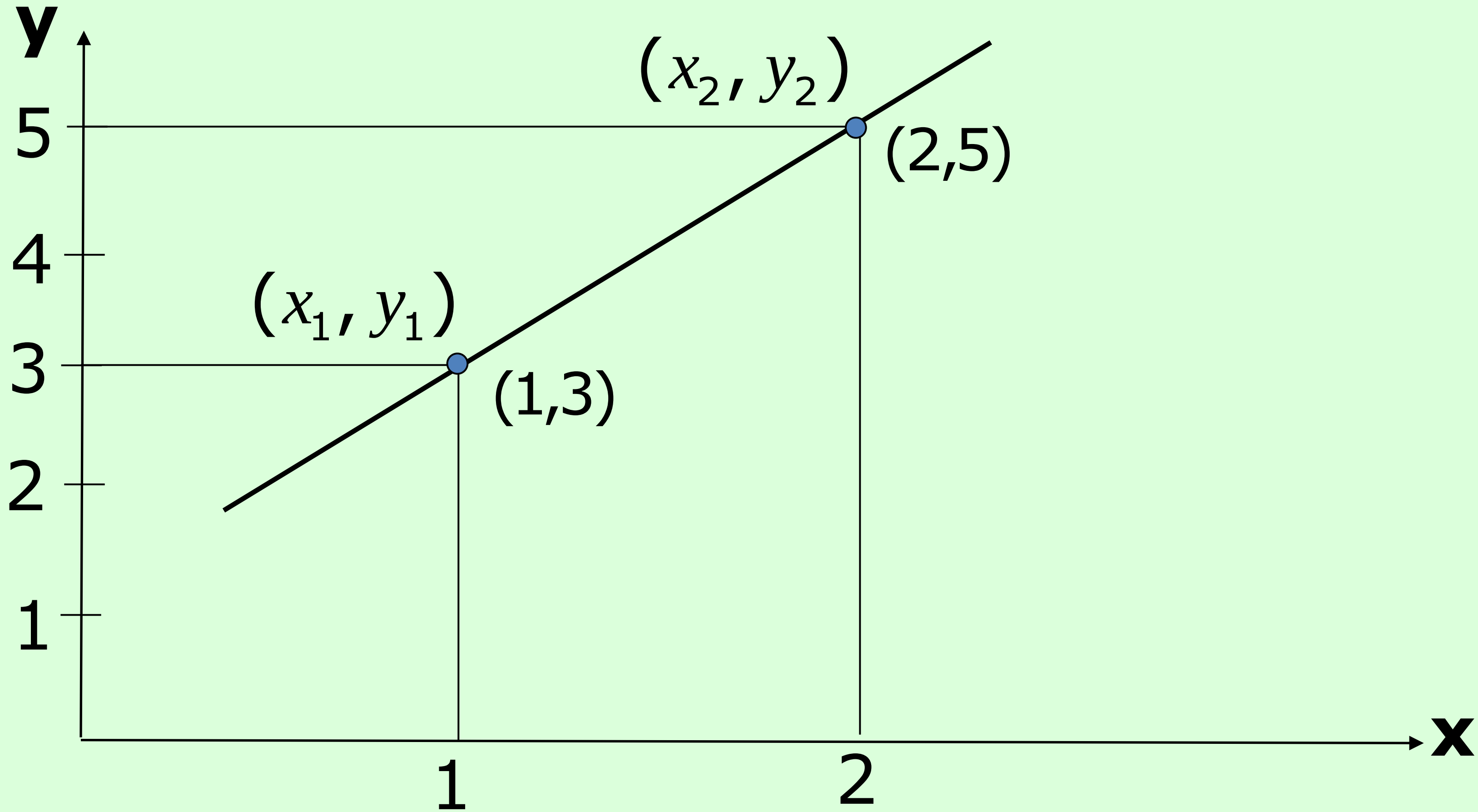


Finding the **slope** of a straight line…

Plot the straight line $y = 2x + 1$ by finding any _ _ _ points on it.

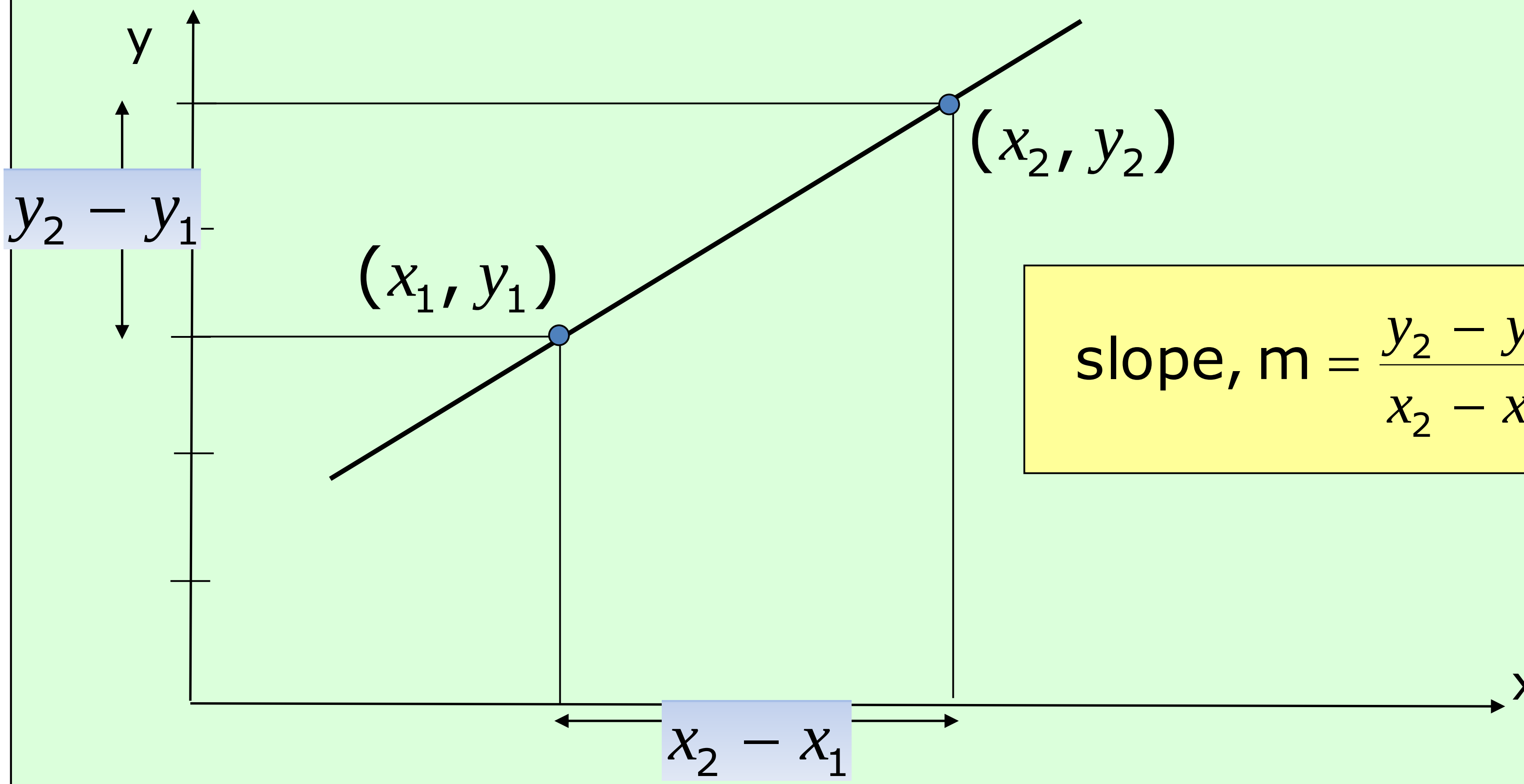
$x = 1$ $\quad y = 2(1) + 1 = 3$ $\quad$ Point is $(x_1, y_1) = (1,3)$

$x = 2$ $\quad y = 2(2) + 1 = 5$ $\quad$ Point is $(x_2, y_2) = (2,5)$



Slope of $y = 2x + 1$ between any two points…

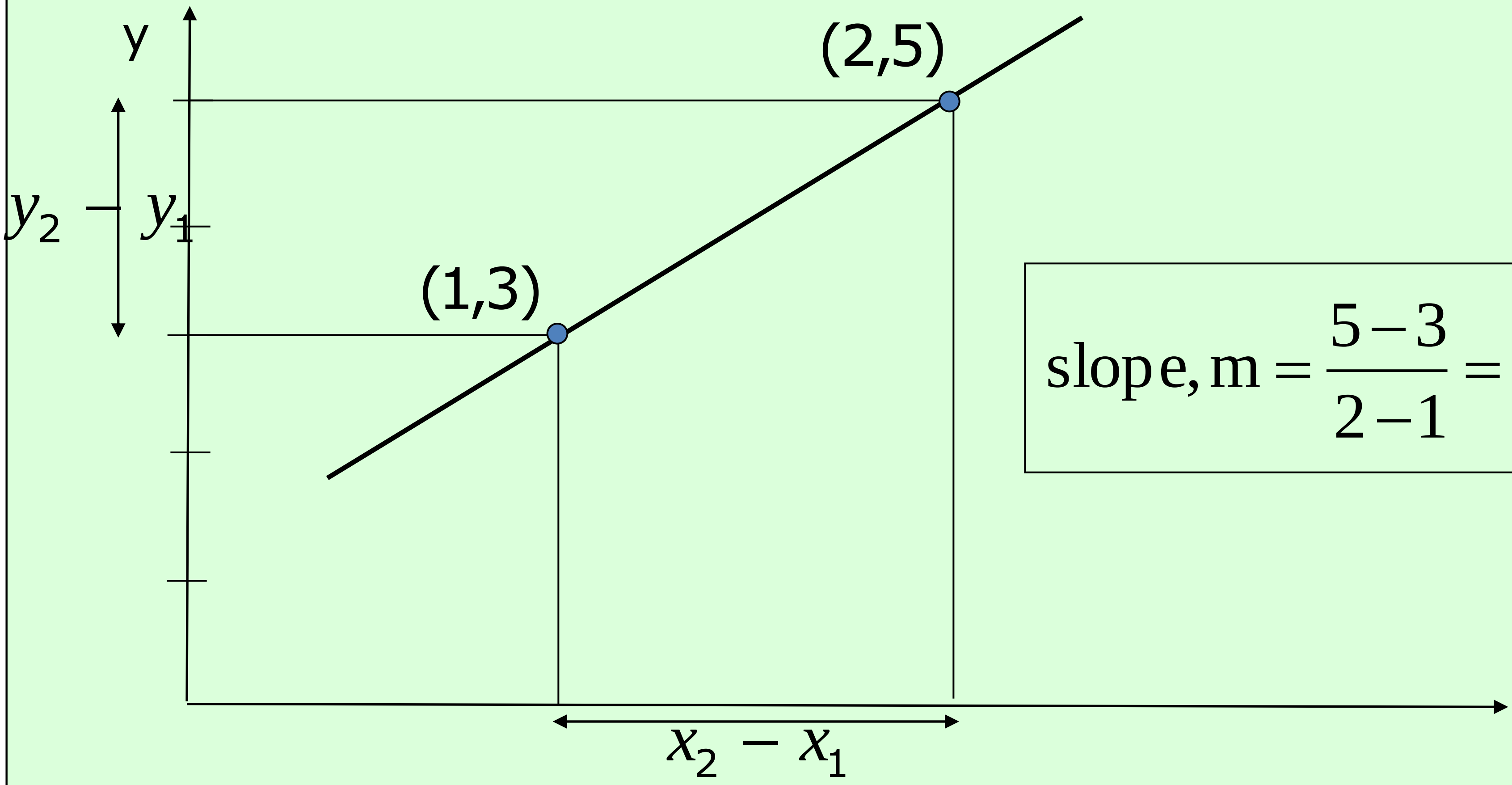
$$\text{slope} = \frac{\text{change in vertical height}}{\text{change in horizontal height}}$$




$$\text{slope, m} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of $y = 2x + 1$ between any two points…

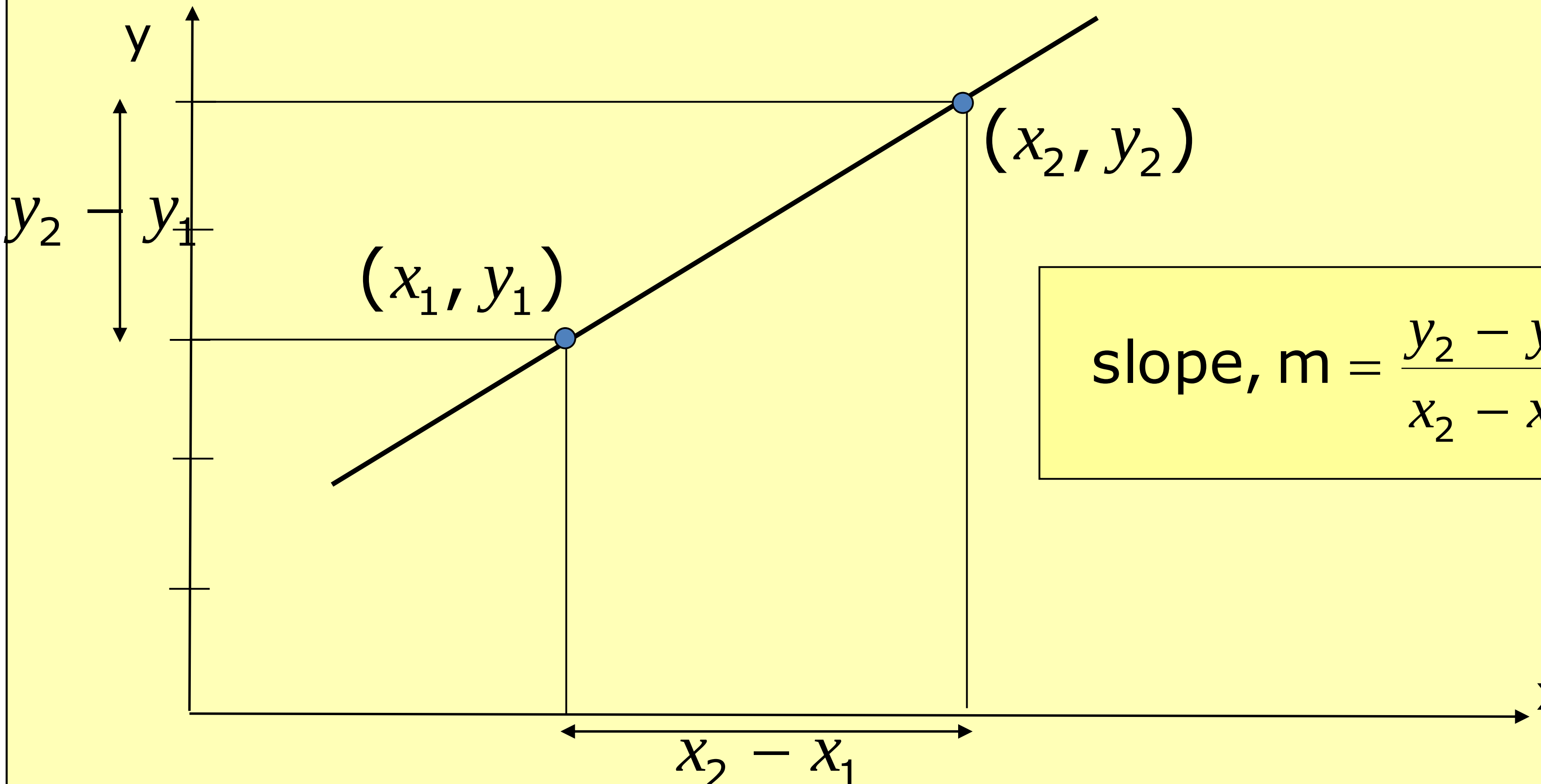
$$\text{slope} = \frac{\text{change in vertical height}}{\text{change in horizontal height}}$$




$$\text{slope, m} = \frac{5-3}{2-1} = 2$$

Slope of any straight line between any two points…

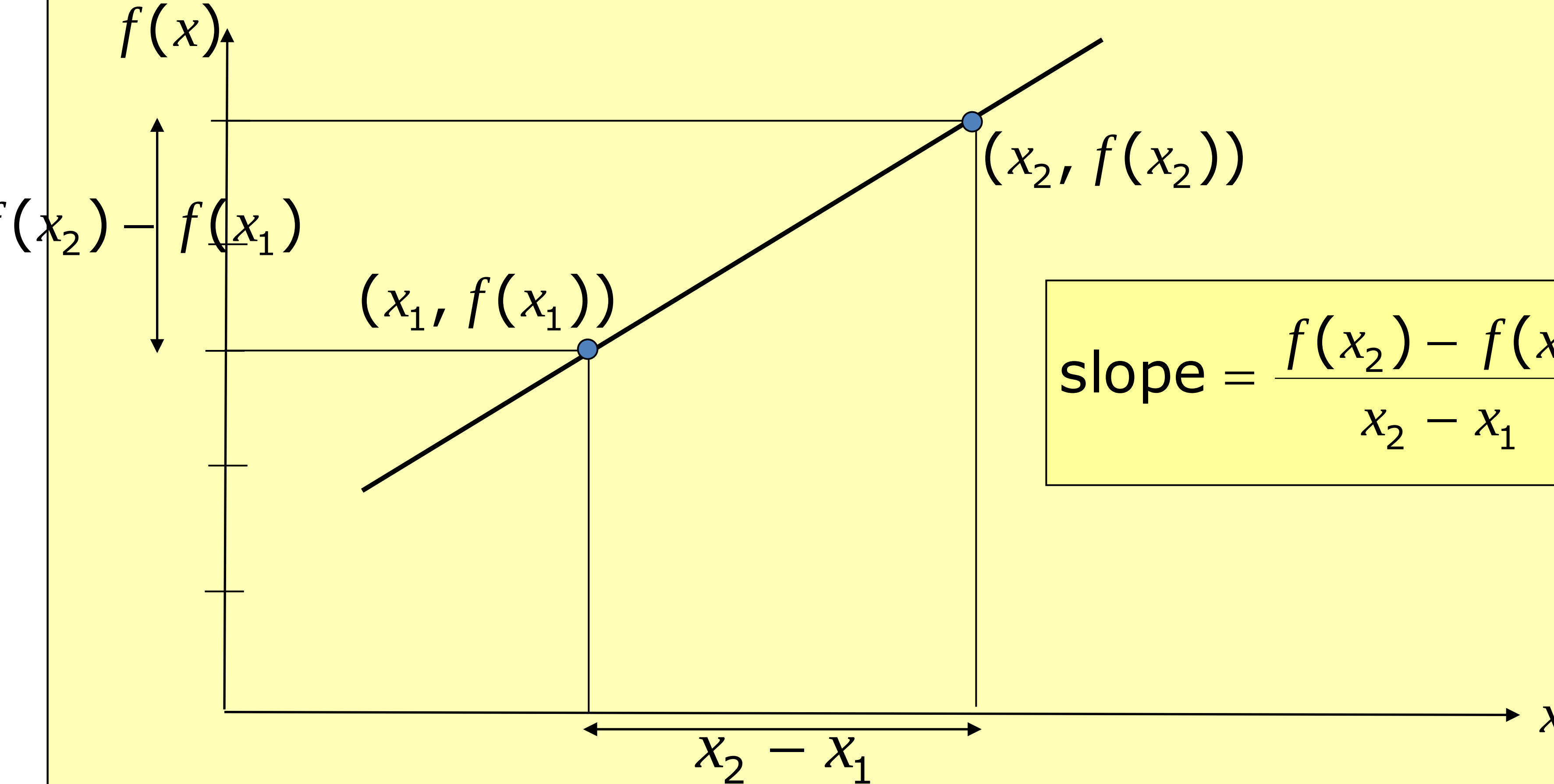
$$\text{slope} = \frac{\text{change in vertical height}}{\text{change in horizontal height}}$$




$$\text{slope, m} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of any straight line between any two points…

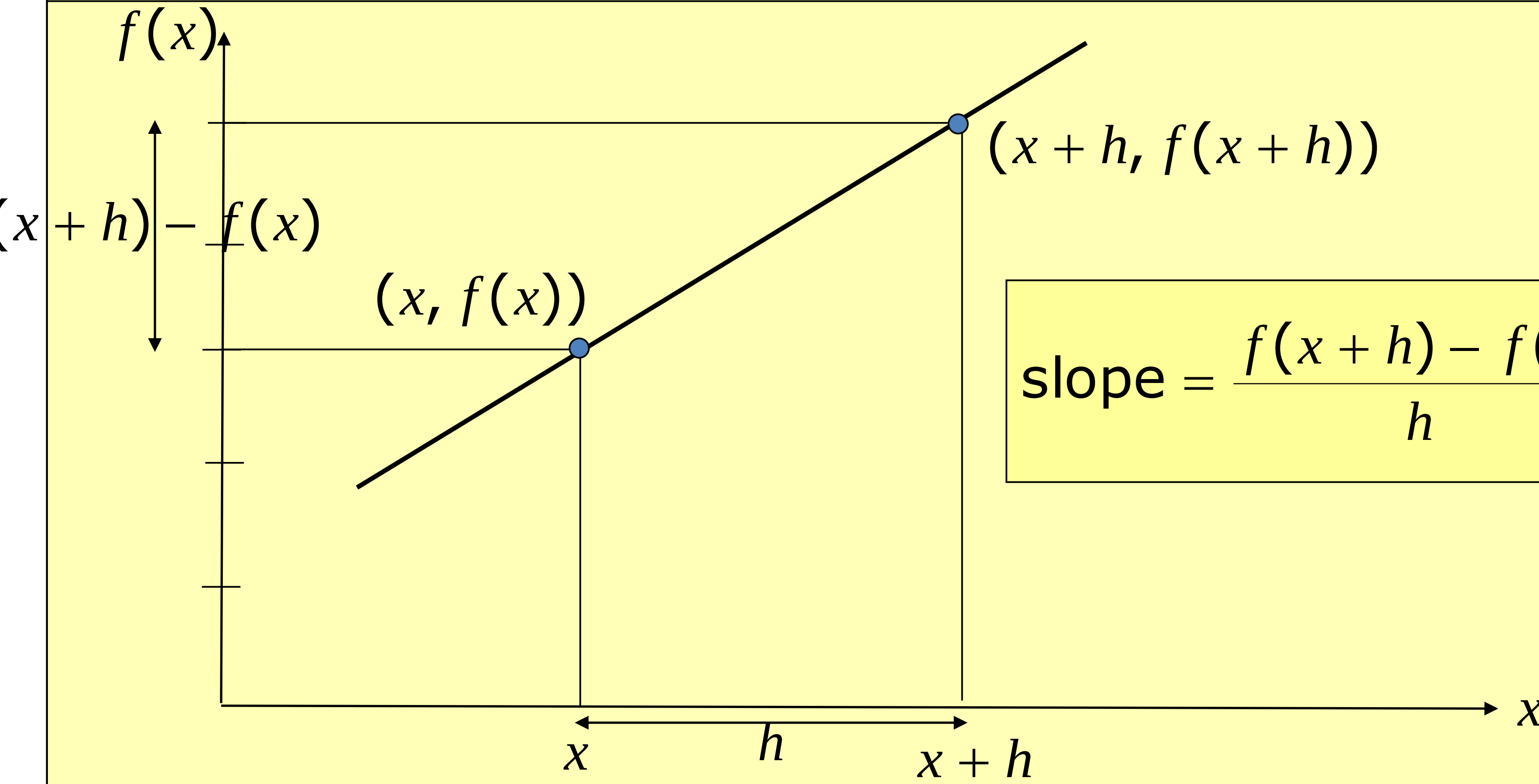
$$\text{slope} = \frac{\text{change in vertical height}}{\text{change in horizontal height}}$$




$$\text{slope} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Slope of any straight line between any two points…

$$\text{slope} = \frac{\text{change in vertical height}}{\text{change in horizontal height}}$$




$$\text{slope} = \frac{f(x+h) - f(x)}{h}$$

**In-class exercise:** Complete the following sentences.

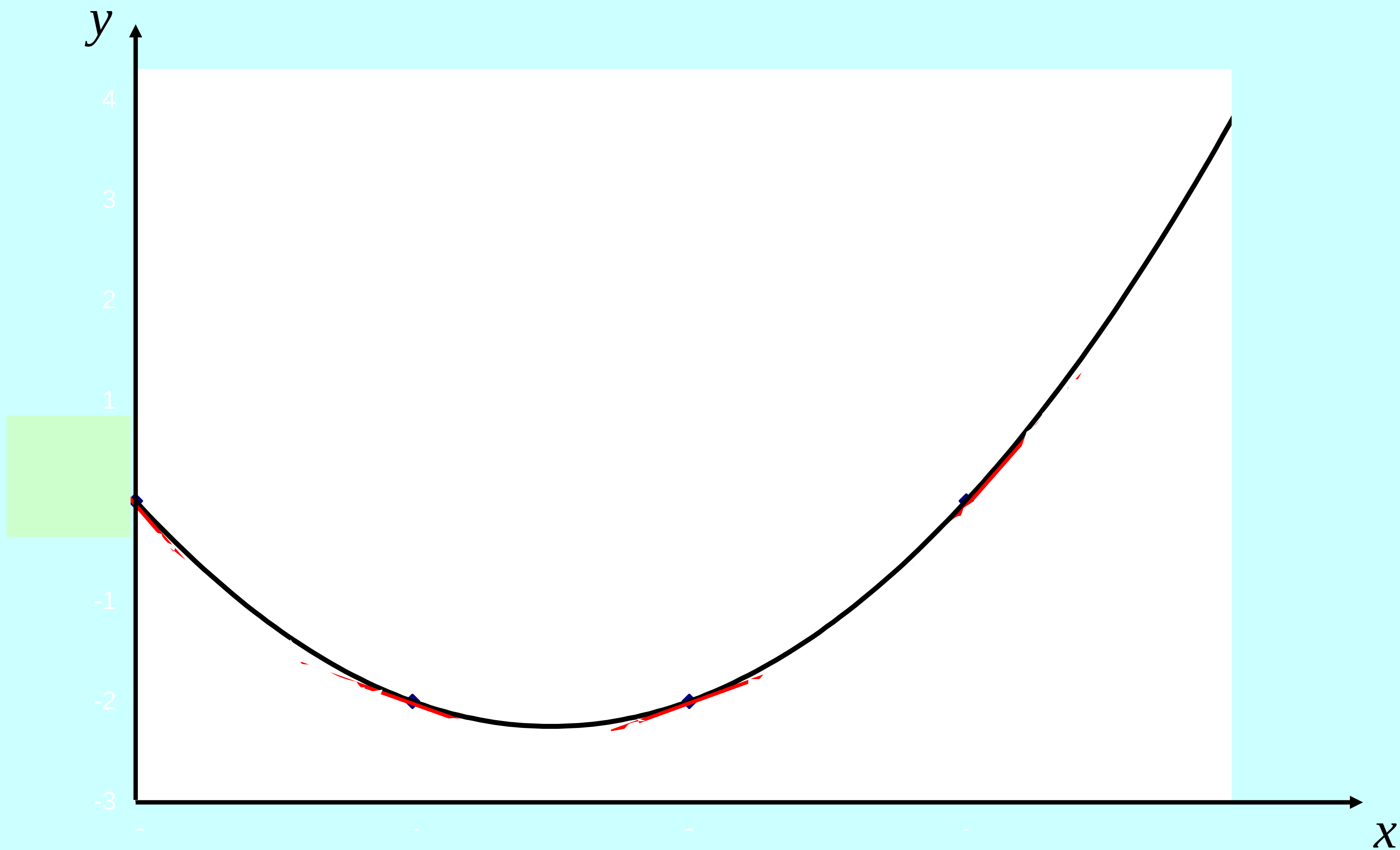
| |
|---|
| The Average R A T E of C H A N G E of a physical quantity which varies linearly with another is found by getting the S L O P E of the line which joins the 2 points defining the relationship. |
| The S L O P E of a straight line is found by getting the change in V E R T I C A L height divided by the change in H O R I Z O N T A L distance between any T W O points. |
| |

**Next finding the instantaneous rate of change**

**…we have looked at how to calculate the average rate of change of one quantity with respect to another (over an interval). But there are times when we wish to know the rate of change of one quantity with respect to another at some particular value i.e. Instantaneous rate of change**

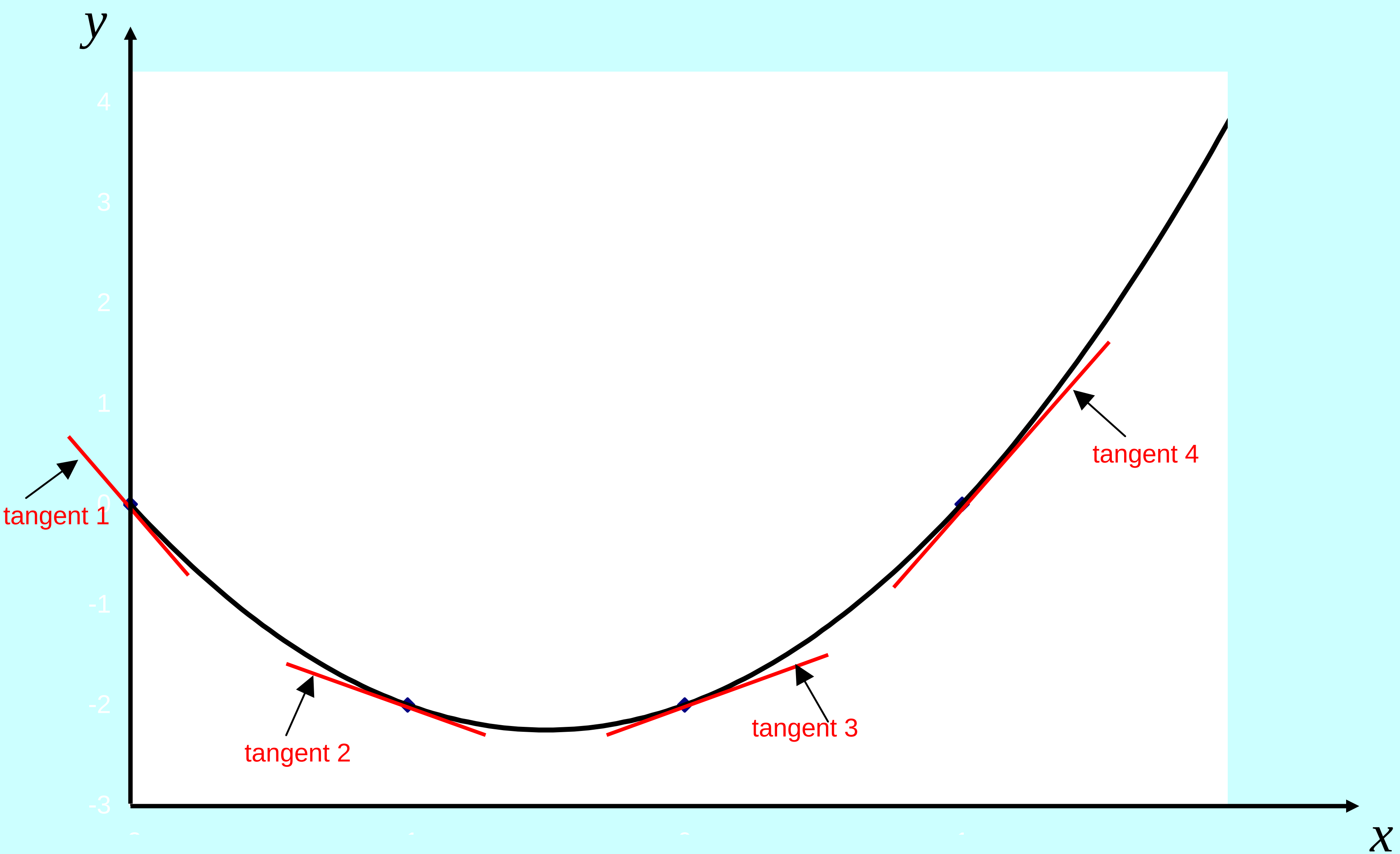
Consider curve below where the value of *y* varies with respect to *x*.

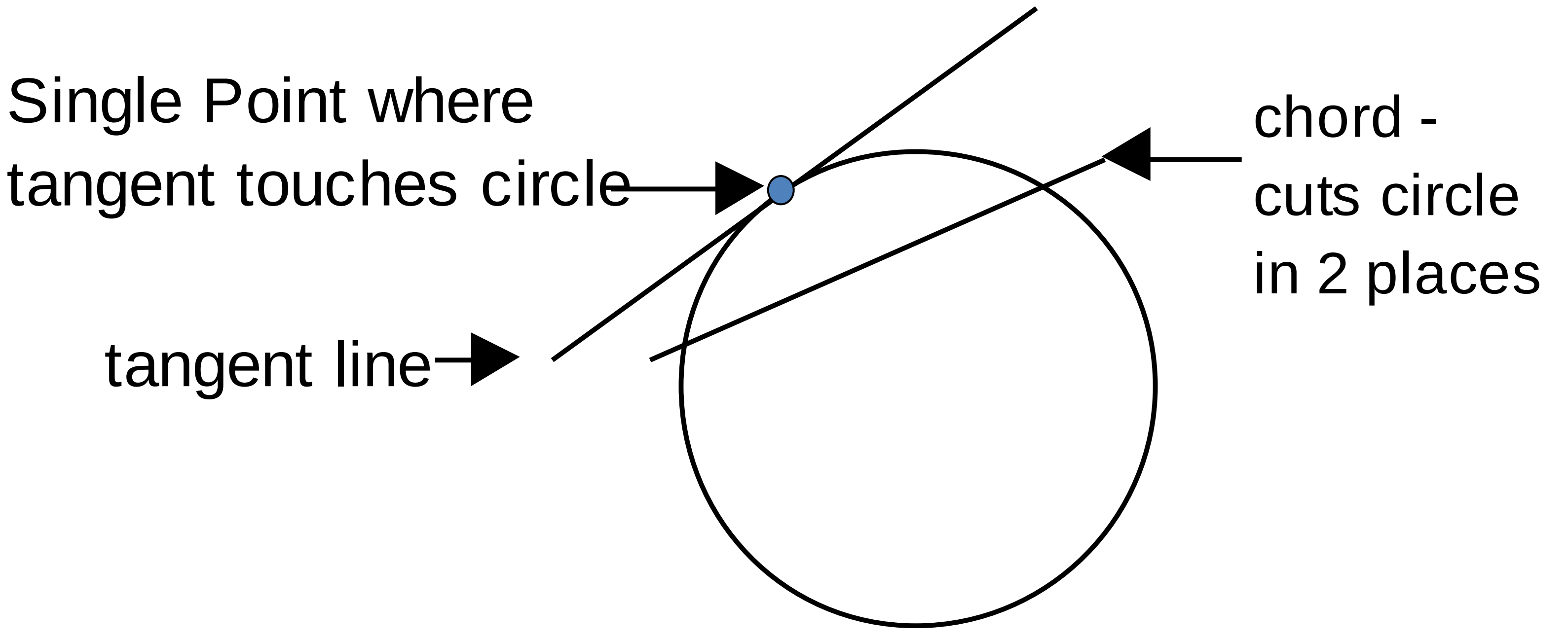
**Question:** Can we find how the value of y varies with respect to x?



A curve is made up of an infinite number of **tangent lines**.

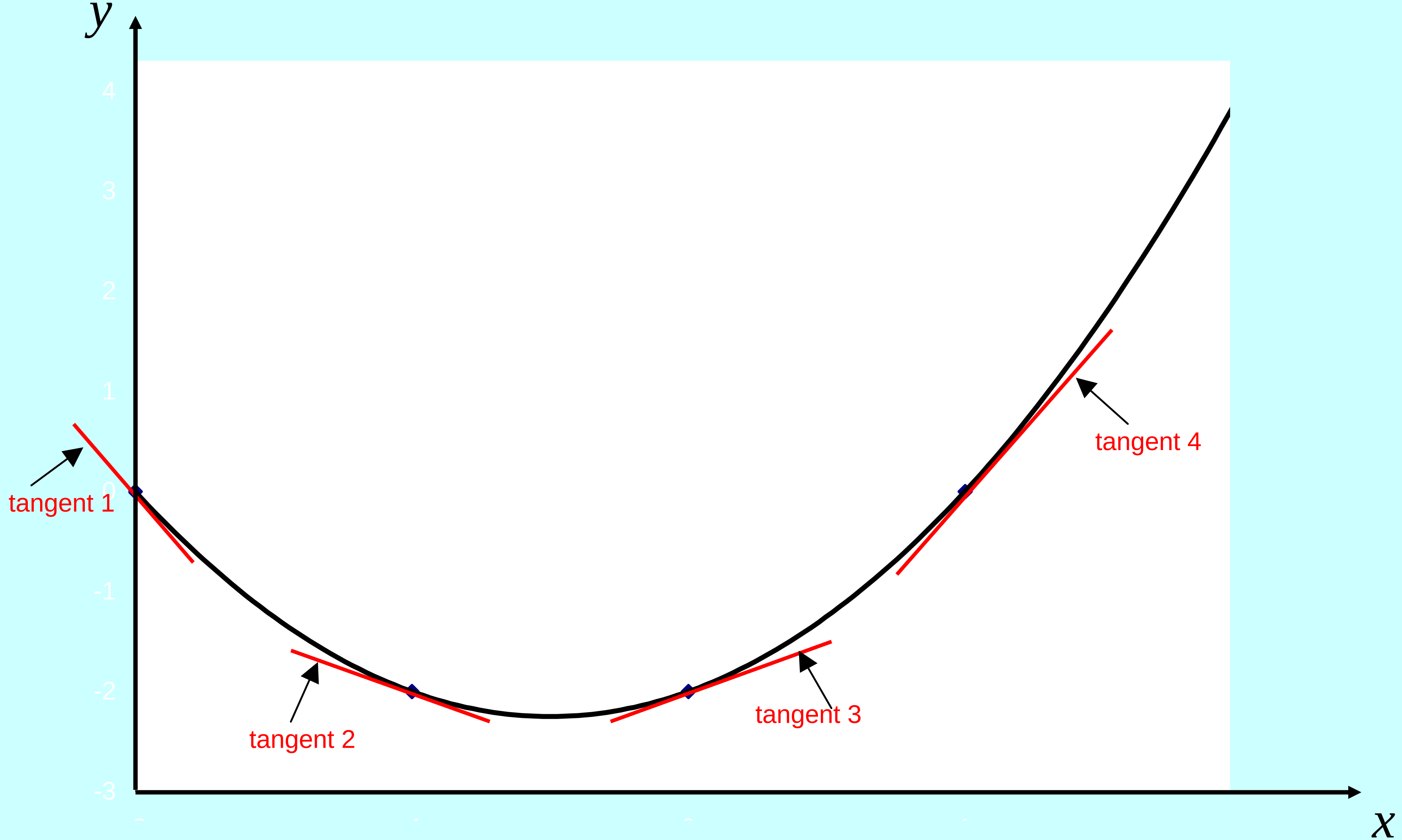
A tangent line is a straight line which touches a curve at one point only.




Single Point where
tangent touches circle
tangent line
chord -
cuts circle
in 2 places

If the **slope** of a tangent line at a point is found…

…then the **rate of change** of the **dependent** variable with respect to the **independent** variable can be found **at that point**.



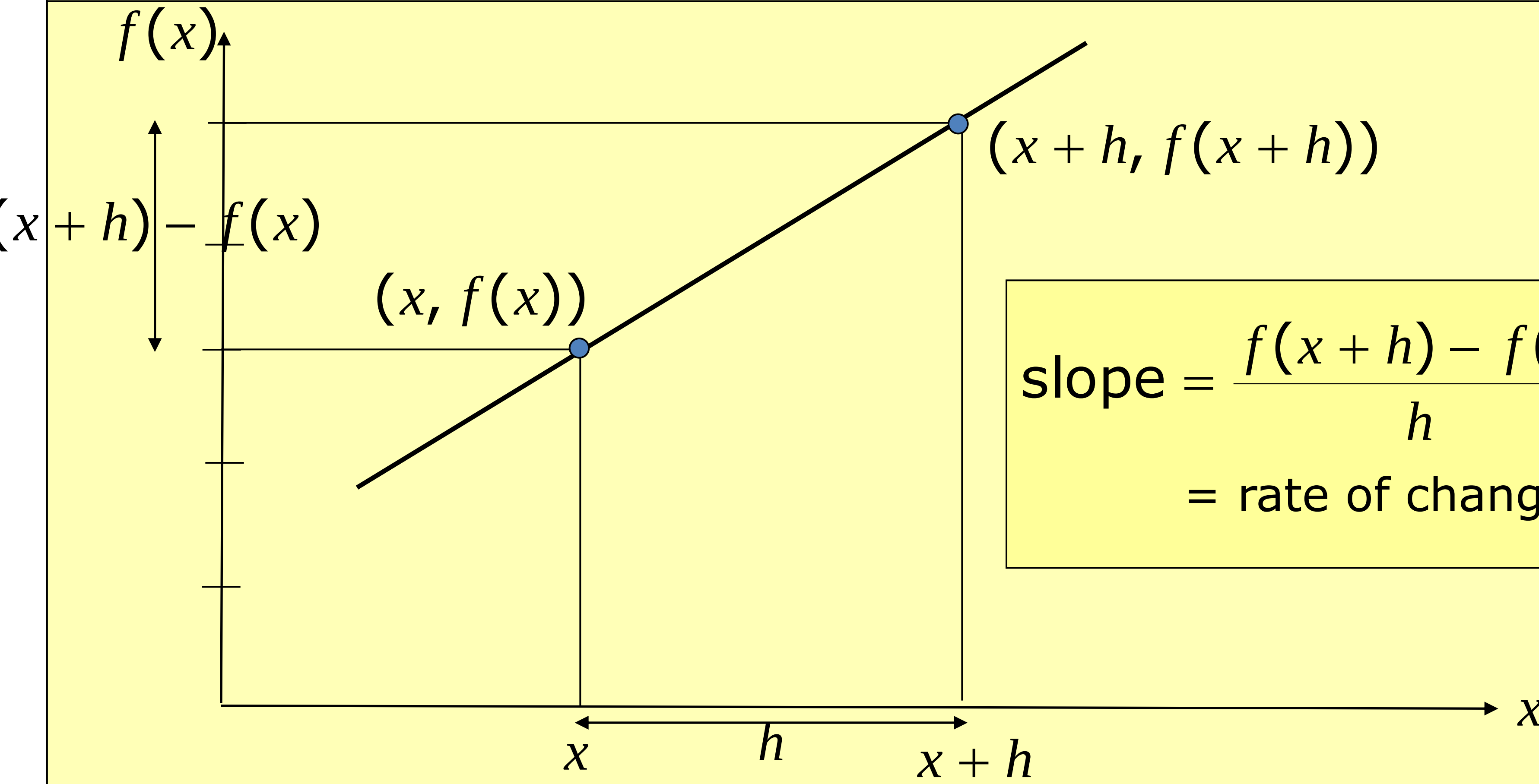
**Next Question…**

**…how to find the slope of a tangent line at a point on a curve?**

**…the solution is to use the formula for finding the slope (rate of change) for a straight line between two point and apply it to the tangent line**

Recap : Slope of any straight line between any two points…

$$\text{slope} = \frac{\text{change in vertical height}}{\text{change in horizontal height}}$$

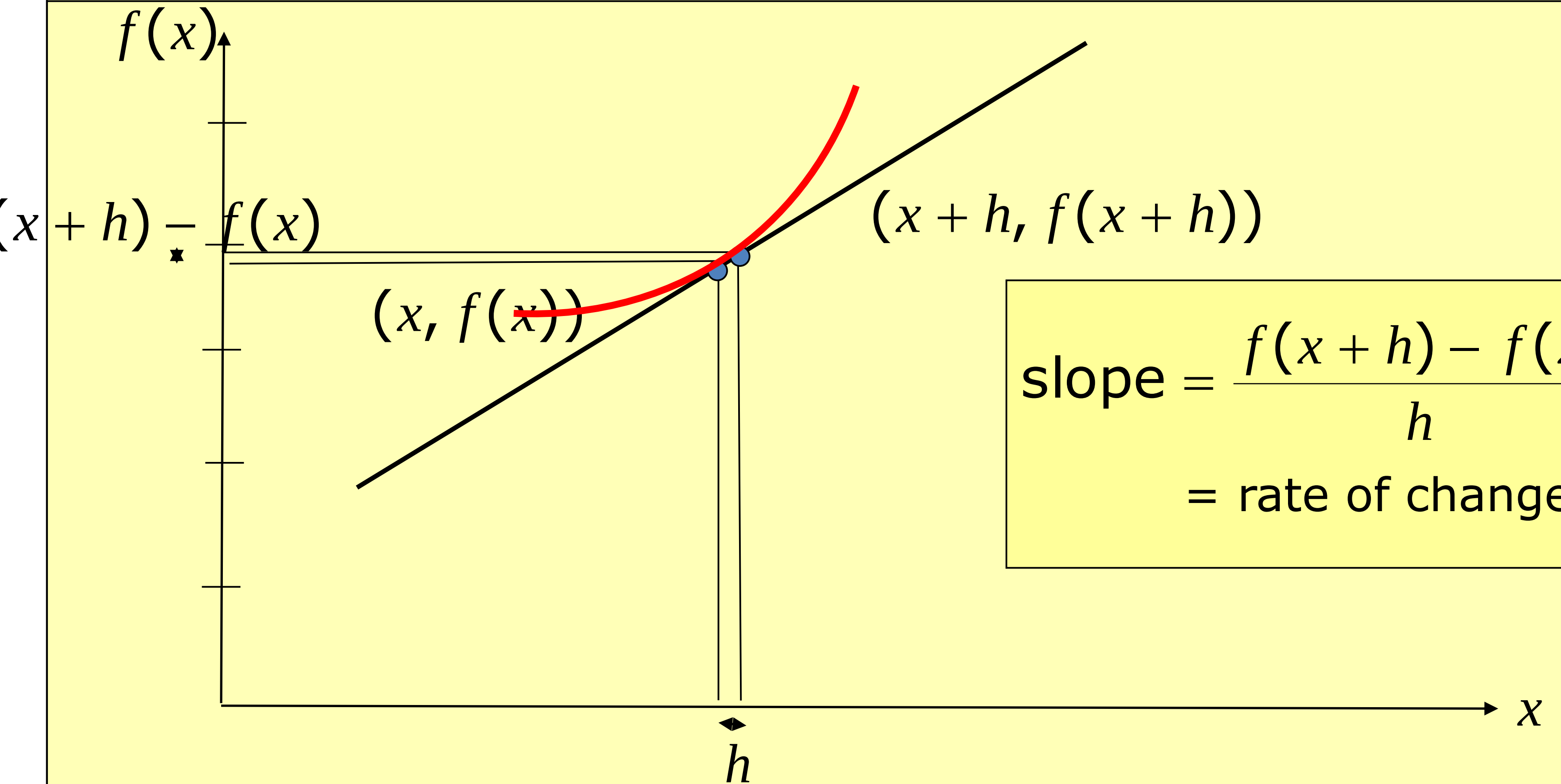



$$\text{slope} = \frac{f(x+h) - f(x)}{h}$$

= rate of change

If 'h' approaches zero…

… then the formula approaches the tangent line case.




$$\text{slope} = \frac{f(x+h)-f(x)}{h}$$

= rate of change

## Approximating the Slope (*cont.*)

As h approaches zero, the chord approaches a tangent to the curve at a single point. The distance between the 2 points above is h.

The mathematical notation for letting h approach zero is:

$$\lim_{h \to 0}$$

Therefore the above expression becomes:

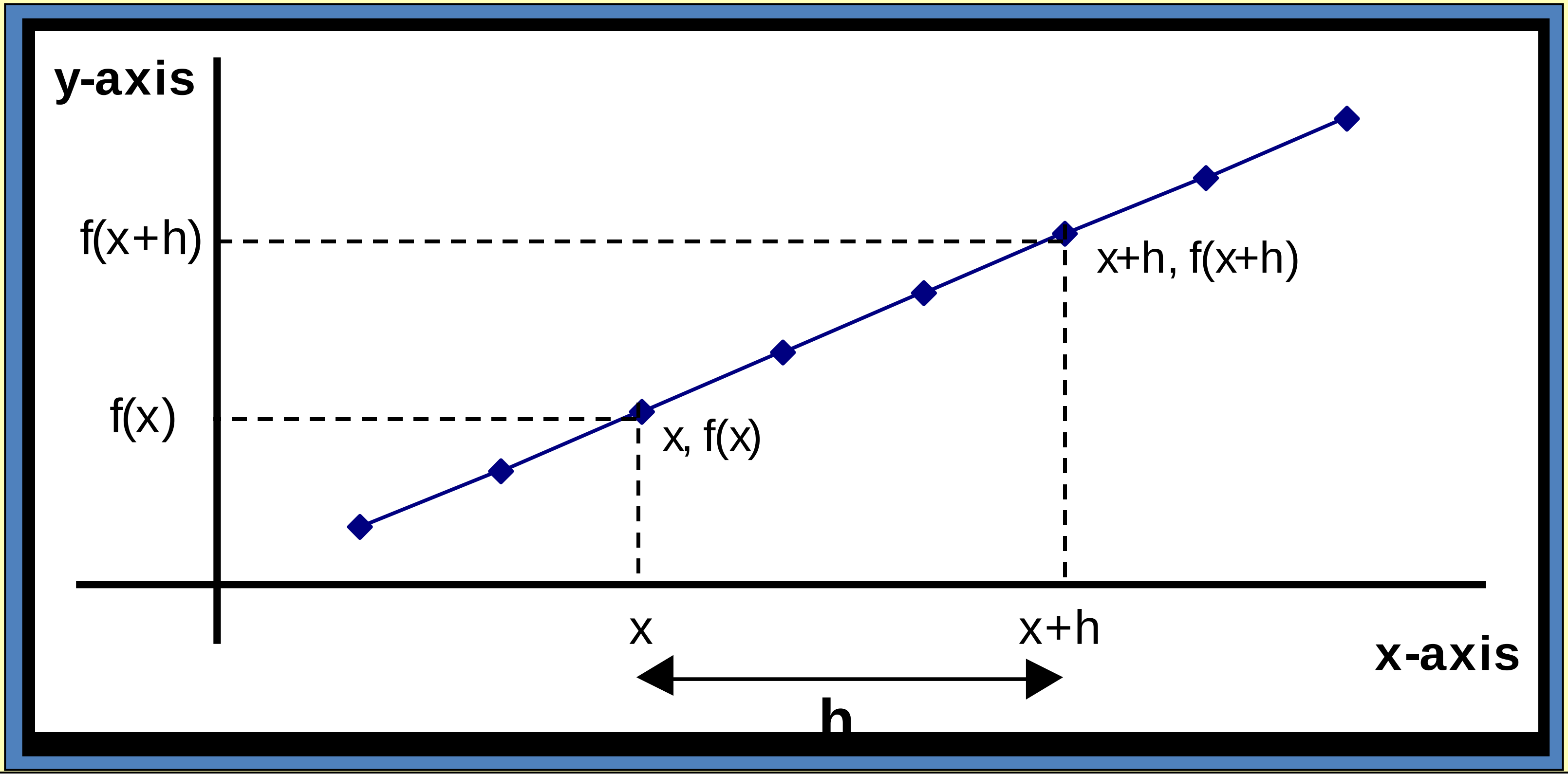
$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This gives us an expression for the slope of a tangent to the curve in terms of x, since h is a distance along the x-axis.

## Summary:

Consider the straight line below as a chord on a curve.
The slope, m, is given by:

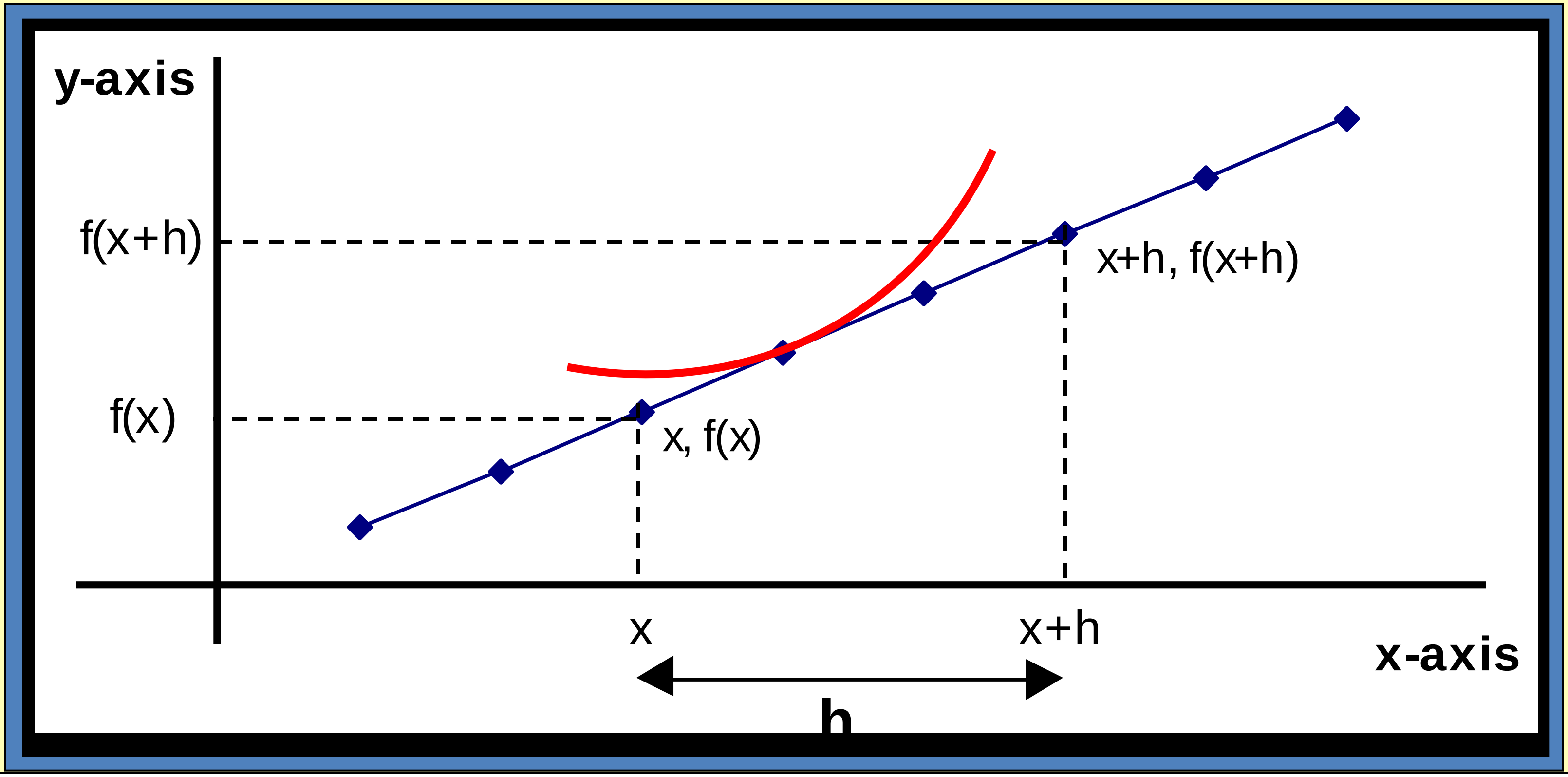
$$m = \frac{f(x+h) - f(x)}{h}$$



**For a curve:**

The rate of change of one variable with the other at a point on the curve is found by getting the slope of the tangent line at that point.

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



**In-class exercise:** Complete the following sentences

| |
|---|
| For each point on a curve there exists a T A N G E N T line which touches the curve at O N E point. **Click for solution** |
| The S L O P E of the T A N G E N T line at a point on a curve is found by using the formula for the S L O P E of a S T R A I G H T line between two points but letting the difference between the I N D E P E N D E N T variable approach zero. **Click for solution** |
| Finding the S L O P E of the T A N G E N T line at a point on a curve is the same as finding the R A T E of C H A N G E of the D E P E N D E N T physical quantity with respect to the I N D E P E N D E N T physical quantity at that point on the curve. **Click for solution** |

## What is differerential calculus?

This is an area of mathematics which is concerned with the **rate of change** of one physical quantity with respect to another…

The **rate of change** of one quantity with respect to another is called its **derivative**.

Therefore if we have a function with 2 variables, then we can get the **derivative** of the function using the formula above.

This is the basis of **differentiation**.

Hence, for a function $y = f(x)$

The rate of change of *y* with respect to *x* is:

$$\boxed{\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$f'(x)$ is the **slope** of the tangent to the curve **at the point *x***.

The **slope** of the tangent is called the derivative…

… and gives the **instantaneous rate of change** of the function **at a point**.

Hence, for a function $y = f(x)$

The rate of change of *y* with respect to *x* is:

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

**Differentiation is the rate of change of one quantity with another.**

**Example:**

The **average** rate of change is that taken over an interval.

For example the average speed is given by:

$$\frac{\textit{Change in distance}}{\textit{Change in time}}$$

The smaller the time interval gets the closer we get to a value for the speed at an instantaneous moment.

In other words this is an approximation to the instantaneous rate of change of distance with respect to time.

**Example:** Consider the following expression:

$$s = 3t - 5t^2$$

where s is the height of a ball thrown into the air (distance) and t is the time.

Here we have an expression with distance and time. How can this be used to get an expression for the velocity of the ball?

Recall velocity is a physical quantity giving the rate of change of distance with time.

Therefore if we differentiate (get $\dfrac{ds}{dt}$ ) , we get an expression for the velocity of the ball in terms of t. Therefore the velocity can be found for any value of t. (t = independent variable & approaches 0)

When this formula is used to find the rate of change of one physical quantity with respect to another:

$$\boxed{\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

then this is **differentiating from first principles.**

**Example:**

Find an expression for the rate of change with respect to *x* of:

$$y = x^2$$

$$f(x) = x^2$$

$$f(x+h) = x^2 + 2xh + h^2$$

**Summary:**

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\boxed{\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

$$= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - (x^2)}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} 2x + h$$

$$= 2x$$

If $y = x^2$ then $\frac{dy}{dx} = 2x$

**Slope** of function $y = x^2$ at any point, $x$ , on the curve is:

$$\text{slope at any point x} = 2x$$

**Rate of change** of $y$ with respect to $x$ at any point on the curve is given by:

$$\text{rate of change at any point x} = 2x$$

slope at any point x $= 2x$

**is another way of saying...**

rate of change at any point x $= 2x$

If $y = f(x) = x^2$

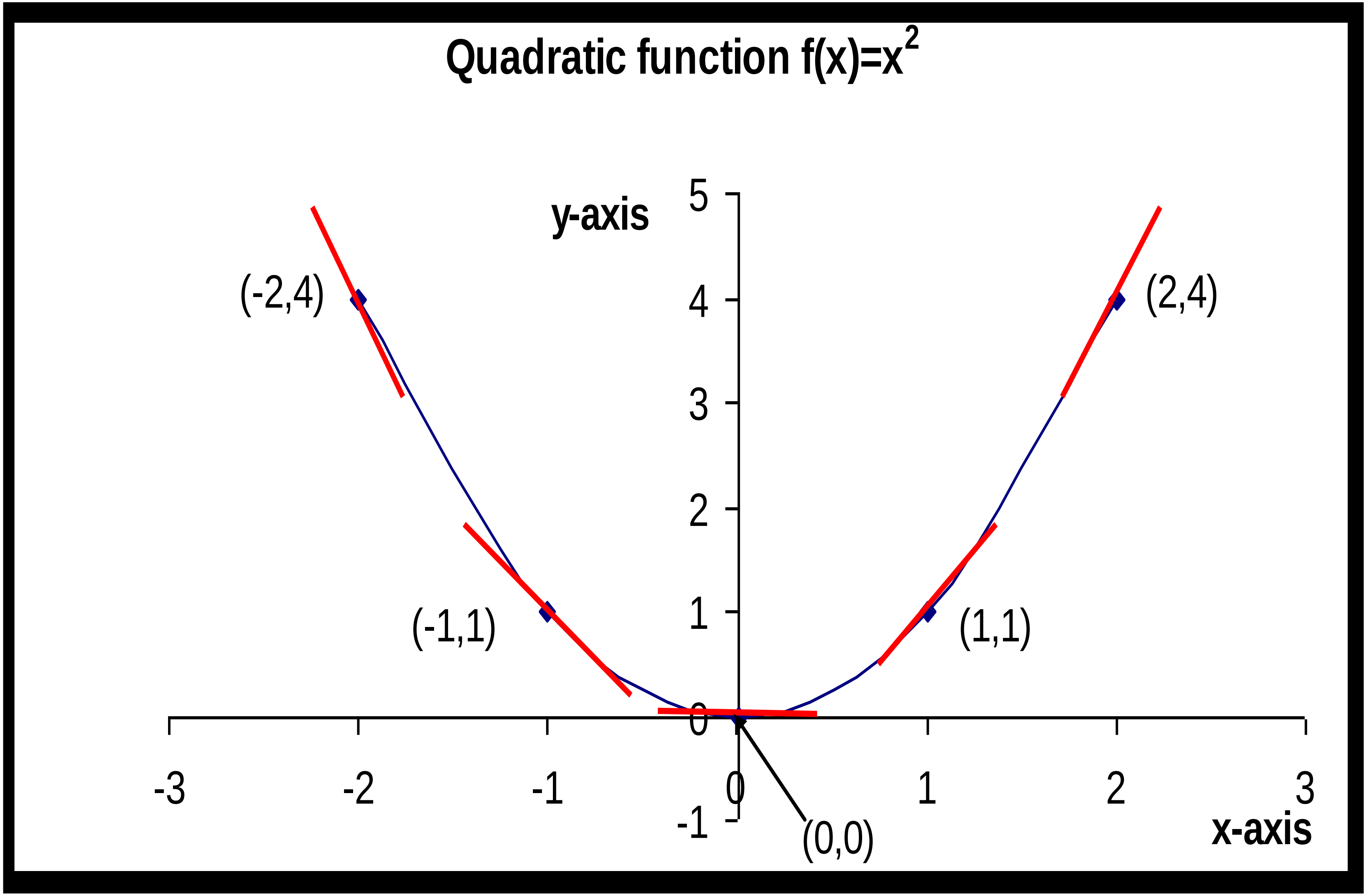
then $\dfrac{dy}{dx} = f'(x) = 2x$

slope at any point $= 2x$

**In-class exercise:**
complete following table

**Try yourself first & then click for solution**

| | Finding slope | Finding point on curve | |
|---|---|---|---|
| $x$ | slope $= 2x$ | $y = f(x)$ | point on curve |
| $x = -2$ | $2(-2) = -4$ | $(-2)^2 = 4$ | $(-2,4)$ |
| $x = -1$ | $2(-1) = -2$ | $(-1)^2 = 1$ | $(-1,1)$ |
| $x = 0$ | $2(0) = 0$ | $(0)^2 = 0$ | $(0,0)$ |
| $x = 1$ | $2(1) = 2$ | $(1)^2 = 1$ | $(1,1)$ |
| $x = 2$ | $2(2) = 4$ | $(2)^2 = 4$ | $(2,4)$ |



Quadratic function f(x)=x2
y-axis
5
4
3
2
1
0
-1
(-2,4)
(2,4)
(-1,1)
(1,1)
(0,0)
-3
-2
-1
0
1
2
3
x-axis

## Some conclusions:

1. there is different slope at each different value of $x$

2. the slope of the lowest point (minima) is zero

3. the slope of the points where $x$ is less than the minima all have a different sign to those after the minima. Recall that a negative slope means the slope is decreasing from left to right and vice versa for a positive slope.

**In-class Exercise:** Differentiate the following function from first principles: $y = f(x) = x^2 - 1$

Try yourself first & then click for solution

$$f(x) = x^2 - 1$$

$$f(x+h) = (x+h)^2 - 1$$

**Short way using Power Rule:**

$$y = x^2 - 1$$

$$\frac{dy}{dx} = f'(x) = 2x$$

**Long way using first principles:**

$$\boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$= \lim_{h \to 0} \frac{\left((x+h)^2 - 1\right) - \left(x^2 - 1\right)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 1 - x^2 + 1}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} \qquad = \lim_{h \to 0} 2x + h$$

$= 2x$ as h goes to zero

# Short Cut!

Check your answer using the Power Rule of differentiation

$$y = x^n$$

$$\frac{dy}{dx} = nx^{n-1}$$

$$y = x^2$$

$$\frac{dy}{dx} = 2x^{2-1} = 2x$$

**In-class exercise:**

Differentiate the following using the power rule:

**Try yourself first & then click for solution**

$$y = ax^n$$
$$\frac{dy}{dx} = an\,x^{n-1}$$

Worked Example: $y = 2x^2$ $\quad \frac{dy}{dx} = (2 \times 2)x^{2-1} = 4x^1$

| | | | |
|---|---|---|---|
| $y = x^3$ | $\frac{dy}{dx} = 3x^2$ | $y = 2x^{-3} + 5$ | $\frac{dy}{dx} = -6x^{-4} + 0$ |
| $y = 2x^4$ | $\frac{dy}{dx} = 8x^3$ | $y = 2 + 4x + 2x^2$ | $\frac{dy}{dx} = 0 + 4 + 4x$ |
| $y = -2x^5$ | $\frac{dy}{dx} = -10x^4$ | $y = 3x^3 - 4x^6$ | $\frac{dy}{dx} = 9x^2 - 24x^5$ |

## Differentiation = the Rate of Change

If $s$ is the distance of a particle from a point $p$, then the rate at which this distance increases is given by, $\frac{ds}{dt}$ .

In other words, $\frac{ds}{dt}$ , is the velocity.

In the same way, $\frac{dv}{dt}$, is the rate of change of velocity, that is, acceleration.

It then follows that acceleration $= \frac{dv}{dt} = \frac{d^2 s}{dt^2}$

**In-class exercise:**

A boy hits a tennis ball straight up above him. The height *h* of the ball at any time *t* seconds after the ball is hit is given by,

$$h(t) = 1 + 30t - 5t^2$$

Find, 1) The height of the ball 1 second after it is hit

2) The rate at which the ball is rising after 2 seconds

Solution: 1) The height of the ball 1 second after it is hit

$$h(1) = 1 + 30(1) - 5(1)^2 = 26$$

2) The rate at which the ball is rising after 2 seconds

$$h'(t) = 0 + 30 - 10t$$

$$h'(2) = 30 - 10(2) = 10$$

**Try yourself first & then click for solution**

Differentiating from first principles becomes very tedious for more complex problems…

…there are a number of rules which allow this to be done quickly

…next part of differentiation is to look at some of the rules of differentiation…