Probability

Contents

2 Probability

Probability is the area of mathematics which deals with chance. There are several ideas central to the topic, which will be defined first; everything we come to study in this area can only be properly understood if these fundamental definitions are kept in mind.

We will start by defining events, experiments and probability.

2.1 Basic definitions

2.1.1 Definition: An Event

An *event* is a well-defined occurrence that may or may not happen.

A very important part of this definition is the idea that the event is *well-defined*. To be discussed formally in mathematics, an event must be exactly described.

A good example; throwing dice.

For an example of a more ambiguous event, consider a “White Christmas.”

Now that we know what an event is, the next step is to define exactly what its probability is.

We will define this idea in two steps, the first giving a general idea of probability, the second giving a quantifiable version.

2.1.2 First Definition: Probability

The *probability* of an event is a measure of how likely it is to occur. It is a number between 0 and 1.

A probability of 0 for an event means it will not occur, while a probability of 1 means it definitely will occur.

2.1.3 Notation for probability

Mathematically, an event can be denoted by any symbol. Thus the event E could be the event of a particular number showing up on a dice.

With this in mind, the probability of event E is written mathematically as

$$P[E].$$

2.1.4 An Example of the Notation

The meteorological service has stated that ‘there is a 35% chance of rain in Dublin today’.

Set R to be the event that there is rain in Dublin today. Then we write:

$$P[R] = 0.35.$$

Take note of the fact that the ‘35% chance’, written in layman’s language, becomes a probability of 0.35.

The probability of an event is best defined if it may occur many times. To allow us to follow this line of thought for the definition, we need another definition, that of an experiment.

2.1.5 Definition: Experiment

An action, with a well-defined set of outcomes, that can be repeated a large number of times, is called an *experiment*.

The importance of this idea of an experiment is that we can now think of an event as the result of an experiment, defined in terms of one or more of the outcomes of the experiment.

With this idea of an event happening as the result of an experiment, we can go back to the definition of the probability of an event and quantify it as the proportion of times it occurs if the experiment is repeated many times.

2.1.6 Second Definition: Probability

Let E denote an event, which is the possible outcome of an experiment.

The probability $P[E]$ is the proportion of times event E occurs, as the experiment is repeated a very large number of times.

Mathematically, if the experiment is repeated N times, and E turns up N_E times, then the probability of E occurring, is the fraction

$$P[E] = \frac{N_E}{N} \ ,$$

evaluated as the number N gets bigger. Strictly speaking this is the limit:

$$P[E] = \lim_{N \to \infty} \frac{N_E}{N} \ .$$

2.1.7 The Example of the Dice

In the example of the throw of one dice, when the experiment is done, that is, the dice has been thrown, we are interested in the number left facing up. The set of possible outcomes is

$$\{1, 2, 3, 4, 5, 6\}$$

It would be expected that after a large number of throws, each side of a fair dice would come up a roughly equal number of times.

In the case of the throw of one dice, we will now look at the probabilities of getting the numbers 1, 2, 3 and so on. Let us denote these probabilities as $P[1]$, $P[2]$, and so on.

Let the dice be thrown N times, where N is a large number. Each face will come up approximately the same number of times. Call this number n. Then we would expect that

$$6n = N.$$

From the new, quantitative definition of probability the number 1 comes up is

$$P[1] = \frac{n}{N} = \frac{1}{6}\,.$$

The calculation for all the other numbers is the same, so that

$$P[E] = \frac{1}{6}\,,$$

where E is the event of a particular number coming up when the dice is thrown.

2.1.8 Rolling Two Dice

We will now look at this definition of a probability with the example of the throw of two dice.

For this case, we will calculate the probability of the result adding up to 5. Firstly, let E be the event that we roll two dice and we get a sum of five.

To find the probability, we need the number N, the number of times the experiment is repeated, and N_E, the number of times the event E turns up.

Then the probability of E occurring, is the fraction

$$P[E] = \frac{N_E}{N}$$, evaluated as the number N gets bigger.

Firstly, we will find how many possible outcomes are there in total.

For a pair of numbers, with either number gong from 1 to 6, the possibilities are

$$(1,1), (1,2), \ldots, (1,6), (2,1), (2,2), \ldots, (2,6),$$

etc…

$$(6,1), (6,2), \ldots, (6,6).$$

With each die having 6 possible outcomes, there are 36 possible outcomes in all.

All of these are equally likely to come up. Only 4 have the sum of 5:

(1, 4), (2, 3), (3, 2) and (4, 1)

This list of outcomes is the event.

The proportion of times the sum of 5 comes up is then

$$P[E] = \frac{N_E}{N} = \frac{4}{36} \, .$$

The probability is then

4/36 = 1/9.

Note the distinction between the outcome of the experiment; the numbers which turn up, and the event, which is they add up to 5.

2.1.9 The Product of Two Numbers

In the same experiment, rolling two dice, calculate the probability that the two numbers, when multiplied, give 6.

The possible ways to get a product of 6 are:

$$(1,6), (2,3), (3,2), (6,1).$$

There are four possible outcomes that give 6, so the probability is

$$4/36 = 1/9.$$

2.1.10 The Event Space

For the case of two dice, calculate the probability that the sum is below 6.

The most methodical way of counting the number of outcomes which satisfy the definition of the event is to calculate the sum for each possible outcome in a table.

Here it is:

	1	*2*	*3*	*4*	*5*	*6*
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Such a table is called an *event space*, since every possible event is a subset of this table.

This table is the event space for ‘the sum’.

If the probability required was that of a product or difference, a similar event space could be drawn up, but with the product or difference calculated.

There are 10 results where the sum is below 6, so the probability is 10/36 = 0.277778.

2.1.11 Definition of Event Space

The event space is the list of all possible outcomes of an experiment. Any event which arises as a result of that experiment is a subset of the event space, in other words, it can be built up from the event space.

2.1.12 Difference Event Space

In the same experiment, rolling two dice, calculate the probability that the difference of the two numbers is 2.

This calculation is done by drawing up the event space for the experiment, and for ‘the difference’. Then find the probability by counting, as before. Here is the event space for ‘the difference’:

	1	*2*	*3*	*4*	*5*	*6*
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

There are 8 results where the difference is 2, so the probability is :

$$8/36 = 2/9 = 0.22222222.$$

2.1.13 Two Complementary Events

We will now look at a related question. Calculate the probability that the sum is equal to or greater than 6.

It is known that there are 10 outcomes with the sum less than 6, therefore the number of outcomes equal to or greater than 6 is

$$36 - 10 = 26.$$

The probability of the sum being equal to or greater than 6 is then 26/36.

In an experiment with a number of possible outcomes, let A be an event and let B be the exactly opposite event. The event B is called the complement of A. Then

$$P[A] + P[B] = 1.$$

In other words, when two distinct events between them cover all possibilities eventualities, the sum of their probabilities is 1.

2.1.14 Three or More Complementary Events

Calculate the probability of the following events;

- the sum is less than 7,
- the sum is equal to 7,
- And then the sum is greater than 7.

Based on counting the outcomes the probabilities are then

15/36, 6/36 and 15/36.

Note that these all add up to 1, as they should from the counting process.

A generalisation of the rule for complementary events can now be based on this example of the dice.

Say an experiment has n distinct (in other words, not overlapping) possible events:

$$E_1, E_2, E_3, \ldots, E_n,$$

and no others. Then

$$P[E_1] + P[E_2] + \ldots + P[E_n] = 1.$$

In other words, when all distinct possibilities have been taken care of, the sum of the probabilities is 1.

2.2 The Birthday Enigma

If an event is the result of a very simple experiment, then finding its probability is a matter of seeing how many possible outcomes the experiment can produce, and then how many it can produce which satisfy the criteria of the event.

If all the outcomes are equally likely, the probability is then the ratio of these two numbers.

Here is an example which gives what seems like an unusual result.

> "In a group of 25 people, calculate the chances that two or more share the same birth date."

Consider a slightly more abstract, but simpler, version of the same question.

2.2.1 A Simpler Question

20 people are asked to choose a number from 1 to 100. Calculate the probability that two or more people pick the same number.

To work on this question, find the probability that no two people pick the same number. Let S the event that one or more of the numbers are the same. Let T be the opposite event.

The probability $P[T]$ can be found by seeing how many ways in which the 20 people can pick their numbers, with no restrictions.

This is the total number of outcomes, the quantity N.

Then find the number of outcomes which satisfy the criteria for the event, in other words, the number of ways they can pick the numbers *with no two the same*. This is the quantity N_E.

The probability is then the second figure divided by the first,

$$P[T] = \frac{N_T}{N}, \text{ evaluated as the number } N \text{ gets bigger.}$$

Step 1:

Count how many ways 20 people can pick a number between 1 and 100.

Each of the 20 people can pick any of 100 numbers. So the number of possible combinations of the 20 numbers is 100 multiplied by itself 20 times:

$$100 \times 100 \times \ldots \times 100 = 100^{20}.$$

With the laws of indices, this is

$$100^{20} = (10^2)^{20} = 10^{40}.$$

This number is 1 with 40 zeros.

Step 2

Now find how many ways the numbers can be chosen so none are the same:

- Let the first person pick their number. They have a choice of the full 100.
- In order that the first number be not picked again, the second person has the choice of the other 99 numbers. The number of ways the two numbers can be chosen is therefore: 100 x 99.

- The third person will now have a choice of the remaining 98 numbers. The total possible combinations are now: 100 x 99 x 98.
- Continuing like this, the last person will have a choice of 81 numbers, so the full number of combinations will be 100 x 99 x 98 x …x 81

This number works out to be roughly $1.3x10^{39}$:

$$100 \times 99 \times 98 \times \ldots \times 81 = 1.3x10^{39}.$$

We will see in the next section of this topic that this number is written as

$$^{100}P_{20}.$$

This makes it quick to calculate.

Step 3

Then we have just worked out that

$$P[T] = \frac{100 \times 99 \times 98 \times ... \times 81}{100 \times 100 \times 100 \times ... \times 100} = 0.13$$

This then means that

$$P[S] = 1 - 0.13 = 0.87.$$

2.2.2 Birthdays

Now return to the original question of birthdays. In a group of 25, calculate the probability that two or more people share a birthday.

Let S be the event that two or more of the people share the same birthday, and let N be the opposite event, that none are the same.

Then, from the same considerations as in the previous example, the probability of nobody having the same birthday among this group is

$$P[N] = \frac{365\times 364\times 363\times ...\times 341}{365\times 365\times 365\times ...\times 365} = \frac{{}^{365}P_{25}}{365^{25}}\ .$$

The figures are:

$$365^{25} = 1.14\text{x}10^{64},$$

$${}^{365}P_{25} = 365\text{x}364\text{x}363\text{x}\ldots\text{x}341 = 4.92\text{x}10^{63}.$$

The probability of nobody having a birthday in common is then

$$P[N] = 0.43,$$

So

$$P[S] = 0.57.$$

Therefore in a group of 25 people, the probability that two or more people share the same birthday is 0.57.

2.3 The Laws of Probability

In this section we will look at ways in which probabilities involving two or more events can be calculated, depending on how the events are related to one another. Recall the rule for complementary events; if A is an event and B is the exact opposite event, then

$$P[A] + P[B] = 1.$$

In other words, the probabilities of two distinct events, which cover all possibilities, add up to 1.

A generalisation of the above rule concerns n distinct possible events:

$$E_1, E_2, E_3, \ldots, E_n,$$

and no others. Then

$$P[E_1] + P[E_2] + \ldots + P[E_n] = 1.$$

In other words, when all distinct possibilities have been taken care of, the sum of the probabilities of the events is 1.

2.3.1 The Addition Law of Probabilities

The ‘addition law’ concerns how we calculate the probability of the event ‘A or B’.

Before we look at this, we have to define the type of events we are looking at.

2.3.2 Definition – Mutually Exclusive Events

Two events *A* and *B*, the possible results of the same experiment, are said to be *mutually exclusive* if it is impossible for them to happen together. If two events *A* and *B* are not mutually exclusive, that is, *A* and *B* can occur together, they are said to be mutually *non-exclusive* events.

With this definition, we can now write down two laws governing how the probabilities of the event *A* or *B*.

2.3.3 The Addition Law

If two events A and B are mutually exclusive, then the following law holds:

$$P[A \text{ or } B] = P[A] + P[B].$$

If the two events are not mutually exclusive, the law above is broadened so that the probability of A or B occurring is given by:

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B].$$

2.3.4 An Example of the Addition Law

If a single dice is thrown, determine the probability of getting a multiple of 3 or a multiple of 2, and then the probability of one or the other.

To start, the events should be defined.

Let A be the event of getting a multiple of 3, and B be the event of getting a multiple of 2.

The probability of scoring a multiple of 3 is that of getting a 3 or a 6. Then

$$P[A] = 2/6 = 1/3.$$

The probability of scoring a multiple of 2 is:

$$P[B] = 3/6 = ½.$$

Now both of these events could happen at the same time; a number can be a multiple of 2 and 3, and one of these is one the die: 6.

It then follows that A and B are non-exclusive events, so

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B].$$

The event 'A and B' means that a number is a multiple of 2 and 3, so this is equivalent to getting a 6. Thus

$$P[A \text{ and } B] = 1/6.$$

The final probability is then

$$P[A \text{ or } B] = 2/6 + 3/6 - 1/6 = 4/6 = 2/3.$$

2.3.5 An Example of the Addition Law

With a full deck of cards, determine the probability of drawing an ace or a red card.

Let A be the event of drawing an ace and B the event of drawing a red card. The question we are studying is to determine

$$P[A \text{ or } B].$$

To solve this problem, first find the probabilities of drawing an ace and then the probability of drawing a red card.

There are 4 aces in a pack, so $P[A] = 4/52$.
There are 26 red cards in a pack, so $P[B] = 26/52$.

These are mutually non-exclusive events (ace of hearts or aces of diamonds both satisfy the criteria for both events)

This means the law

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B]$$

must be applied again.

Finding the probability: there are 2 red aces in a pack, so

$$P[A \text{ and } B] = 2/52.$$

Substituting the values found;

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B] =$$

$$= 4/52 + 26/52 - 2/52 = 28/52 = 7/13.$$

As with the last case, this probability could have been calculated by simply counting out the number of ways the event could happen.

The number of ways of taking out an ace or a red card is clearly the 26 red cards plus the remaining two aces, giving 28 in total. Therefore the probability is:

$$P[A \text{ or } B] = 14/52 = 7/13.$$

In fact, this is also ‘computationally’ the same.

2.3.6 Independent and Dependent Events

The next concept in probability concerns which events do or don't affect the occurrence of other events. An example would be the rolling of a dice on two occasions.

The outcome of the first throw will not affect the probabilities for the second throw; the dice is picked up and thrown again with no link to the last throw.

Now consider the event of drawing a king from a full deck of cards without replacing it, and the second event of drawing a king after this.

Clearly whether the first event *does* or *does not* happen, this will alter the probability of the second event. We will see the details of this soon.

2.3.7 Definition – Independent Events

Two events are *independent* when the occurrence of one event **does not** affect the probability of the occurrence of the second event. If the outcome of one event **does** affect the probability of the second event, they are said to be *dependent*.

To go back to the case of two dependent events, consider the event F defined as drawing a king from a full deck of cards, without replacement, and event N, drawing a king after this. The probability of F is:

$$P[F] = 4/52 = 1/13.$$

The card is not replaced, and the same experiment is carried out again.

If the first card *was* a king, there are 3 kings in the 51 remaining cards, so then the probability of getting a king the second time is:

$$P[N] = 3/51.$$

If the first card *was not* a king, there are 4 kings in the 51 remaining cards. Therefore the probability is:

$$P[N] = 4/51.$$

The two events are not independent, in other words they are dependent.

2.3.8 The Multiplication Law of Probability (1)

For two *independent events* A and B, the probability of the occurrence of both events A and B, is given by

$$P[A \text{ and } B] = P[A].P[B]$$

The probability of the occurrence of both events is the product of the two individual probabilities.

2.3.9 An Example of Independent Events

A single fair dice is thrown 4 times. Calculate the probability of getting 4 5's in a row.

The event of rolling the die each time is an independent event. Each time, the probability of getting as 5 is 1/6. Thus using the multiplicative law, the probability is

$$(1/6)^4 = 1/6^4.$$

2.3.10 Dependant Events – Conditional Probability

Consider the situation of two events A and B, where the occurrence or not of event A does effect the probability of event B. In other words, B is dependent on A. We will set up a notation for this.

2.3.11 Notation – Conditional Probability

Let A and B be two events, where event B is dependent on event A.

The probability of event B, given that event A has already occurred is denoted by the notation:

$$P[B|A].$$

This is 'the probability of B, given A.'

For two independent events A and B, by definition, the fact that A has already occurred does not affect the probability of event B. It then follows that

$$P[B \mid A] = P[B].$$

If the events are dependent, these probabilities are not the same.

2.3.12 The Multiplication Law of Probability (2)

Now consider two events A and B; with event B dependent on A. The probability of the occurrence of both events is given by

$$P[A \text{ and } B] = P[A].\, P[B \mid A]$$

The probability of both events happening is the probability of A times the probability of B, given that A has occurred.

2.3.13 Example of a Conditional Probability

Consider two events A and B, where

- A is throwing a six with a fair die, and
- B is drawing a king from a full deck of cards.

Determine the probability of the occurrence of both events. In other words, find $P[A \text{ and } B]$.

Since these are independent events, we use

$$P[A \text{ and } B] = P[A].P[B]$$

Thus

$$P[A \text{ and } B] = (1/6) \times (4/52) =$$

$$= (1/6) \times (1/13) = 1/78.$$

2.3.14 Example – Conditional Probabilities

A box contains five 10 kΩ resistors and twelve 20 kΩ resistors. Determine

- The probability of randomly picking a 10 kΩ resistor from the box.
- The probability of randomly picking a 10 kΩ resistor from the box and then a 20 kΩ resistor.

Solution

Let event A denote the event of picking a 10 kΩ resistor, and let B denote the event of picking a 20 kΩ resistor.

The first probability is just $P[A]$, and since the total number of resistors is 17, it is

$$P[A] = 5/17.$$

To find the probability of both, that is, $P[A \text{ and } B]$, observe that B depends on A, since A is the event that is happening first. The probability law for dependent events must be used:

$$P[A \text{ and } B] = P[B \mid A].P[A].$$

To find $P[B \mid A]$, we need the probability that a second resistor picked from the box will be a 20 kΩ resistor, providing that the first one was a 10 kΩ resistor. For this case,

$$P[B \mid A] = 12/16 = 3/4.$$

So the probability of both events, picking a 20Ω k resistor after getting a 10 kΩ resistor is:

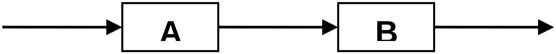
$$P[A \text{ and } B] = P[B \mid A].P[A] = ¾ \text{ x } 5/17 = 15/68.$$

2.4 Some Examples of Complex Systems

In each of the following systems, the probability that each individual component of type A will fail is 0.03, and the probability that each individual component of type B will fail is 0.05. All components A and components B are independent.

For each system, we will calculate the overall probability that it will work.

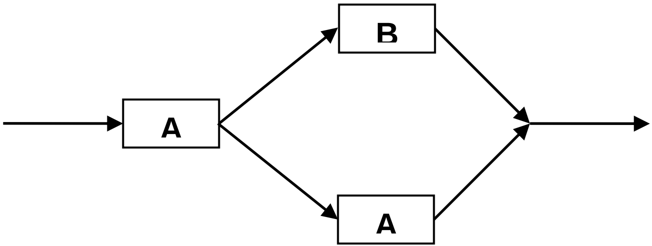
2.4.1 Example 1



For this example, clearly both components have to work so that the system overall works. The two events, *A* and *B*, are independent, so the probability they both work is:

$$0.97 \times 0.95 = 0.9215.$$

2.4.2 Example 2



In this case, going from left to right, the first component A has to work, followed by the second stage, which is *either* the B or the second A.

For the system overall, both stages must work.

$$P[\text{system works}] = P[\text{stage 1 works}] \text{ x } P[\text{stage 2 works}].$$

Firstly,

$$P[\text{stage 1 works}] = P[\text{A}].$$

For the second stage to work, A or B must work. *A* working and *B* working are not mutually exclusive. Then

$$P[\text{stage 2 works}] = P[\text{A or B works}]$$

So to find the probability this part of the system works, we have to use the second version of the addition law, this is just

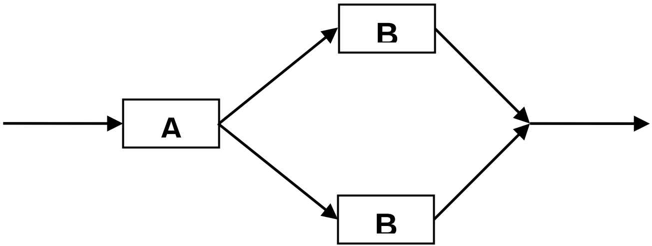
$$P[\text{A or B}] = P[\text{A}] + P[\text{B}] - P[\text{A}]P[\text{B}].$$

$$= 0.97 + 0.95 - 0.97 \text{x} 0.95 = 0.9985.$$

The laws then mean that the probability the whole system works is

$$P[\text{system works}] = P[\text{A}] \text{ x } 0.9985 = 0.97 \text{x} 0.9985 = 0.968{,}545.$$

2.4.3 Example 2 (a)



In this case, going from left to right, the first component A has to work, followed by the second stage, which is *either* one B or the other.

For the system overall, both stages must work. Therefore:

$$P[\text{system works}] = P[\text{stage 1 works and stage 2 works}] =$$

$$= P[\text{stage 1 works}] \times P[\text{stage 2 works}].$$

Firstly, stage 1 is equivalent to component A:

$$P[\text{stage 1 works}] = P[\text{A}].$$

For stage 2:

$$P[\text{stage 2 works}] = P[\text{B or B}]$$

In the second stage, *B* working and *B* working are not mutually exclusive. So to find the probability this part of the system works, we have to use the second version of the addition law, this is just

$$P[\mathrm{B\ or\ B}] = P[\mathrm{B}] + P[\mathrm{B}] - P[\mathrm{B}]P[\mathrm{B}].$$

Putting in the known probabilities:

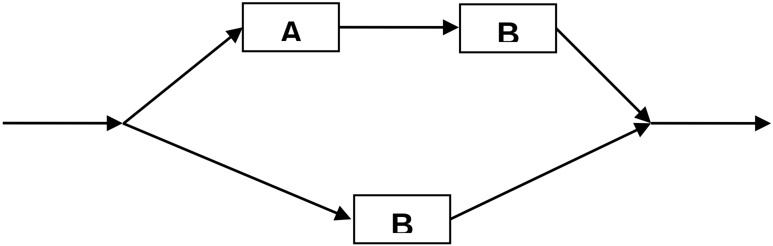
$$P[\mathrm{B\ or\ B}] = 0.95 + 0.95 - 0.95\mathrm{x}0.95 = 0.9975.$$

The laws then mean that the probability the whole system works is

$$P[\text{system works}] = P[\text{A}]\,(\,P[\text{B}] + P[\text{B}] - P[\text{B}]P[\text{B}]\,) =$$

$$= 0.97\text{x}0.9975 = 0.9676.$$

2.4.4 Example 3



In this example, for the system to work;

The lower B has to work, *or* the upper A and the B together, *or* both scenarios.

For the probabilities, this is then:

$$P[\text{A and B}] + P[\text{B}] - P[\,[\text{A and B}] \text{ and B}].$$

The part P[A and B] is given by

$$P[\text{A and B}] = P[\text{A}] \text{x} P[\text{B}].$$

The last term can be broken down as:

P[[A and B] and B] = P[A and B]xP[B] = P[A]xP[B]xP[B].

The probability of the system working is then:

$$P[A] \times P[B] + P[B] - P[A] \times P[B] \times P[B].$$

Then the probability the system works is

$$0.97 \times 0.95 + 0.95 - 0.97 \times 0.95 \times 0.95 =$$

$$= (0.97 + 1 - 0.97 \times 0.95) \times 0.95 =$$

$$= 0.996075.$$