Matrices

1 Contents

1 Matrices – Definitions

Matrix algebra is an abstract way of presenting numerical information, which has many applications throughout engineering mathematics.

In any situation where a linear system is being studied, or where large numbers of linear equations arise, matrices come into play. As we will see, they are essentially a generalisation of ordinary algebra.

This section covers the following topics.

- The definition of a matrix
- Matrix algebra: addition, multiplication by scalars, matrix multiplication.
- The transpose of a matrix and symmetry.
- The idea of a determinant for 2 by 2 matrices
- The idea of an inverse for 2 by 2 matrices

The first few topics are naturally the most important for studying matrices.

All the following ideas on matrices and their uses come from the definitions of addition and multiplication, the core of matrix algebra.

In fact, the definition of matrix multiplication arose from these applications in engineering and science.

1.1 Definition of a Matrix

A matrix consists of a set of numerical values arranged in a rectangular grid, surrounded by one set of brackets. The numbers are not separated by commas or individual brackets.

If a matrix has r rows and c columns, then it is said to be an r by c matrix.

Here are two examples of matrices:

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The matrix A is a 2 by 2 matrix, and B is a 2 by 1 matrix.

These numbers describe the size of the matrix.

Here are two matrices, C is a 4 by 5 matrix, and D is a 5 by 4 matrix:

$$C = \begin{pmatrix} 1 & 2 & 9 & -3 & 0 \\ 0 & -3 & 6 & 10 & 2 \\ -1 & 4 & 5 & 3 & -1 \\ 7 & 5 & -2 & 1 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 4 & 6 & 3 \\ 0 & 5 & 9 & 5 \\ 7 & -1 & 6 & -5 \\ 12 & 6 & 8 & 4 \\ -2 & 3 & 4 & 0 \end{pmatrix}.$$

1.1.1 Definition – A Scalar

Note that an ‘ordinary’ number, a 1 by 1 matrix, is now referred to as a scalar.

1.1.2 Definition – Elements

The individual numbers within a matrix are called the elements of the matrix. For example, the elements of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}$$

are 1, 3, –5, and 3.

Matrices have a well-defined algebra.

- They can always be multiplied by scalars.
- They can be added, subtracted and multiplied by other matrices, given certain conditions on their order.
- Given stricter conditions, they can be inverted, and so one matrix can be ‘divided’ by another.

1.2 The Algebra of Matrices

In this section, we will define how matrices are added and subtracted, and multiplication by ordinary numbers, which we are now referring to as scalars.

We will show how matrices multiply together in the next section.

1.2.1 Addition of Matrices

As might be expected, two matrices are added by adding each of the corresponding elements.

This means that two matrices can only be added if they are of identical size. Thus none of the matrices seen so far could be added, as they are all of different size.

The following two matrices can be added, since both are 2 by 2 matrices:

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix}.$$

For each element of A there is a corresponding element of B to add to it. The result is

$$A + B = \begin{pmatrix} 1 + (-1) & 3 + (-2) \\ -5 + 4 & 3 + (-3) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Each of the corresponding elements has been added.

Subtraction is done in an analogous way.

$$A - B = \begin{pmatrix} 1-(-1) & 3-(-2) \\ -5-4 & 3-(-3) \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -9 & 6 \end{pmatrix}.$$

1.2.2 Multiplication by a scalar

A matrix is multiplied by a scalar by multiplying each element of the matrix by the scalar. For example, if

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix},$$

then multiplying A by the number 5 gives:

$$5A = \begin{pmatrix} 5 & 15 \\ -25 & 15 \end{pmatrix}.$$

Dividing this matrix by 2 is the same as multiplying by ½, so that:

$$A/2 = \frac{1}{2}A = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix}.$$

These ideas are all combined for the first part of matrix algebra.

1.2.3 Example of matrix algebra

For the matrices

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix},$$

Calculate:

(i) $3A - 2B$

(ii) $B - 5A$

(iii) $B + \frac{1}{2}A$

For the first calculation,

$$3A - 2B = 3\begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix} - 2\begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & 9 \\ -15 & 9 \end{pmatrix} - \begin{pmatrix} -2 & -4 \\ 8 & -6 \end{pmatrix} = \begin{pmatrix} 5 & 13 \\ -23 & 15 \end{pmatrix}.$$

For the second:

$$B - 5A = \begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix} - 5\begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix} - \begin{pmatrix} 5 & 15 \\ -25 & 15 \end{pmatrix} = \begin{pmatrix} -6 & -17 \\ 29 & -18 \end{pmatrix}.$$

For the third calculation:

$$B + \tfrac{1}{2}A = \begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} 0.5 & 1.5 \\ -2.5 & 1.5 \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ 1.5 & -1.5 \end{pmatrix}.$$

1.3 Matrix multiplication

Consider these two matrices:

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}.$$

The mechanism for multiplication will be based around the case where the number of columns of A are the same as the number of rows of B.

This means that each row in A has the same number of elements as each column of B.

Each element in a row in A can be lined up with an element in a column of B.

This forms the basis for the definition of matrix multiplication.

Take a look at the first row in *A*, which is 1, 3, and the first column in *B*, which is 2, –1.

Multiply each of the corresponding elements in this row and column, and add the results:

$$1x2 + 3x(-1) = 2 - 3 = -1.$$

This will be an element in the resulting product matrix. It came from row 1 in *A* and column 1 in *B*, suggesting it should be in row 1 column 1 of the resulting product.

When this process is repeated with row 2 in A and column 1 in B, the result is –5.

Again, this number will be the element in row 2 column 1.

Repeating this procedure with the first row of A and the second column of B, gives 3. This will then occupy row 1, column 2 of the product matrix.

Taking the second row of A and the second column of B, gives 3, which occupies row 2, column 2 of the product matrix.

Putting these results together gives the product:

$$A \times B = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -5 & 3 \end{pmatrix}.$$

An important fact to notice is that the result of this type of multiplication is a 2 by 2 matrix.

For this method of multiplication to be possible, the number of elements in the rows of the first matrix must be the same as the number of elements in the columns of the second matrix.

This means the number of columns in the first matrix must be the same as the number of rows in the second.

This is the essential condition for multiplication of matrices; if this is not true, then the matrices cannot be multiplied.

The same procedure is applied for all matrices. Consider the following example:

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}.$$

The product *AB* here will be calculated exactly as before.

Each row of *A* is taken with each column of *B*, and a number produced.

There are 3 columns in *B*, and the first row of *A* produces 3 numbers with the columns of *B*.

Here are the calculations:

Row 1 of *A* with column 1 of *B*:

$$1 \text{ x } 1 + 3 \text{ x } (-1) = -2.$$

Row 1 of *A* with column 2 of *B*:

$$1 \text{ x } 0 + 3 \text{ x } 2 = 6.$$

Row 1 of *A* with column 3 of *B*:

$$1 \text{ x } (-1) + 3 \text{ x } (-2) = -7.$$

This means the first row of the product is –2, 6 and –7.

Taking the second row:

Row 2 of *A* with column 1 of *B*:

$$(-5) \text{ x } 1 + 3 \text{ x } (-1) = -8.$$

Row 2 of *A* with column 2 of *B*:

$$(-5) \text{ x } 0 + 3 \text{ x } 2 = 6.$$

Row 2 of *A* with column 3 of *B*:

$$(-5) \times (-1) + 3 \times (-2) = -1.$$

The second row of the product is then –8, 6 and –1.

The result is then

$$\begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 6 & -7 \\ -8 & 6 & -1 \end{pmatrix}$$

What size is the resulting matrix? The number of rows produced is the same as the number of rows in the first matrix *A*.

One column is produced in the resulting matrix for each one in *B*.

The result is a 2 by 3 matrix.

This can be generalised to the following rule. Say two matrices, call them *A* and *B*, are to be multiplied.

Let A be an m by n matrix, where m and n are positive integers.

The number of columns in A must be the same as the number of rows in B, so if the multiplication is to happen B must be an n by k matrix, where k is a positive integer.

The number of rows in the product matrix will be the same as those of the first matrix, and the number of columns the same as those of the second matrix.

Thus the result will be an m by k matrix. So in summary the rule for multiplication is

$$(m \text{ by } n) \text{ times } (n \text{ by } k) \text{ gives } (m \text{ by } k).$$

1.3.1 An Example of matrix multiplication

Here are two larger matrices:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix}.$$

We will calculate *AB* and then *BA.* Both of these calculations are possible:

AB is a 2 by 3 times a 3 by 2, giving a 2 by 2.

BA is a 3 by 2 times a 2 by 3, giving a 3 by 3.

Product AB:

$$AB = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix}.$$

The details are:

First row from A, first column from B:

$$1 \times 2 + 0 \times 3 + (-1) \times (-1) = 2 + 0 + 1 = 3$$

Put this in row 1, column 1 of the answer.

First row from A, second column from B:

$$1 \times 1 + 0 \times (-4) + (-1) \times 0 = 1 + 0 + 0 = 1$$

Put this in row 1, column 2 of the answer.

Finished with the first row, move on to the second:

Second row with first column:

$$(-1) \times 2 + 2 \times 3 + (-2) \times (-1) = -2 + 6 + 2 = 6$$

Put this in row 2, column 1 of answer matrix.

Second row with second column:

$$(-1)\text{x}1 + 2\text{x}(-4) + (-2)\text{x}0 = -1 - 8 + 0 = -9$$

Put this in row 2, column 2 of the answer.

$$AB = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 6 & -9 \end{pmatrix}.$$

And also for *BA*, the calculations are:

$$BA = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}.$$

First row with first column:

2x1 + 1x(–1) = 1, put in row 1 column 1.

First row with second column:

2x0 + 1x2 = 2, put in row 1 column 2.

First row with third column:

$$2x(-1) + 1x(-2) = -2 - 2 = -4, \text{ put in row 1 column 3.}$$

Second row with first column:

$$3x1 + (-4)x(-1) = 7, \text{ drop in row 2 column 1.}$$

Second row with second column:

$$3x0 + (-4)x2 = -8, \text{ drop in row 2 column 2.}$$

Second row with third column:

$$3x(-1) + (-4)x(-2) = -3 + 8 = 5, \text{ drop in row 2 column 3.}$$

Third row with first column:

(–1)x1 + 0x(–1) = –1 + 0 = –1, drop in row 3 column 1.

Third row with second column:

(–1)x0 + 0x2 = 0, drop in row 3 column 2.

Third row with third column:

(–1)x(–1) + 0x(–2) = 1 + 0 = 1, drop in row 3 column 3.

$$BA = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -4 \\ 7 & -8 & 5 \\ -1 & 0 & 1 \end{pmatrix}.$$

1.3.2 Another Example of matrix multiplication

Multiply the following two matrices:

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The result is:

$$AB = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\times(-1)+3\times 1 \\ -5\times(-1)+3\times 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

Looking at the sizes of the matrices, a 2 by 2 matrix by a 2 by 1 matrix is a 2 by 1 matrix.

1.3.3 Definition – column matrix, vector

A column matrix is a matrix of size n by 1, where n is a positive integer. This is also known as a vector.

The reason for the use of this name is that a 2 by 1 column matrix can be taken as a point in the $X - Y$ plane, and a 3 by 1 column matrix can be taken as a point in the three dimensional $X - Y - Z$ space. There will be more on this later on.

1.4 The Order of multiplication

In matrix multiplication, the order in which two matrices are multiplied does matter, unlike for ordinary numbers.

The two potential products may not even be well defined, or they may be different size. This is shown in the following example.

1.4.1 Example

Recall the two matrices multiplied before:

$$A \times B = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -5 & 3 \end{pmatrix}.$$

How does the product *B*x*A* compare with *A*x*B*?

It is also well defined, but will give a different answer:

$$B \times A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ -2 & 0 \end{pmatrix}.$$

Thus in this case

$$A \times B \neq B \times A.$$

In this way, matrix multiplication is different from ordinary multiplication of scalars.

The order in which two matrices are multiplied affects the answer.

The same will be true of the two matrices we saw above.

The results of the calculations were:

$$AB = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 6 & -9 \end{pmatrix}.$$

$$BA = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -4 \\ 7 & -8 & 5 \\ -1 & 0 & 1 \end{pmatrix}.$$

Not only are these results not the same, they are not the same size.

1.5 Some further Definitions

There are certain matrices which come up often in calculations because of their useful properties.

1.5.1 Definition – identity matrix

The most important example is the identity matrix, so-called because it is the equivalent for matrices of the number 1 in ordinary numbers.

The 2 by 2 identity matrix is:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If this matrix is pre-multiplied by any 2 by n matrix, the result is the same matrix again.

If it is post-multiplied by any n by 2 matrix, again, the result is the same matrix.

1.5.2 The Identity matrix – Example

Let X be the matrix

$$X = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}.$$

This is a 2 by 3 matrix.

This matrix can therefore be pre-multiplied by I, the 2 by 2 identity, in other words, we can calculate I times X.

When this calculation is done, we see that *IX* = *X*:

$$IX = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}.$$

1.5.3 Another Example of the Identity Matrix

Let *T* be the matrix given by

$$T = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix}.$$

This is a 3 by 2 matrix, so it is possible to calculate *TI*. It is:

$$TI = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix}.$$

So then $TI = T$.

For 3 by 3 matrices, the identity is

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can multiply X times I:

$$XI = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}.$$

In the same way, this matrix leaves 3 by n or n by 3 matrices unchanged. The result of multiplying X times I is X:

$$XI = X.$$

1.5.4 Some Notation

We will use a subscript to denote the size of the identity matrix, as follows. The 2 by 2 identity matrix is I_2:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For 3 by 3 matrices, the identity is I_3:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And so on…

1.5.5 The Commutator

Consider the following two matrices, both 3 by 3 matrices.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -2 \\ 3 & -4 & -1 \\ -1 & 0 & 3 \end{pmatrix}.$$

We can calculate *AB* and *BA*, the results are:

$$AB = \begin{pmatrix} 3 & 1 & -5 \\ 6 & -9 & -6 \\ -1 & 11 & -1 \end{pmatrix}, \quad BA = \begin{pmatrix} -5 & 6 & -6 \\ 4 & -6 & 4 \\ 8 & -6 & 4 \end{pmatrix}.$$

Because the matrices A and B are square, the results of these calculations can be directly compared.

More particularly, they can be subtracted:

$$AB - BA = \begin{pmatrix} 8 & -5 & 1 \\ 2 & -3 & -10 \\ -9 & 17 & -5 \end{pmatrix}$$

This matrix is called the commutator of A and B.

1.6 The Transpose of a Matrix

Consider the following 2 by 3 matrix:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}.$$

Consider the second, related matrix formed by switching this matrix around on its diagonal.

This means that the element in row 1, column 2 (which has the value 0), now sits in row 2, column 1.

Overall, the rows of this matrix become columns, the columns become rows. For A the matrix found by doing this is:

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ -1 & -2 \end{pmatrix}.$$

This matrix is known as the transpose of A, and is written as A^{T}.

The two matrices here, A and A^{T}, have the converse or opposite size; A is a 2 by 3 matrix and A^{T} is a 3 by 2 matrix.

This means that the products

$$A \text{ by } A^{\mathrm{T}} \text{ and } A^{\mathrm{T}} \text{ by } A$$

are well defined, and will be square matrices.

It can be seen that this is true of any matrix.

1.6.1 Definition – symmetry

If a matrix is swapped around its diagonal, and the matrix remains unchanged, it is said to be symmetric. More formally, a matrix X is said to be symmetric if $X^{\mathrm{T}} = X$.

Here is an example of a symmetric matrix:

$$F = \begin{pmatrix} 1 & -2 & 4 \\ -2 & 0 & 5 \\ 4 & 5 & 9 \end{pmatrix}.$$

For the following two matrices:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix}.$$

Consider, as examples, the matrices $A^{\mathrm{T}}A$, BB^{T} and $B^{\mathrm{T}}B$. These matrices are:

$$A^T A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 4 & -4 \\ 1 & -4 & 5 \end{pmatrix},$$

$$BB^T = \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 1 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 & -2 \\ 2 & 25 & -3 \\ -2 & -3 & 1 \end{pmatrix},$$

$$B^T B = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -4 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & -4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 14 & -10 \\ -10 & 17 \end{pmatrix}.$$

Note that all of the results are square symmetric matrices.

2 Matrix Inverses

The matrices with the most interesting algebra are those where the number of rows is the same as the number of columns. These are called square matrices.

With square matrices, it is possible to consider the inverse of a matrix. Essentially the use of the inverse is analogous to division with scalar numbers.

For a scalar number, such as 4, the inverse is ¼, or 0.25; this means that if we multiply 4 times ¼, 4 times 0.25, the result is 1. Then dividing by 4 is the same as multiplying by ¼, 0.25.

We will look for something similar with matrices, where instead of the number 1 we will be dealing with the identity matrix.

Firstly, use the notation that I_n is the identity matrix for square matrices of this size.

This is in keeping with our notation of before, where the 2 by 2 identity matrix is I_2:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and the 3 by 3 matrices is: } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And so on…

We can now define an inverse of a square matrix.

2.1.1 Definition – inverse of a matrix

Let A be a known square matrix. Call n the number of elements in each row or column, so that A is a n by n matrix. Let B be a second square matrix of the same size. The products BA and AB are well defined.

If B has the property that

$$BA = I_n \text{ and } AB = I_n,$$

then it is said to be the inverse of A.

As in ordinary arithmetic, this can be denoted using negative powers:

$$A^{-1} \text{ is the inverse of } A.$$

A rule is required for finding the inverse of the general matrix. First look at the relatively simple case of 2 by 2 matrices.

Let A be the most general 2 by 2 matrix.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Consider the second 2 by 2 matrix

$$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We will see what happens when these two matrices are multiplied, in both possible orders.

The results are:

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}, \text{ and}$$

$$BA = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}.$$

Both of these results are simply the 2 by 2 identity matrix multiplied by the factor $ad - bc$.

Therefore if I_2 is the 2 by 2 identity matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then $BA = AB = (ad - bc)I_2$.

This means that if the matrix B is divided by the factor $ad - bc$, it will satisfy exactly the requirements of an inverse.

We have therefore proved that the inverse of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This equation provides a straightforward way of writing down the inverse of a 2 by 2 matrix.

The result was produced here and shown to work, but in fact it could be derived from the two defining equations for the inverse.

These equations also mean that once an inverse is found, it is the only one.

2.1.2 Examples of Inverses

Calculate the inverses of the following matrices:

$$B = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix},$$

$$D = \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix}, \; E = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}.$$

For each one, demonstrate that it does work as an inverse.

The inverse of the matrix B:

$$B = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \text{ is } B^{-1} = \frac{1}{3 \times 3 - 4 \times 2} \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}.$$

For this case, check that the inverse does as it should: calculate B times B^{-1} and B^{-1} times B, to see that it gives I_2. Firstly:

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 3 \times 3 - 4 \times 2 & 3 \times (-2) + 2 \times 3 \\ 4 \times 3 - 3 \times 4 & 4 \times (-2) + 3 \times 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is also true that

$$\begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows that the matrix we have produced is the inverse of *B*.

The inverse of the matrix *C*:

$$C = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix},$$

$$C^{-1} = \frac{1}{1\times 3 - 4\times 1}\begin{pmatrix} 3 & -1 \\ -4 & 1 \end{pmatrix} = -1\times\begin{pmatrix} 3 & -1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix},$$

The inverse of the matrix D:

$$D = \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix}, \text{ is}$$

$$D^{-1} = \frac{1}{-1\times(-4) - 2\times 3}\begin{pmatrix} -4 & -3 \\ -2 & -1 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -4 & -3 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} \\ 1 & \frac{1}{2} \end{pmatrix}.$$

Finally look at the matrix E:

$$E = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}.$$

The very first step of the calculation gives a problem:

$$ad - bc = 3\text{x}4 - 2\text{x}6 = 0.$$

This means finding the inverse of matrix E involves a division by 0, which cannot happen. Therefore the inverse of E does not exist if this number is 0.

2.2 The Determinant

Clearly the quantity $ad - bc$ is of paramount importance in this definition. If it is zero, as was the case in the last example above, then the inverse is not defined, and so does not exist for A.

This quantity is known as the determinant.

So we have seen that the determinant of the 2 by 2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is given by $ad - bc$.

There are several ways of denoting the determinant of a matrix A.

It can be denoted by

det A, $|A|$ or by a delta symbol: $\Delta(A)$.

Using this notation

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \ .$$

We will be using the first notation, det(*A*).

The equation for the by 2 inverse can be written as

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

2.3 Matrices and Linear Equations

One of the most important applications of matrix algebra comes when it is applied to systems of linear equations.

Let x and y be two variables, and B a 2 by 1 column matrix formed from them:

$$B = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Consider what happens if we multiply this matrix by the square matrix A given by

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}.$$

The result is:

$$AB = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 3y \\ -5x + 3y \end{pmatrix}.$$

This is a 2 by 1 column matrix whose elements are linear expressions in x and y.

Now look at the matrix equation

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

We saw that the left hand side became

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ -5x+3y \end{pmatrix}.$$

This means that the matrix equation is the same as:

$$\begin{pmatrix} x+3y \\ -5x+3y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix},$$

In other words, the system of equations:

$$x + 3y = 2,$$

$$-5x + 3y = 8.$$

Now go in reverse. The system of equations

$$x + 3y = 2,$$

$$-5x + 3y = 8.$$

can be written as

$$\begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

Note that this works because the equation was written with all the x and y on the left hand side and in the same order.

So writing down the matrix was a matter of reading off the coefficients in the order they appear. Then the system of equations:

$$x + 3y = 2,$$

$$-5x + 3y = 8.$$

can be written as

$$\begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

Now the inverse of the matrix comes into play. Write the equation as:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}, \text{ where } A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}.$$

Multiply across by the inverse of A:

$$A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

Now using the definition of the inverse for the 2 by 2 case,

$$A^{-1}A = I_2,$$

and so the equation is now

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

If the inverse of A exists, in other words det(A) is not 0, the equation above can be solved immediately for the matrix of unknowns.

So solving this system of equations is equivalent to finding the inverse of A.

The advantage of this is that is a more methodical system, rather than relying on intuition.

2.3.1 Summary

To summarise: Given a system of equations

$$x + 3y = 2,$$

$$-5x + 3y = 8.$$

Make sure that the variables are written in the same order in each equation.

Then read off the coefficients of the system for a matrix:

$$A = \begin{pmatrix} 1 & 3 \\ -5 & 3 \end{pmatrix}.$$

Rewrite the system of equations as

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

If the inverse of A exists, the equation above can be solved immediately for the matrix of unknowns:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

2.3.2 An Example of linear equations as matrices

Solve the system of equations

$$4x - 3y = 1, \; y - 2x + 1 = 0.$$

The first step is to write the equations in the correct form, with x and y terms in the same order, and all constant terms on the right-hand side.

This is

$$4x - 3y = 1,$$

$$-2x + y = -1.$$

This is now written as a matrix equation:

$$\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Set A to be the 2 by 2 matrix.

The solution will be:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Finding the inverse of the matrix:

Firstly, det(A) = 4x1 – (–2) (–3) = 4 – 6 = –2.

This is non-zero, so proceed:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = -\tfrac{1}{2} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

So now the solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1-3 \\ 2-4 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \text{ so } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In other words, $x = 1$ and $y = 1$.

3 General Determinants and Inverses

Before we can calculate the inverses of matrices larger than 2 by 2, we need to discuss determinants of such matrices.

We saw that the determinant of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

was $\det(A) = ad - bc$. The determinants of higher order matrices are defined in terms of the 2 by 2 case.

In this section, we will cover the following material:

- A definition for determinants for higher order matrices.
- Methods for calculating inverses for higher order matrices.
- A method called Cramer’s Rule for solving linear systems of equations.

3.1 Determinants

To define determinants (and subsequently inverses) for larger matrices, we will need a notation for the individual elements in a matrix.

Let a_{ij} be the element of matrix A in row i and column j. This is the variable a with two subscripts for the row and column.

Here is an example of a 3 by 3 matrix A:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}.$$

For this case of matrix A, for example,

$$a_{13} = 3,\ a_{23} = 5 \text{ and } a_{31} = 6.$$

For a given element of A, consider the sub-matrix of A formed by leaving out the row and column containing this element.

For example, for the element $a_{12} = 2$, in the 1st row and 2nd column, the corresponding sub-matrix is found by leaving out the first row and the second column. The elements left are:

$$\begin{pmatrix} 4 & 5 \\ 6 & 2 \end{pmatrix}.$$

The three by three has gone to a two by two. We know how to get the determinant of this matrix; it is:

$$4 \times 2 - 6 \times 5 = -22.$$

For each row i and column j of the matrix A, let M_{ij} be the value of the determinant formed in this way.

Thus M_{ij} is the determinant of the matrix formed by leaving out row i and column j.

This leads to the idea of the *cofactors* of the matrix A.

3.1.1 Definition – Cofactors of a matrix

For a square matrix A, of size n by n, let M_{ij}, be the determinant of the matrix formed by leaving out row i and column j. Then the cofactor for i, j is given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

If the M_{ij} are calculated, this equation means that if the value of $i + j$ is even, then the cofactor is

$$C_{ij} = M_{ij},$$

and if the value of $i + j$ is odd, then the cofactor is

$$C_{ij} = -M_{ij}.$$

We now have a cofactor for each value of i and j, in other words, for each element of the matrix.

This allows us to define the determinant for any matrix above the 2 by 2. Let us look at the matrix A first.

The determinant for the 3 by 3 matrix A is found by expanding one row of the matrix into cofactors:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

This equation means that we follow the top row across, multiply each element in the row by its corresponding cofactor, and then take the total.

The result is the value of the determinant. Here are the calculations.

For $a_{11} = 1$, the sub-matrix is found by omitting the 1st row and 1st column, giving sub-matrix:

$$\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}.$$

The determinant of this matrix is 1x2 – 0x5 = 2. Thus

$$M_{11} = 2.$$

The value of $i + j$ is 2, which is even, so the cofactor is

$$C_{11} = 2.$$

For the 2 in the 1st row and 2nd column, the sub-matrix is

$$\begin{pmatrix} 4 & 5 \\ 6 & 2 \end{pmatrix}.$$

The determinant of this sub-matrix is –22, so $M_{12} = -22$.

The value of $i + j$ is 3, which is odd, so the cofactor is

$$C_{12} = 22.$$

Finally, for the 3 in the 1st row and 3rd column, the sub-matrix is

$$\begin{pmatrix} 4 & 1 \\ 6 & 0 \end{pmatrix}.$$

The determinant here is –6, so $M_{13} = -6$.

The value of $i + j$ is 4, which is even, so the cofactor is

$$C_{13} = -6.$$

These values can now all be put in the equation

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

This gives

$$\det A = 1 \text{x} 2 + 2 \text{x} 22 + 3 \text{x}(-6) = 28.$$

In fact, we could use any row, or column, as long as we find the appropriate cofactors. It will always give us the same value for the determinant. Here is the general case.

3.1.2 Definition – Determinant of a matrix

Let A be a square matrix. Let a_{ij} be the elements of the matrix, where i and j are integers between 1 and n.

The determinant of the matrix is given by

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots + a_{1n}C_{1n}.$$

This is the elements times the corresponding cofactors summed across the top row.

In fact, any row or column can be used:

Row i

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \ldots + a_{in}C_{in}.$$

Column j:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \ldots + a_{nj}C_{nj}.$$

3.1.3 The Diagonal Method for 3 by 3 Determinants

There is an alternative method of calculating the determinant of a 3 by 3 matrix without using the cofactor method. It can sometimes be quicker. Consider the following example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}.$$

The method is to write a second copy of the matrix out beside it:

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 1 & 5 & 4 & 1 & 5 \\ 6 & 0 & 2 & 6 & 0 & 2 \end{pmatrix}.$$

Start at each element of the top row, and take the product of the diagonal coming down from it:

$$1x1x2 = 2,\ 2x5x6 = 60,\ 3x4x0 = 0.$$

Take the sum of these products, which is 62.

Now repeat this with the diagonals coming *up* from the elements on the bottom row:

6x1x3 = 18, 0x5x1 = 0, 2x4x2 = 16.

The sum here is 34.

The determinant of this matrix is the first sum, minus the second:

62 – 34 = 28.

In summary, when calculating the determinant of a 3 by 3 matrix, the products of the diagonals coming down from the first row are added, those coming up from the 3rd row are subtracted.

Example

Find the determinant of the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}.$$

This determinant can be calculated quickly with the second method, especially as there is a zero in the first row.

Writing in the ‘second copy’ of the matrix:

$$\begin{pmatrix} 1 & -1 & 0 & \vdots & 1 & -1 & 0 \\ 1 & 2 & 1 & \vdots & 1 & 2 & 1 \\ -2 & 1 & -1 & & -2 & 1 & -1 \end{pmatrix}$$

The determinant is:

Downward diagonals: $1.2.(-1) + (-1).1.(-2) + 0.1.1 = -2 + 2 + 0 = 0$.

Upward diagonals: $(-2).2.0 + 1.1.1 + (-1).1.(-1) = 0 + 1 + 1 = 2$.

The determinant is then $\det A = 0 - 2 = -2$.

Example

Solve the determinant equation

$$\det\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & x \\ 0 & -3 & 4 \end{pmatrix} = -20\,.$$

Writing in the 'second copy' of the matrix, the determinant is:

$$\left(\begin{array}{ccc:ccc} -1 & 2 & -3 & -1 & 2 & -3 \\ 2 & 1 & x & 2 & 1 & x \\ 0 & -3 & 4 & 0 & -3 & 4 \end{array}\right).$$

Working this out:

The downward diagonals are:

$$(-1).1.4 + 2.x.0 + (-3).2.(-3) = -4 + 0 +18 = 14$$

The upward diagonals are:

$$0\text{x}1\text{x}(-3) + (-3)x(-1) + 4\text{x}2\text{x}2 = 0 + 3x + 16 = 3x + 16$$

So the determinant is:

$$14 - [3x + 16] = 14 - 3x - 16 = -2 - 3x.$$

The determinant is to be equal to 20, so this is a matter of solving a simple linear equation:

$$-2 - 3x = -20,$$

$$-3x = -18,$$

$$x = 6.$$

3.2 Some properties of determinants

There are several properties of determinants which can sometimes make calculating them slightly easier.

These properties link the determinant of a matrix with the determinant of a matrix linked to it.

In other words, the matrix can be changed, and this produces a known change in the value of the determinant.

Hopefully the matrix can be written in a handy form, for example, with plenty of zeros.

The usefulness of this comes from the equation defining the determinant of a general matrix.

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots + a_{1n}C_{1n}.$$

Here the a_{ij} are the elements of the matrix and the C_{ij} are the corresponding cofactors generated from them.

If any of these elements is zero, then there is no need to find its cofactor, since it will be multiplied by 0. This makes the calculation quicker.

Our aim in studying the properties of determinants is to see if we can transform a matrix to bring in more zeros, while keeping track of the effect this has on the determinant.

These properties are also vital for programming determinant calculations.

All the properties listed here follow, with a bit of algebra, from this defining equation for the determinant.

3.2.1 Property 1

If two rows or columns of a matrix are interchanged, the determinant changes in sign.

In the example here, we switch column one and two:

$$\det\begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix} = -\det\begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & -2 \\ -2 & 3 & 1 \end{pmatrix}.$$

3.2.2 Property 2

If the elements of one row or column are all multiplied by the same factor, the determinant is multiplied by that factor.

In this example, all the elements in the first column are divisible by 2, so this factor can be taken out:

$$\det\begin{pmatrix} 2 & 0 & -1 \\ -2 & 2 & -2 \\ 6 & -2 & 1 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix}.$$

This means that for a matrix A of order n, if every element is multiplied by a scalar k, then the value of the determinant is multiplied by k for every row, or every column, in other words, n times:

$$\det (kA) = k^n \det A.$$

This is because the factor of k must be taken out for each row or column.

3.2.3 Property 3

If the elements of one row or column are added to another, the determinant remains unchanged.

Here the elements of row 1 are added onto the elements of row 2:

$$\det\begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \\ 3 & -2 & 1 \end{pmatrix}.$$

3.2.4 Property 4

If any two rows or columns are the same, the determinant is 0. For example:

$$\det\begin{pmatrix} 2 & 2 & -1 \\ -2 & -2 & -2 \\ 6 & 6 & 1 \end{pmatrix} = 0\,.$$

Combining this with property 3, if any row or column of a matrix is a multiple of any other, the determinant is also 0.

In the following example matrix, one of the rows or columns here is a multiple of another:

$$\det\begin{pmatrix} 2 & 1 & -2 \\ 3 & -4 & 8 \\ 0 & -1 & 2 \end{pmatrix} = 0\,.$$

These rules are useful, especially the last one, since they can be used to bring in zeros or reduce the size of calculations.

3.2.5 Example

Find the determinant of the following matrix:

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix}.$$

Solution:

Try and see which rows or columns can be added to others to bring in more zeros.

The first step would be to add column 2 to column 3:

$$\det\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 3 & -2 & -1 \end{pmatrix}$$

This has immediately improved the situation.

If we expand along the third column, the first two elements are zero, so there is no need to work out the cofactors.

Only the last need be done:

$$\det\begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 3 & -2 & -1 \end{pmatrix} = -1 \times \det\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = -3$$

The calculation of this determinant therefore involved only one cofactor. Even if only one zero were produced, it would still be an improvement.

3.2.6 Example

Find the determinant of the following matrix:

$$\begin{pmatrix} -3 & 5 & 1 \\ 4 & 4 & -2 \\ 3 & 7 & 1 \end{pmatrix}.$$

Solution.

The first step is to use the fact that the 2nd row is divisible by 2.

If we take out this factor it leaves a very handy –1.

So taking out the factor, the result is:

$$\det\begin{pmatrix} -3 & 5 & 1 \\ 4 & 4 & -2 \\ 3 & 7 & 1 \end{pmatrix} = 2\det\begin{pmatrix} -3 & 5 & 1 \\ 2 & 2 & -1 \\ 3 & 7 & 1 \end{pmatrix}.$$

Now add row 2 to row 1:

$$= 2\det\begin{pmatrix} -3 & 5 & 1 \\ 2 & 2 & -1 \\ 3 & 7 & 1 \end{pmatrix} = 2\det\begin{pmatrix} -1 & 7 & 0 \\ 2 & 2 & -1 \\ 3 & 7 & 1 \end{pmatrix},$$

And adding row 2 to row 3:

$$= 2\det\begin{pmatrix} -1 & 7 & 0 \\ 2 & 2 & -1 \\ 3 & 7 & 1 \end{pmatrix} = 2\det\begin{pmatrix} -1 & 7 & 0 \\ 2 & 2 & -1 \\ 5 & 9 & 0 \end{pmatrix}.$$

The third column now only has one non-zero element, so use this column to work the determinant:

$$= 2\det\begin{pmatrix} -1 & 7 & 0 \\ 2 & 2 & -1 \\ 5 & 9 & 0 \end{pmatrix} = 2\times(-1)\left((-1)\times\det\begin{pmatrix} -1 & 7 \\ 5 & 9 \end{pmatrix}\right).$$

This is worked out as:

$$= 2 \times \det\begin{pmatrix} -1 & 7 \\ 5 & 9 \end{pmatrix} = 2\big((-1) \times 9 - 5 \times 7\big) = 2 \times 9 - 23) = -46 \,.$$

3.3 Calculating general Inverses

Let A be a matrix, and let a_{ij} be the element in row i and column j. Recall that there is a cofactor C_{ij} associated with each element a_{ij}.

We saw that the determinant was given by the equation

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

As mentioned before, we could use any row, or column, as long as we find the appropriate cofactors. It will always give us the same value for the determinant.

However, if we used the elements of one row and the determinant of another row, we would get 0.

This suggests a way of forming an inverse.

Consider the matrix C, formed with elements given by the cofactors C_{ij}, placed in row i column j.

If the elements of the first row of A are multiplied by the elements of the first row of this new matrix, and the sum taken, we know the answer will be det A.

If the first row is taken with any other row, this will be zero.

These operations are equivalent to multiplying A by the transpose of this new matrix of cofactors C (it has to be the transpose for the elements to match up as required).

Thus for any matrix A, with matrix of cofactors C, the result of multiplying A by C^T is

$$AC^T = \begin{pmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{pmatrix} = (\det A)I \,.$$

The transposed matrix of cofactors, the matrix C^T, is known as the *adjoint* of A.

It immediately follows that

$$A^{-1} = \frac{1}{\det A} C^T .$$

The inverse of a matrix is its transposed matrix of cofactors, divided by its determinant. We have explained this with a 3 by 3 matrix, but it is true of any size square matrix.

3.3.1 Summary of the General Inverse

Let A be a matrix, and let C be its matrix of cofactors. Then the inverse of A is given by

$$A^{-1} = \frac{1}{\det A} C^T .$$

3.3.2 Example

Find the inverse of the 3 by 3 matrix A:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}.$$

To find this inverse, first work out the matrix of cofactors.

For $a_{11} = 1$, the sub-matrix is:

$$\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}.$$

The determinant of this matrix is 1x2 – 0x5 = 2.

The value of $i + j$ is 2, which is even, so the cofactor is $C_{11} = 2$.

For the 2 in the 1st row and 2nd column, the sub-matrix is

$$\begin{pmatrix} 4 & 5 \\ 6 & 2 \end{pmatrix}.$$

The determinant of this sub-matrix is –22, and the value of $i + j$ is 3, which is odd, so the cofactor is

$$C_{12} = 22.$$

Finally, for the 1st row and 3rd column, the sub-matrix is

$$\begin{pmatrix} 4 & 1 \\ 6 & 0 \end{pmatrix}.$$

The determinant here is –6, so $M_{13} = –6$. The value of $i + j$ is 4, which is even, so the cofactor is

$$C_{13} = –6.$$

These values can now all be used to get the determinant, which we will need later:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13},$$

$$= 1x2 + 2x22 + 3x(-6) = 28.$$

Continuing on with the calculations, the other cofactors are:

$$C_{21} = -4,\ C_{22} = -16,\ C_{23} = 12,$$

$$C_{31} = 7,\ C_{32} = 7,\ C_{33} = -7.$$

The matrix of cofactors is

$$C = \begin{pmatrix} 2 & 22 & -6 \\ -4 & -16 & 12 \\ 7 & 7 & -7 \end{pmatrix},$$

and the adjoint, the transposed matrix of cofactors, is

$$C^T = \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}.$$

The inverse is then this matrix divided by the determinant, which was 28:

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{28}\begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}.$$

3.3.3 Solving a 3 by 3 set of equations

Solve the following system of equations:

$$2x - y - z = 1,$$

$$3x + y + 2z = 2,$$

$$x - 2y - 3z = 1.$$

Solution:

The first step is to check that this system of equations is written with all the variables in the same order; x, y and then z.

Then we can write this system in matrix form by simply reading off the coefficients of the variables. The result is:

$$\begin{pmatrix} 2 & -1 & -1 \\ 3 & 1 & 2 \\ 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Note that the arrangement of the values in the matrix is the same as that of the coefficients in the system of equations.

The matrix of cofactors is then calculated; it is:

$$C = \begin{pmatrix} 1 & 11 & -7 \\ -1 & -5 & 3 \\ -1 & -7 & 5 \end{pmatrix}.$$

and the determinant is –2.

The inverse can now be calculated from the equation:

$$A^{-1} = \frac{1}{\det A} C^T .$$

It is

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -1 & -1 \\ 11 & -5 & -7 \\ -7 & 3 & 5 \end{pmatrix}.$$

It is convenient to take in the minus sign, but not the division by 2:

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -1 & -1 \\ 11 & -5 & -7 \\ -7 & 3 & 5 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ -11 & 5 & 7 \\ 7 & -3 & -5 \end{pmatrix}.$$

The solution to the system of equations is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ -11 & 5 & 7 \\ 7 & -3 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 2 \\ 6 \\ -18 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -9 \end{pmatrix}.$$

3.3.4 Another Example

Solve the following system of equations:

$$7x = 11 + 2y,$$

$$3y + 4z = 4x + 2,$$

$$3x = y + 5.$$

In order to write this system in matrix form, the equations must be written so that the unknowns come in the same order.

This is:

$$7x - 2y = 11,$$

$$-4x + 3y + 4z = 2,$$

$$3x - y = 5.$$

Now the system of equations can be easily written as a matrix equation. The result is:

$$\begin{pmatrix} 7 & -2 & 0 \\ -4 & 3 & 4 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \\ 5 \end{pmatrix}.$$

To find the inverse of the 3 by 3 matrix, the first step is calculating the cofactors. We will start with the third column – the two zero's mean we only need one cofactor. The cofactor C_{23} is

$$C_{23} = (–1)^5 [7x(–1) – (–2)x3] = 1.$$

Then the determinant is

$$0x(-5) + 4x1 + 0x13 = 4.$$

Now that we know it is not zero, we will go ahead and find all the rest of the cofactors.

The matrix of cofactors is:

$$C = \begin{pmatrix} 4 & 12 & -5 \\ 0 & 0 & 1 \\ -8 & -28 & 13 \end{pmatrix}.$$

The inverse is then the transposed matrix of cofactors, divided by the determinant.

$$A^{-1} = \frac{1}{\det A} C^T .$$

The result is:

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{4}\begin{pmatrix} 4 & 0 & -8 \\ 12 & 0 & -28 \\ -5 & 1 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 0 & -7 \\ -1.25 & 0.25 & 3.25 \end{pmatrix}.$$

Using this matrix inverse, the solution is then:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 0 & -7 \\ -1.25 & 0.25 & 3.25 \end{pmatrix}\begin{pmatrix} 11 \\ 2 \\ 5 \end{pmatrix}.$$

Working this out, the solution is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11-10 \\ 33-35 \\ -13.75+0.5+16.25 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}.$$

Thus $x = 1$, $y = -2$ and $z = -3$.

4 Cramer’s Rule

When we solve a system of equations in this way, we are effectively doing the same calculation repeatedly. Is there any pattern we can see that would make our calculations quicker?

Consider a general system of equations, and see what happens when it is solved using matrix inverses.

Label the system as shown:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2,$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3.$$

As a matrix equation this is

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

where A is the matrix with elements a_{ij}.

Denoting the cofactor from row i, column j by c_{ij}.

Then when the inverse is found, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The matrix has been transposed, so the cofactors are in different positions than the indices would indicate.

When these matrices are multiplied out to give the solutions, the solution for x_1 is:

$$x_1 = (c_{11}b_1 + c_{21}b_2 + c_{31}b_3)/\det A.$$

The expression in brackets is the same as a determinant calculated for the matrix A, using column 1, but with the elements a_{11}, a_{21} and a_{31} replaced by b_1, b_2, b_3.

In other words, x_1 is given by

$$x_1 = \frac{1}{\det A} \det \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix},$$

and in a similar way,

$$x_2 = \frac{1}{\det A} \det \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}, \; x_3 = \frac{1}{\det A} \det \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}.$$

We have got to the following procedure for solving a linear system.

4.1 The method for Cramer's rule

In summary, let A be an n by n square matrix, $\boldsymbol{x}$ a column matrix of unknowns, and $\boldsymbol{b}$ a column matrix of known constants. The linear system

$$A\boldsymbol{x} = \boldsymbol{b},$$

is solved as follows: Let A_i be the matrix formed by replacing the column i with the column matrix $\boldsymbol{b}$.

Then the value of the unknown x_i is:

$$x_i = {}^{\det A_i}\!\Big/_{\det A}.$$

This method is known as Cramer's rule.

4.1.1 An application of Cramer's Rule

We saw the following system of equations:

$$2x - y - z = 1,$$

$$3x + y + 2z = 2,$$

$$x - 2y - 3z = 1.$$

written in matrix form as:

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \text{ where } A = \begin{pmatrix} 2 & -1 & -1 \\ 3 & 1 & 2 \\ 1 & -2 & -3 \end{pmatrix}.$$

We will now solve it using Cramer's rule.

The first step in solving this matrix equation should always be to find the determinant of the matrix A, since if it is zero then there is no solution to the system of equations.

For this example we saw already that the determinant of A was equal to -2.

To solve for the variable x, replace the first column with the column matrix of constant values, and call this new matrix A_1:

$$A_1 = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 2 \\ 1 & -2 & -3 \end{pmatrix}.$$

The solution for x is then

$$x = {\det A_1}\big/{\det A}.$$

The determinant of A_1 is simplified by adding the first column to the second column and then the first column to the third:

$$\det A_1 = \det\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 2 \\ 1 & -2 & -3 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 1 & -1 & -2 \end{pmatrix}.$$

This is a good result; the top row now has two zeros. The Determinant of this matrix can be quickly found using the top row, the element in row 1, column 1, with its cofactor.

The cofactor for this element is:

$$\det\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} = 3\text{x}(-2) - (-1)\text{x}4 = -6 - (-4) = -2.$$

There is no change of sign (1 + 1 = 2, an even number), so:

$$\det A_1 = 1\text{x}(-2) = -2.$$

The value of x is then

$$x = \det A_1/\det A = -2/(-2) = 1.$$

To find y, replace the second column by the column matrix of constants:

$$A_2 = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 2 \\ 1 & 1 & -3 \end{pmatrix}.$$

The determinant of this matrix can be simplified by adding the second column to the third.

This gives:

$$\det A_2 = \det\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 2 \\ 1 & 1 & -3 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 4 \\ 1 & 1 & -2 \end{pmatrix}.$$

Then add twice the third row to the second row:

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 4 \\ 1 & 1 & -2 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 & 0 \\ 5 & 4 & 0 \\ 1 & 1 & -2 \end{pmatrix}.$$

This determinant is then the element in row 3, column 3, by its cofactor:

$$\det\begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix} = 2x4 – 5x1 = 8 – 5 = 3.$$

There is no change of sign (3 + 3 = 6, an even number) so

$$\det A_2 = (–2)x3 = –6.$$

The value of y is then

$$y = –6/(–2) = 3.$$

The advantage of Cramer's rule is that it allows the calculation of any one of the unknowns, without having to calculate all of them together.

In this example, we can now just use on of the original equations for z:

$$2x - y - z = 1 \text{ so } z = 2x - y - 1,$$

Then $z = 2\text{x}1 - 3 - 1 = -2$.

A further advantage of this method is that each calculation is two n by n determinants, and this can be made easier using row and column addition for determinants.

This comes into its own with larger size systems.

4.2 Example – a fourth order case

Solve the following system of equations for the second variable and the last:

$$x - y + 2z + 3w = 6,$$

$$2x - y + 5z + 4w = 13,$$

$$3x + 2y + z - 4w = 3,$$

$$3x + y + 2z - 3w = 6.$$

This situation calls for Cramer's rule, since only two of the four possible variables are required.

Writing the system as a matrix equation gives:

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 5 & 4 \\ 3 & 2 & 1 & -4 \\ 3 & 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \\ 3 \\ 6 \end{pmatrix}.$$

Let A be the matrix above:

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 5 & 4 \\ 3 & 2 & 1 & -4 \\ 3 & 1 & 2 & -3 \end{pmatrix}.$$

The determinant of A should be calculated first, since if it is zero, then the solution, for all the variables, will be undefined.

It is simplified by adding row 1 to row 4:

$$\det\begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 5 & 4 \\ 3 & 2 & 1 & -4 \\ 3 & 1 & 2 & -3 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 5 & 4 \\ 3 & 2 & 1 & -4 \\ 4 & 0 & 4 & 0 \end{pmatrix}.$$

Then subtract column 1 from column 3:

$$\det\begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 5 & 4 \\ 3 & 2 & 1 & -4 \\ 4 & 0 & 4 & 0 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 3 & 2 & -2 & -4 \\ 4 & 0 & 0 & 0 \end{pmatrix}.$$

The 4^{th} row now has three 0's, so use this row to calculate the determinant. The element by cofactor is:

$$4 \times \left((-1) \times \det \begin{pmatrix} -1 & 1 & 3 \\ -1 & 3 & 4 \\ 2 & -2 & -4 \end{pmatrix} \right).$$

It is multiplied by a –1 since the element '4' is in row 4, column 1.

Some row addition can be done on the 3 by 3 cofactor determinant – adding the middle column to the first gives two zero's in the first column, so the determinant can be reduced to a 2 by 2 determinant.

The calculations are:

$$-4 \times \det\begin{pmatrix} -1 & 1 & 3 \\ -1 & 3 & 4 \\ 2 & -2 & -4 \end{pmatrix} = -4 \times \det\begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & 4 \\ 0 & -2 & -4 \end{pmatrix}.$$

Now break down the calculation for the 3 by 3:

$$-4\times\det\begin{pmatrix}0 & 1 & 3\\ 2 & 3 & 4\\ 0 & -2 & -4\end{pmatrix} = -4\times 2\left(-\det\begin{pmatrix}1 & 3\\ -2 & -4\end{pmatrix}\right) =$$

$$= 8\times\det\begin{pmatrix}1 & 3\\ -2 & -4\end{pmatrix}.$$

= 8x(1x(–4) – (–2)x3) = 8x2 = 16.

The first part of the problem is to calculate y, the second variable. Its value is given by replacing the second column with the column matrix on the right-hand side. Call this A_2:

$$A_2 = \begin{pmatrix} 1 & 6 & 2 & 3 \\ 2 & 13 & 5 & 4 \\ 3 & 3 & 1 & -4 \\ 3 & 6 & 2 & -3 \end{pmatrix}.$$

Then the value of y is given by:

$$y = \det A_2/\det A.$$

The upper determinant must be calculated next. Here add the 1st row to the last, and the 2nd row to the 3rd:

$$\det\begin{pmatrix} 1 & 6 & 2 & 3 \\ 2 & 13 & 5 & 4 \\ 3 & 3 & 1 & -4 \\ 3 & 6 & 2 & -3 \end{pmatrix} = \det\begin{pmatrix} 1 & 6 & 2 & 3 \\ 2 & 13 & 5 & 4 \\ 5 & 16 & 6 & 0 \\ 4 & 12 & 4 & 0 \end{pmatrix}.$$

A factor of 4 can now be taken out from the 4th row. This step is not, in itself, essential, but it does mean the figures involved in the calculation are lower in magnitude.

The result is:

$$\det\begin{pmatrix} 1 & 6 & 2 & 3 \\ 2 & 13 & 5 & 4 \\ 5 & 16 & 6 & 0 \\ 4 & 12 & 4 & 0 \end{pmatrix} = 4 \times \det\begin{pmatrix} 1 & 6 & 2 & 3 \\ 2 & 13 & 5 & 4 \\ 5 & 16 & 6 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix}.$$

Subtract 3 times the 1st column from the 2nd, and following this the 1st from the 3rd.

The calculations are:

$$4 \times \det \begin{pmatrix} 1 & 6 & 1 & 3 \\ 2 & 13 & 3 & 4 \\ 5 & 16 & 1 & 0 \\ 1 & 3 & 0 & 0 \end{pmatrix} = 4 \times \det \begin{pmatrix} 1 & 3 & 1 & 3 \\ 2 & 7 & 3 & 4 \\ 5 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This determinant can now be reduced to one 3 by 3 determinant by using the 4^{th} row to work it out.

The result is:

$$4 \times (-1) \times \det\begin{pmatrix} 3 & 1 & 3 \\ 7 & 3 & 4 \\ 1 & 1 & 0 \end{pmatrix} = (-4) \times \det\begin{pmatrix} 3 & 1 & 3 \\ 7 & 3 & 4 \\ 1 & 1 & 0 \end{pmatrix}.$$

Subtracting the 1st column from the 2nd brings one more 0, and then the determinant is reduced to a 2 by 2:

$$-4 \times \det\begin{pmatrix} 3 & -2 & 3 \\ 7 & -4 & 4 \\ 1 & 0 & 0 \end{pmatrix} = (-4) \times \det\begin{pmatrix} -2 & 3 \\ -4 & 4 \end{pmatrix}.$$

This is:

$$= 4x[\ (-2)x4 - (-4)x3\ ] = 4 \text{ x } 4 = 16.$$

The final result for *y* is then

$$y = 16/16 = 1.$$

The next variable to calculate is the fourth, *w*.

The original matrix system was:

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 5 & 4 \\ 3 & 2 & 1 & -4 \\ 3 & 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \\ 3 \\ 6 \end{pmatrix}.$$

The variable w, is given by replacing the fourth column with the column matrix on the right-hand side.

So then:

$$A_4 = \begin{pmatrix} 1 & -1 & 2 & 6 \\ 2 & -1 & 5 & 13 \\ 3 & 2 & 1 & 3 \\ 3 & 1 & 2 & 6 \end{pmatrix},$$

And then $w = \det A_4 / \det A$.

Calculating the determinant of A_4, it is simplified by subtracting row 4 from row 1.

The result is:

$$\det A_4 = \det\begin{pmatrix} 1 & -1 & 2 & 6 \\ 2 & -1 & 5 & 13 \\ 3 & 2 & 1 & 3 \\ 3 & 1 & 2 & 6 \end{pmatrix} = \det\begin{pmatrix} -2 & -2 & 0 & 0 \\ 2 & -1 & 5 & 13 \\ 3 & 2 & 1 & 3 \\ 3 & 1 & 2 & 6 \end{pmatrix}.$$

Now subtracting column 1 from column 2 results in 3 0's in the top row, and the determinant can be reduced to a 3 by 3.

The details are:

$$\det\begin{pmatrix} -2 & -2 & 0 & 0 \\ 2 & -1 & 5 & 13 \\ 3 & 2 & 1 & 3 \\ 3 & 1 & 2 & 6 \end{pmatrix} = \det\begin{pmatrix} -2 & 0 & 0 & 0 \\ 2 & -3 & 5 & 13 \\ 3 & -1 & 1 & 3 \\ 3 & -2 & 2 & 6 \end{pmatrix} =$$

$$= -2 \times \det\begin{pmatrix} -3 & 5 & 13 \\ -1 & 1 & 3 \\ -2 & 2 & 6 \end{pmatrix}.$$

Now row 3 is twice row 2, so $\det A_4 = 0$ and so $w = 0$.

5 Eigenvalues and Eigenvectors

Many applications of matrices to problems involving coupled oscillations and vibrations involve matrix equations of the form

$$A\mathbf{x} = \lambda\mathbf{x},$$

where A is an n by n square matrix, $\mathbf{x}$ is an n by 1 column matrix and λ is a scalar.

This equation means that for a particular square matrix A, if the column matrix $\boldsymbol{x}$ and scalar λ can be found, then the effect of multiplying matrix $\boldsymbol{x}$ by matrix A is the same as if it has been simply multiplied by a scalar number λ.

This also applies to any multiple of matrix $\boldsymbol{x}$.

Such situations crop up throughout engineering mathematics, and understanding of them can be particularly useful in many different applications of linear systems analysed by matrix methods.

The number λ is called an *eigenvalue*, and the column matrix $\boldsymbol{x}$ is called an *eigenvector*.

A method must be produced for finding the solution to this equation. The first step is to find the possible values of scalar λ.

From the original equation

$$A\mathbf{x} = \lambda\mathbf{x},$$

start by bringing $\lambda\mathbf{x}$ to the left-hand side. The equation is now

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}.$$

The boldface $\mathbf{0}$ indicates that the right-hand side is the n by 1 column matrix of zeros.

Looking at this equation, it seems that it should be possible to take out the common factor of column matrix $\boldsymbol{x}$.

However, since a matrix and a scalar cannot be added or subtracted, the n by n identity matrix must be introduced.

The equation can be written as

$$A\boldsymbol{x} - \lambda I\boldsymbol{x} = \mathbf{0},$$

where I is the n by n identity matrix.

It is now possible to take out the common factor of column matrix $\boldsymbol{x}$:

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}.$$

If the matrix $A - \lambda I$, multiplying column matrix $\boldsymbol{x}$ on the left-hand side, can be inverted, then the solution for matrix $\boldsymbol{x}$ is

$$\boldsymbol{x} = (A - \lambda I)^{-1}\, \mathbf{0},$$

so that it appears that the first and only solution that can be produced from this equation is the trivial solution $\boldsymbol{x} = \mathbf{0}$.

But there is a loophole, since it was assumed that there is an inverse for the matrix $A - \lambda I$. This is of course not necessarily true; it might be singular.

This would allow a non-trivial solution of the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

If matrix $A - \lambda I$ is to be singular, then it must follow that its determinant is 0:

$$\det(A - \lambda I) = 0.$$

This gives an equation for the scalar λ.

This is called the *characteristic equation*.

If there are solutions to this equation, which will be a polynomial of order n, then the matrix

$$A - \lambda I$$

is singular, and so there are values of λ for which a non-trivial solution exists for

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

5.1.1 Example

Find the solution of the above equation for the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix}.$$

Solution: The characteristic equation is

$$\det(A - \lambda I) = 0.$$

The first task is to simplify the matrix we are taking the determinant of. It becomes

$$A-\lambda I = \begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & 1 \\ 5 & -\lambda \end{pmatrix}.$$

The determinant equation is now

$$\det\begin{pmatrix} 4-\lambda & 1 \\ 5 & -\lambda \end{pmatrix} = 0$$

This determinant can be easily worked out in terms of the unknown scalar λ; it is

$$(4 - \lambda)(- \lambda) - 5\text{x}1 = 0.$$

Simplifying the left-hand side gives the quadratic

$$\lambda^2 - 4\lambda - 5 = 0.$$

Factorise:

$$(\lambda + 1)(\lambda - 5) = 0, \text{ and so } \lambda = -1 \text{ and } \lambda = 5.$$

Thus the eigenvalues of the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix}$$

are $\lambda = -1$ and $\lambda = 5$.

5.1.2 Example

Repeat this procedure for the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}.$$

The characteristic equation is

$$\det(A - \lambda I) = 0.$$

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 0 & 1 \\ -1 & 4-\lambda & -1 \\ -1 & 2 & -\lambda \end{pmatrix}.$$

The determinant equation is now

$$\det \begin{pmatrix} 2-\lambda & 0 & 1 \\ -1 & 4-\lambda & -1 \\ -1 & 2 & -\lambda \end{pmatrix} = 0.$$

Use the cofactor method to find the determinant.

Use the top row, since it has a zero.

For element in row 1, column 1, the element is 2 – λ.

Leaving out row 1 and column 1, the sub-matrix is:

$$\begin{pmatrix} 4-\lambda & -1 \\ 2 & -\lambda \end{pmatrix}.$$

The determinant of this sub-matrix is

$$(4 - \lambda)(- \lambda) + 2 = \lambda^2 - 4\lambda + 2.$$

This is row 1, column 1, so 1 + 1 = 2, an even number, so no sign change. Therefore cofactor for row 1 column 1 is $\lambda^2 - 4\lambda + 2$.

The element in row 1, column 2 is 0, so we don’t the cofactor here.

The next element is row 1, column 3. Leaving out row 1 and column 3, the sub-matrix is:

$$\begin{pmatrix} -1 & 4-\lambda \\ -1 & 2 \end{pmatrix}.$$

The determinant of this sub-matrix is:

$$(-1)\text{x}2 - (-1)\,(4 - \lambda) = -2 + 4 - \lambda = 2 - \lambda.$$

This is row 1, column 3, 1 + 3 = 4 an even number, so there is no sign change. The cofactor is $2 - \lambda$.

The determinant is then each element by its corresponding cofactor: for a 3 by 3, using the top row:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

So

$$\det(A - \lambda I) = (2 - \lambda)(\lambda^2 - 4\lambda + 2) + 0 + 1(2 - \lambda).$$

This in turn was to be set to zero, giving the equation:

$$(2 - \lambda)(\lambda^2 - 4\lambda + 2) + 2 - \lambda = 0.$$

There is a common factor of $2 - \lambda$, so simplifying the LHS gives the cubic equation

$$(2 - \lambda)[(\lambda^2 - 4\lambda + 2) + 1] = 0.$$

$$(2 - \lambda)[\lambda^2 - 4\lambda + 3] = 0.$$

We have kept in the factor $(2 - \lambda)$, so this is our first factor to solve: this is 0 when $\lambda = 2$.

The remaining roots come from the quadratic:

$$\lambda^2 - 4\lambda + 3$$

The roots of this quadratic are:

$$\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

1 and 3 are the other two roots.

The solutions to this equation are: from the outside factor; $\lambda = 2$, and from the quadratic; $\lambda = 1$ and $\lambda = 3$.

This problem is reworked here using the diagonal method:

$$\det\begin{pmatrix} 2-\lambda & 0 & 1 \\ -1 & 4-\lambda & -1 \\ -1 & 2 & -\lambda \end{pmatrix} = 0 \,.$$

$$(2 - \lambda)(4 - \lambda)(-\lambda) + 0 - 2 - [- (4 - \lambda) - 2(2 - \lambda) - 0] = 0.$$

$$(2 - \lambda)(4 - \lambda)(-\lambda) + 0 - 2 - [- 4 + \lambda - 4 + 2\lambda] = 0.$$

$$(2 - \lambda)(4 - \lambda)(-\lambda) - 2 + 8 - 3\lambda = 0.$$

$$(2 - \lambda)(4 - \lambda)(-\lambda) + 6 - 3\lambda = 0.$$

Take out the common factor of the last term:

$$(2 - \lambda)(4 - \lambda)(-\lambda) + 3(2 - \lambda) = 0.$$

$$(2 - \lambda)[(4 - \lambda)(-\lambda) + 3] = 0.$$

$$(2 - \lambda)(\lambda^2 - 4\lambda + 3) = 0, \text{ as before.}$$

5.1.3 Example

For the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}.$$

Check that $\lambda = 2$ is an eigenvalue. The characteristic equation is

$$\det(A - \lambda I) = 0.$$

The matrix $A - 2I$ is:

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & -1 \\ -1 & 2 & -2 \end{pmatrix}.$$

The first column times –2 gives the second column, therefore the determinant is 0. Alternatively, the first row added to the third row is equal to the second.

5.1.4 Example

For the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}.$$

Check that $\lambda = 1$ is an eigenvalue. The characteristic equation is

$$\det(A - \lambda I) = 0.$$

The matrix $A - 1.I = A - I$ is:

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}.$$

The first and third columns are the same, so the determinant is 0.

Example

Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}.$$

This determinant can be calculated quickly with the cofactor method, especially as there is a zero in the first row.

The characteristic equation is

$$\det\begin{pmatrix} 1-\lambda & -1 & 0 \\ 1 & 2-\lambda & 1 \\ -2 & 1 & -1-\lambda \end{pmatrix} = 0.$$

This becomes

$$(1 - \lambda)(2 - \lambda)(-1 - \lambda) + (-1).1.\,(-2) + 0 - 0 - 1.1(1 - \lambda)$$

$$- (-1 - \lambda).1.(-1) = 0.$$

The LHS simplifies to

$$(1 - \lambda)(2 - \lambda)(-1 - \lambda) + 2 - 1 + \lambda - 1 - \lambda = 0.$$

The last terms all cancel out, giving

$$(1 - \lambda)(2 - \lambda)(-1 - \lambda) = 0.$$

The roots immediately follow, they are:

$$\lambda = 1, 2, \text{ and } -1.$$

5.1.5 Example

Find the eigenvalues of the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -1 & 2 & 3 \end{pmatrix}.$$

The characteristic equation is

$$\det\begin{pmatrix} 2-\lambda & 0 & 0 \\ -1 & 4-\lambda & 0 \\ -1 & 2 & 3-\lambda \end{pmatrix} = 0$$

Note that calculating the determinant of a matrix with all zeros above or below the diagonal is particularly easy; it is the product of the diagonal elements.

Thus the characteristic equation here is

$$(2 - \lambda)(4 - \lambda)(3 - \lambda) = 0.$$

The roots immediately follow, they are:

$$\lambda = 2, 3 \text{ and } 4.$$

5.1.6 Rotation Matrices – Eigenvalues

Let the value A be an angle. Find the eigenvalues of the following matrix, interpreting your result:

$$R(A) = \begin{pmatrix} \cos A & -\sin A & 0 \\ \sin A & \cos A & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The characteristic equation means:

$$\text{Det}[\, R(A) - \lambda I \,] = 0.$$

For this matrix, we have:

$$\det\begin{pmatrix} \cos A - \lambda & -\sin A & 0 \\ \sin A & \cos A - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0\,.$$

Apply the cofactor method, for row 3 (or column 3); the determinant is:

$$(1-\lambda)\det\begin{pmatrix} \cos A - \lambda & -\sin A \\ \sin A & \cos A - \lambda \end{pmatrix} = 0.$$

Working out the two-by-two sub-determinant,

$$(1 - \lambda)(\,(\cos A - \lambda)^2 + \sin^2 A) = 0.$$

The first factor here is $1 - \lambda$, this is 0 at $\lambda = 1$.

The other roots are found from the quadratic:

$$(\cos A - \lambda)^2 + \sin^2 A = 0.$$

Bring over the $\sin A$ term: $(\cos A - \lambda)^2 = -\sin^2 A$.

Take the square root of both sides, noting the two possibilities:

$$\cos A - \lambda = \pm \mathrm{j}\sin A.$$

Then $\lambda = \cos A \pm \mathrm{j}\sin A = \exp(\pm \mathrm{j}A)$.

One real root, 1, and two complex roots: $e^{\mathrm{j}A}$ and $e^{-\mathrm{j}A}$.

Note that the product of these three matrices is 1, which is the determinant of the original matrix.

5.2 The Eigenvalues and the Determinant

The eigenvalues can be linked to the determinant of the original matrix itself. We'll look at the 2 by 2 and then the 3 by 3 cases.

Let the 2 by 2 matrix A have eigenvalues λ_1 and λ_2. Say its characteristic equation is

$$\det(A - \lambda I) = 0,$$

which will work out to be a quadratic of the form

$$\det(A - \lambda I) = \lambda^2 + b\lambda + c = 0.$$

The eigenvalues λ_1 and λ_2 are the roots of the characteristic equation, so the factors of the characteristic equation are:

$$\lambda - \lambda_1,\ \lambda - \lambda_2.$$

The characteristic equation can then be written as:

$$\det(A - \lambda I) = \lambda^2 + b\lambda + c =$$

$$= (\lambda - \lambda_1)(\lambda - \lambda_2).$$

This equation is true for all λ; substitute $\lambda = 0$:

$$\det(A) = c = (-\lambda_1)(-\lambda_2) = \lambda_1\lambda_2.$$

The first connection found between the determinant of A, the characteristic equation and the eigenvalues is:

$$\det(A) = c = \lambda_1\lambda_2.$$

Therefore the determinant of the matrix A itself is the same value as the constant term in the quadratic, and the product of the two eigenvalues. We will see this is true for all cases.

A second connection can be found by looking at the factorised characteristic:

$$\lambda^2 + b\lambda + c = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

When we multiply out the RHS,

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \lambda_1\lambda - \lambda_2\lambda + \lambda_1\lambda_2$$

The coefficient multiplying the λ term in the characteristic equation will be

$$b = -(\lambda_1 + \lambda_2).$$

5.2.1 The 3 by 3 Case

Let matrix A have eigenvalues λ_1, λ_2, and λ_3.

Say its characteristic equation is given by

$$-\lambda^3 + a\lambda^2 + b\lambda + c,$$

which is

$$\det(A - \lambda I) = -\lambda^3 + a\lambda^2 + b\lambda + c.$$

The eigenvalues λ_1, λ_2, and λ_3 are the roots of the characteristic equation.

This means that the factors of the characteristic equation are:

$$\lambda - \lambda_1,\ \lambda - \lambda_2,\ \lambda - \lambda_3.$$

The characteristic equation can then be written as the product of these factors:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda).$$

As before, substitute $\lambda = 0$ into this equation:

$$\det(A) = c = \lambda_1\lambda_2\lambda_3.$$

The first connection found between the determinant of A, the characteristic equation and the eigenvalues is:

$$\det(A) = c = \lambda_1\lambda_2\lambda_3.$$

5.3 The Eigenvectors

The next step is to calculate the eigenvectors. The original defining equation for λ and $\boldsymbol{x}$ was

$$A\boldsymbol{x} = \lambda\boldsymbol{x},$$

and in order that there be non-trivial solutions, the characteristic equation must be satisfied:

$$\det(A - \lambda I) = 0.$$

The first matrix equation

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0},$$

represents a set of equations for the n unknown values of the elements of the column matrix $\boldsymbol{x}$.

It can be multiplied out, to produce a system of n linear equations. But the second equation,

$$\det(A - \lambda I) = 0,$$

means that the system cannot be fully solved.

To see why this is, recall that a matrix is singular if two rows or columns are the same. For the system of equations coming from this matrix, this means two of the equations are the same.

Thus there will be one less equation than the number of unknowns. It can be seen from the defining equation what this means.

The original equation for λ and $\boldsymbol{x}$ was

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}.$$

Now multiply both sides of this equation by a fixed number, a to get

$$a(A - \lambda I)\boldsymbol{x} = \mathbf{0}.$$

Take the a through to the $\boldsymbol{x}$ gives

$$(A - \lambda I)a\boldsymbol{x} = \mathbf{0}.$$

This result means that if the column matrix $\boldsymbol{x}$ is a solution, then so is the column matrix $a\boldsymbol{x}$, for any number a.

The column matrix $\boldsymbol{x}$ can therefore only be found up to a multiplicative factor, and this will be seen when solutions are found from the resulting set of equations.

5.3.1 Example – A 2 by 2 Case

Recall the eigenvalues of the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix}$$

were found to be 5 and –1.

Now the eigenvectors can be found by going back to the equation for column matrix $\boldsymbol{x}$. For $\lambda = 5$, the equation is

$$(A - 5I)\boldsymbol{x} = \boldsymbol{0}.$$

Let the elements of $\boldsymbol{x}$ be x_1 and x_2.

The equation above is then

$$\begin{pmatrix} 4-5 & 1 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Multiplying out gives

$$-x_1 + x_2 = 0,$$

$$-5x_1 + 5x_2 = 0.$$

These are both the same equation, $x_1 = x_2$.

No more information about x_1 and x_2 can be found. The convention we will adopt is to choose a value of 1 for x_1, if possible, and then find the remaining values from this.

Thus $x_2 = 1$ and the column matrix $\boldsymbol{x}$ associated with the eigenvalue $\lambda = 5$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The second convention employed now is to ensure that the eigenvalue is of length 1; this is done by dividing by the length of the vector found so far. This eigenvector is called the *unit* eigenvector.

The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is of length $\sqrt{(1^2 + 1^2)} = \sqrt{2}$.

The unit eigenvector for $\lambda = 5$ is then $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For the second eigenvalue, $\lambda = -1$, the equation is

$$(A + I)\boldsymbol{x} = \mathbf{0}.$$

The system of equations above is then

$$\begin{pmatrix} 4+1 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In this case, it is immediately apparent that there is only one equation:

$$5x_1 + x_2 = 0.$$

This means that

$$x_2 = -5x_1.$$

Following the same convention and taking $x_1 = 1$, the column matrix $\boldsymbol{x}$ associated with the eigenvalue $\lambda = -1$ has then been shown to be

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}.$$

Since the value of x_1 was taken to be 1, this eigenvector is only unique up to a multiplicative factor.

If the value $x_1 = 2$ was taken, the eigenvector would then have been

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \end{pmatrix}.$$

This is just the previous matrix, times 2. Essentially what this means is that instead of thinking of the eigenvector as a normal vector of a particular magnitude and direction, it should be thought of as just a direction.

5.3.2 Example

Find the eigenvectors of the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -1 & 2 & 3 \end{pmatrix}.$$

The eigenvalues were found to be $\lambda = 2$, 3 and 4.

The system of equations is

$$(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}.$$

Setting the elements of $\boldsymbol{x}$ be x_1, x_2 and x_3, and subtracting the λ from the diagonals for $A - \lambda I$, the matrix equation for $\boldsymbol{x}$ is:

$$\begin{pmatrix} 2-\lambda & 0 & 0 \\ -1 & 4-\lambda & 0 \\ -1 & 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 2$, the matrix equation becomes

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives the system of equations

$$0 = 0,$$

$$-x_1 + 2x_2 = 0,$$

$$-x_1 + 2x_2 + x_3 = 0.$$

Subtracting the second equation from the third gives a value

$$x_3 = 0.$$

The second equation is

$$x_1 = 2x_2,$$

so take $x_1 = 1$ and then $x_2 = ½$.

Thus the eigenvector for $\lambda = 2$ is

$$\begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

For $\lambda = 3$, the matrix equation becomes

$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives the system of equations

$$-x_1 = 0,$$

$$-x_1 + x_2 = 0,$$

$$-x_1 + 2x_2 = 0.$$

It follows immediately from these equations that

$$x_1 = 0 \text{ and } x_2 = 0.$$

There is no equation with x_3, so take $x_3 = 1$.

Thus the eigenvector for $\lambda = 3$ is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 4$, the matrix equation becomes

$$\begin{pmatrix} -2 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

This gives the system of equations

$$-2x_1 = 0,$$

$$-x_1 = 0,$$

$$-x_1 + 2x_2 - x_3 = 0.$$

The first two equations are simply stating that $x_1 = 0$.

The third equation is then $x_3 = 2x_2$.

Taking $x_2 = 1$, the eigenvector for $\lambda = 4$ is

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

5.3.3 Example

Find the eigenvectors of the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}.$$

The eigenvalues were $\lambda = 1, 2,$ and 3.

The system of equations is

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}.$$

Setting the elements of $\boldsymbol{x}$ be x_1, x_2 and x_3, the matrix equation for $\boldsymbol{x}$ is

$$\begin{pmatrix} 2-\lambda & 0 & 1 \\ -1 & 4-\lambda & -1 \\ -1 & 2 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 1$, the matrix equation becomes

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is multiplied out to give the system of equations

$$x_1 + x_3 = 0,$$

$$-x_1 + 3x_2 - x_3 = 0,$$

$$-x_1 + 2x_2 - x_3 = 0.$$

Subtracting the third equation from the second gives a value

$$x_2 = 0.$$

The other equations are all equivalent to

$$x_1 + x_3 = 0,$$

in other words, $x_3 = -x_1$, so with $x_1 = 1$ then $x_3 = -1$.

Thus the eigenvector for $\lambda = 1$ is

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda = 2$, the matrix equation becomes

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & -1 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives the system of equations

$$x_3 = 0,$$

$$-x_1 + 2x_2 - x_3 = 0,$$

$$-x_1 + 2x_2 - 2x_3 = 0.$$

Putting in the value $x_3 = 0$ into the second or third equation gives

$$x_1 = 2x_2,$$

so with $x_1 = 1$ then $x_2 = ½$.

Thus the eigenvector for $\lambda = 2$ is

$$\begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

For $\lambda = 3$, the matrix equation becomes

$$\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives the system of equations

$$-x_1 + x_3 = 0,$$

$$-x_1 + x_2 - x_3 = 0,$$

$$-x_1 + 2x_2 - 3x_3 = 0.$$

The first equation is $x_1 = x_3$. Using this to remove x_3 in the second or third equation gives

$$-2x_1 + x_2 = 0, \text{ so } x_2 = 2x_1.$$

With $x_1 = 1$ then $x_2 = 2$. Thus the eigenvector for $\lambda = 3$ is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

5.3.4 Example – Unit Eigenvectors

Find unit the eigenvectors of the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix},$$

and choose the eigenvectors so that they are all of length 1.

The calculations before gave the values $\lambda = 1, 2$, and -1.

For $\lambda = 1$, the matrix equation becomes

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives the system of equations

$$-x_2 = 0,$$

$$x_1 + x_2 + x_3 = 0,$$

$$-2x_1 + x_2 - 2x_3 = 0.$$

Substituting the first equation, $x_2 = 0$, into the second or third gives $x_3 = -x_1$. With the value $x_1 = 1$, then $x_2 = -1$.

With these values, the eigenvector for eigenvalue $\lambda = 1$ would be

$$x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

This time we will produce an eigenvector of length 1. The length of the vector given is

$$|\mathbf{x}|^2 = 1^2 + 1^2 = 2,$$

so that $|\mathbf{x}| = \sqrt{2}$. Divide out by this factor of $\sqrt{2}$ to give

$$x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

This is now a vector of length 1, that is, of unit magnitude, and will be taken as the eigenvector for $\lambda = 1$.

For $\lambda = 2$, the matrix equation becomes

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives the system of equations

$$-x_1 - x_2 = 0,$$

$$x_1 + x_3 = 0,$$

$$-2x_1 + x_2 - 3x_3 = 0.$$

The first and second equations give

$$x_3 = -x_1, \text{ and } x_2 = -x_1,$$

Then taking $x_1 = 1$, the eigenvector for eigenvalue $\lambda = 2$ is

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

The length of this eigenvector is $|\boldsymbol{x}|^2 = 1^2 + (-1)^2 + (-1)^2 = 3$, so that $|\boldsymbol{x}| = \sqrt{3}$.

Divide out by this factor of √3 to give

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

For $\lambda = -1$, the matrix equation is

$$\begin{pmatrix} 3 & -1 & 0 \\ 1 & 4 & 1 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives the system of equations

$$3x_1 - x_2 = 0,$$

$$x_1 + 4x_2 + x_3 = 0,$$

$$-2x_1 + x_2 + x_3 = 0.$$

The first equation is simply $x_2 = -3x_1$, and substituting this in the third equation gives

$$x_3 = -x_1.$$

Thus with $x_1 = 1$, the eigenvector for $\lambda = -1$ is

$$\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}.$$

The length of this eigenvector is $|\boldsymbol{x}|^2 = 1^2 + (-3)^2 + (-1)^2 = 11$, so that $|\boldsymbol{x}| = \sqrt{11}$. Divide out by this factor of $\sqrt{11}$ to give

$$\frac{1}{\sqrt{11}}\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}.$$

The three eigenvalues for the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix},$$

were the values $\lambda =$ 1, 2, and –1, and the corresponding eigenvectors were

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{11}}\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}.$$

5.3.5 Rotation Matrices – Eigenvectors

Let the value A be an angle, and the rotation matrix $R(A)$ is given by:

$$R(A) = \begin{pmatrix} \cos A & -\sin A & 0 \\ \sin A & \cos A & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Recall the eigenvalues were $\lambda = 1$, and then the other two were

$$\lambda = \cos A \pm \mathrm{j}\sin A = \exp(\pm \mathrm{j}A).$$

One real root, 1, and two complex roots: $e^{\mathrm{j}A}$ and $e^{-\mathrm{j}A}$.

Note that the product of these three matrices is 1, which is the determinant of the original matrix.

We will find the eigenvector of the real root. To do this, return to the system of equations

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}.$$

To find the eigenvector of the real root $\lambda = 1$, subtract one from the diagonal, as usual.

This gives:

$$\begin{pmatrix} \cos A - 1 & -\sin A & 0 \\ \sin A & \cos A - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Multiplying out this matrix equation would not 'pick up' the x_3 variable, and there would only be two equations in x_1 and x_2. So we can leave it as a set of equations in x_1 and x_2 :

$$\begin{pmatrix} \cos A - 1 & -\sin A \\ \sin A & \cos A - 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Can this two by two matrix be inverted? Its determinant is:

$$(\cos A - 1)^2 + \sin^2 A.$$

Multiply this out:

$$\cos^2 A - 2\cos A + 1 + \sin^2 A.$$

This is:

$$2 - 2\cos A = 2(1 - \cos A).$$

Then so as long as A is not any multiple of 2π, we can say that the equation has a solution in the normal way of solving any matrix equation; the inverse will exist.

It can be solved so that

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

leaving x_3 as the only non-zero value. Thus the eigenvector for eigenvalue $\lambda = 1$ is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which is the z axis.