itb Institute of Technology Blanchardstown
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Integration

Integration

- Integral calculus – concerned with how things accumulate, and used in modelling
  - Machine Learning, Signal Processing & Image processing
    - e.g. Fourier Transform gives a way of filtering and manipulating signals. The Fourier transform defines a relationship between a signal in the time domain and its representation in the frequency domain.

$$X(f) = \int_{-\infty}^{\infty} x(t)\; e^{-j2\pi f t}\; dt$$

Lecture Content

- integration – the reverse of differentiation
- constant of integration, C
- definition: indefinite integrals
- application of integration: finding areas under curves
- definition: definite integrals

Integration – the reverse of differentiation!

Worked Example: For the following function:

$$y = 3x^2 - 2x + 5$$

the independent variable is…. x

the dependent variable is…. y

The first derivative $\frac{dy}{dx}$ is…. $\frac{dy}{dx} = 6x - 2$

Worked Example:

$$y = 3x^2 - 2x + 5$$

$$\frac{dy}{dx} = 6x - 2$$

Power rule: differentiation

...multiply by power

...reduce power by 1

Worked Example:

$$\frac{dy}{dx} = 6x - 2 \qquad \text{Find } y=$$

Reverse the previous process!

Power rule: integration

...increase power by 1

...divide by new power

$$y = \int (6x - 2)\,dx$$

$$= \frac{6x^2}{2} - \frac{2x^1}{1} + C$$

$$= 3x^2 - 2x + C$$

Worked Example:

$$y = 3x^2 - 2x + 5$$

$$\frac{dy}{dx} = 6x - 2$$

Power rule: **differentiation**

...multiply by power

...reduce power by 1

Worked Example:

$$y = \int (6x - 2)\,dx$$

$$= \frac{6x^2}{2} - \frac{2x^1}{1} + C$$

$$= 3x^2 - 2x + C$$

Power rule: **integration**

...increase power by 1

...divide by new power

Indefinite Integral

What does the term $+C$ stand for?

Worked Examples: Compare following examples

$y = 3x^2 - 2x + 5$	$y = 3x^2 - 2x - 3$	$y = 3x^2 - 2x + 8$
$\frac{dy}{dx} = 6x - 2$	$\frac{dy}{dx} = 6x - 2$	$\frac{dy}{dx} = 6x - 2$

Therefore if $\frac{dy}{dx} = 6x - 2$ is integrated the $+C$ term indicates that there could have been *any* 'constant' value in original expression.

$$y = \int (6x - 2)\, dx$$

$$= 3x^2 - 2x + C$$

C is the **'constant of integration'**

Formal Rule: Integration of a polynomial function (variable raised to some power)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$\int x^3 dx$	$\int \left(x^6 - 2\right)dx$	$\int 6x^2 dx$
$= \frac{x^4}{4} + C$	$= \frac{x^7}{7} - 2x + C$	$= \frac{6x^3}{3} + C \quad = \frac{6}{3}x^3 + C$
		$= 2x^3 + C$

In-class exercises:

Formal Rule: Integration of a polynomial function (variable raised to some power)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$\int 12x^5 dx$	$\int (x^2 + 5x^4) dx$	$\int (3x^2 - 2x + 5) dx$

This type of integral which results in a ‘constant of integration’ in the solution is an **‘indefinite integral’**

Worked example:

$$\int (6x)\,dx$$

$$= \frac{6x^2}{2} + C$$

$$= 3x^2 + C$$

In-class exercise:

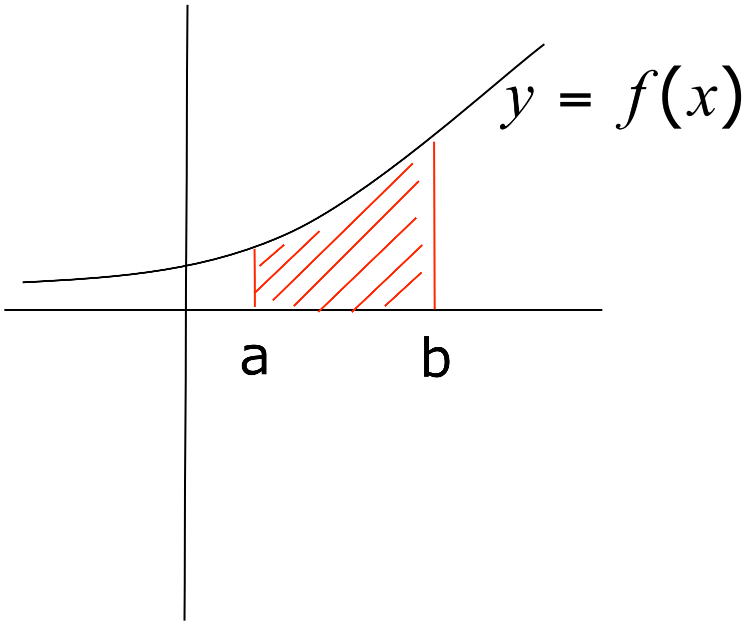
$$\int (9x^2 + 4x + 3)\,dx$$

Definite Integral

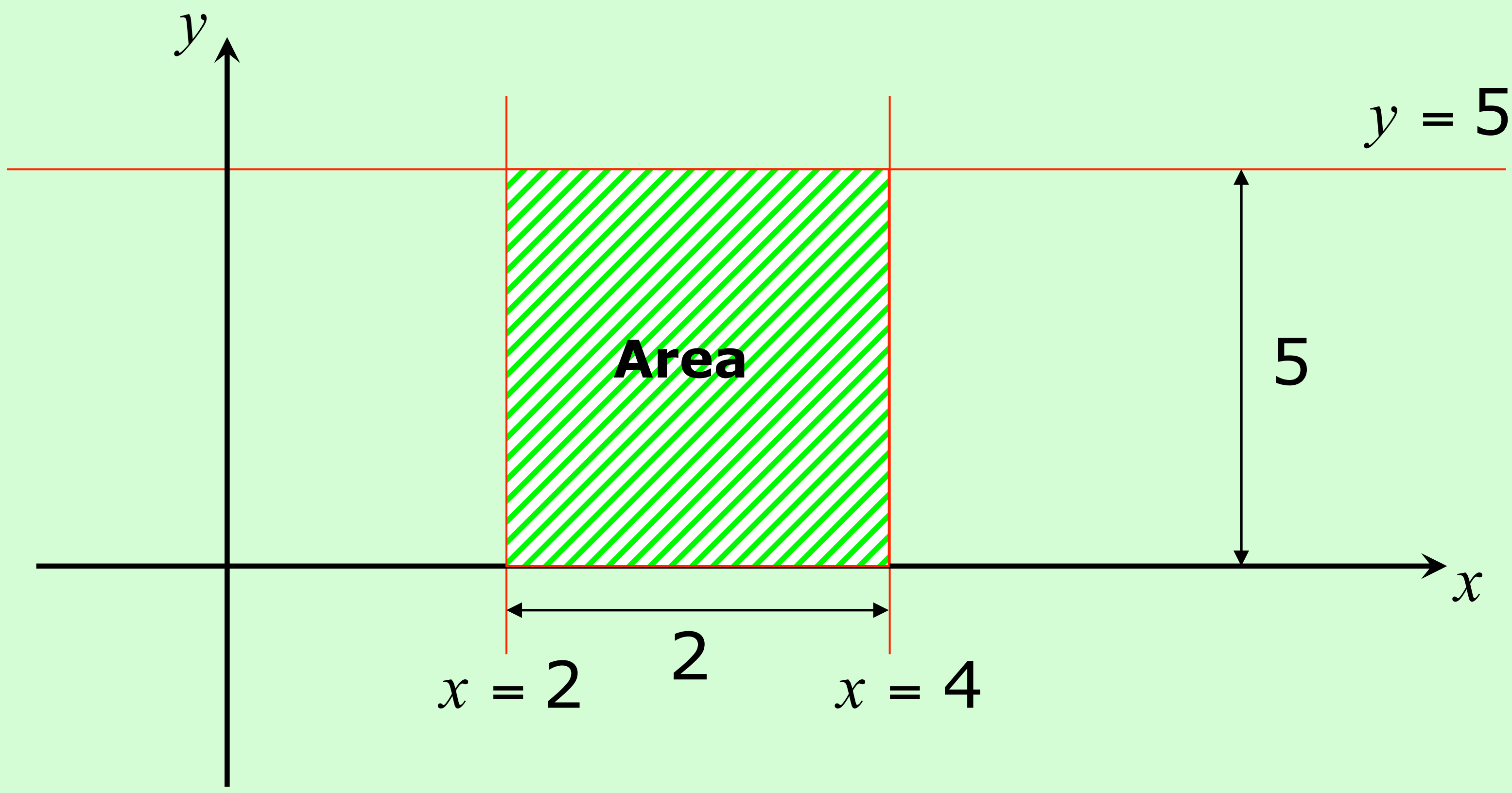
When you integrate a function between upper & lower bounds, you get a 'definite' answer. (a number)

This type of integral is a 'definite integral'

An application of integration is in finding areas under curves...



Worked Example: First look at case of area enclosed by line y=5 in the domain x=2 and x=4.



$$\text{Area} = 5 \times 2 = 10 \,\text{square units}$$

The same result can be found by integrating the function:

$$y = 5$$

between the bounds $x=2$ and $x=4$

This is written mathematically as:

Replace x by upper bound

Replace x by lower bound

$$\int_{x=2}^{x=4} 5.\,dx = 5x\Big]_{x=2}^{x=4} = (5 \times 4) - (5 \times 2)$$

$$= 20 - 10$$

$$= 10 \text{ square units}$$

Properties of Definite Integrals

- the integral is evaluated between ‘bounds’
- result is a numeric value
- there is no constant of integration

Worked Example: evaluate the following definite integral

$$\int_0^1 (2x)\,dx = \left.\frac{2x^2}{2}\right]_0^1 = \left. x^2 \right]_0^1$$

$$= (1)^2 - (0)^2 = 1$$

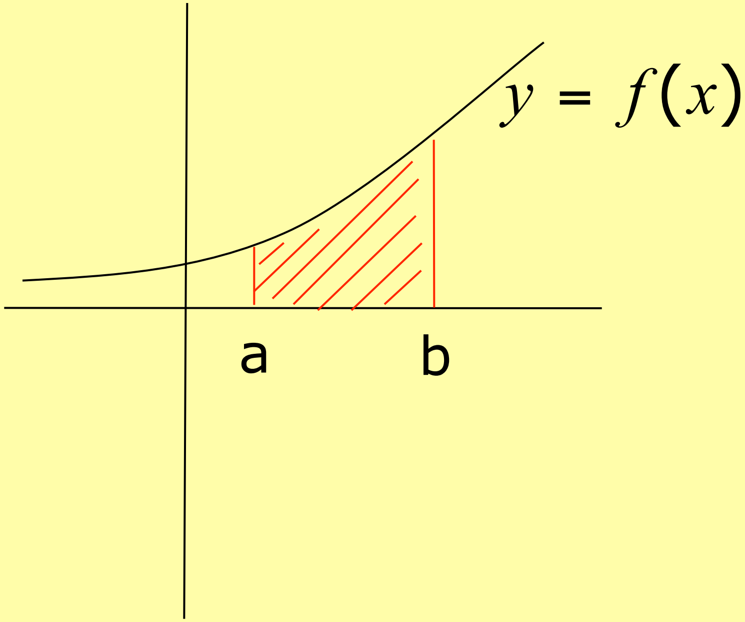
Formal way to state how definite integral between upper & lower bounds is evaluated

$$\int_a^b f'(x)\,dx = f(x)\Big]_a^b = f(b) - f(a)$$

- Integrate function
- Put in upper limit into integrated function
- Then put in lower limit and subtract

The area enclosed between a curve $y = f(x)$ and the x-axis between x = a and x = b, is given by the formula

$$A = \int_a^b f(x)\,dx$$

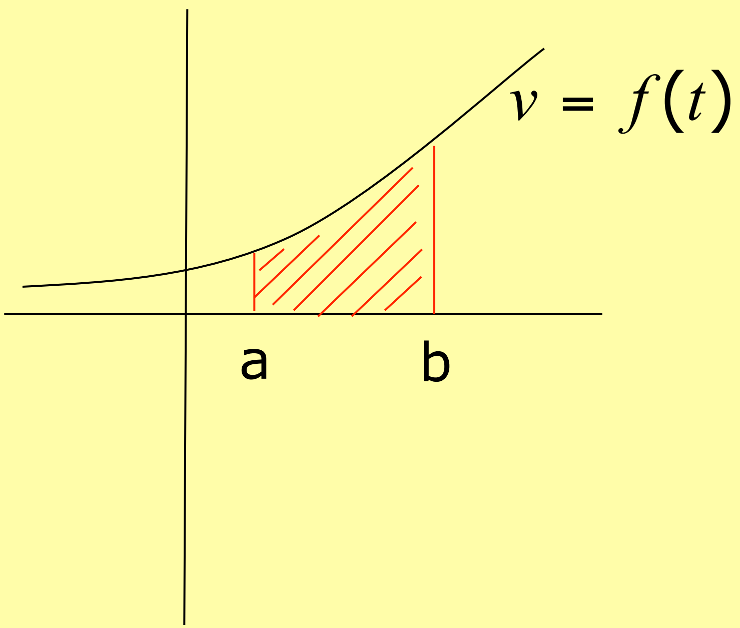


Note: x is not always independent variable & y the dependent i.e. $v = f(t)$

$$A = \int_a^b f(t)\,dt$$

The area enclosed between a curve $v = f(t)$ and the t-axis between t = a and t = b, is given by the formula

$$A = \int_a^b f(t)\,dt$$



Worked Example: Find the area enclosed between the curve $y = x^2$ and the x-axis, from x = 1 to x = 3.

Solution:

$$Area = \int_1^3 x^2 dx$$

$$= \left.\frac{x^3}{3}\right]_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{27}{3} - \frac{1}{3}$$

$$= \frac{26}{3} \text{ square units}$$

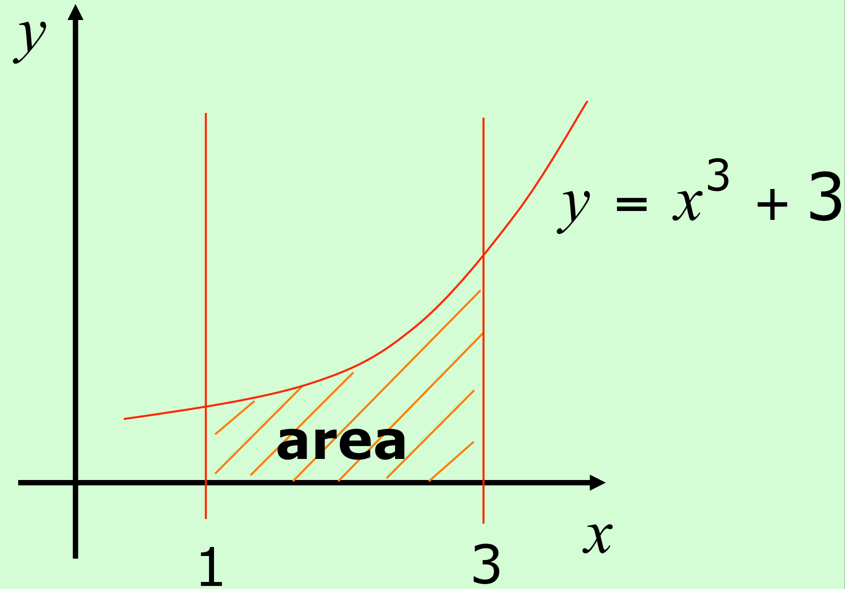
Worked Example: Find the area bounded by the curve $y = x^3 + 3$ the x-axis and the lines $x = 1$ and $x = 3$

Solution:

$$\text{area} = \int_1^3 (x^3 + 3)\,dx$$

$$= \left[\frac{x^4}{4} + 3x\right]_1^3$$

$$= \left(\frac{3^4}{4} + (3 \times 3)\right) - \left(\frac{1^4}{4} + (3 \times 1)\right) = 26 \text{ square units}$$



To verify this result the area will also be found using a numeric method called Simpson's Rule

Lets prove this result by finding the area under the curve using numeric methods…

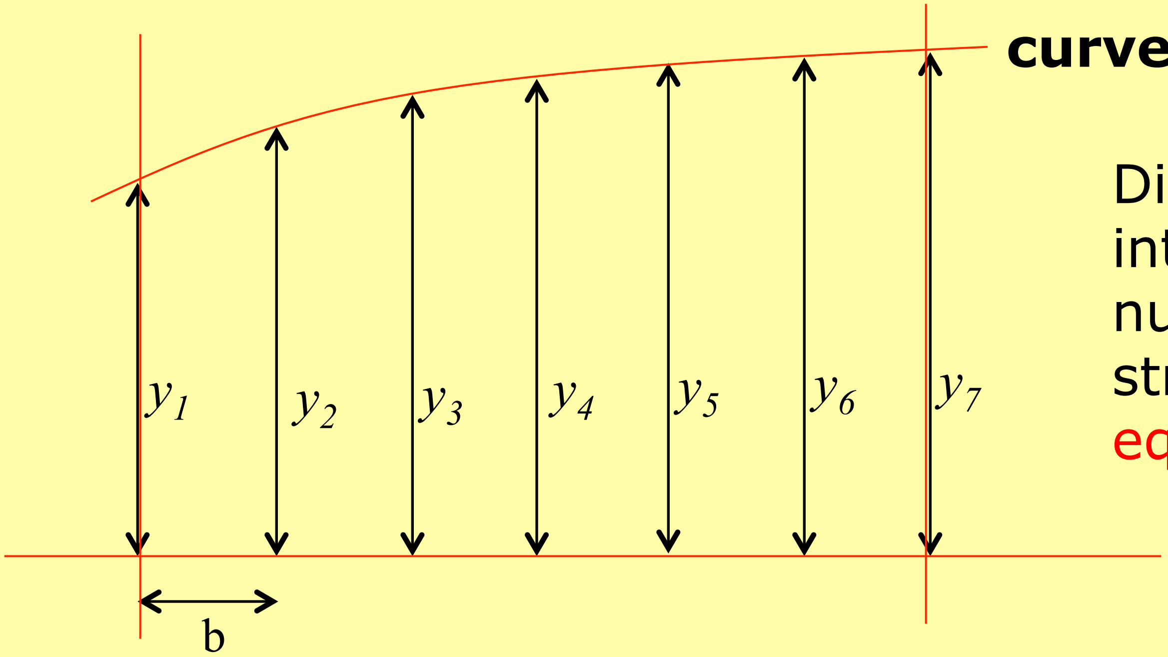
Numeric Integration methods include:

- Trapezoidal rule
- Mid-ordinate rule
- Simpson’s rule

Here Simpson’s Rule will be used to find areas under curves and verify the solutions found by the method of integration.

Simpson's Rule for finding the area under a curve or an irregular shape…

Divide area into an even number of strips of equal width, b



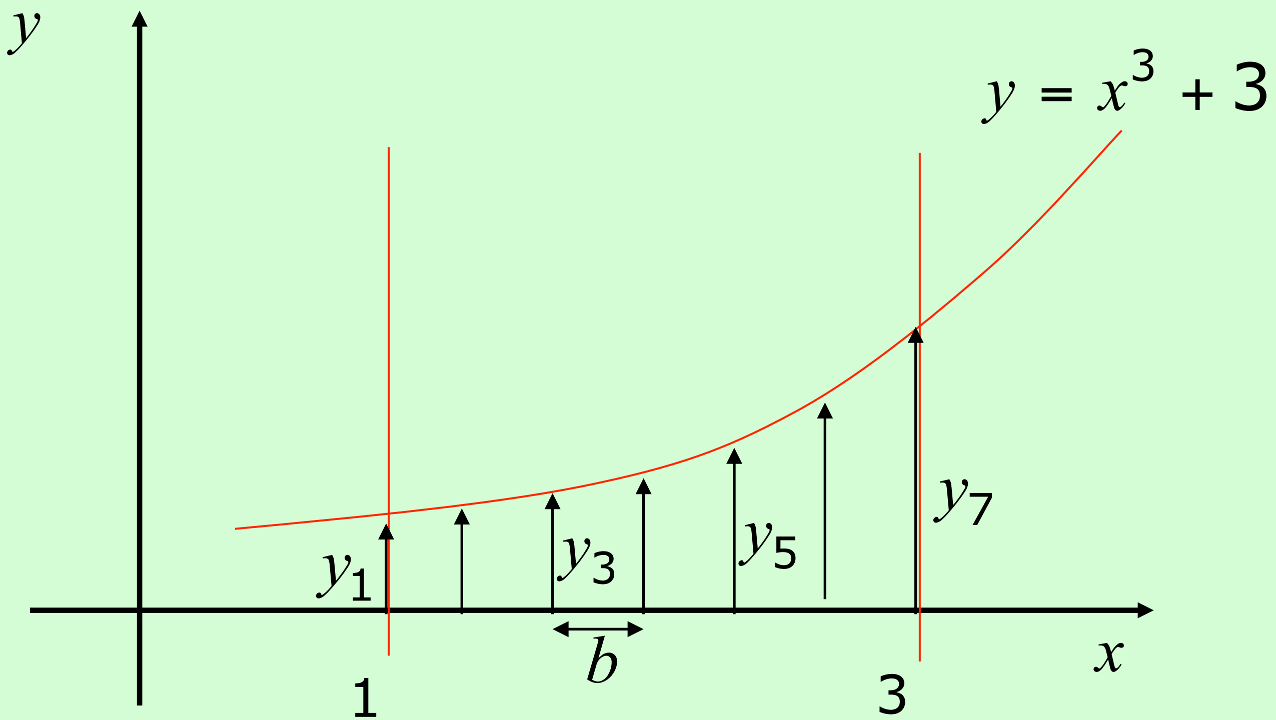
Divide area into an even number of strips of equal width:

$$\text{Area} = \frac{1}{3}b\left[y_1 + y_7 + 4(y_2 + y_4 + y_6) + 2(y_3 + y_5)\right]$$

$$= \frac{b}{3}\left[\left(\sum \text{1st \& last ordinate}\right) + 4\left(\sum \text{even ordinates}\right) + 2\left(\sum \text{odd ordinates}\right)\right]$$

sum of

Worked Example: Find the area bounded by the curve $y = x^3 + 3$ the x-axis and the lines $x = 1$ and $x = 3$



$$\text{Area} = \frac{1}{3}b\left[y_1 + y_7 + 4(y_2 + y_4 + y_6) + 2(y_3 + y_5)\right]$$

= 26 square units

$$y = x^3 + 3$$

$$y_1 = (1)^3 + 3 = 4$$

$$x_1 = 1$$

$$y_2 = \left(1\frac{1}{3}\right)^3 + 3 = 5.37$$

$$x_2 = 1\frac{1}{3}$$

$$y_3 = \left(1\frac{2}{3}\right)^3 + 3 = 7.63$$

$$x_3 = \frac{2}{3}$$

$$x_4 = 2$$

$$y_4 = (2)^3 + 3 = 11$$

$$x_5 = 2\frac{1}{3}$$

$$y_5 = \left(2\frac{1}{3}\right)^3 + 3 = 15.70$$

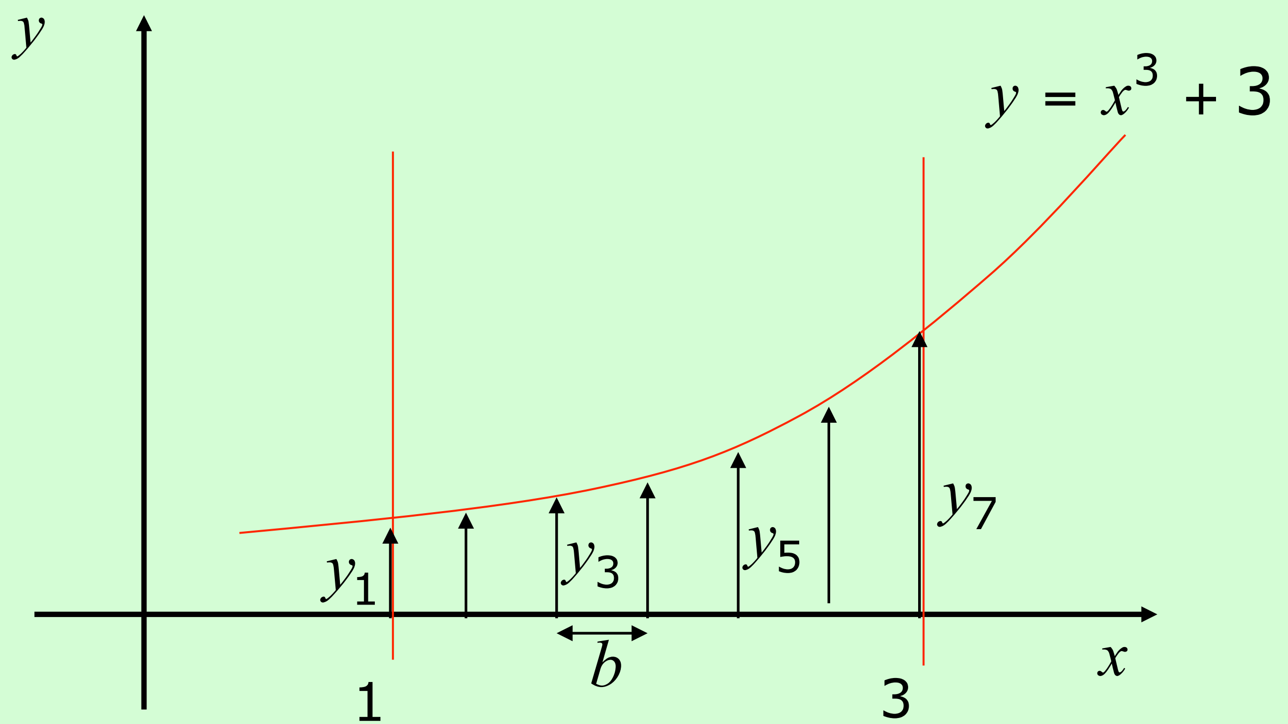
$$x_6 = 2\frac{2}{3}$$

$$y_6 = \left(2\frac{2}{3}\right)^3 + 3 = 21.96$$

$$x_7 = 3$$

$$y_7 = (3)^3 + 3 = 30$$

Worked Example: Find the area bounded by the curve $y = x^3 + 3$ the x-axis and the lines $x = 1$ and $x = 3$



$$\text{Area} = \frac{1}{3}\left(\frac{1}{3}\right)\left[4 + 30 + 4(5.37 + 11 + 21.96) + 2(7.63 + 15.70)\right]$$

$= 26$ square units **...as before**