One Degree-of-Freedom (DoF) Hamiltonian Bifurcation of Equilibria

We will now consider two examples of bifurcation of equilibria in two dimensional Hamiltonian system; in particular, the Hamiltonian saddle-node and Hamiltonian pitchfork bifurcations.

Hamiltonian saddle-node bifurcation

We consider the Hamiltonian:

$$H(q,p) = \frac{p^2}{2} - \lambda q + \frac{q^3}{3}, \quad (q,p) \in \mathbb{R}^2. \tag{1}$$

where λ is considered to be a parameter that can be varied. From this Hamiltonian, we derive Hamilton's equations:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = p, \\ \dot{p} &= -\frac{\partial H}{\partial q} = \lambda - q^2. \end{aligned} \tag{2}$$

The fixed points for (2) are:

$$(q,p) = (\pm\sqrt{\lambda}, 0), \tag{3}$$

from which it follows that there are no fixed points for $\lambda < 0$, one fixed point for $\lambda = 0$, and two fixed points for $\lambda > 0$. This is the scenario for a saddle-node bifurcation.

Next we examine stability of the fixed points. The Jacobian of (2) is given by:

$$\begin{pmatrix} 0 & 1 \\ -2q & 0 \end{pmatrix}. \tag{4}$$

The eigenvalues of this matrix are:

$$\lambda_{1,2} = \pm\sqrt{-2q}.$$

Hence $(q,p) = (-\sqrt{\lambda}, 0)$ is a saddle, $(q,p) = (\sqrt{\lambda}, 0)$ is a center, and $(q,p) = (0,0)$ has two zero eigenvalues. The phase portraits are shown in Fig. 1.

Hamiltonian pitchfork bifurcation

We consider the Hamiltonian:

$$H(q,p) = \frac{p^2}{2} - \lambda\frac{q^2}{2} + \frac{q^4}{4}, \tag{5}$$

where λ is considered to be a parameter that can be varied. From this Hamiltonian, we derive Hamilton's equations:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = p, \\ \dot{p} &= -\frac{\partial H}{\partial q} = \lambda q - q^3. \end{aligned} \tag{6}$$

The fixed points for (6) are:

$$(q,p) = (0,0),\ (\pm\sqrt{\lambda}, 0), \tag{7}$$

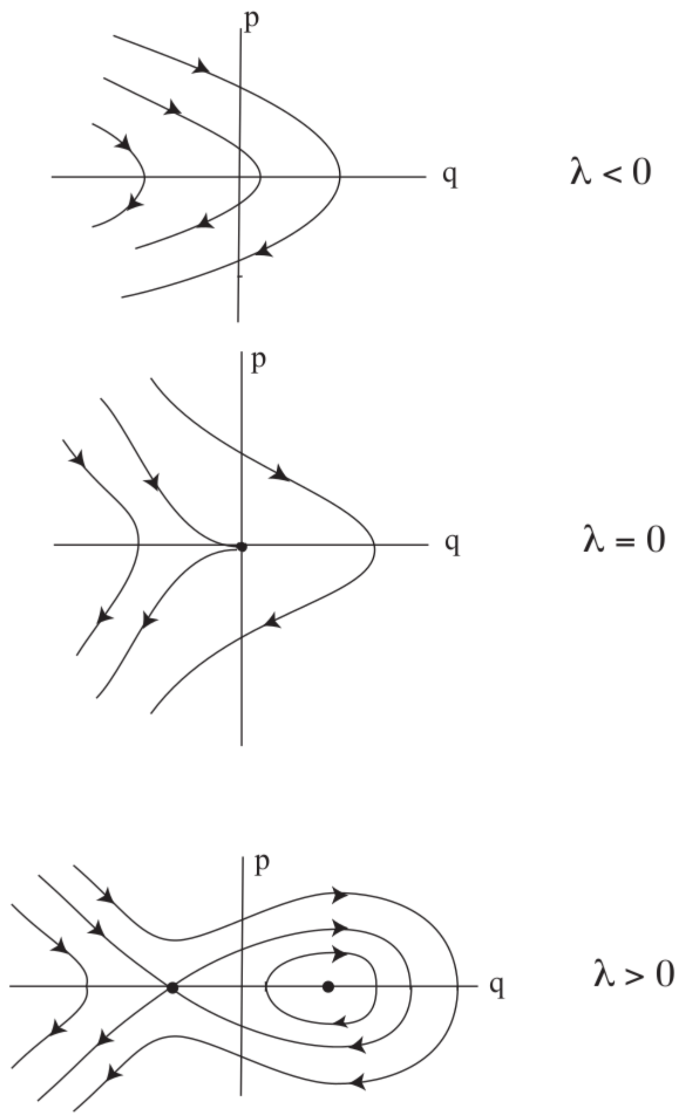

Figure 1: The phase portraits for the Hamiltonian saddle-node bifurcation.

from which it follows that there is one fixed point for $\lambda < 0$, one fixed point for $\lambda = 0$, and three fixed points for $\lambda > 0$. This is the scenario for a pitchfork bifurcation.

Next we examine stability of the fixed points. The Jacobian of (6) is given by:

$$\begin{pmatrix} 0 & 1 \\ \lambda - 3q^2 & 0 \end{pmatrix}. \tag{8}$$

The eigenvalues of this matrix are:

$$\lambda_{1,2} = \pm\sqrt{\lambda - 3q^2}.$$

Hence $(q, p) = (0, 0)$ is a center for $\lambda < 0$, a saddle for $\lambda > 0$ and has two zero eigenvalues for $\lambda = 0$. The fixed points $(q, p) = (\sqrt{\lambda}, 0)$ are centers for $\lambda > 0$. The phase portraits are shown in Fig. 2.

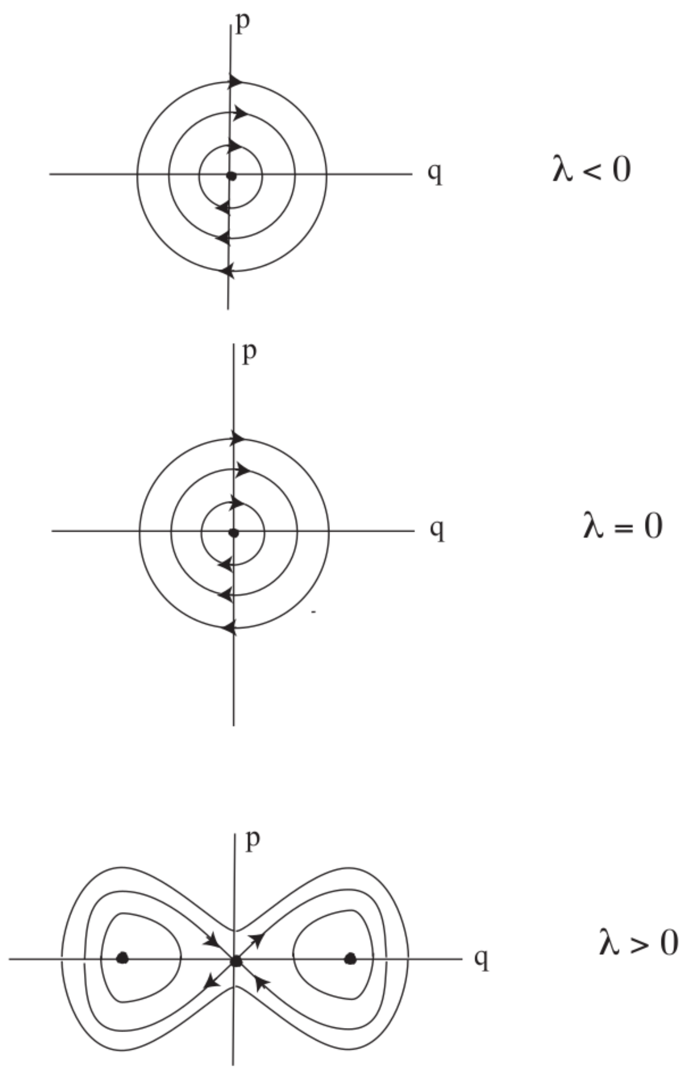

Figure 2: The phase portraits for the Hamiltonian pitchfork bifurcation.