Homework 4: Practical Linear and Exponential Frequency Modulation for Digital Music Synthesis

Introduction

Frequency Modulation (FM) synthesis was introduced by John Chowning in 1973. FM synthesis focuses on synthesizing naturally sounding sounds by dynamically controlling the frequency modulation. Paper by Kasper Nielsen [1] provides a very good insight overview of various FM synthesis methods accompanied by the implementation details, it also provides practical antialiasing solutions. I will give a brief overview of the provided methods along with an implementation example of the phase FM followed by the exponential frequency modulation both of which result in interesting spectral characteristics.

Simple FM Synthesis

FM synthesis can be expressed in time-domain by using sinusoids:

$$y(t) = sin(\omega_c t + Isin(\omega_m t))$$

where ω_c is the carrier angular frequency and ω_m is the modulator angular frequency. I denotes the modulation index which controls the amount of modulation, by increasing modulation index we increase the amount of modulation present in the output signal. Altering modulation index can also add or remove harmonics, controlling the timbre as a result.
In order to synthesize harmonic spectra the ratio r between carrier and modulator frequency should be rational:

$$r = \frac{\omega_c}{\omega_m} = \frac{N_1}{N_2}$$

which implies that N_1 and N_2 should be integers. Moreover, fundamental frequency of the resulting sound can be described as:

$$\omega_0 = \frac{\omega_c}{N_1} = \frac{\omega_m}{N_2}$$

.

Linear FM

Linear FM allows the frequency to sweep up and down from the carrier frequency by a deviation Δw and can be expressed as:

$$\omega_{lin}(t) = \omega_c + \Delta\omega sin(\omega_m t)$$

and the time-domain signal as:

$$y(t) = sin((\omega_c + \Delta\omega sin(\omega_m t))t).$$

In Linear FM the modulation index is defined as:

$$I = \frac{\Delta\omega}{\omega_m}$$

.
It is important to note that we want to avoid using modulating waves that have a DC offset when performing linear FM as they can cause the sound to go out of tune. Moreover, to handle going through zero, i.e. when $\omega_c - \Delta\omega = 0$, we allow frequencies to go to negative values which results in sound generation with reverse phase.

Phase Modulation

Modulating frequency through phase modulation is possible due to the fact that frequency is the derivative of phase. Therefore, frequency can be viewed as the angular velocity of the phase angle in time. When performing phase modulation frequency is kept constant while the phase changes. This can be written as:

$$\omega(t) = \frac{d\theta}{dt}$$

$$\theta(t) = \int \omega(t)dt$$

where modulating phase function can also be described as:

$$\theta(t) = \Delta\theta sin(\omega_m t)$$

with the resulting time-domain signal as:

$$y(t) = sin(\omega_c t + \Delta\theta sin(\omega_m t)).$$

The modulation index in this case is just $I = \Delta\theta$. Since PM has the ability to make the carrier signal run backwards and altering the modulation index doesn't cause detuning, PM is generally favoured over Linear FM.

Exponential FM

In exponential FM the modulation range follows the musical spacing of noes and octaves, i.e. going up or down an octave means doubling or halving the frequency, respectively. This can mathematically be described as:

$$\omega_{exp}(t) = \omega_c 2^{V sin(\omega_m t)}$$

where V denotes the amplitude of the modulation. Time-domain signal is:

$$y(t) = sin(\omega_c 2^{V sin(\omega_m t)} t).$$

In exponential FM the DC offset is dependent on the modulating waveform, therefore to avoid the harsh sounding waveform that is produced by the naïve implementation a DC correction term ins introduced. The condition for harmonic spectrum in exponential FM oscillator is given by:

$$\frac{\omega_c}{\omega_m} = \frac{1}{I_0(V ln(2))} \frac{N_1}{N_2}$$

where I_0 is a modified Bessel function equivalent to the regular Bessel function of first kind and evaluated only for the imaginary arguments. From here, the harmonic modulation frequency can be expressed as:

$$\omega_m = \frac{I_0(V ln(2))}{r} \omega_c.$$

Furthermore, V is derived to be

$$V = \frac{asinh(\frac{I}{r})}{ln(2)}.$$

This makes the exponential FM span the same size as the frequency range of linear FM, making it easier to perform side by side comparison of the two.

Antialiasing

Aliasing is introduced due to the fact that we are unable to represent the frequencies higher than the Nyquist limit. In order to reduce aliasing we can limit the modulation index. The expression can be derived using Carlosn's rule for estimated bandwidth of sinusoidal Linear FM given by:

$$B_{FM} \approx 2(\Delta f + f_m) = 2f_m(I + 1).$$

We can then approximate the maximum possible modulation index:

$$I_{max} = \frac{F_s/2 - f_c}{f_m} - 1$$

where F_s is the sampling rate, f_c the carrier frequency and f_m modulation frequency.
Another way of reducing aliasing is to perform frequency modulation at an N times higher sample rate and then perform downsampling using a lowpass filter and decimation. Ideally we want the lowpass filter to omit the frequencies above Nyquist limit and leave the rest (i.e. the pass-band) flat and unaffected.

Reflection and Conclusion

The paper provided interesting insights into how to perform FM synthesis along with possible pitfalls of each of FM synthesis methods. It also explained details of how to deal with each of the possible problems that might occur when performing FM synthesis using various techniques. Moreover, it dealt with antialising that might occur during FM synthesis. All in all, it enables the reader to gain a broad and general overview of FM synthesis.

References

[1] Kasper Nielsen. Practical Linear and Exponential Frequency Modulation for Digital Music Synthesis. In G. Evangelista, editor, *Proceedings of the 23-rd Int. Conf. on Digital Audio Effects (DAFx2020)*, volume 1, pages 132–139, Sept. 2020-21.