Introduction to Compartmental *(Dynamical/Mechanistic/Mathematical)* Models

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With some material borrowed from :

Sophia Horigan, University of Chicago
Amy Wesolowski, Johns Hopkins University
Jessica Metcalf, Princeton University
Steve Bellan, University of Georgia

Goals for this lecture

- Understand the difference between statistical and mechanistic models
  Comprendre la différence entre les modèles statistiques et mécanistes.

- Understand how to formalize and conceptualize compartmental models
  Comprendre comment on peut formuler et conceptualiser les modèles compartimentés

- Example: population growth, predator prey, SIR models

Compartmental/Mechanistic/Mathematical Models

1. Populations are divided into compartments
   Les populations sont subdivisées en compartiments

2. Individuals within a compartment are homogeneously mixed
   Les individus d'un compartiment sont mélangés de manière homogène

3. Compartments and transition rates are determined by biological systems
   Les compartiments et les taux de transition sont déterminés par les systèmes biologiques

4. Rates of transferring between compartments are expressed mathematically
   Taux de transition entre les compartiments sont exprimés mathématiquement

How are these different from statistical models?

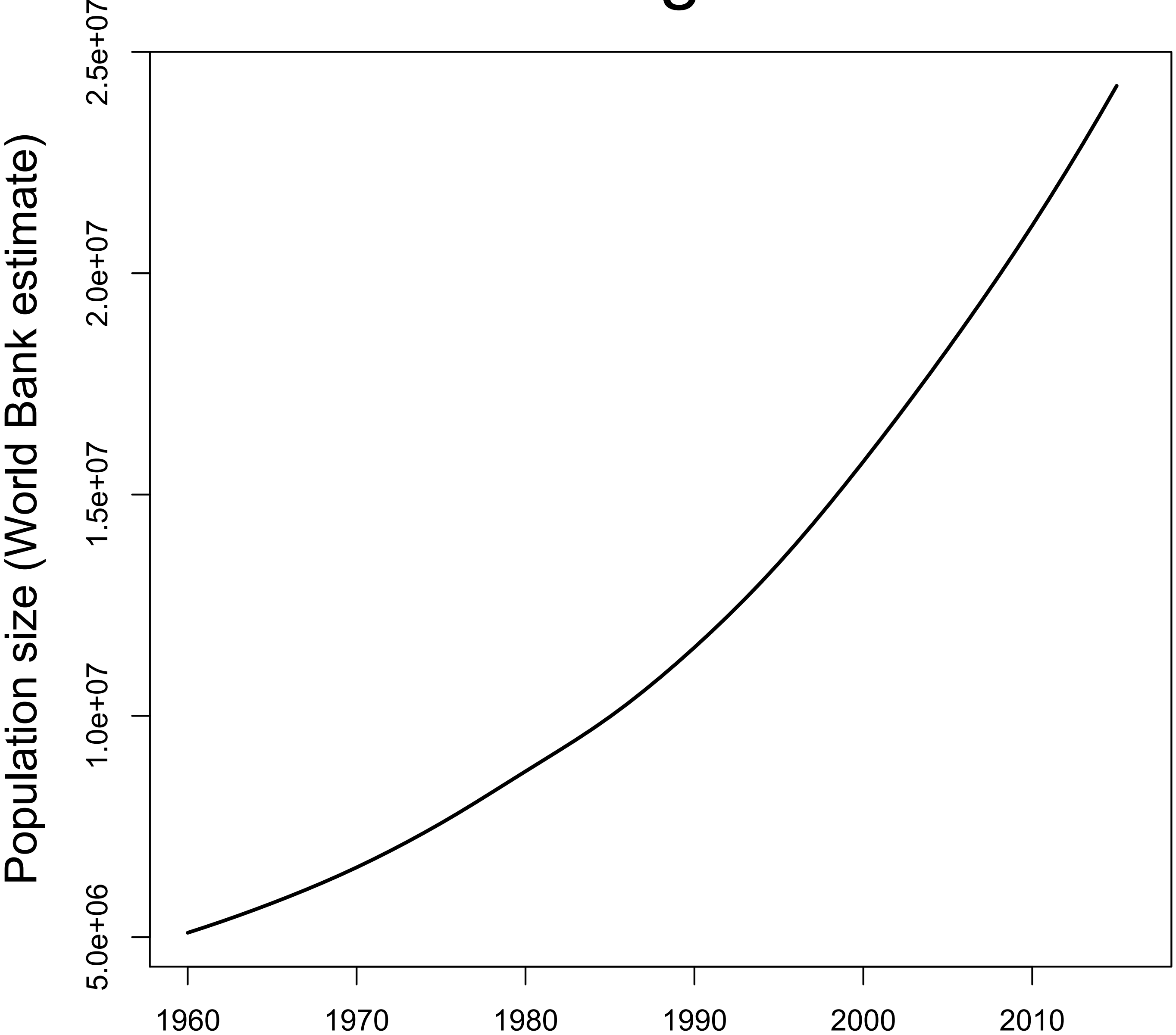
En quoi sont-ils différents des modèles statistiques?

Make explicit hypotheses about biological mechanisms that drive dynamics (may not be realistic, but still explicit)

Faire des hypothèses explicites sur les mécanismes biologiques qui régissent la dynamique de l'infection (peut ne pas être réaliste, mais toujours explicite)

1. Simple Population Models
1. Modèles simples de population

Madagascar



http://databank.worldbank.org

What can we say about the population of Madagascar?

How would a model help us? What kind of model should we use?

The basic population model

Compartmental models (Mechanistic Models)

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The basic population model

<u>Compartmental models (Mechanistic Models)</u>

1. Populations are divided into compartments
2. Compartments and transition rates are determined by biological systems
3. Rates of transferring between compartments are expressed mathematically
4. Individuals within a compartment are homogenously mix

How does the population of Madagascar grow over time?

Comment est-ce que la population de Madagascar s'augmente avec le passage du temps?

The simplest population model

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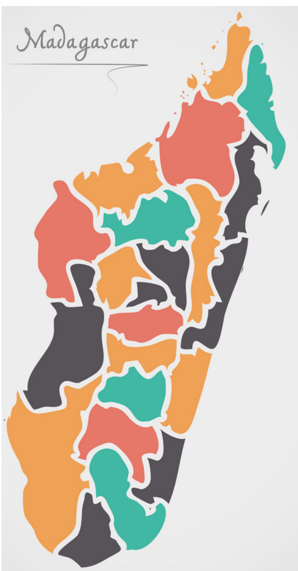

Madagascar (N)

The simplest population model

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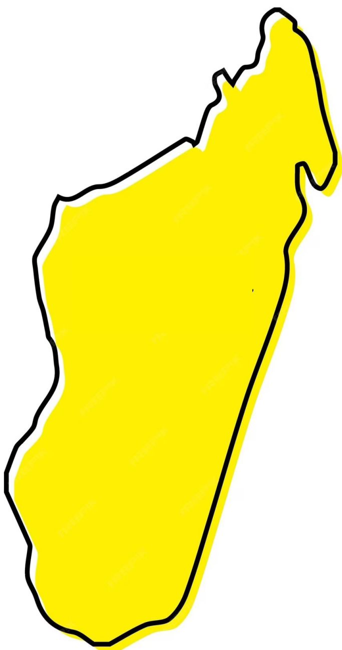

Madagascar (N)

The simplest population model

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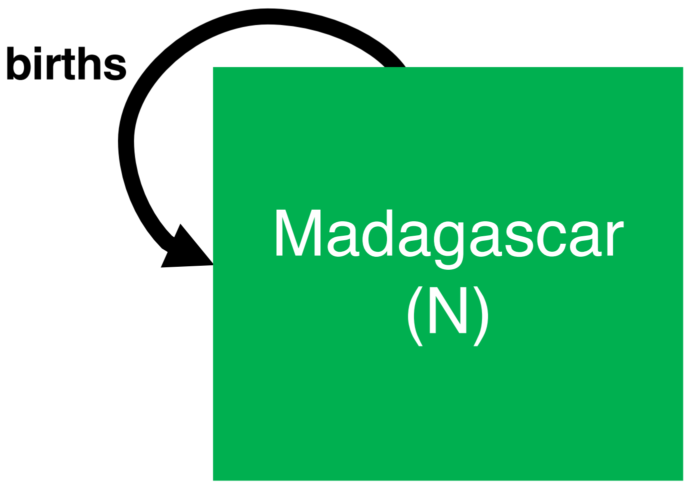
Madagascar
(N)

How does the population grow?

The simplest population model



How does the population grow?

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2. Individuals within a compartment are homogenously mixed
3. **Compartments and transition rates are determined by biological systems**
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The simplest population model

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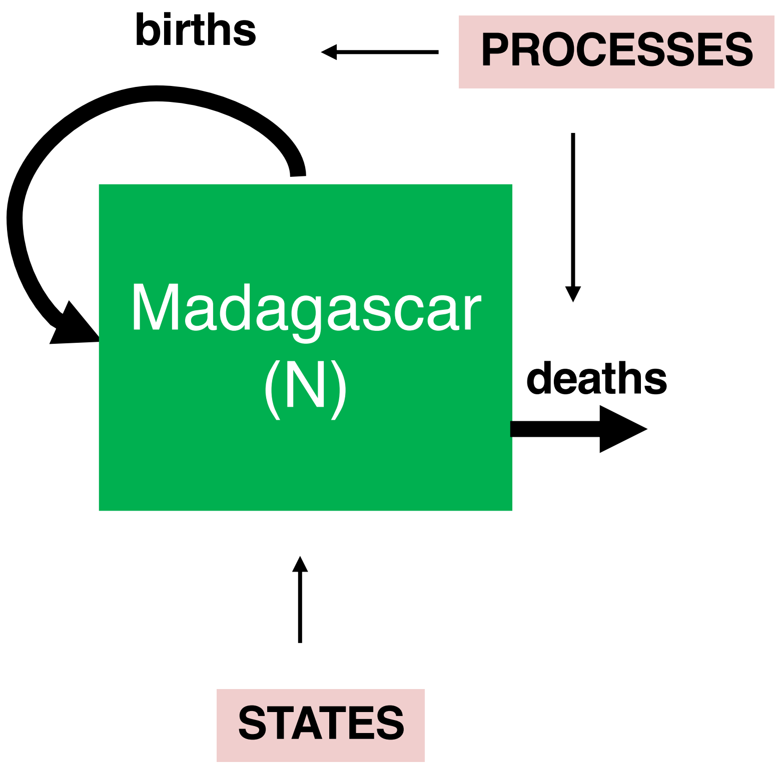
How does the population decrease?

The simplest population model



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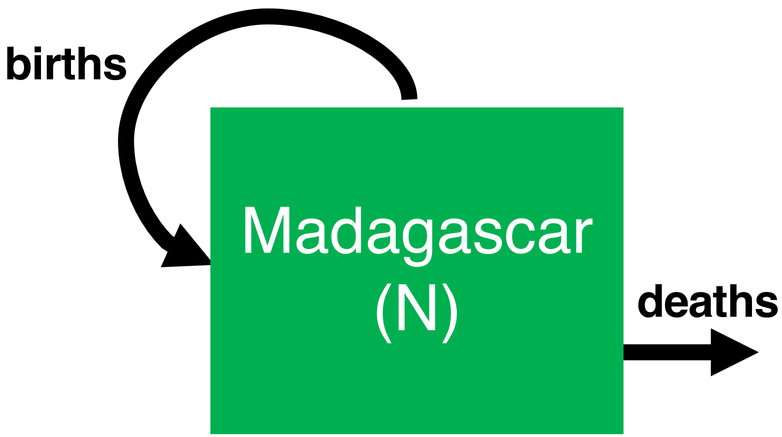

$$N_{t+1} = N_t + \text{births}^*N_t\text{-deaths}^*N_t$$

$$N_{t+1} = N_t + (\text{births-deaths})^*N_t$$

$$N_{t+1} = N_t + \color{red}{R}^*N_t$$

R=geometric rate of increase

R = (births – deaths)

(pop grows @ $R > 0$ & declines @ $R < 0$)

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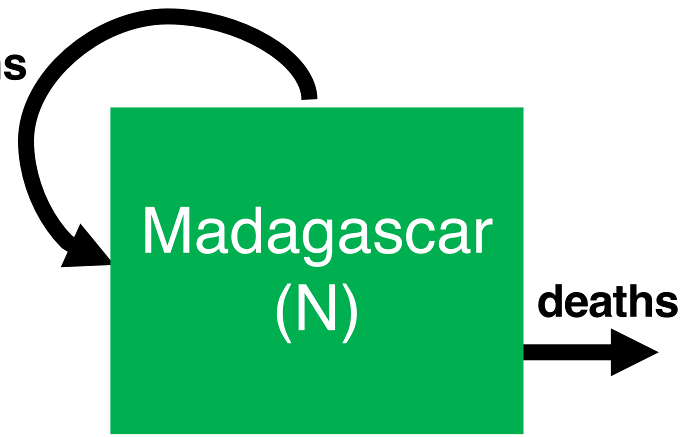
The simplest population model



$$N_{t+1} = N_t + \text{births}*N_t - \text{deaths}*N_t$$

$$N_{t+1} = N_t + R*N_t$$

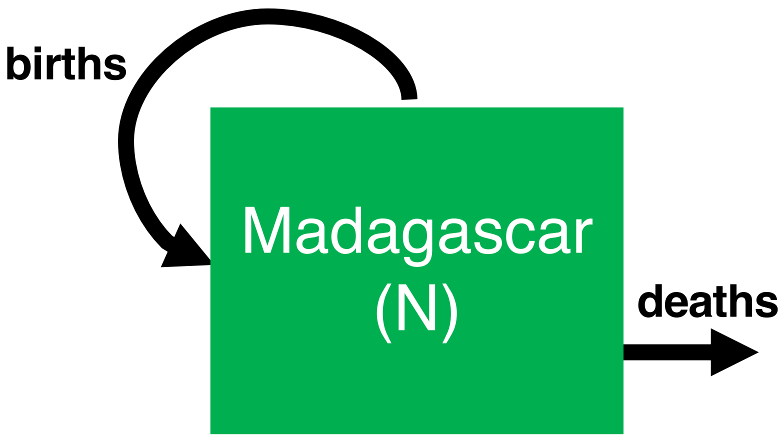
$$N_{t+1} = (1 + R)*N_t$$

$$N_{t+1} = \lambda*N_t$$

λ= population rate of increase (finite growth rate)

$\lambda=1+R$

The simplest population model



$$N_{t+1} = N_t + births*N_t - deaths*N_t$$
$$N_{t+1} = N_t + R*N_t$$
$$N_{t+1} = (1 + R)*N_t$$
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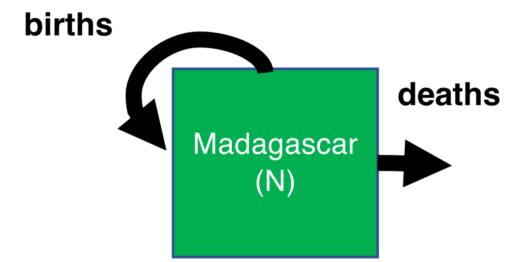
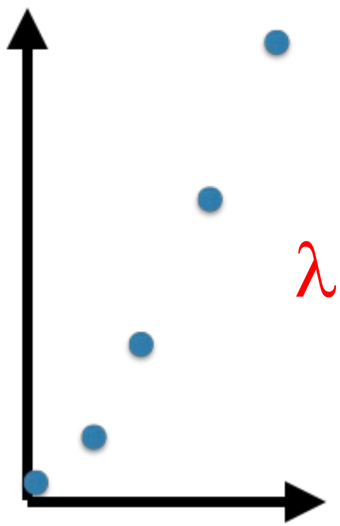
λ= population rate of increase (finite growth rate)

(pop grows @ λ >1 & declines @ λ < 1)



Geometric growth

Geometric growth is measured in <u>discrete</u> time



$$\lambda = N_{t+1}/N_t$$

$N_1 = \lambda N_0$
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Is discrete time realistic for population growth?

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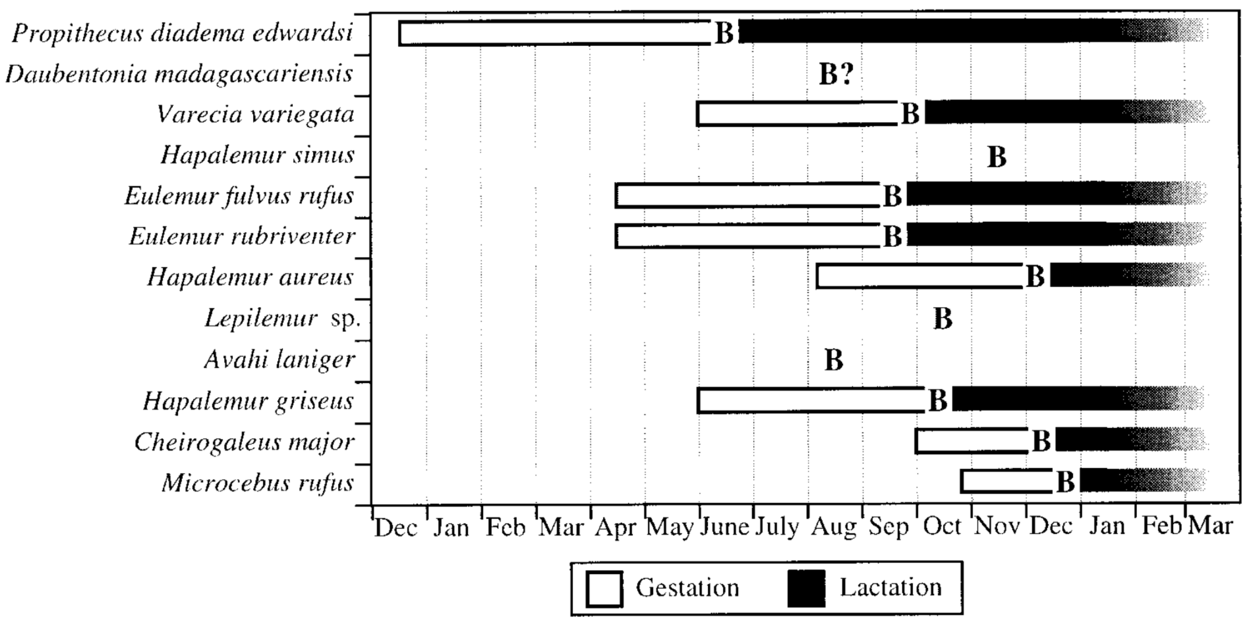
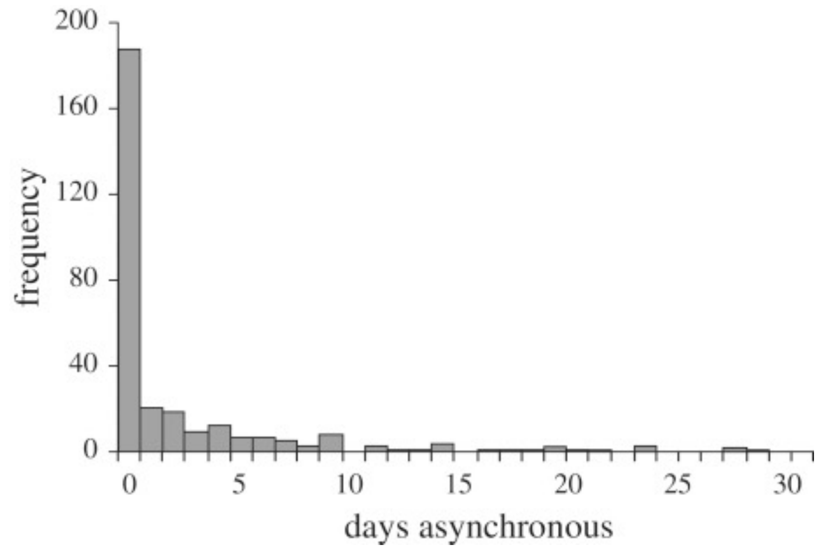


Wright 1999. *Am J of Phys Anthr.*

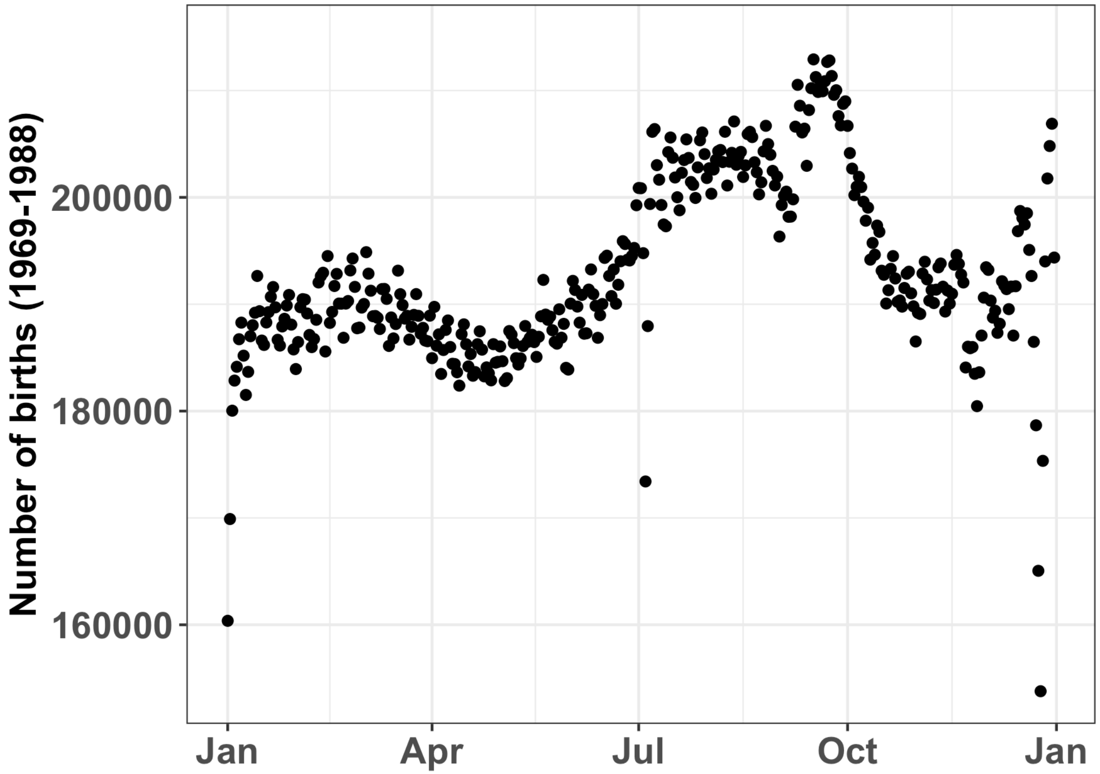
Is discrete time realistic for population growth?



64% of banded mongoose pups are born on the same day.

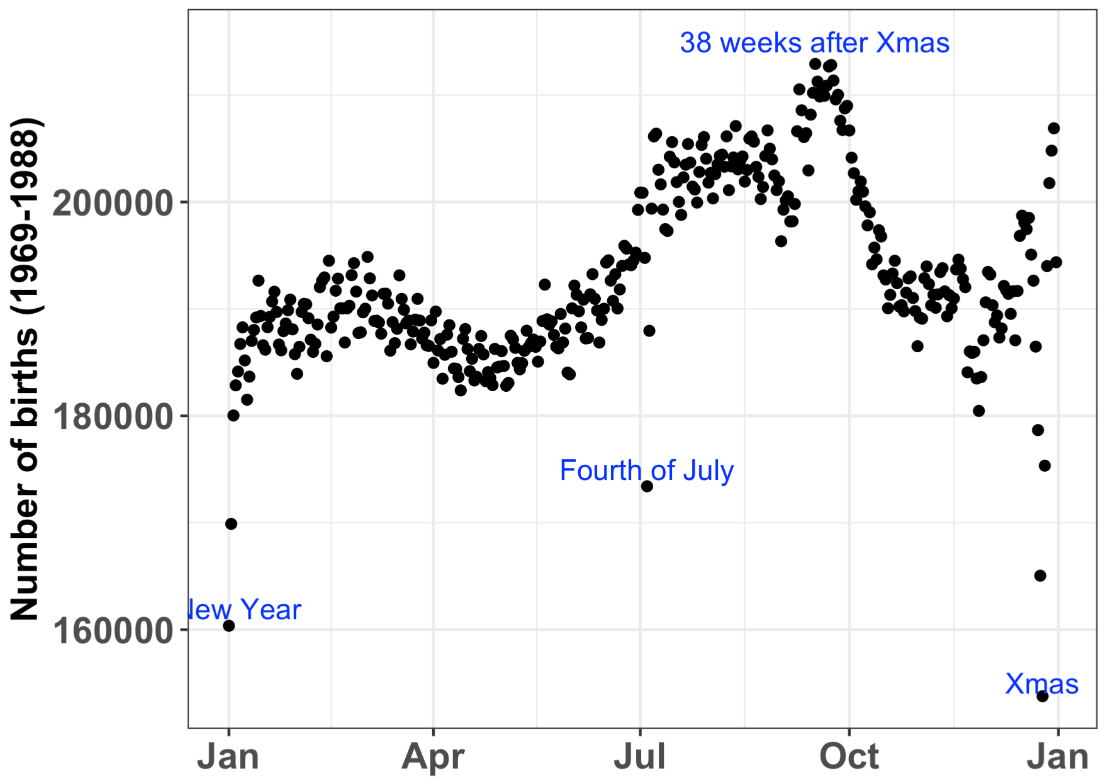
Hodge et al. 2011. *Biology Letters.*

Is discrete time realistic for population growth?



Human births by day in the US (1969-1988)

Is discrete time realistic for population growth?



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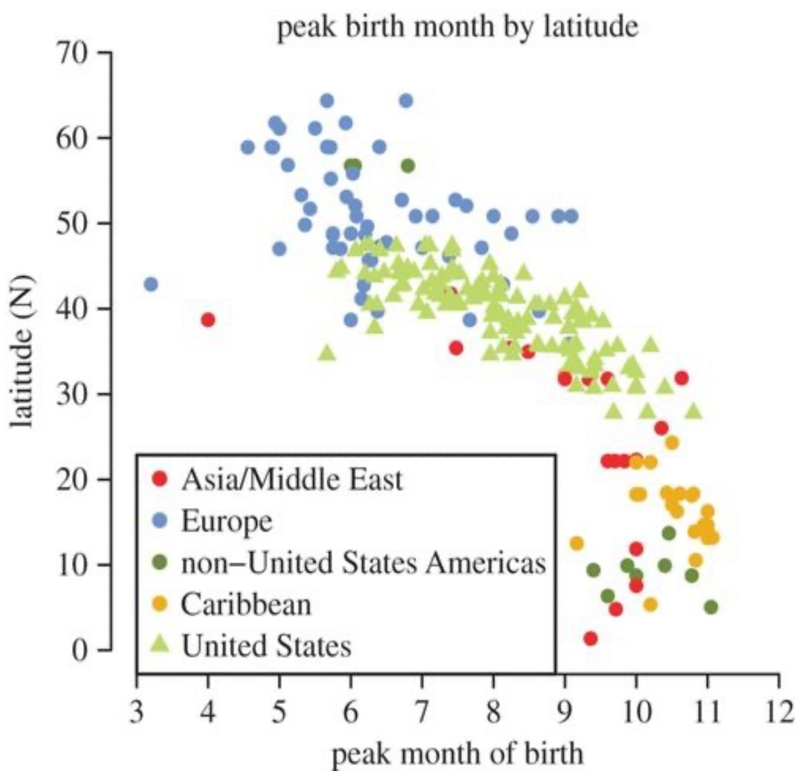


Many populations (including human) are better approximated by an assumption of a continuous birth rate

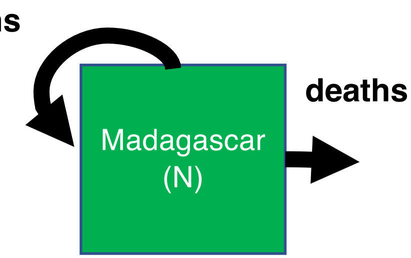
Martinez-Bakker et al. 2014. *Proc Roy Soc B.*

What are the main assumptions of a simple population model?

Closed population
Homogenous mixing
Same birth and death rate for each person

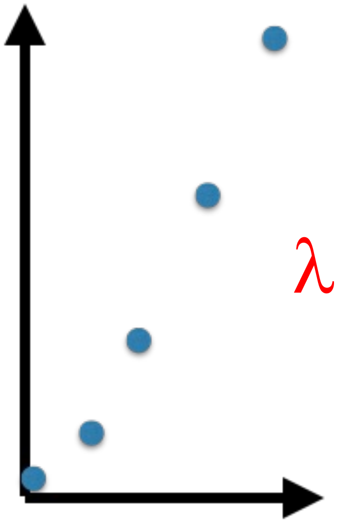
What is lambda?

Population intrinsic growth rate



Geometric vs. exponential growth

Geometric growth



$\lambda = N_{t+1}/N_t$

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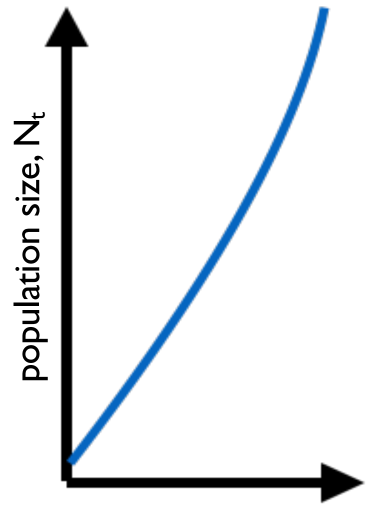
$N_t = \lambda^t N_0$

λ=population rate of increase

Exponential growth is measured in <u>continuous</u> time

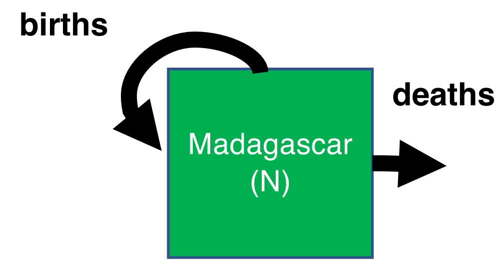


!as $t \to 0$

$$r = \frac{\ln\left(\frac{N_t}{N_0}\right)}{t}$$

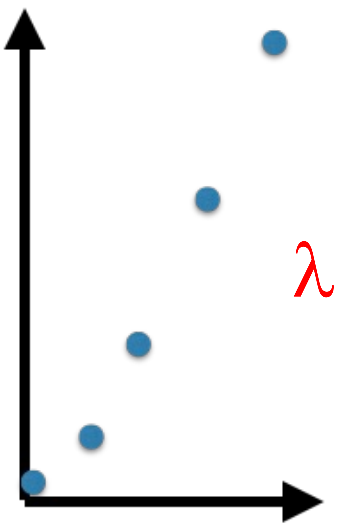
$dN(t)/dt = rN(t)$

r=intrinsic (instantaneous) rate of increase



Geometric vs. exponential growth

Discrete time



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Continuous time

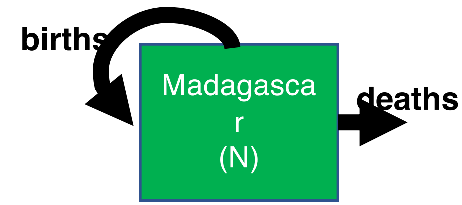
$dN(t)/dt = rN(t)$

Separation of variables:
$dN(t)/N(t) = r\,dt$

Integrate both sides:
$\int dN(t)/N(t) = \int r\,dt$

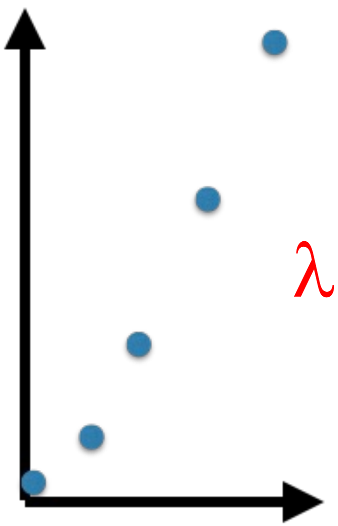
By definition:
$\log(N(t)) = rt + c$

Take exponentials:
$N(t) = e^{rt+c} = Ce^{rt}$
$N(t) = N(0)e^{rt}$



Geometric vs. exponential growth

Discrete time



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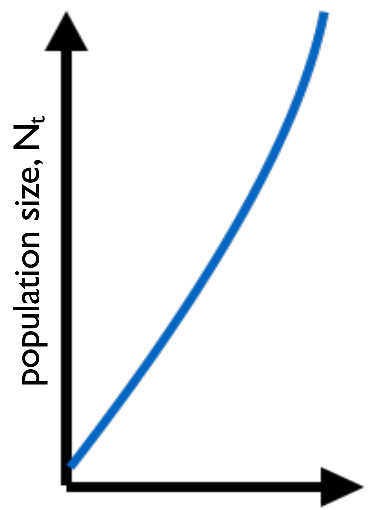

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$N_t = N_0 e^{rt}$

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(pop grows @ λ >1 & declines @ λ < 1)

r=intrinsic (instantaneous) rate of increase
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Geometric vs. exponential growth

geometric
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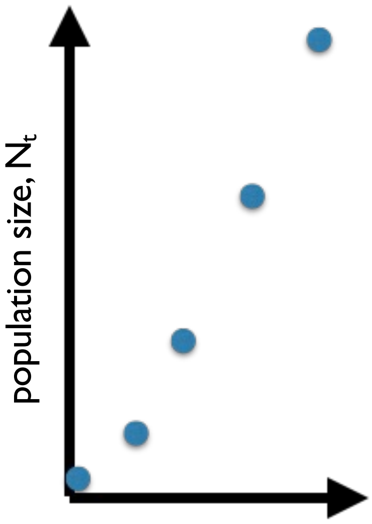
exponential
$N_t = N_0 e^{rt}$

Discrete time



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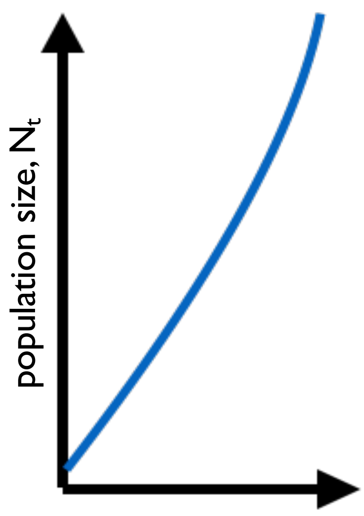

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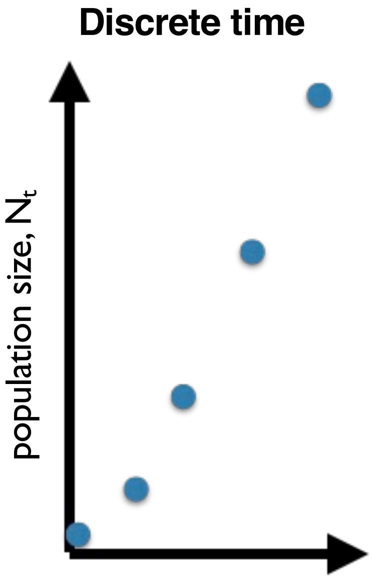

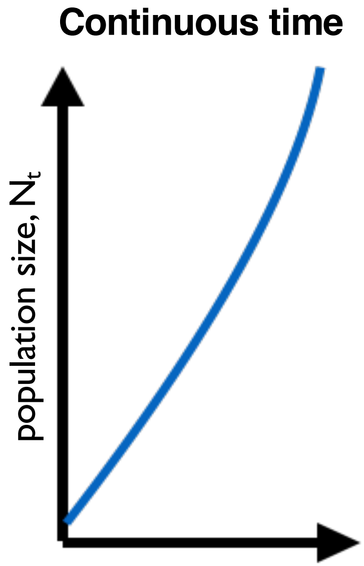

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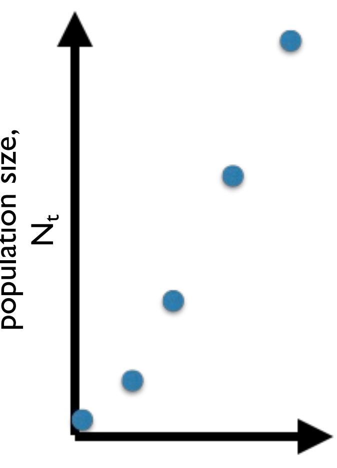

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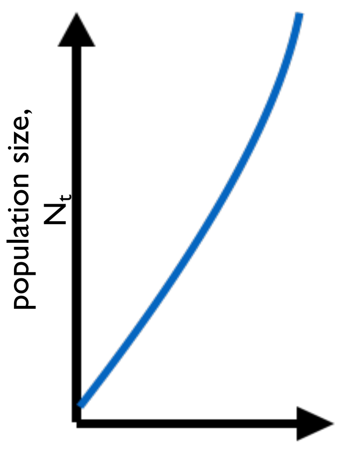

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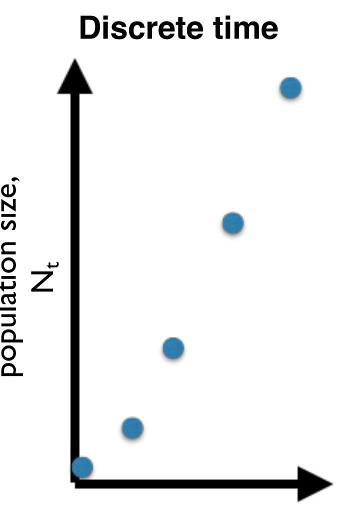

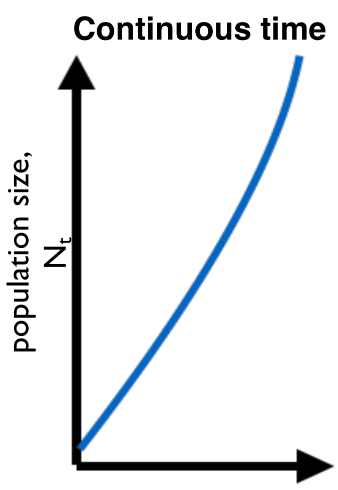

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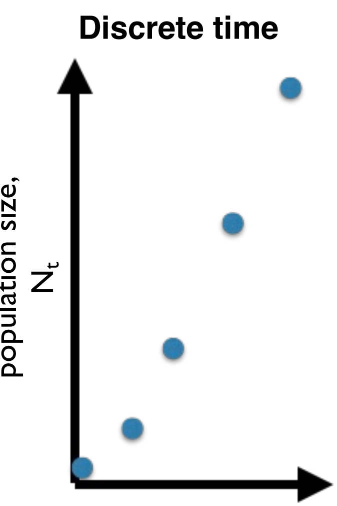

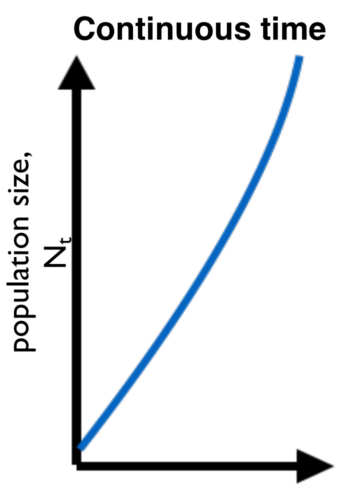

Continuous models can be discretized.

Discrete models can be approximated in continuous time.

Geometric vs. exponential growth

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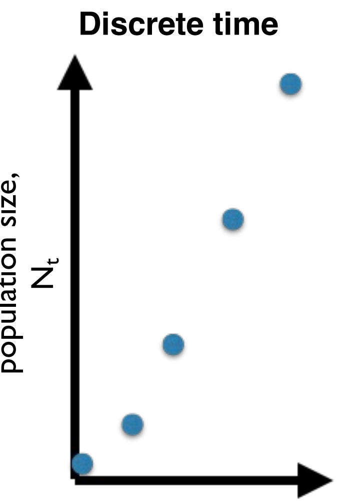

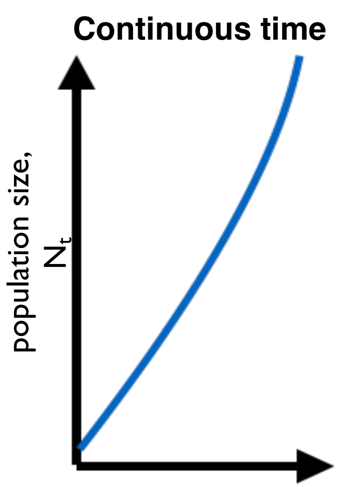

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How to choose what to do?

Geometric vs. exponential growth

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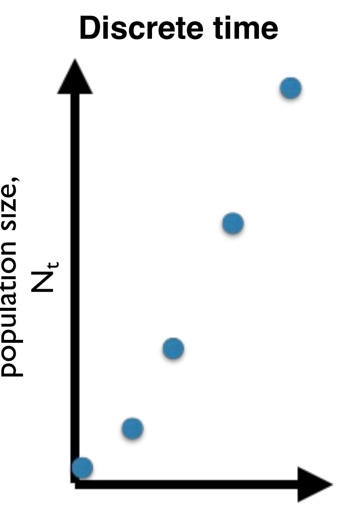
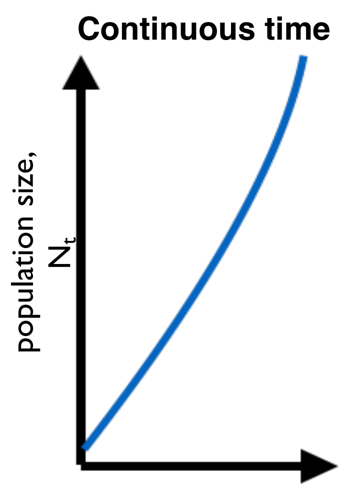




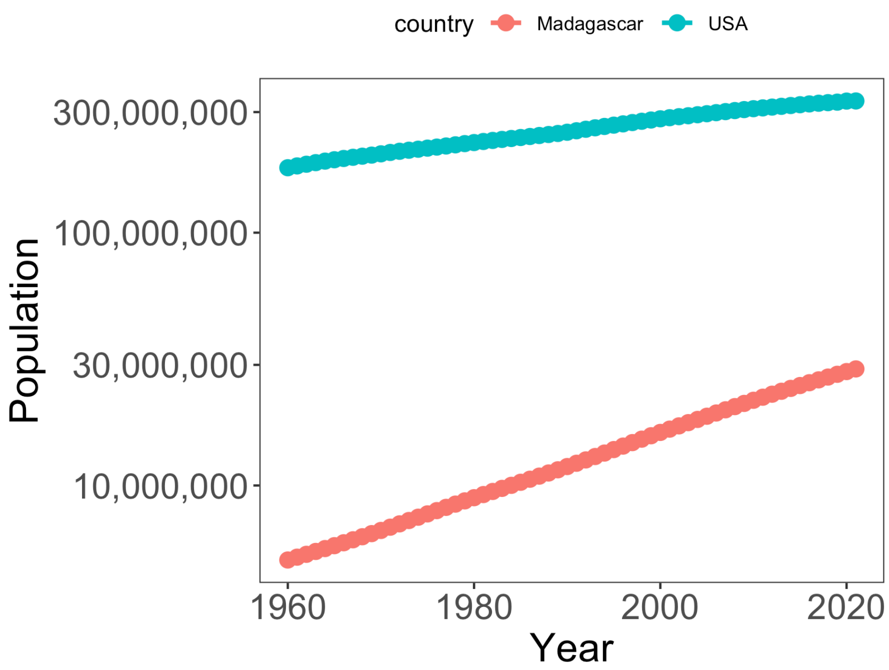
Continuous models can be discretized.

Discrete models can be approximated in continuous time.

How to choose what to do?

The answer depends on the data and the question at hand!

Geometric vs. exponential growth



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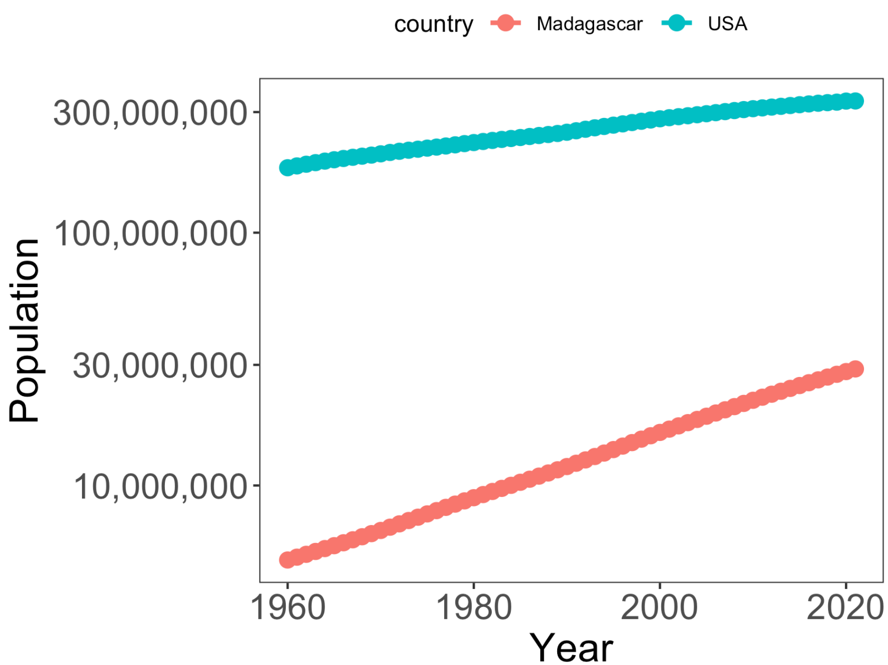
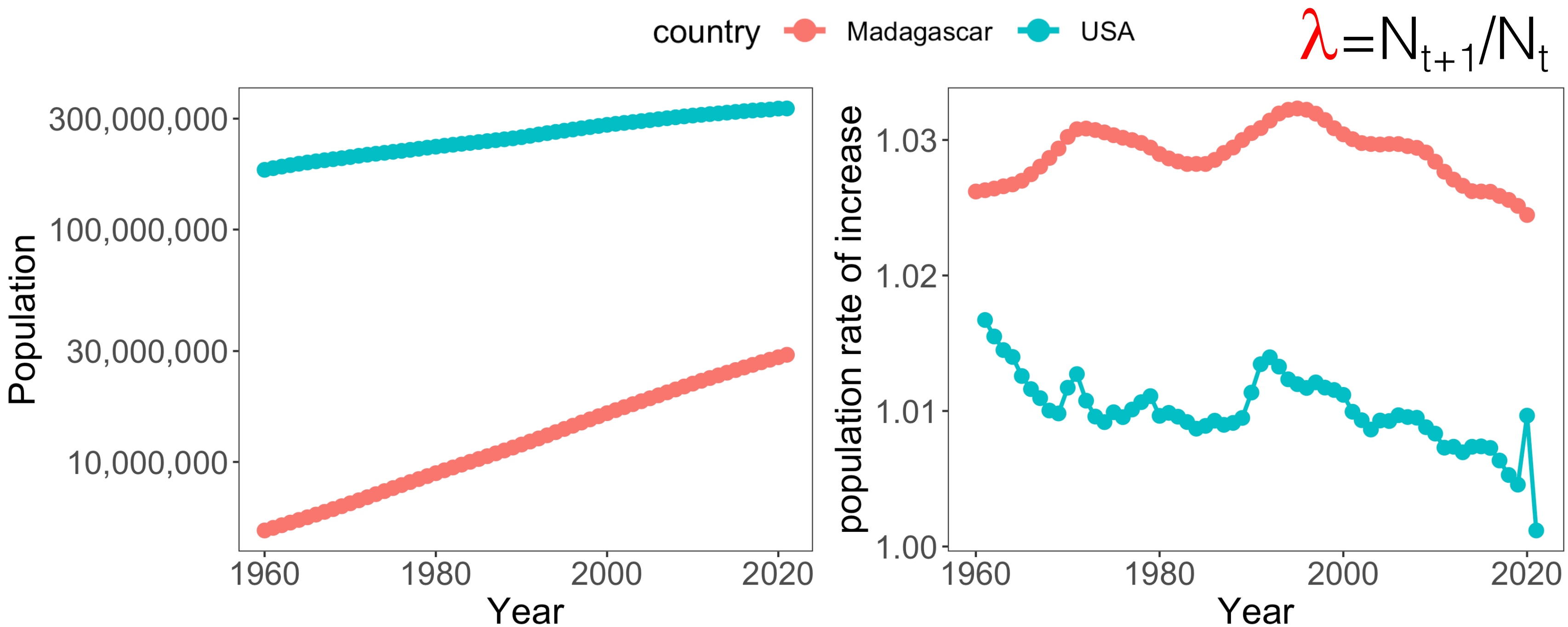


$$\lambda = N_{t+1}/N_t$$

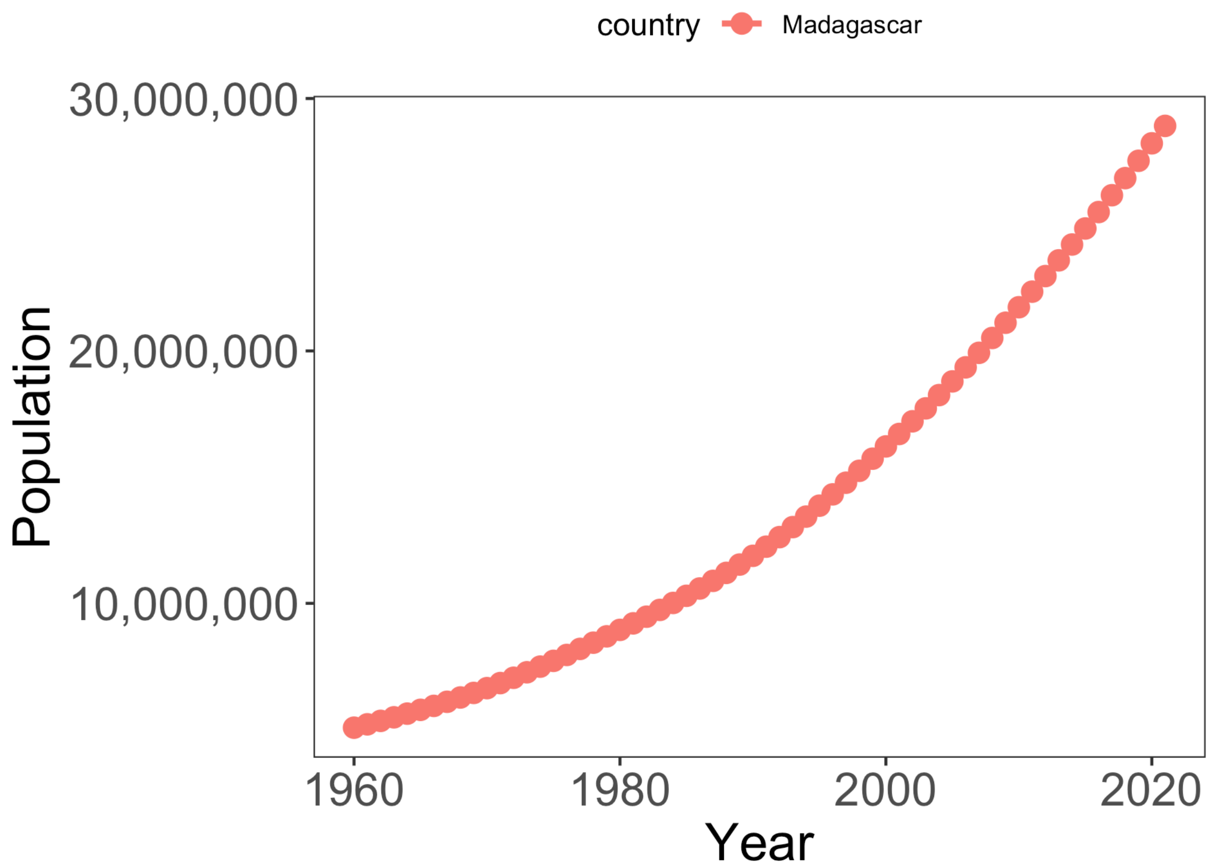
where t = 1 year

Which country has the higher growth rate?

Geometric vs. exponential growth



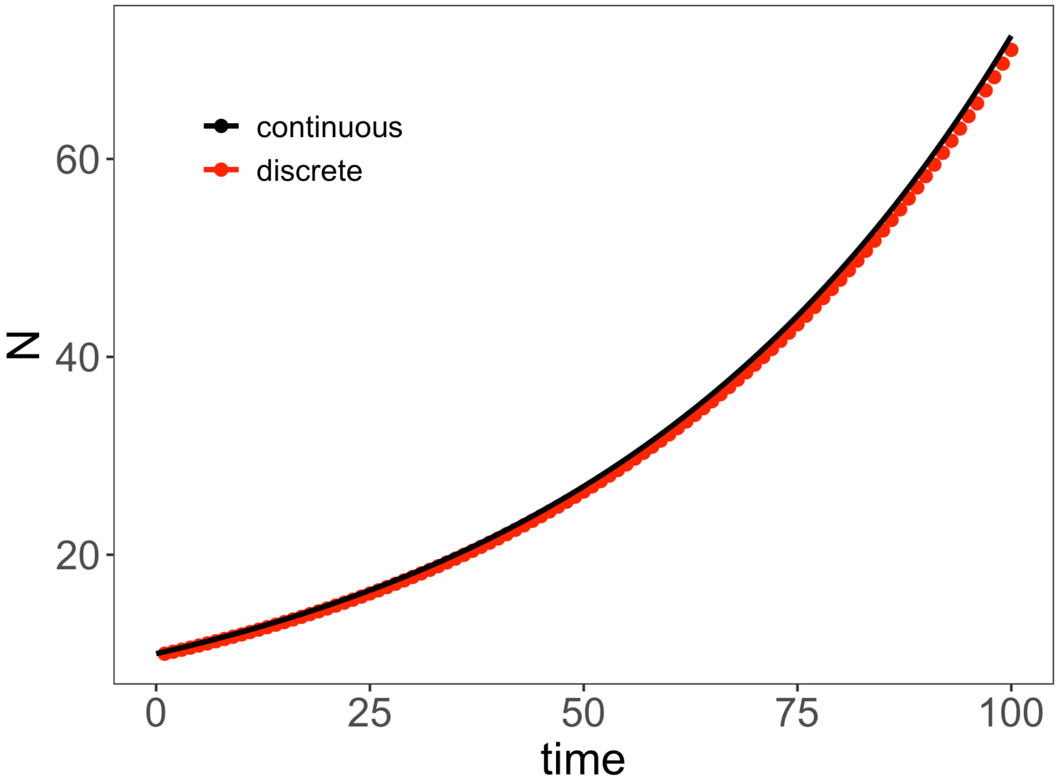
Geometric growth approximation of Madagascar’s population



$\lambda = N_{t+1}/N_t$

mean λ = 1.02

Projecting population under geometric vs. exponential growth



Geometric: λ = 1.02
Exponential: r = log(1.02) = 0.02

Models are similar over short time horizons and/or when discrete timesteps are small!

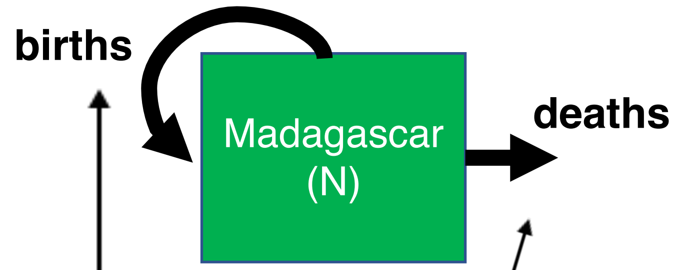
What is the difference between discrete and continuous models?

Discrete: state variable only changes at distinct **geometric** time steps
Continuous: state variables change **continuously** (tiny tiny time steps)

What math is used in discrete pop models? Continuous pop models?

Algebra, calculus

The basic population model

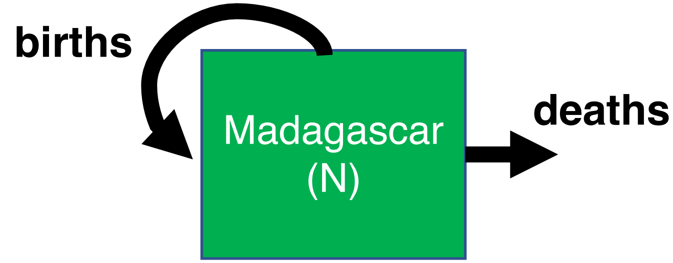


What about those rates?
Are they the same every year?
And in every person? Why might they be different?

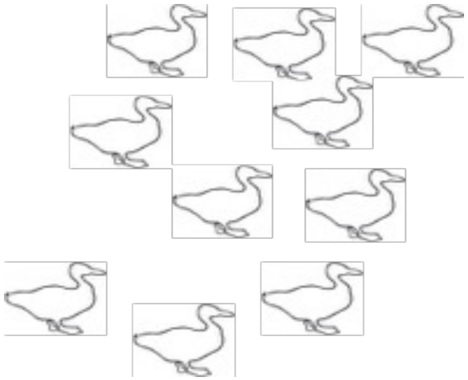
Reproductive age
Death rate increasing with age
Diseases/other health factors

How do we incorporate this ‘randomness’?

The basic population model



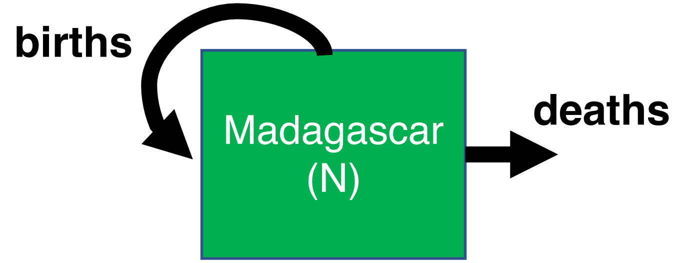
starting population



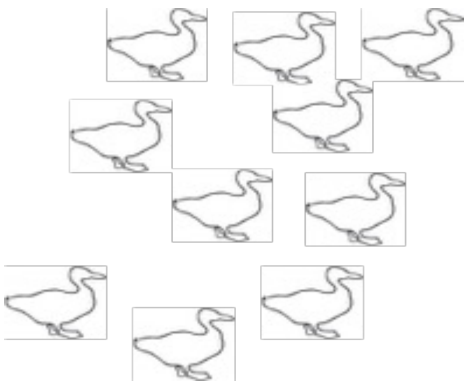
probability of death = 0.5

if deterministic “always the same”

The basic population model



starting population



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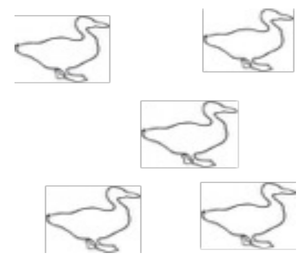

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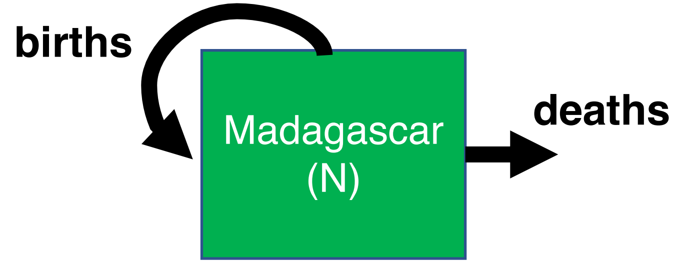

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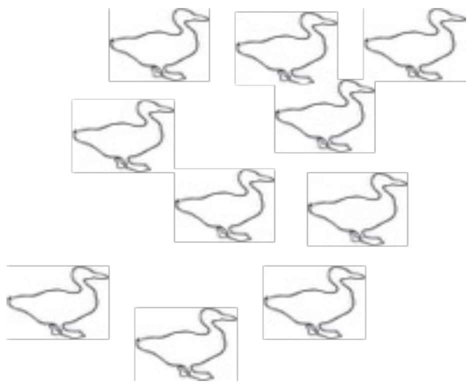

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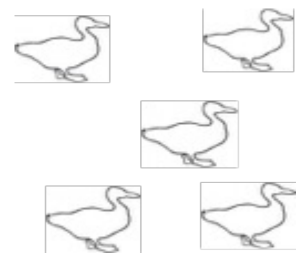

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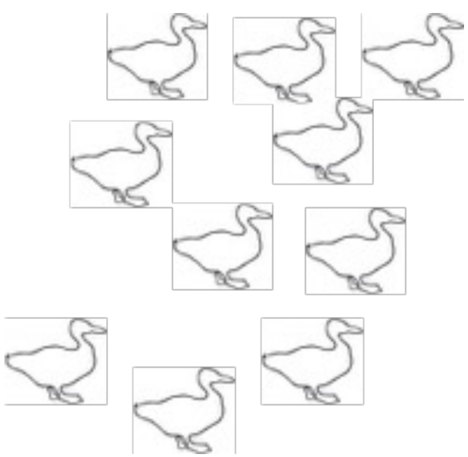

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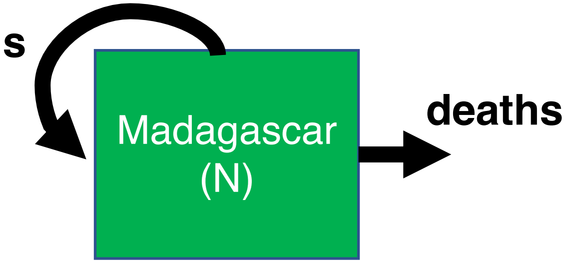

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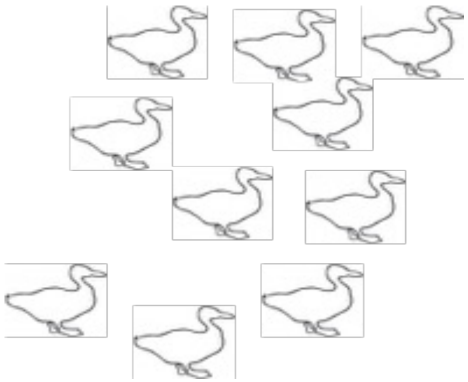

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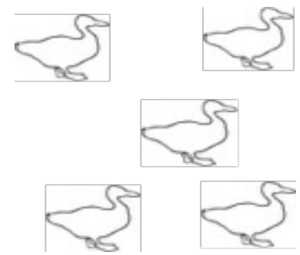

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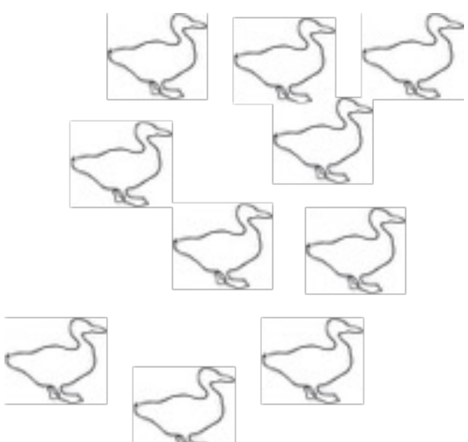

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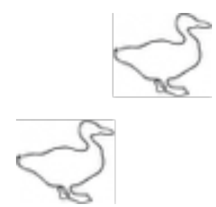

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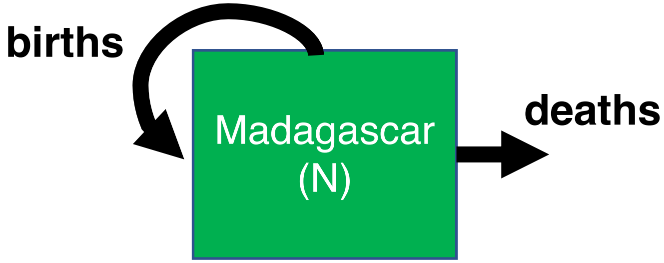

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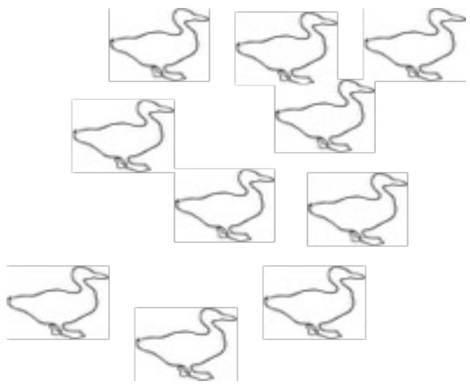

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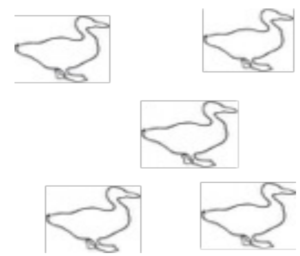

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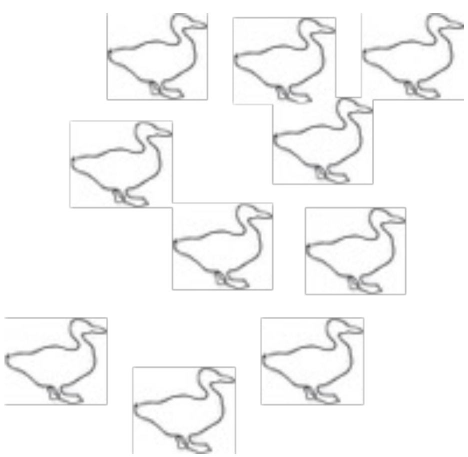

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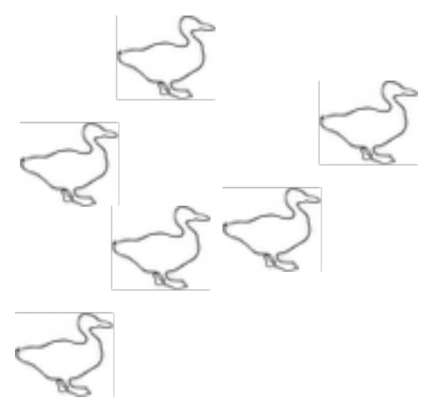

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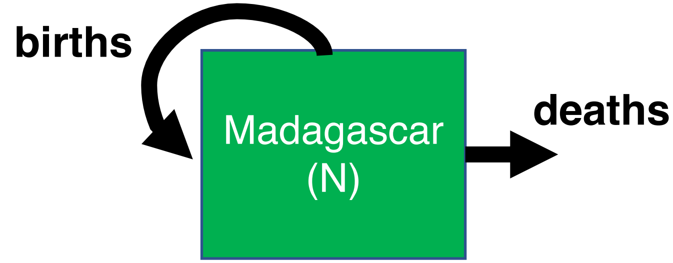

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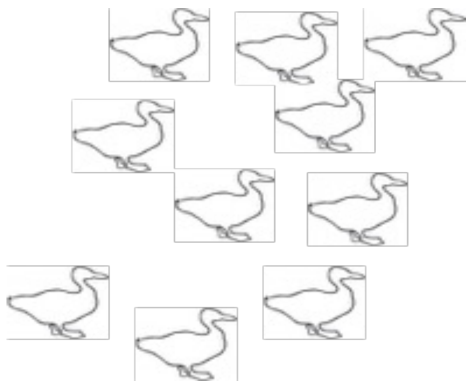

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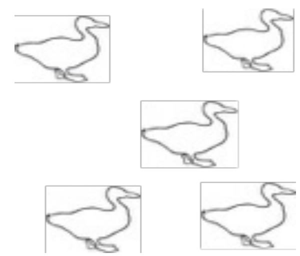
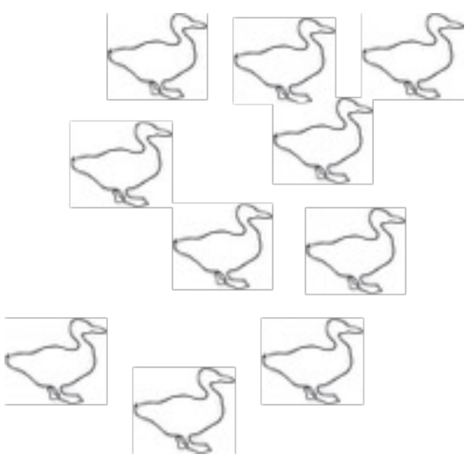


rbinom(200,10,0.5)

starting population



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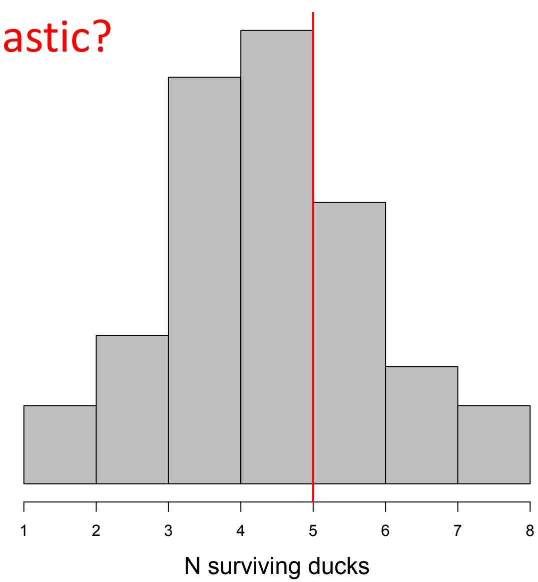

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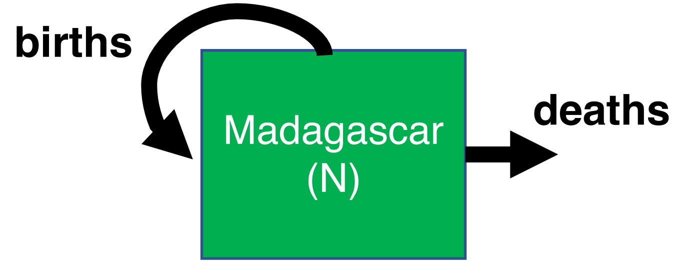

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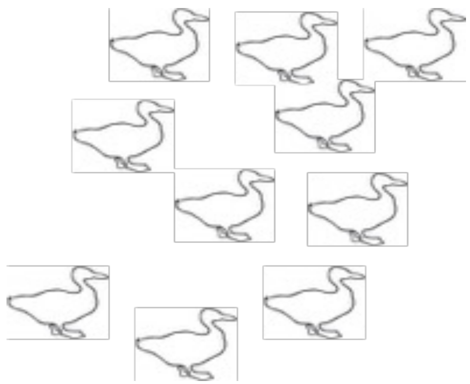

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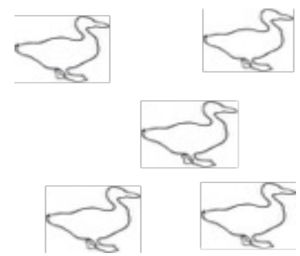
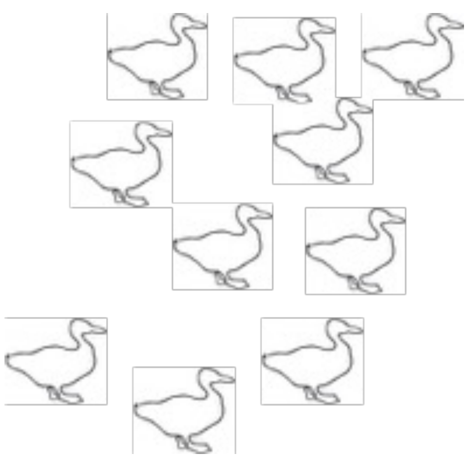


If you test your 10 ducks many times, on average you get 5

rbinom(200,10,0.5)

starting population



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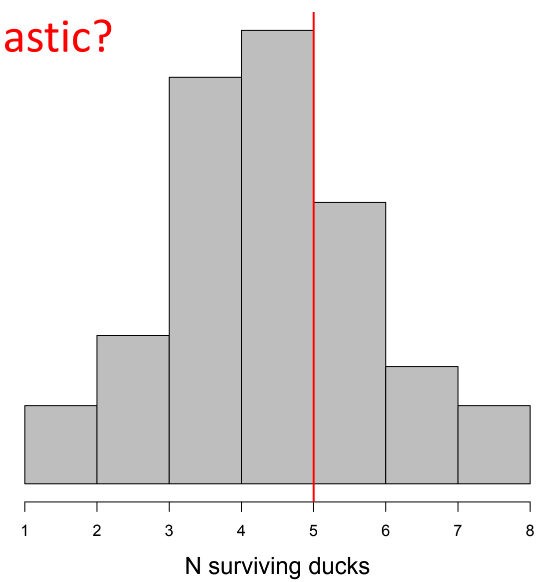

What is the difference between deterministic and stochastic?

Deterministic = always the same
Stochastic = up to chance

Key concepts

- Compartmental/mechanistic/mathematical models

  Modèles en compartiments

- Continuous vs. discrete models

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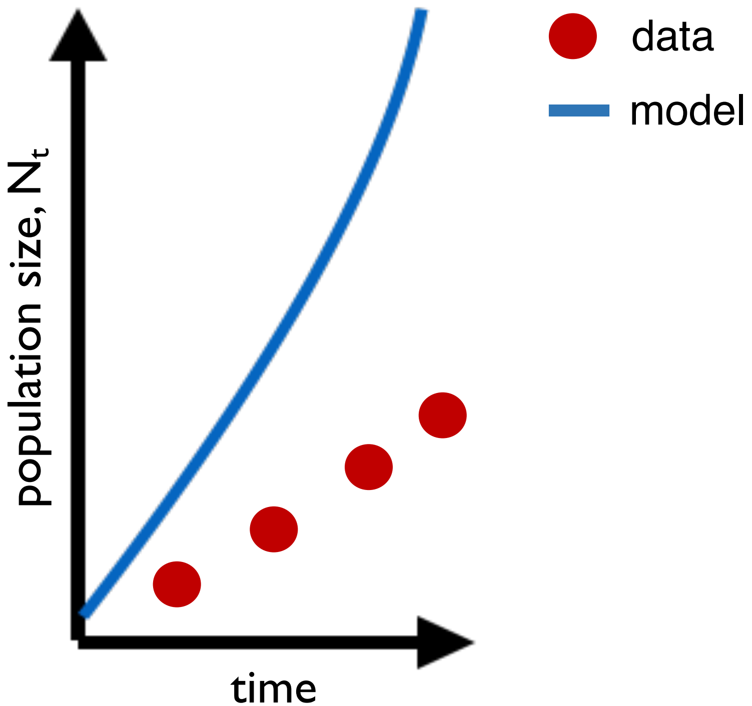
2. Structured Population Models
2. Modèles de la population structurée

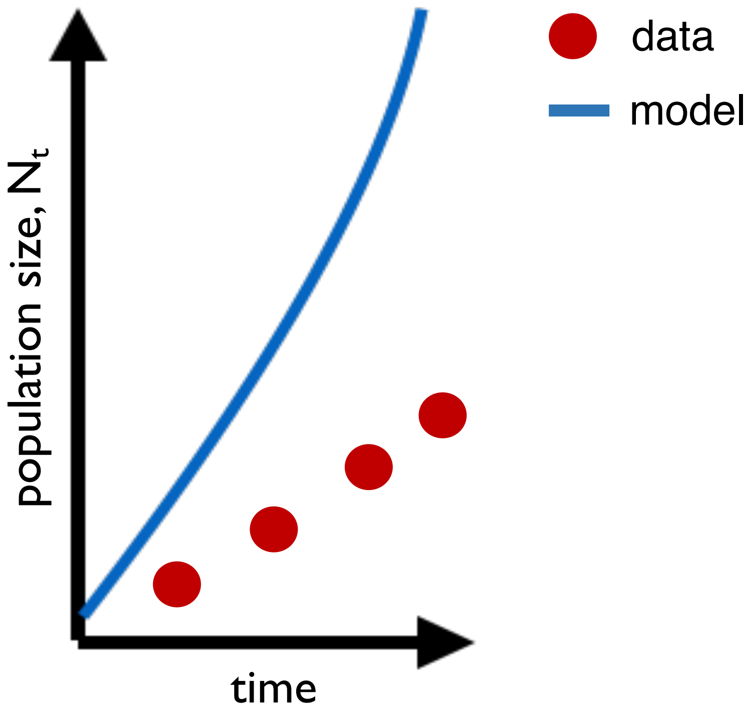
The structured population model



Why does the model perform poorly?

The basic population model

Why does the model perform poorly?



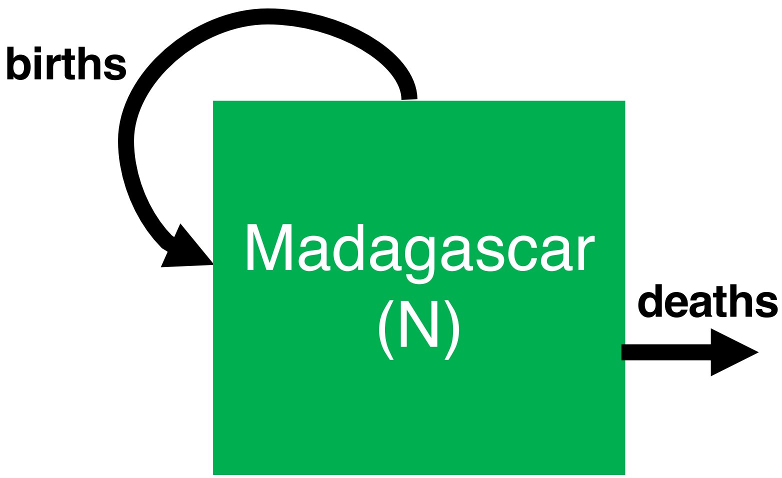
We need population structure!

That means distinguishing babies from adults.

Modeling demographic complexity

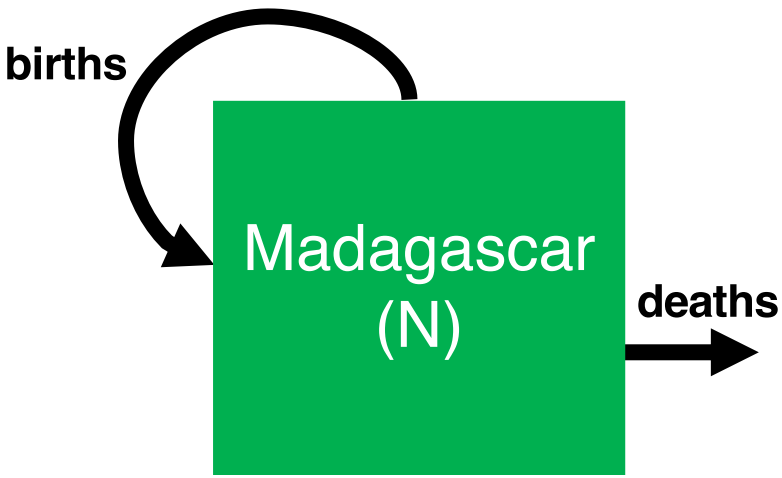
Modeling demographic complexity

1. Populations are divided into compartments
2. Individuals within a compartment are homogenously mixed
3. Compartments and transition rates are determined by biological systems
4. Rates of transferring between compartments are expressed mathematically



The simplest population model

Modeling demographic complexity



What is wrong about this model?

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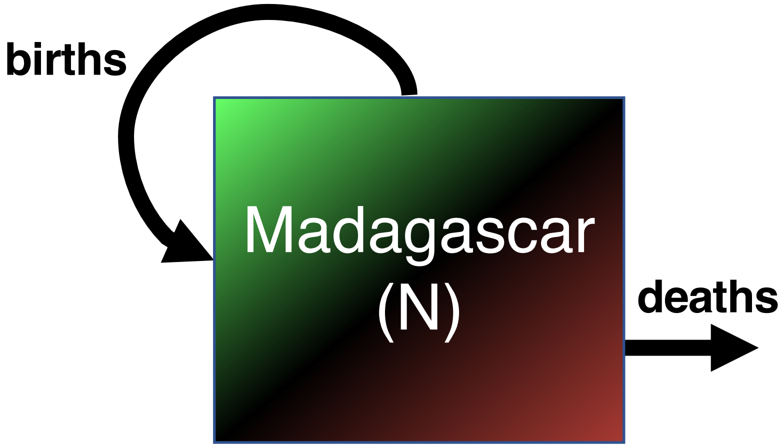
Modeling demographic complexity



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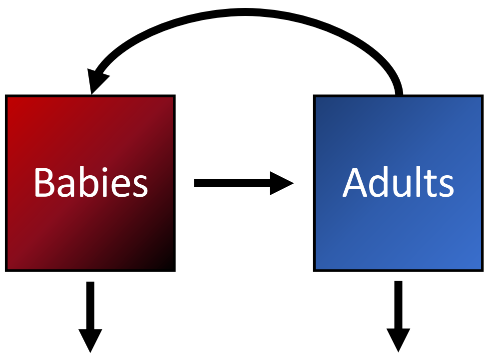

The structured population model

Modeling demographic complexity

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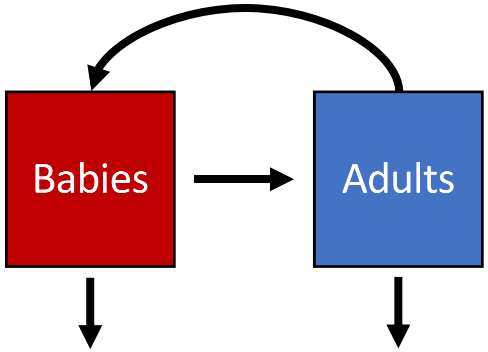

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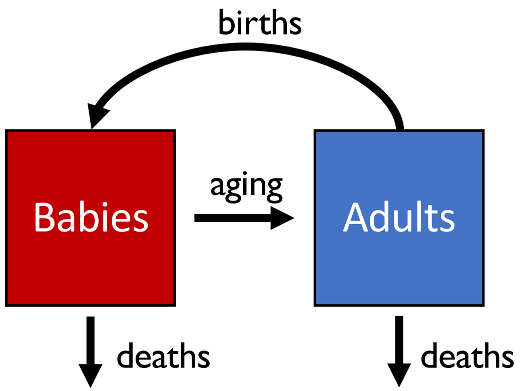


The structured population model

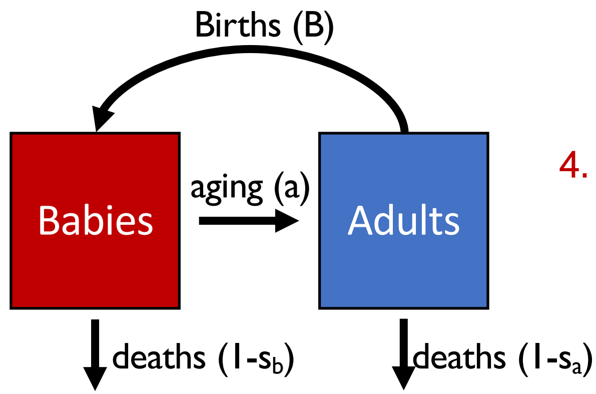
The structured population model

Population rate of increase

$$\lambda = N_{t+1}/N_t$$

pop size at t

pop size at t + 1



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$$N_{t+1} = A^* \, N_t$$

matrix of survival and fecundity rates

vector of population sizes

$s_b(1-a)$	B
$s_b a$	s_a

The structured population model

Population rate of increase

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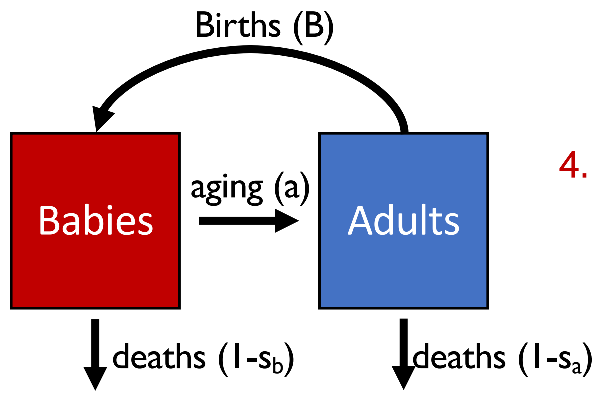

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matrix of survival and fecundity rates

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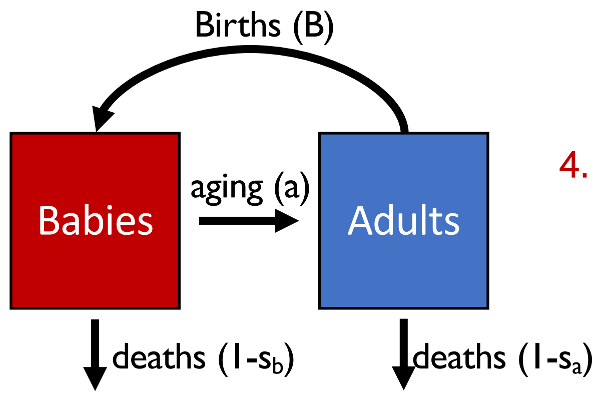
***Discrete time**

The structured population model



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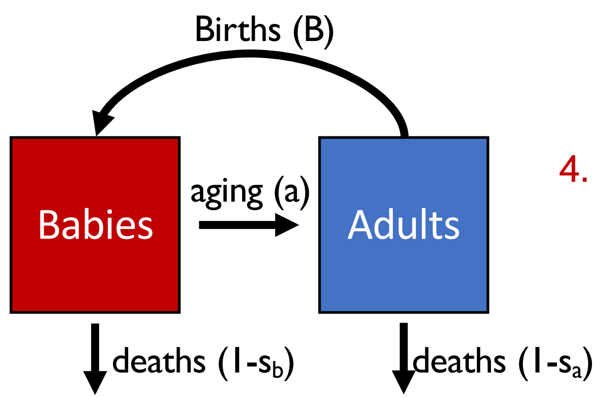
$$N_{t+1} = A^{*}\ N_t$$

$$\underset{\mathbf{A}}{\begin{bmatrix} s_b(1-a) & B \\ s_b a & s_a \end{bmatrix}} \times \underset{\mathbf{N_t}}{\begin{bmatrix} N_b \\ N_a \end{bmatrix}} = \underset{\mathbf{N_{t+1}}}{\begin{bmatrix} s_b(1-a)\ N_b + BN_a \\ s_b\, a\ N_b + s_a N_a \end{bmatrix}}$$

The structured population model



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$$N_{t+1} = A^* \, N_t$$

Population rate of increase

λ=dominant eigenvalue of transition matrix →

$$\overset{\mathbf{A}}{\begin{bmatrix} s_b(1-a) & B \\ s_b a & s_a \end{bmatrix}} \times \overset{\mathbf{N_t}}{\begin{bmatrix} N_b \\ N_a \end{bmatrix}} = \overset{\mathbf{N_{t+1}}}{\begin{bmatrix} s_b(1-a)\,N_b + BN_a \\ s_b\,a\,N_b + s_a N_a \end{bmatrix}}$$

Population growth will depend on population structure!

(pop grows @ λ >1 & declines @ λ < 1)

Key concepts

- Compartmental/mechanistic/mathematical models

  Modèles en compartiments

- Continuous vs. discrete models

  Modèles en temps continue vs. modèles en temps discrète

- Deterministic vs. stochastic models

  Modèles détérministique vs. stochastique

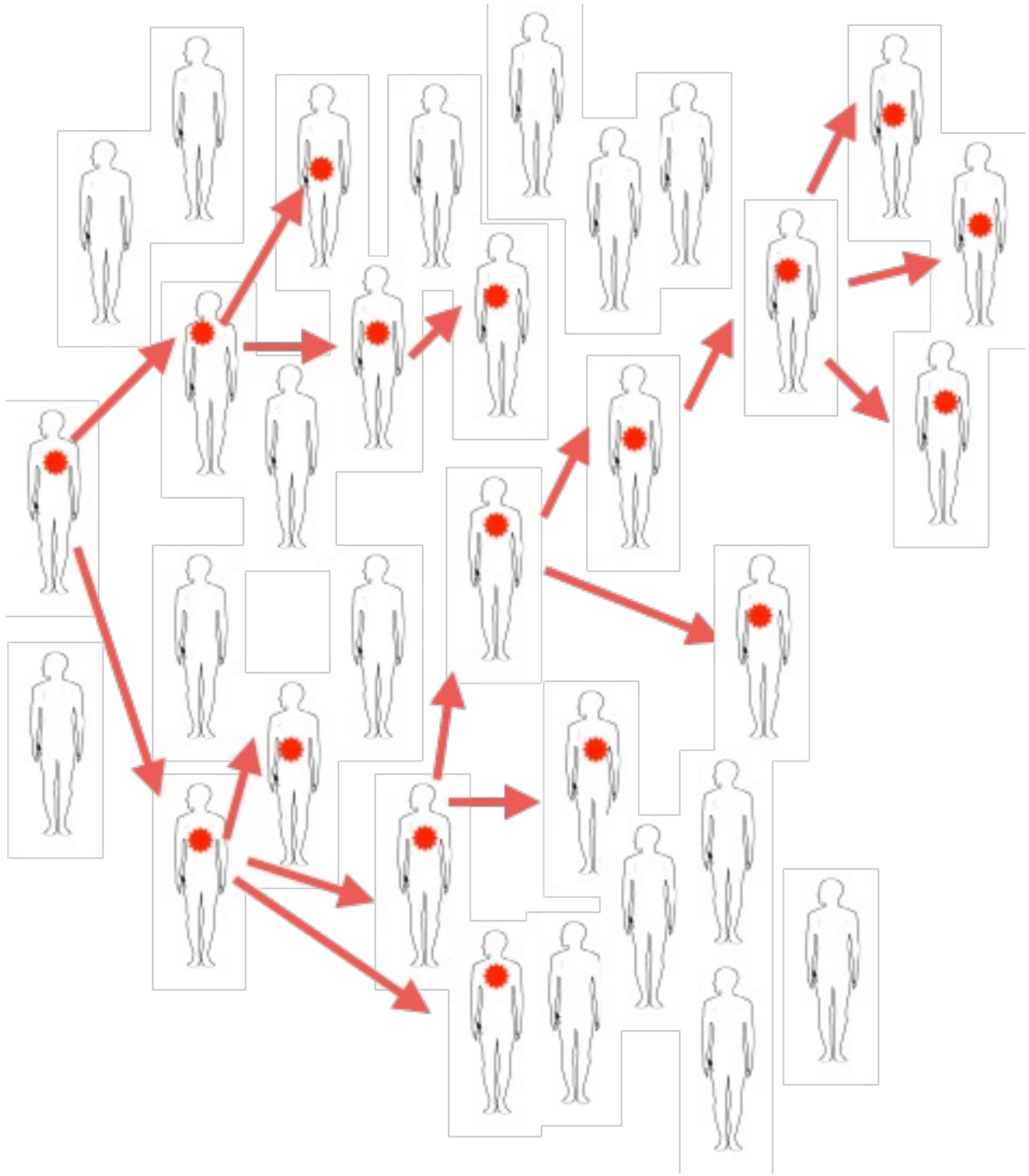
- Structured models

  Modèles structurés.

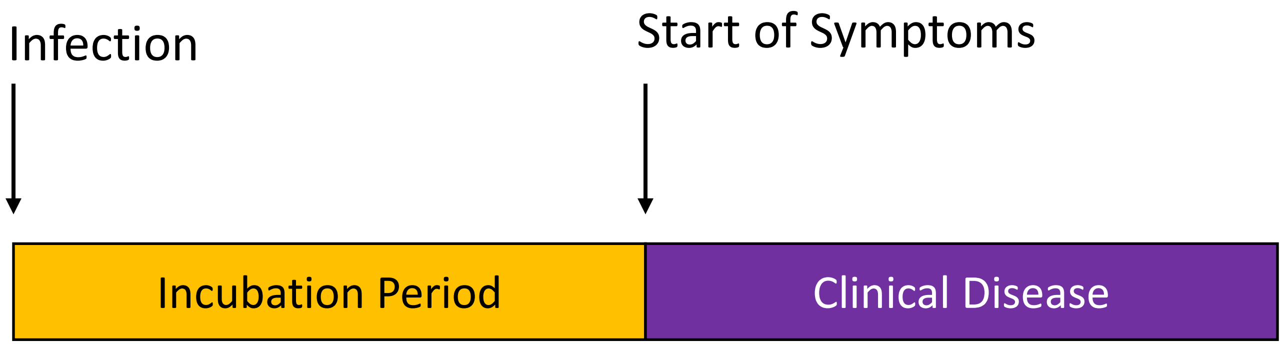
How do we modify a basic population model to make it structured?

- Two compartments (adults and babies)
- Vector/matrix of values

3. SIR Models
3. Les modèles SIR



Infectious diseases: Some background terminology



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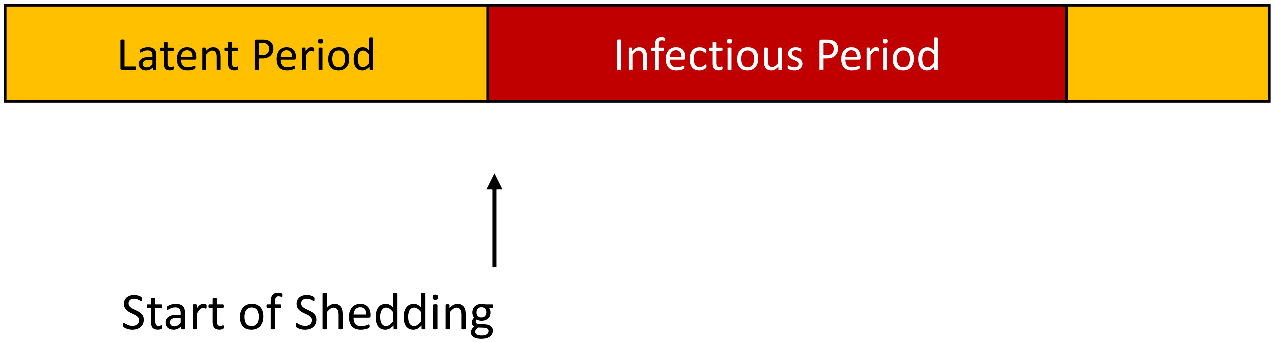

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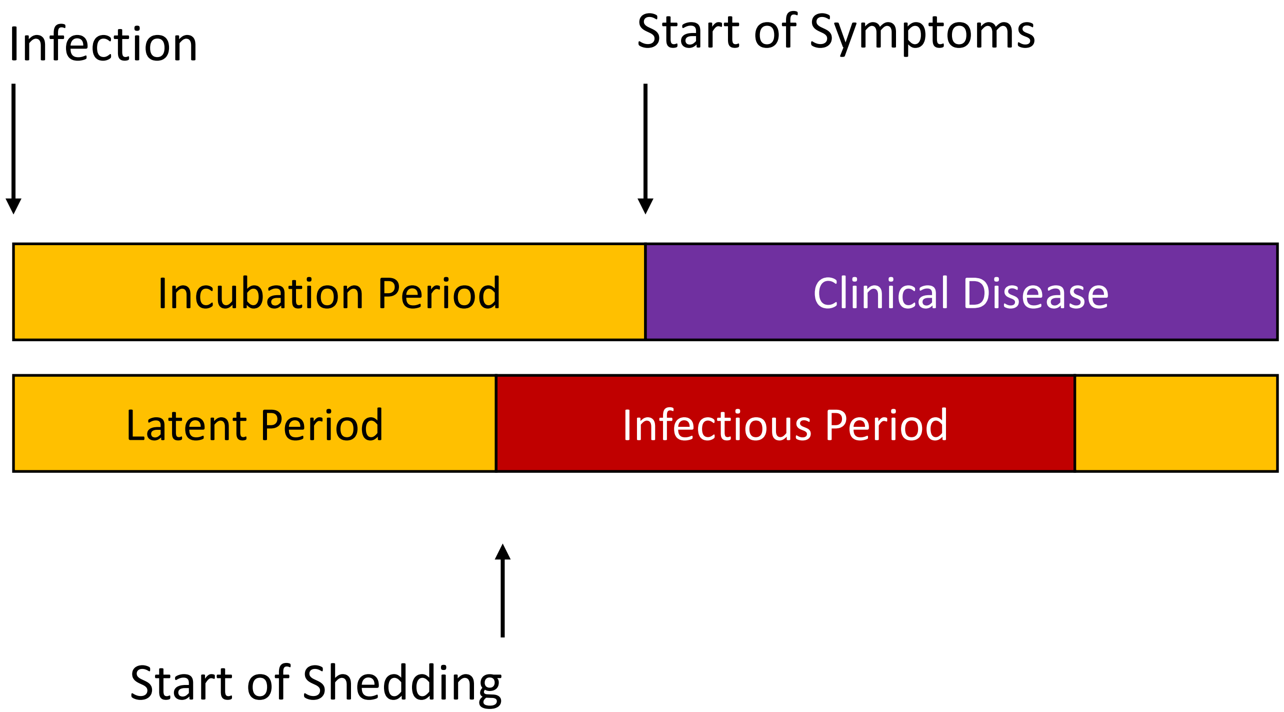

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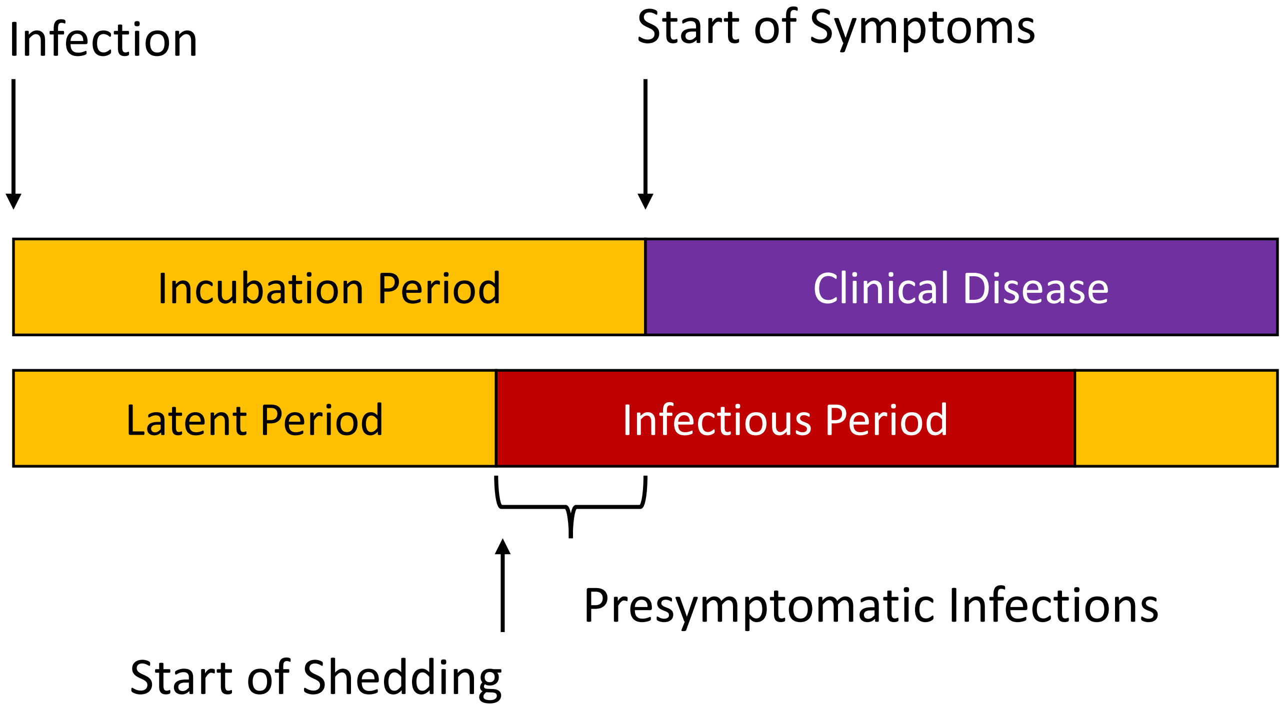

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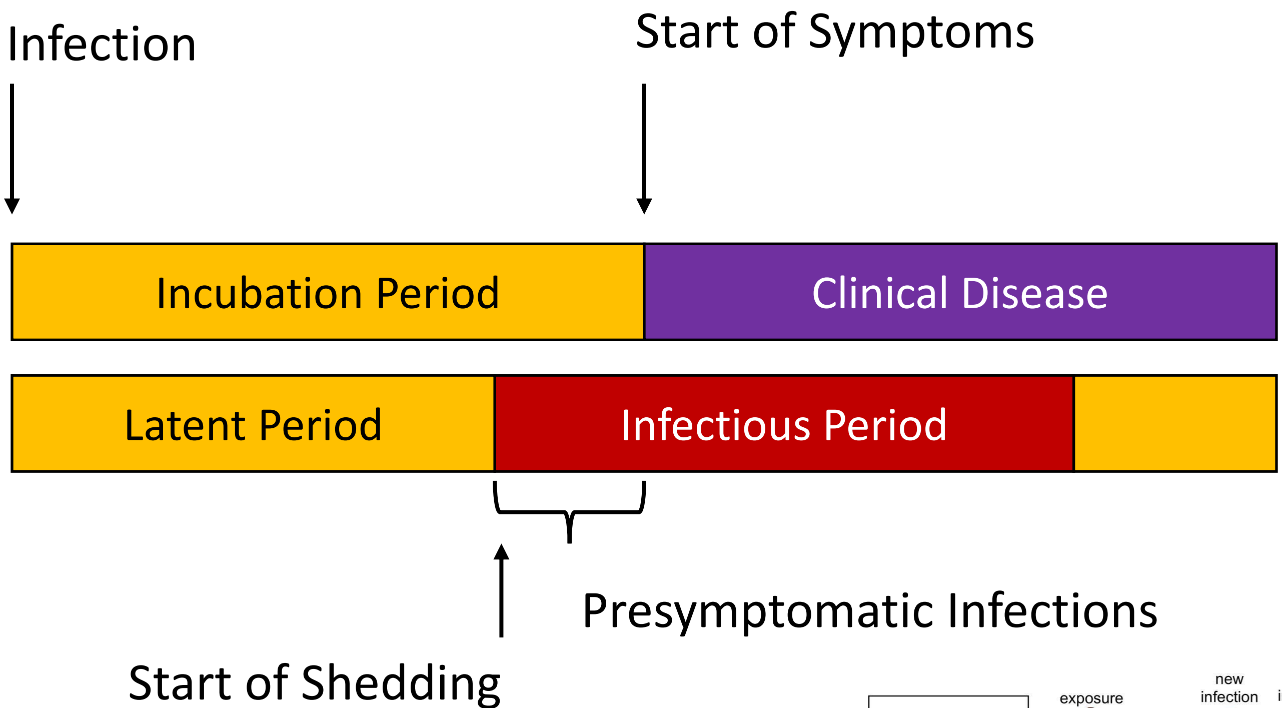
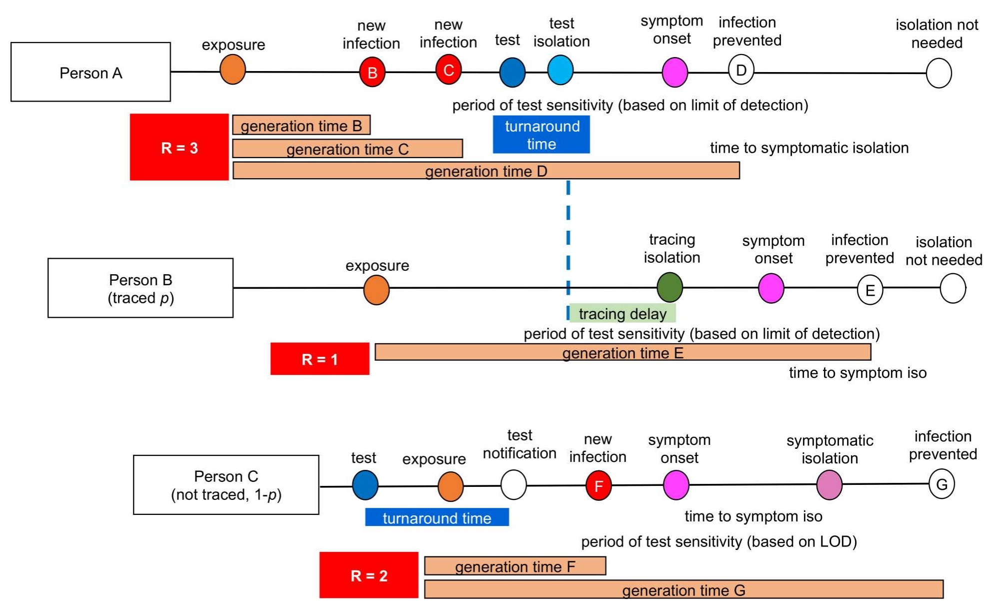
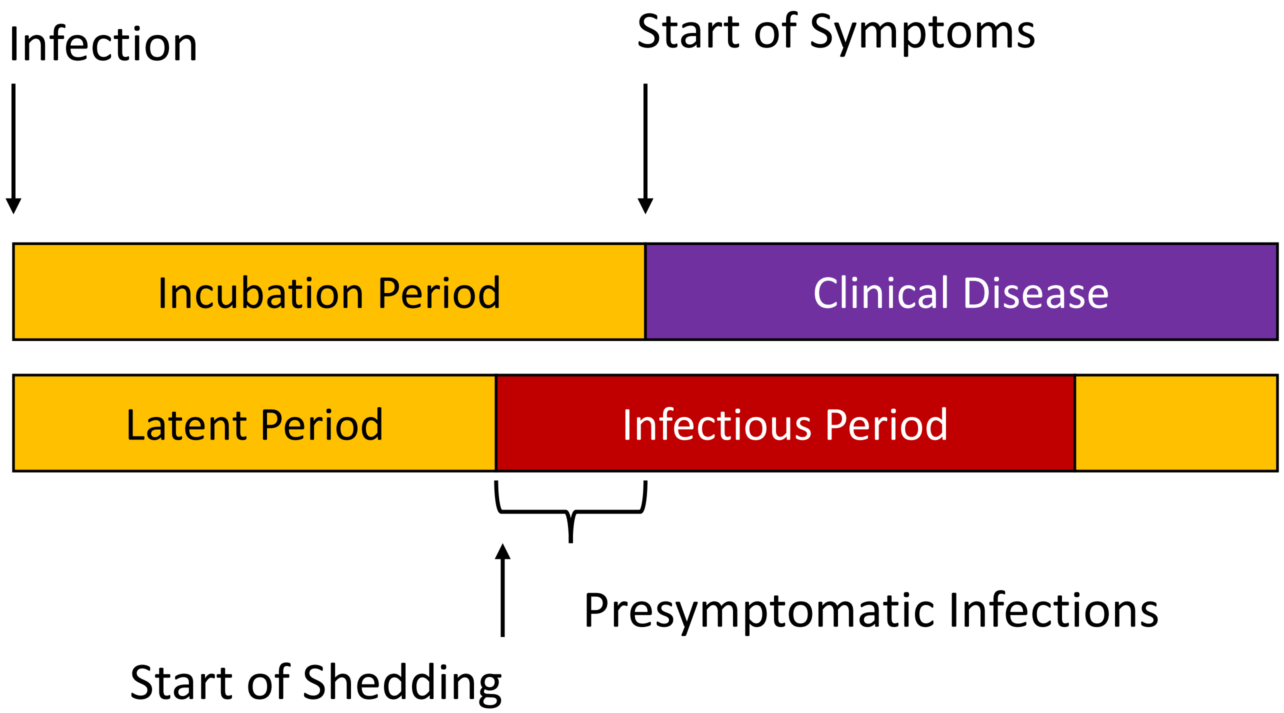


We used a model to understand the impact of surveillance testing on COVID transmission in a university setting.



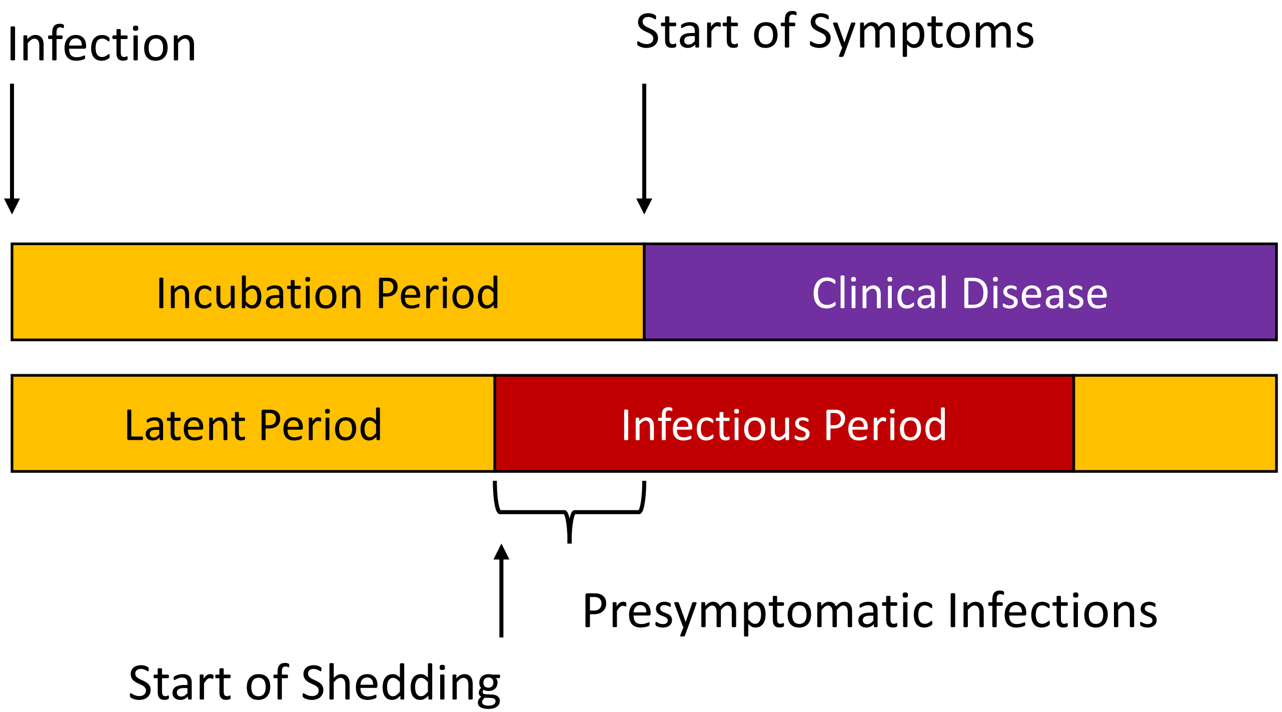
Brook et al. 2021. *Epidemics.*

Infectious diseases: Some background terminology



Let's take a step back and think about acute, immunizing infections.

Infectious diseases: Some background terminology



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pathogen lifespan < < < host lifespan

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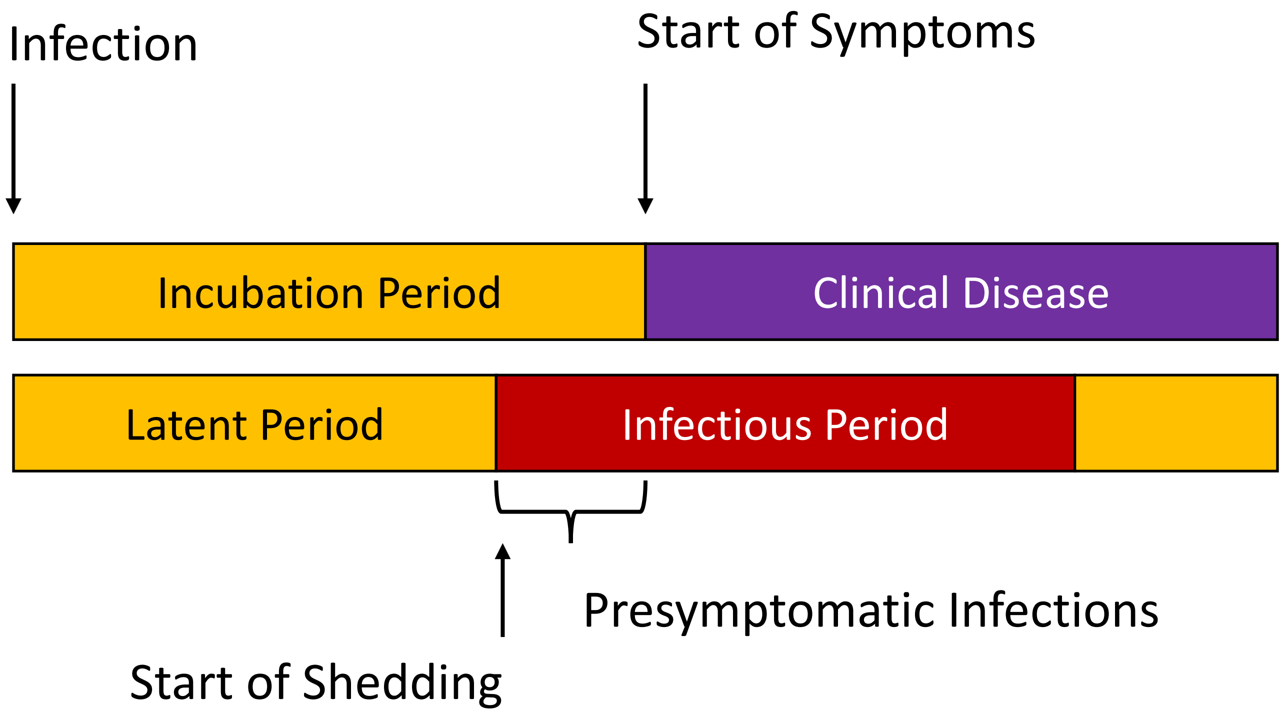

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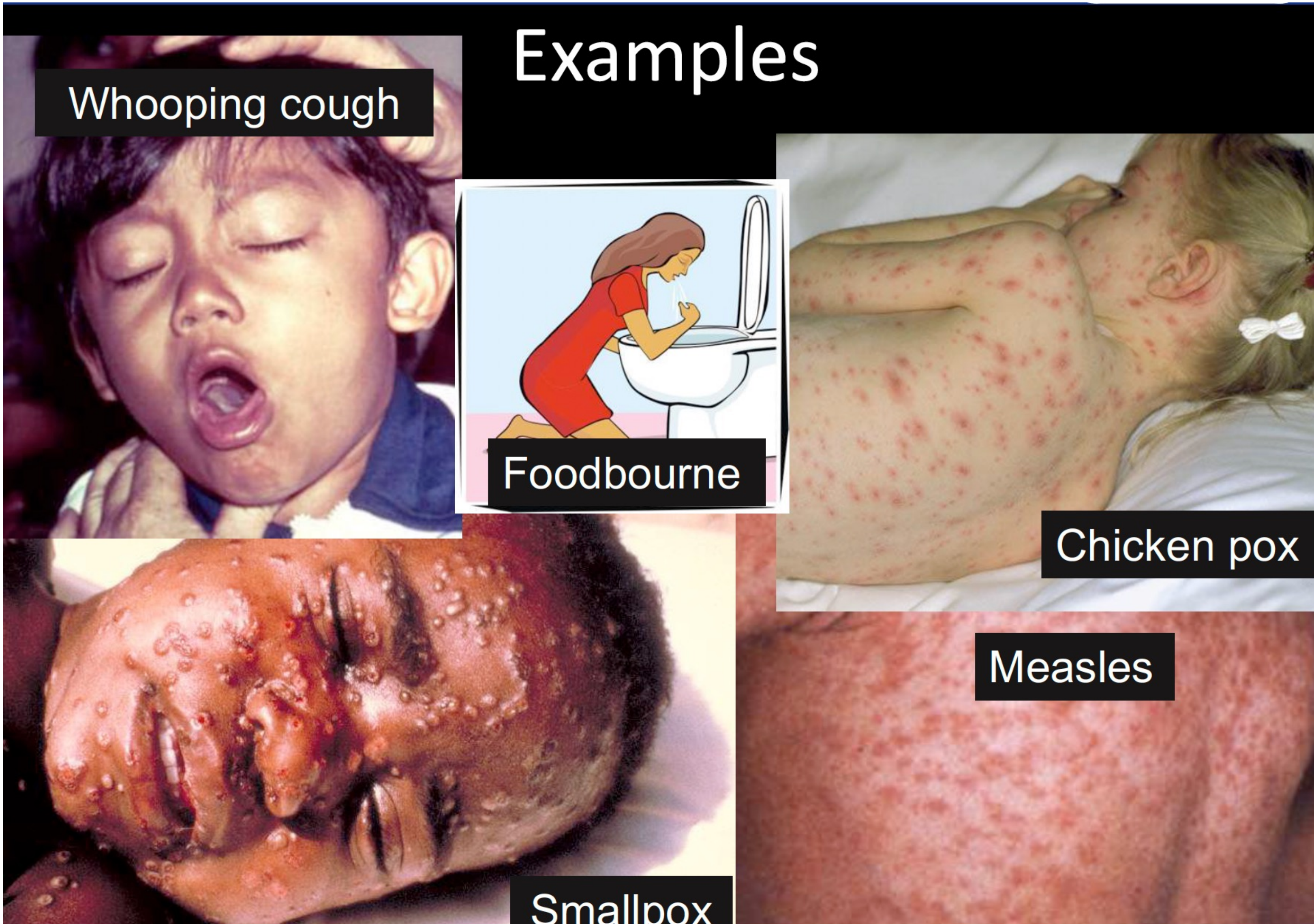
pathogen lifespan < < < host lifespan

infection induces antibody responses that prevent another infection

Acute, Immunizing Infections

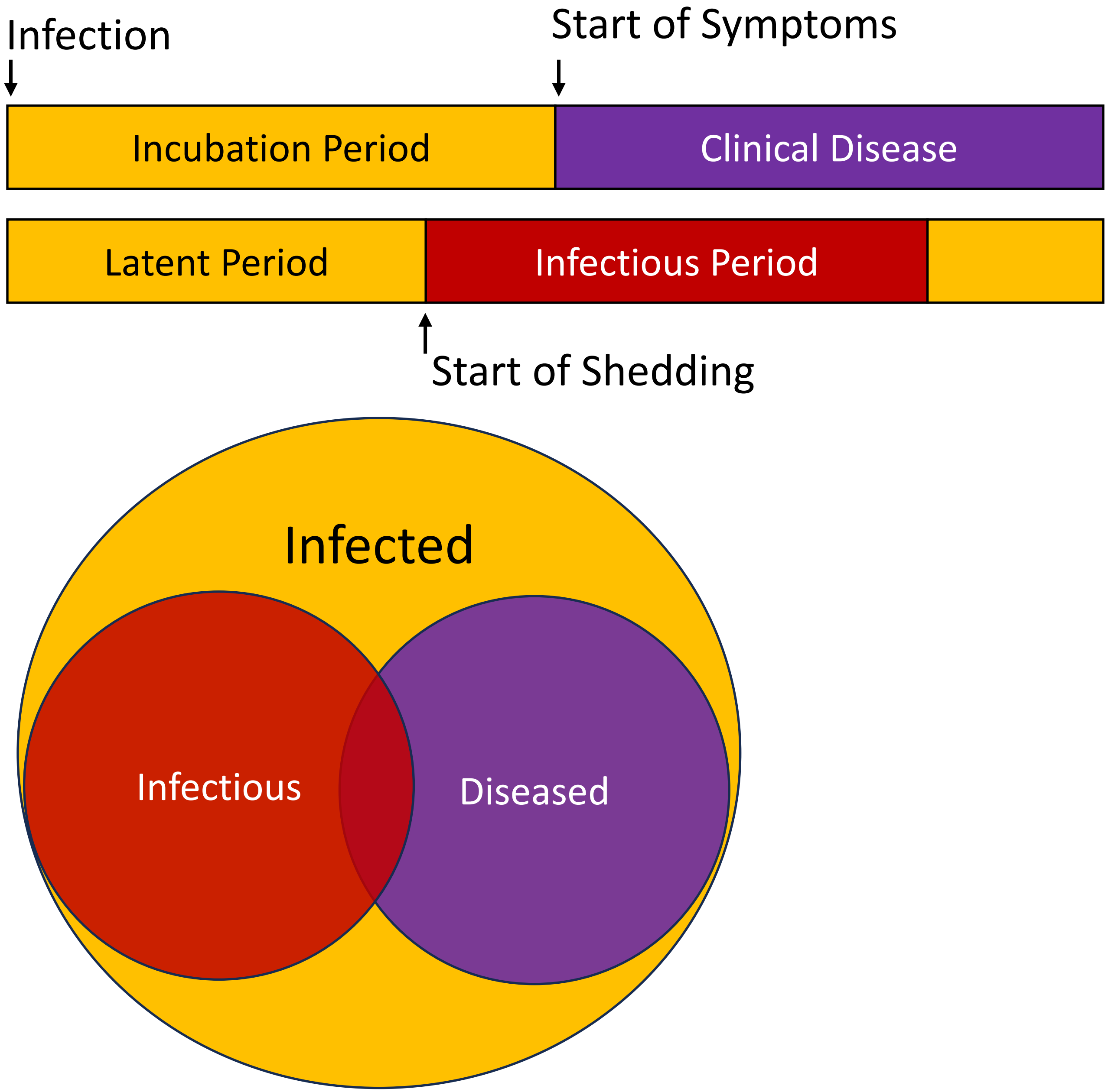


Acute, Immunizing Infections

Table 3.1 Incubation, latent and infectious periods (in days) for a variety of viral and bacterial infections. Data from Fenner and White (1970), Christie (1974), and Benenson (1975)

Infectious disease	Incubation period	Latent period	Infectious period
Measles	8–13	6–9	6–7
Mumps	12–26	12–18	4–8
Whooping cough (pertussis)	6–10	21–23	7–10
Rubella	14–21	7–14	11–12
Diphtheria	2–5	14–21	2–5
Chicken pox	13–17	8–12	10–11
Hepatitis B	30–80	13–17	19–22
Poliomyelitis	7–12	1–3	14–20
Influenza	1–3	1–3	2–3
Smallpox	10–15	8–11	2–3
Scarlet fever	2–3	1–2	14–21

Infectious diseases: Some background terminology



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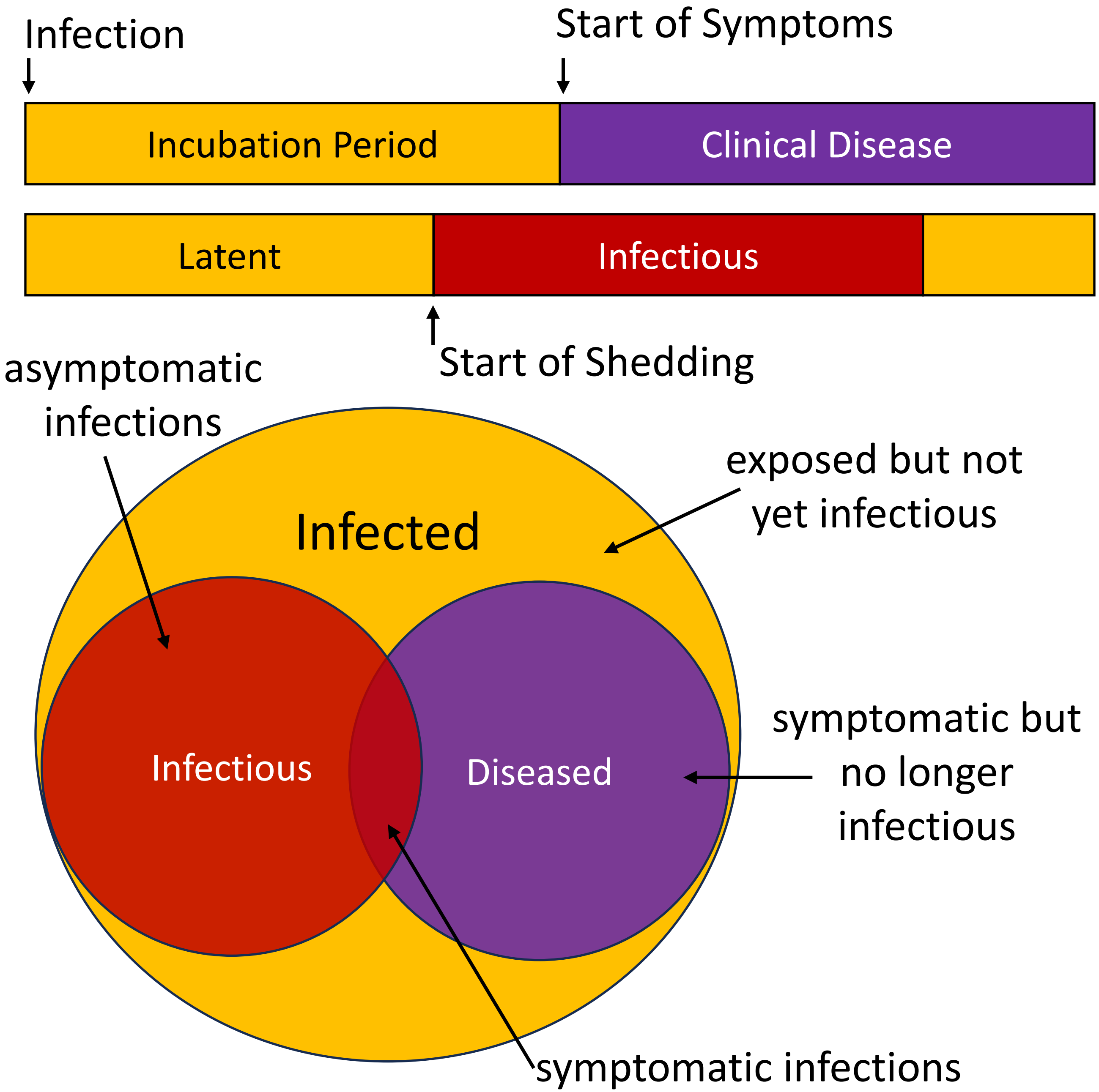
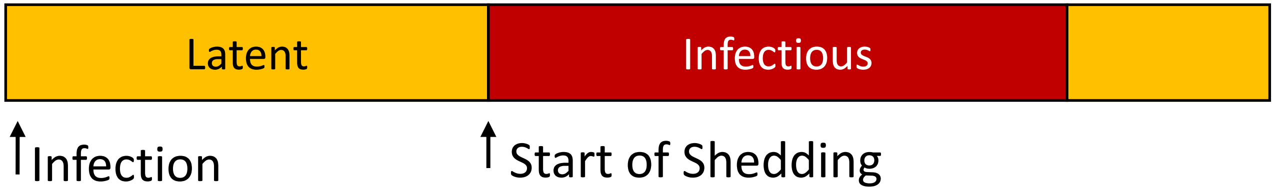
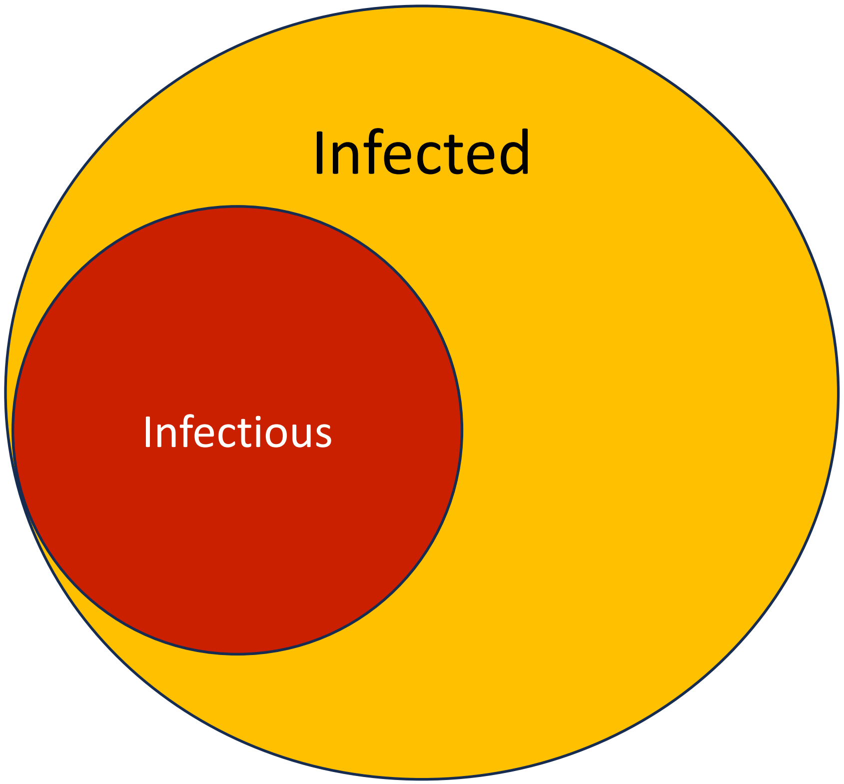


A nuanced view

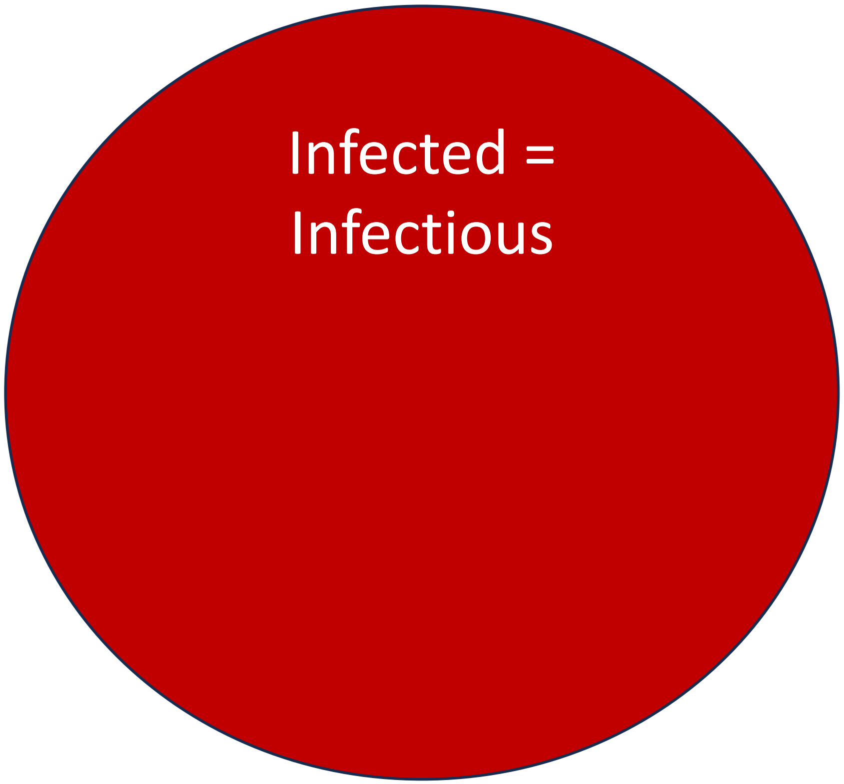
Infectious diseases: Some background terminology





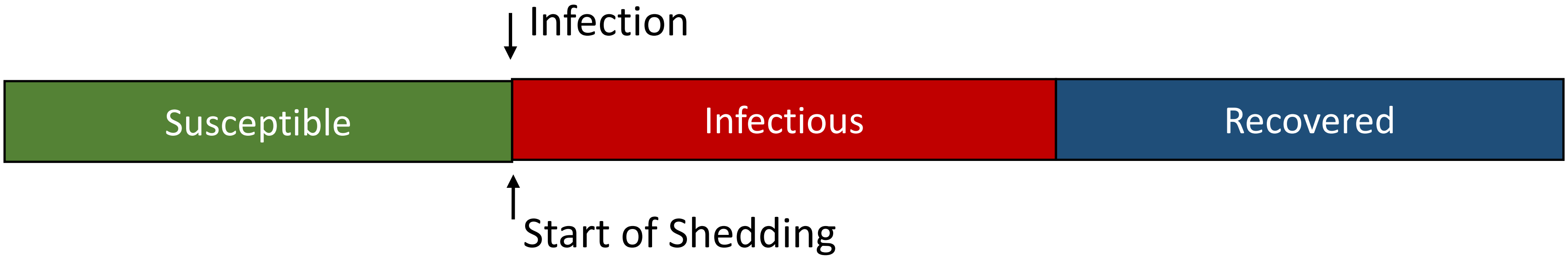
A simpler view – ignore clinical signs

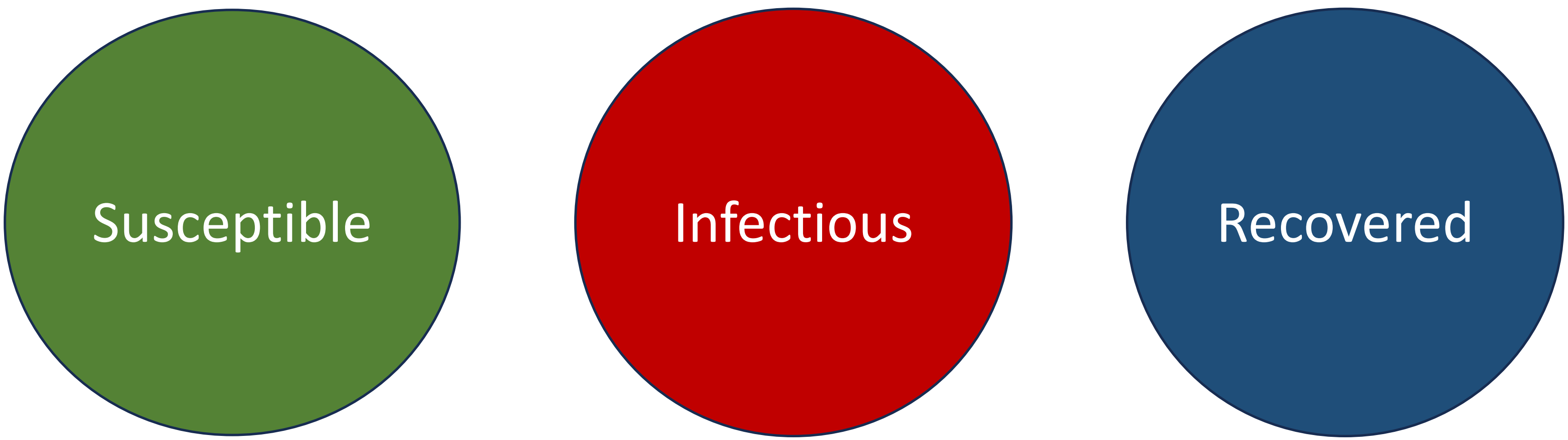
Infectious diseases: Some background terminology



An even simpler view – assume infectious immediately following exposure (contact)

Infectious diseases: Some background terminology





Health-related states = State variables

The SIR model

Compartmental models (Mechanistic Models)

1. Populations are divided into compartments
2. Individuals within a compartment are homogenously mixed
3. Compartments and transition rates are determined by biological systems
4. Rates of transferring between compartments are expressed mathematically

How does measles transmit through Paris?

Comment la rougéole se transmet-elle à Paris?

The SIR model

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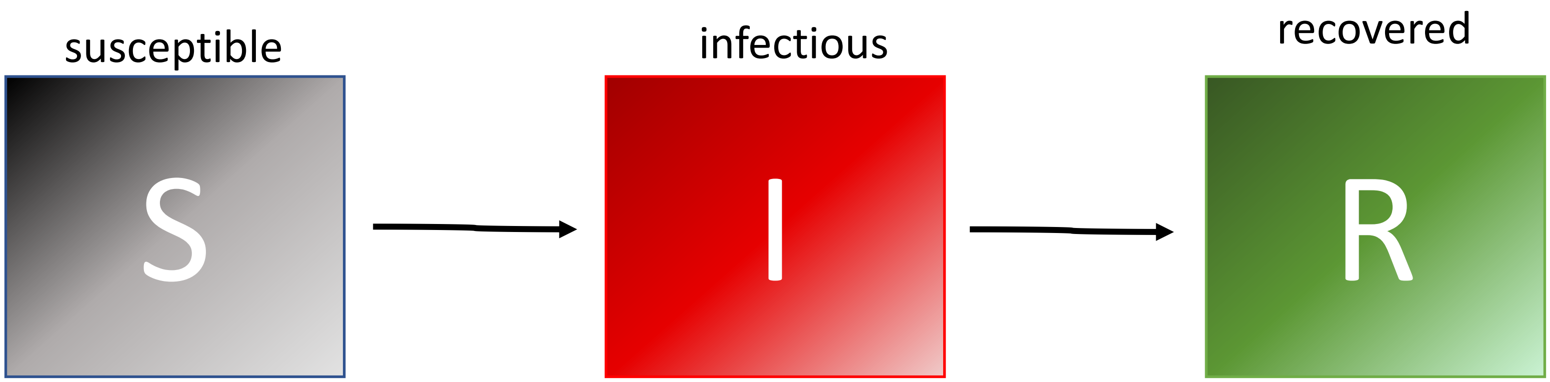

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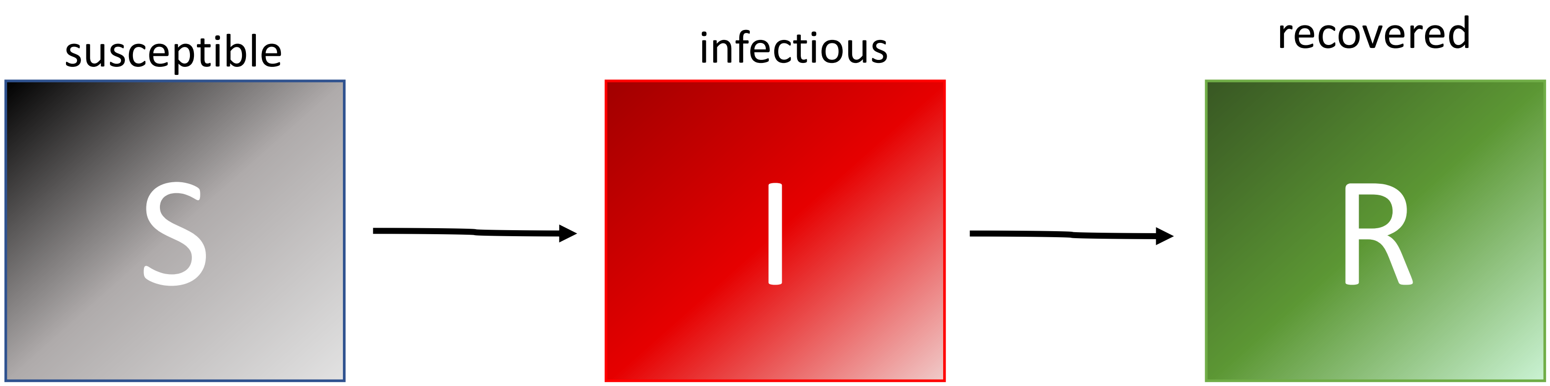

What are the big assumptions here?

The SIR model

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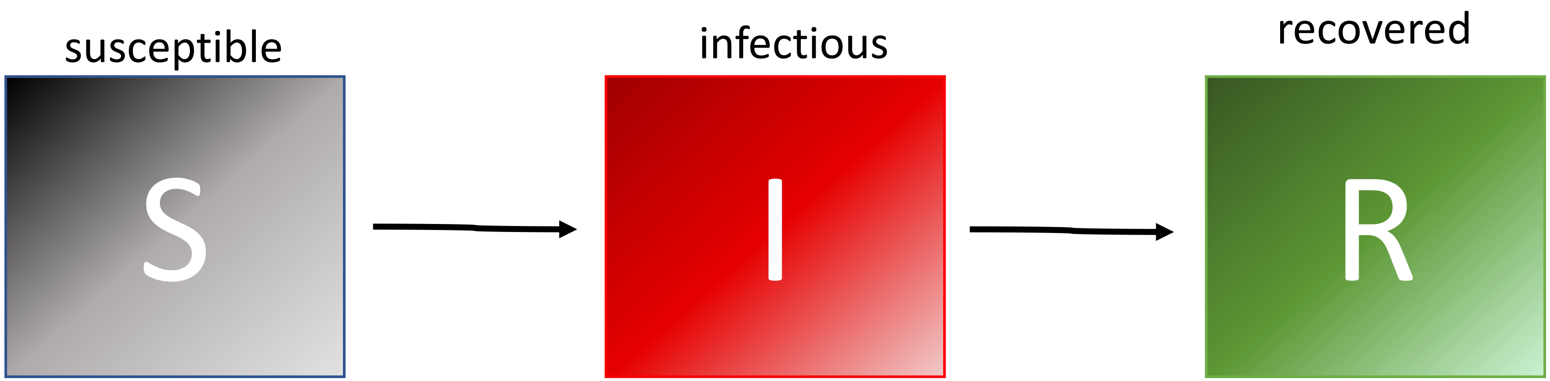

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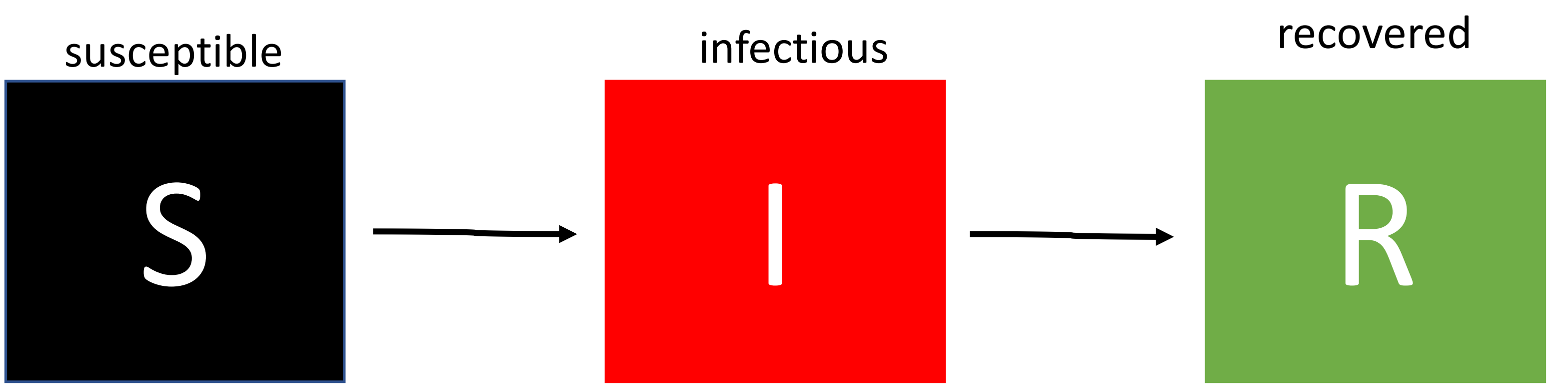

people mix uniformly (mass action)

les gens se mélangent uniformément

The SIR model

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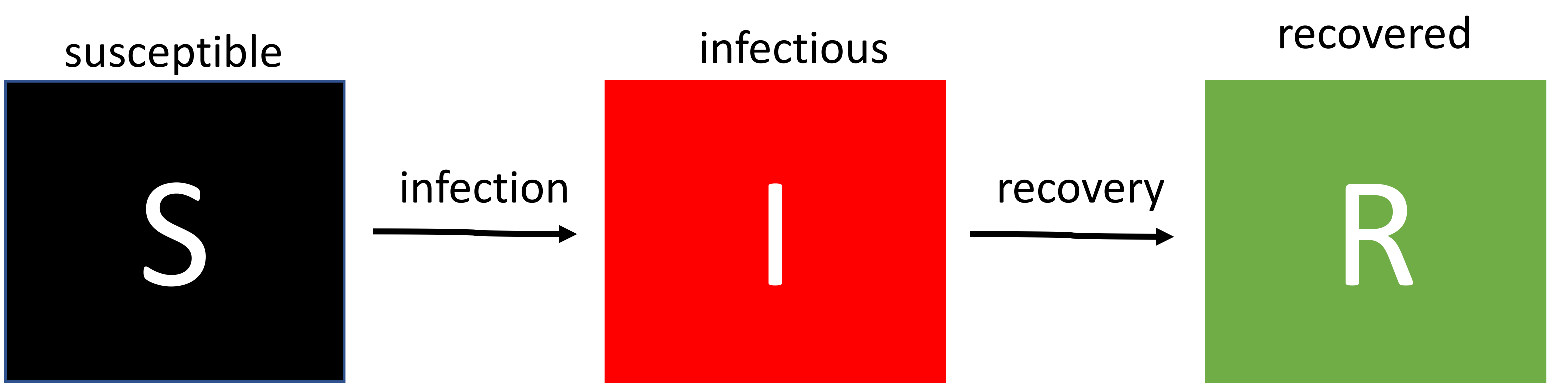

people mix uniformly
(mass action)
les gens se mélangent uniformément

no latent period
(infectious when infected)
pas de période de latence

The SIR model

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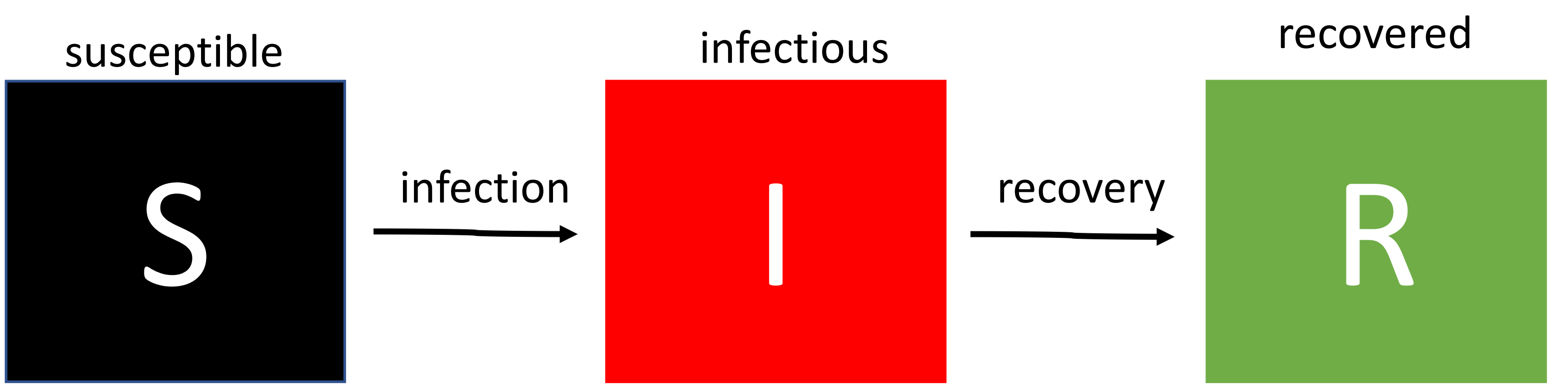

recovery is permanent

la récupération est permanente

people mix uniformly (mass action)

les gens se mélangent uniformément

no latent period (infectious when infected)

pas de période de latence

The SIR model

population size constant - no births or deaths, migration

taille de population constante

Compartmental models (Mechanistic Models)

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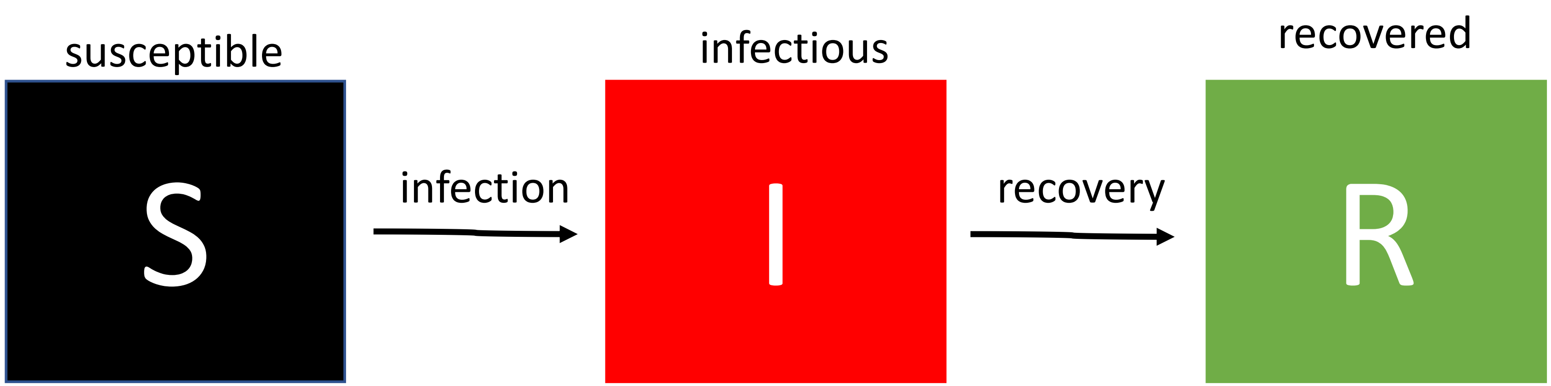

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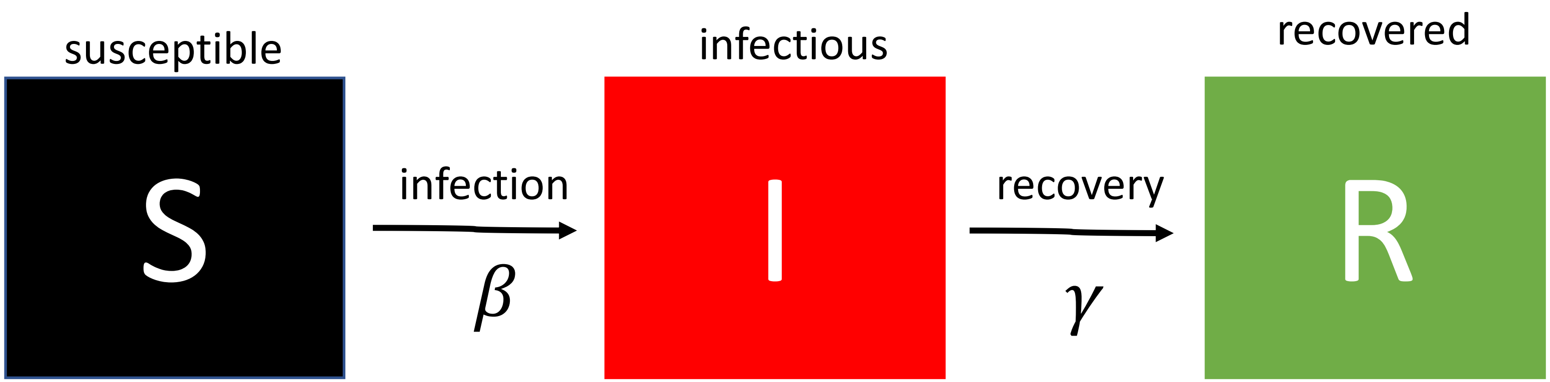
Parameters

β: transmission rate

γ: rate of recovery



$$\frac{dS(t)}{dt} = -\beta S(t)I(t)$$
$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t)$$
$$\frac{dR(t)}{dt} = \gamma I(t)$$

The SIR model

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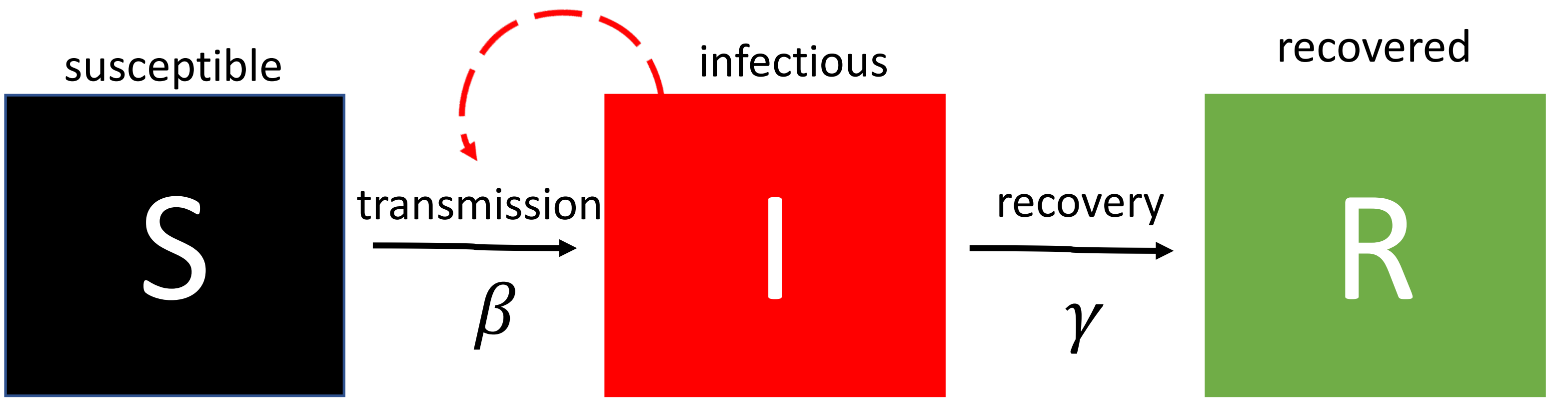

$$\frac{dS(t)}{dt} = -\beta \boxed{S(t)} \boxed{I(t)}$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t)$$

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…infected numbers influence the transmission rate….

The SIR model

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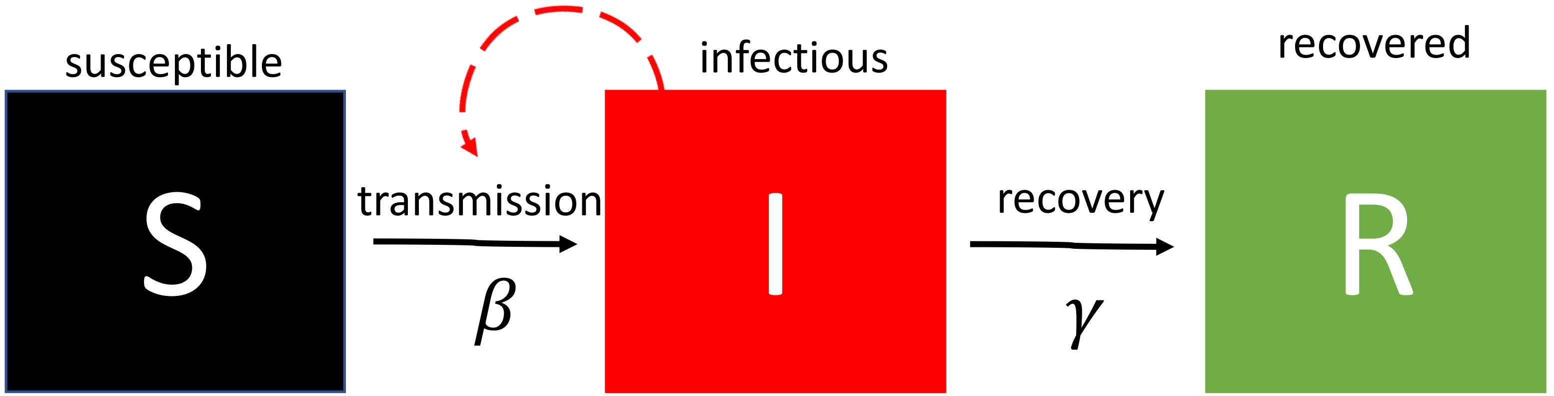
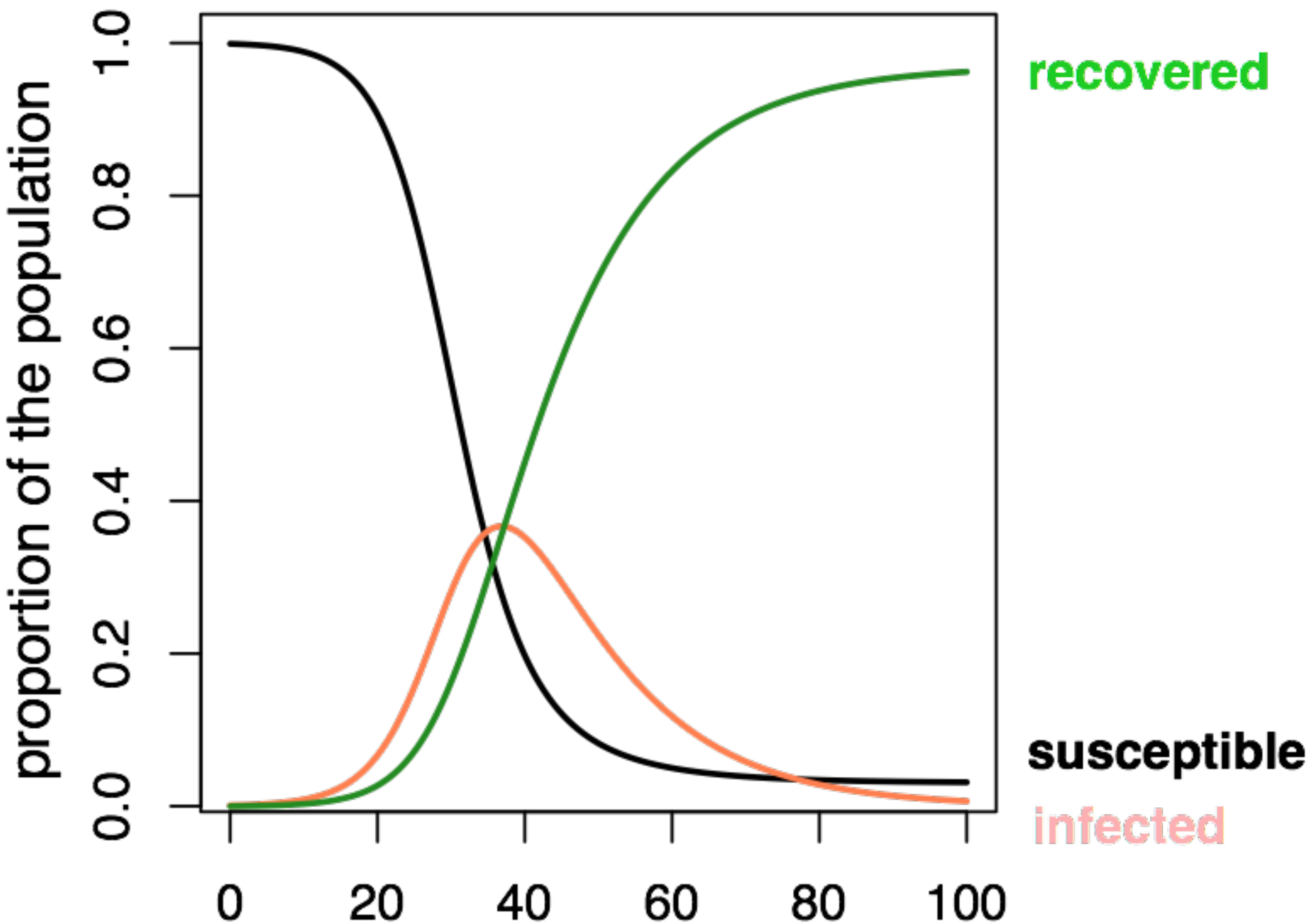
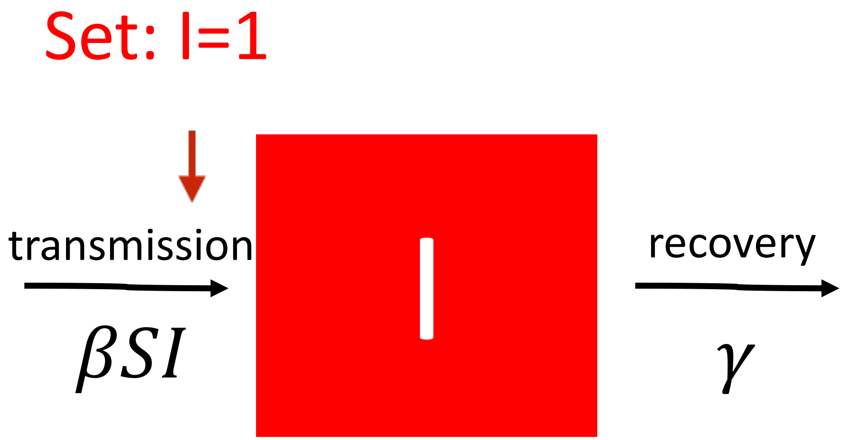


What will the dynamics look like?

The SIR model



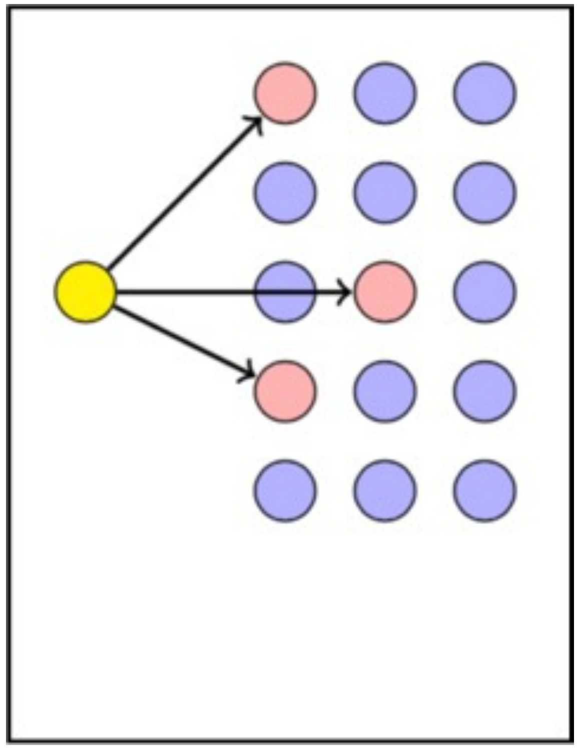
The SIR model



$$R_0 = \beta N/\gamma$$

The average number of persons infected by an infectious individual when everyone is susceptible (S=100%, or S=1, start of an epidemic)

The SIR model



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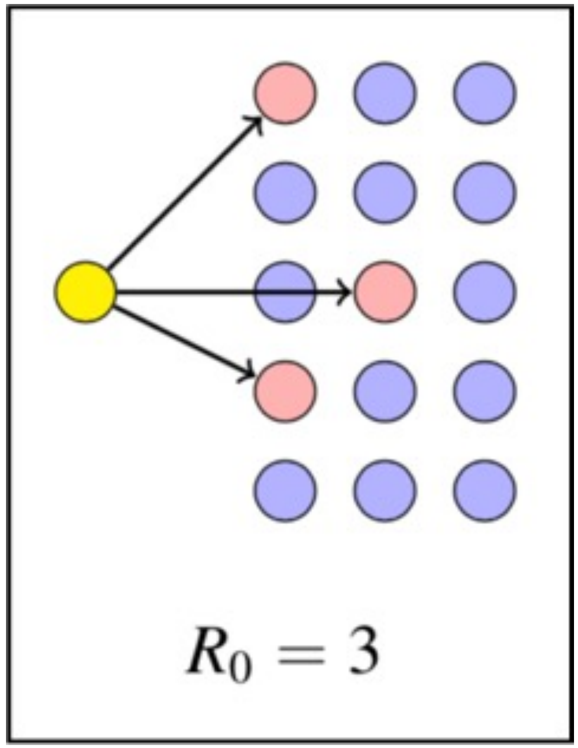
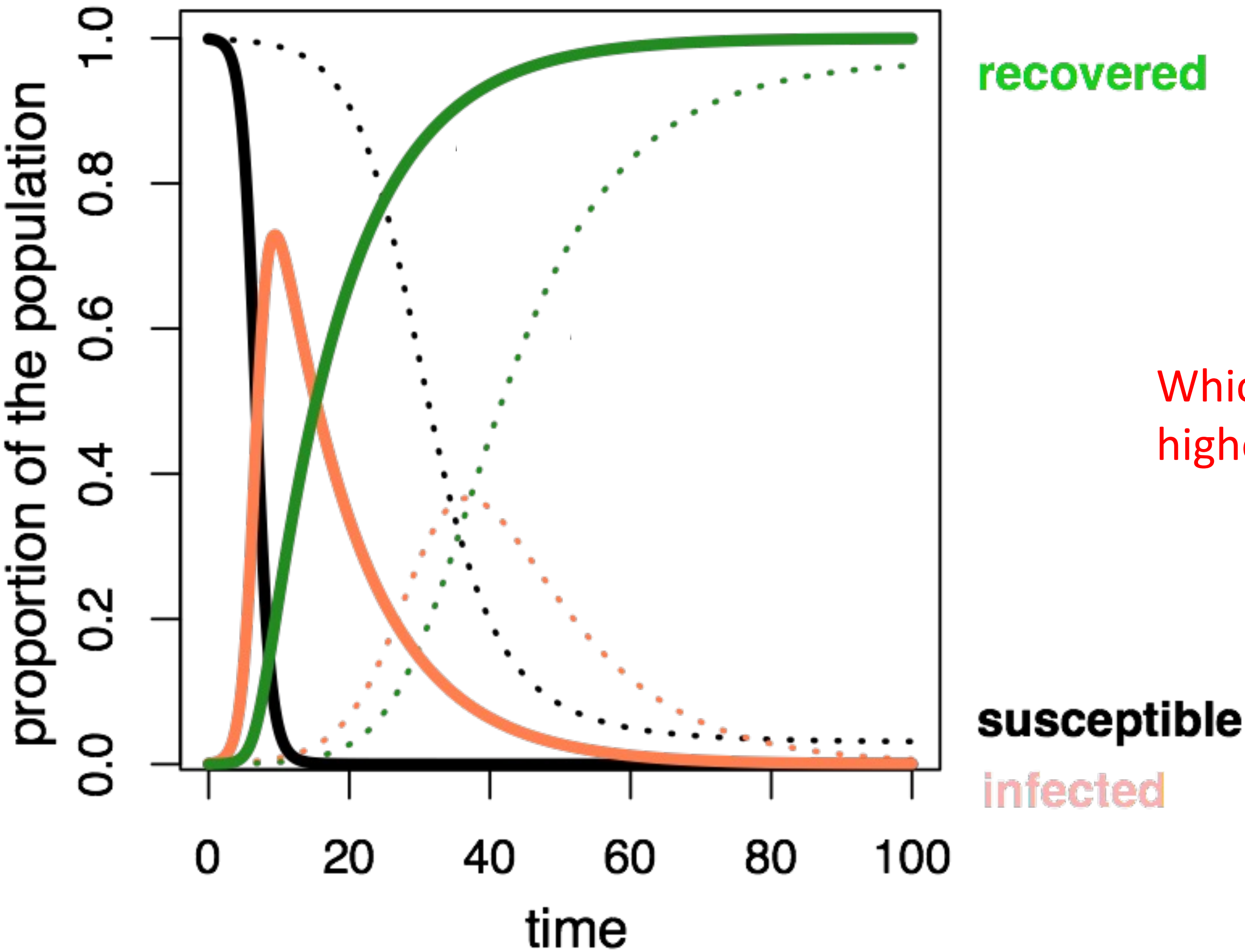
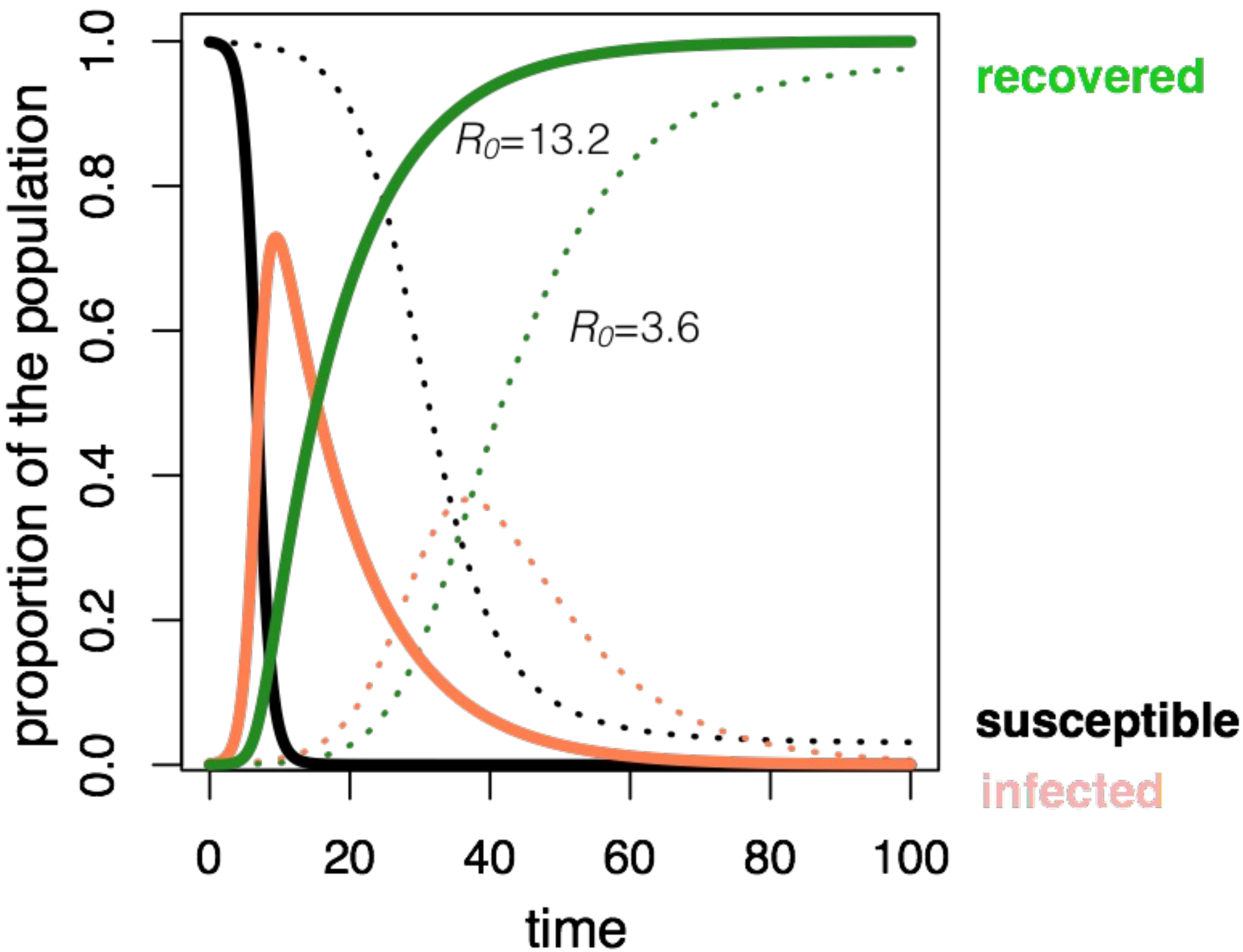


The SIR model



Which has the
higher R0 value?

The SIR model



The SIR model

Set: I=1

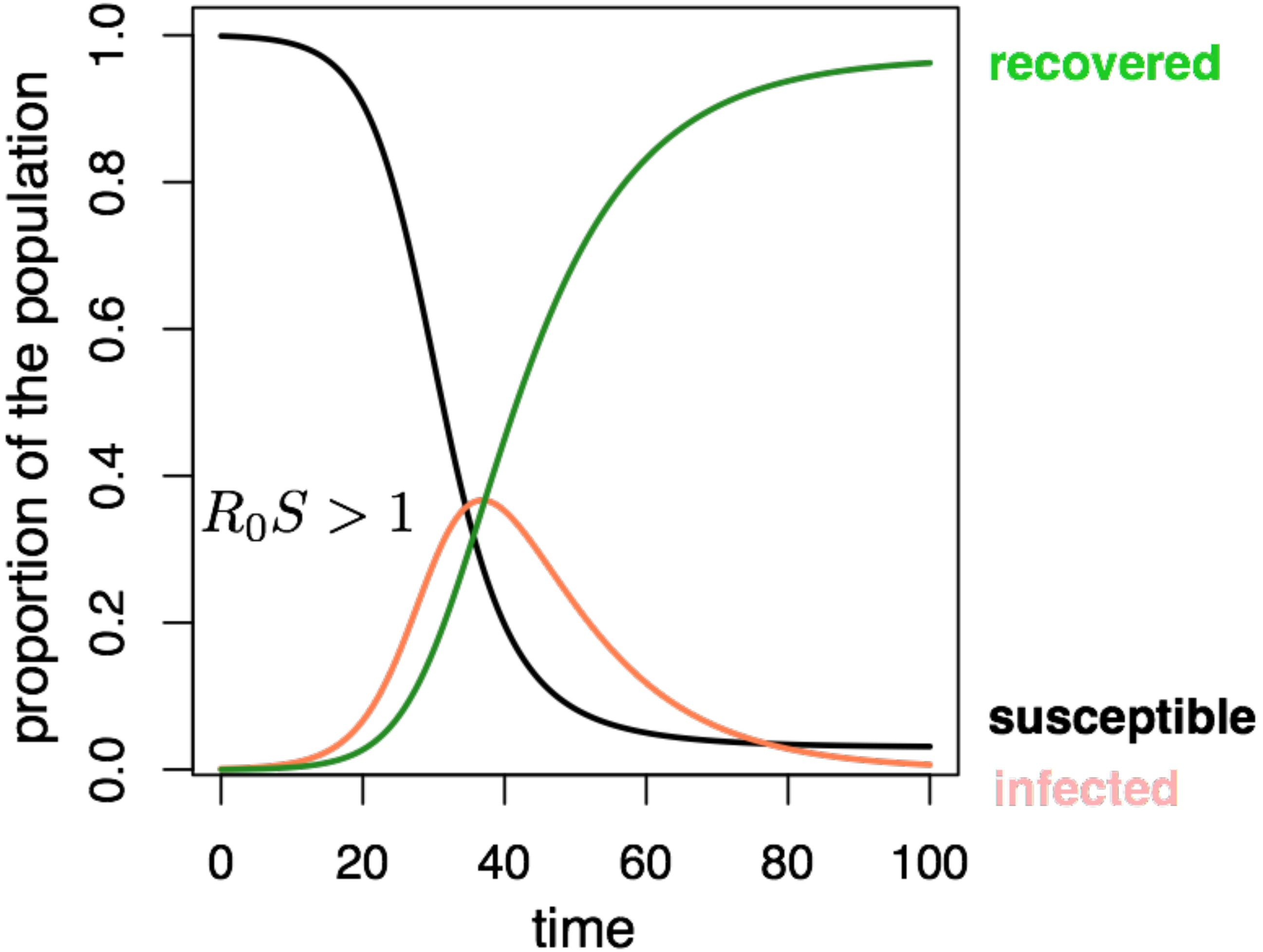
$$R_0 = \beta N / \gamma$$

The average number of persons infected by an infectious individual when everyone is susceptible (S=100%, or S=1, start of an epidemic)

$$R_E = R_0 S \quad \text{“R-effective"}$$

…as the epidemic progresses and S falls

The SIR model



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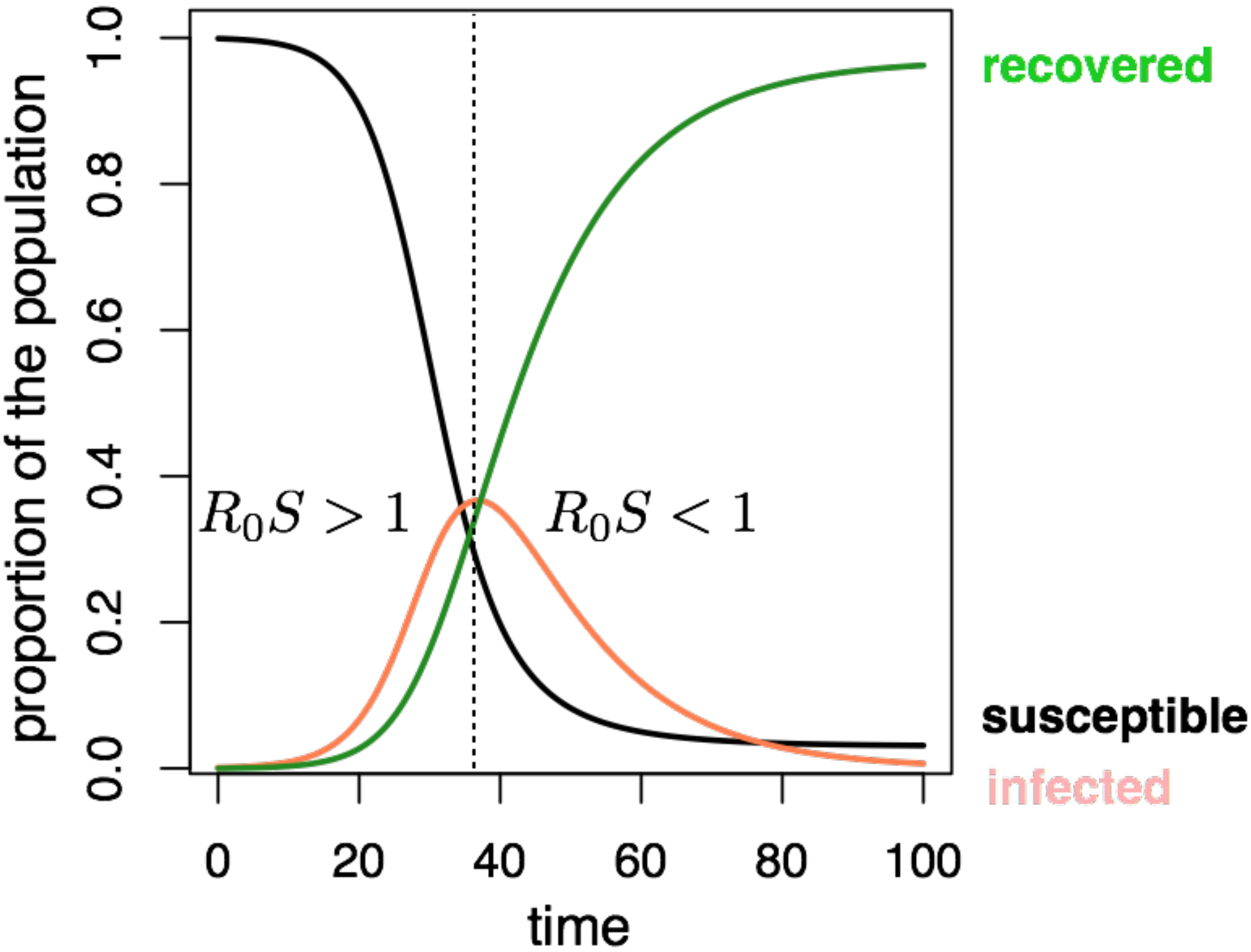



ONE DOES NOT SIMPLY SKIP
A WEEKLY CHECK IN

What is R0?

What is R0?

The average number of secondary infections from the first infectious individual in a naïve population

What is R0?

The average number of secondary infections from the first infectious individual in a naïve population

What is R_E?

What is R0?

The average number of secondary infections from the first infectious individual in a naïve population

What is R_E?

The average number of secondary infections from an infectious individual in a non-naïve population

What is R0?

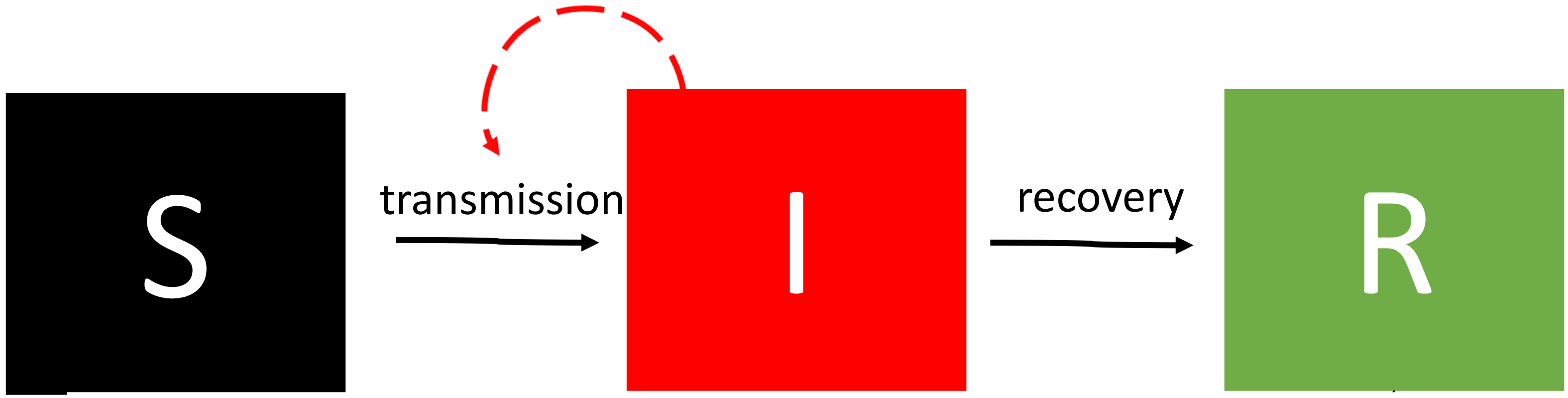
The average number of secondary infections from the first infectious individual in a naïve population

What is R_E?

The average number of secondary infections from an infectious individual in a non-naïve population

How could you modify the simple SIR model to represent COVID-19?

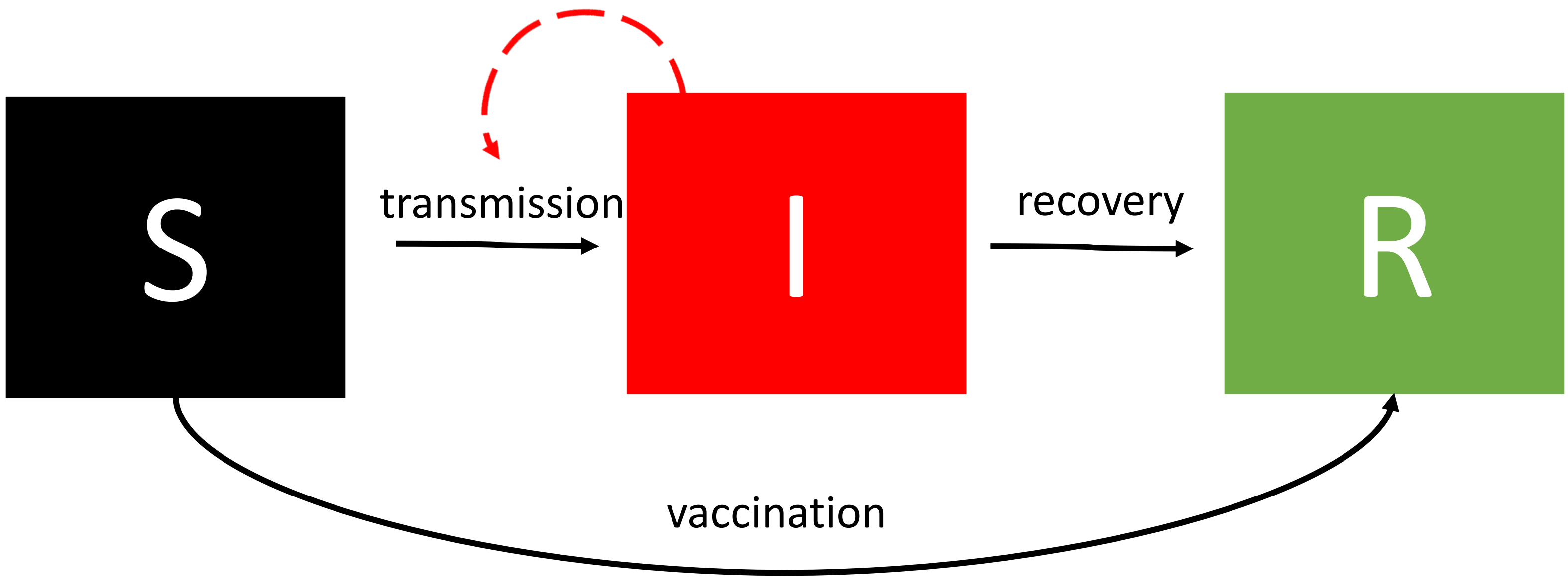
The SIR model : vaccination



Vaccination moves people out of susceptibles into the immune (recovered) class.

La vaccination éloigne les personnes sensibles de la maladie dans la classe immunitaire (rétablie).

The SIR model : vaccination



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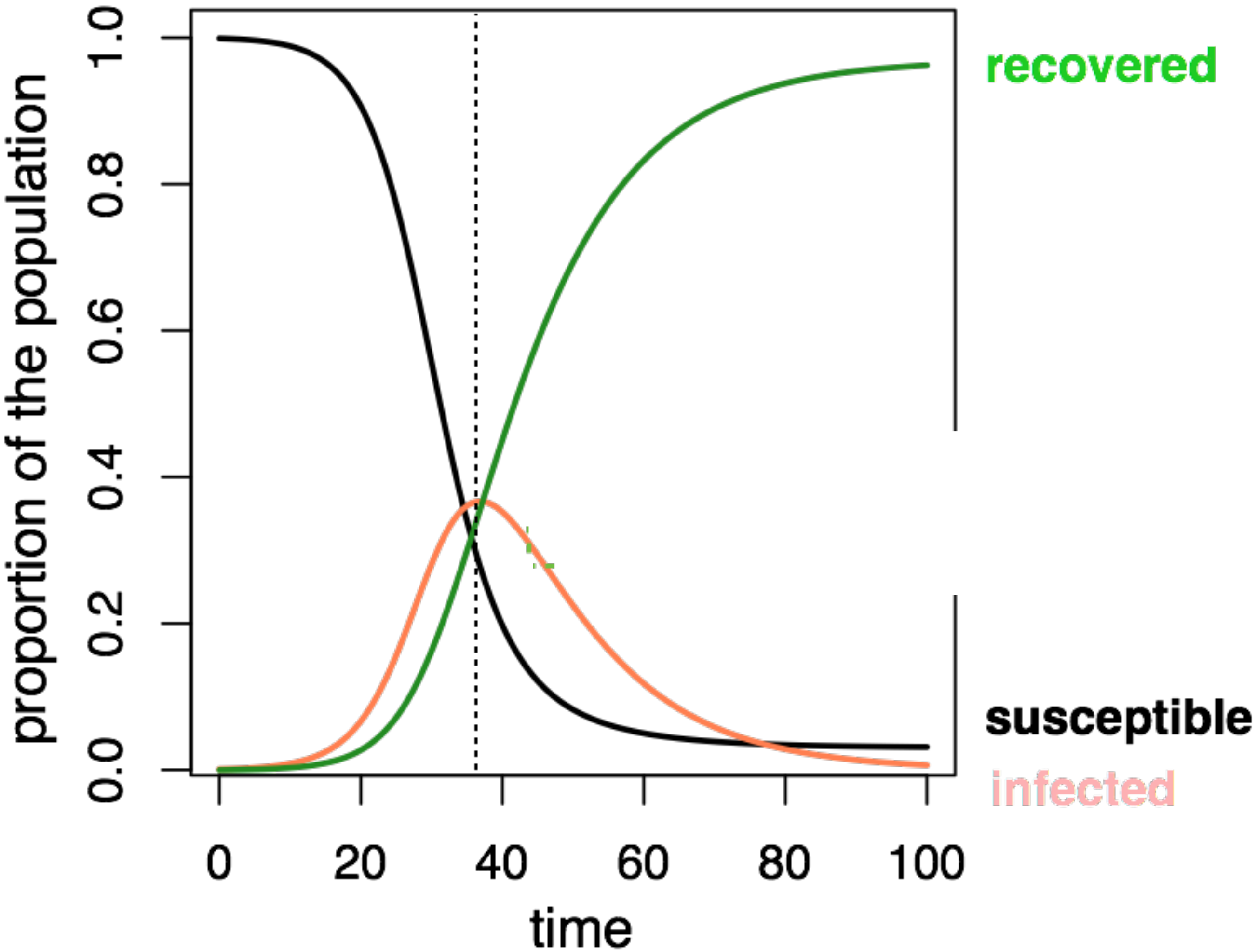
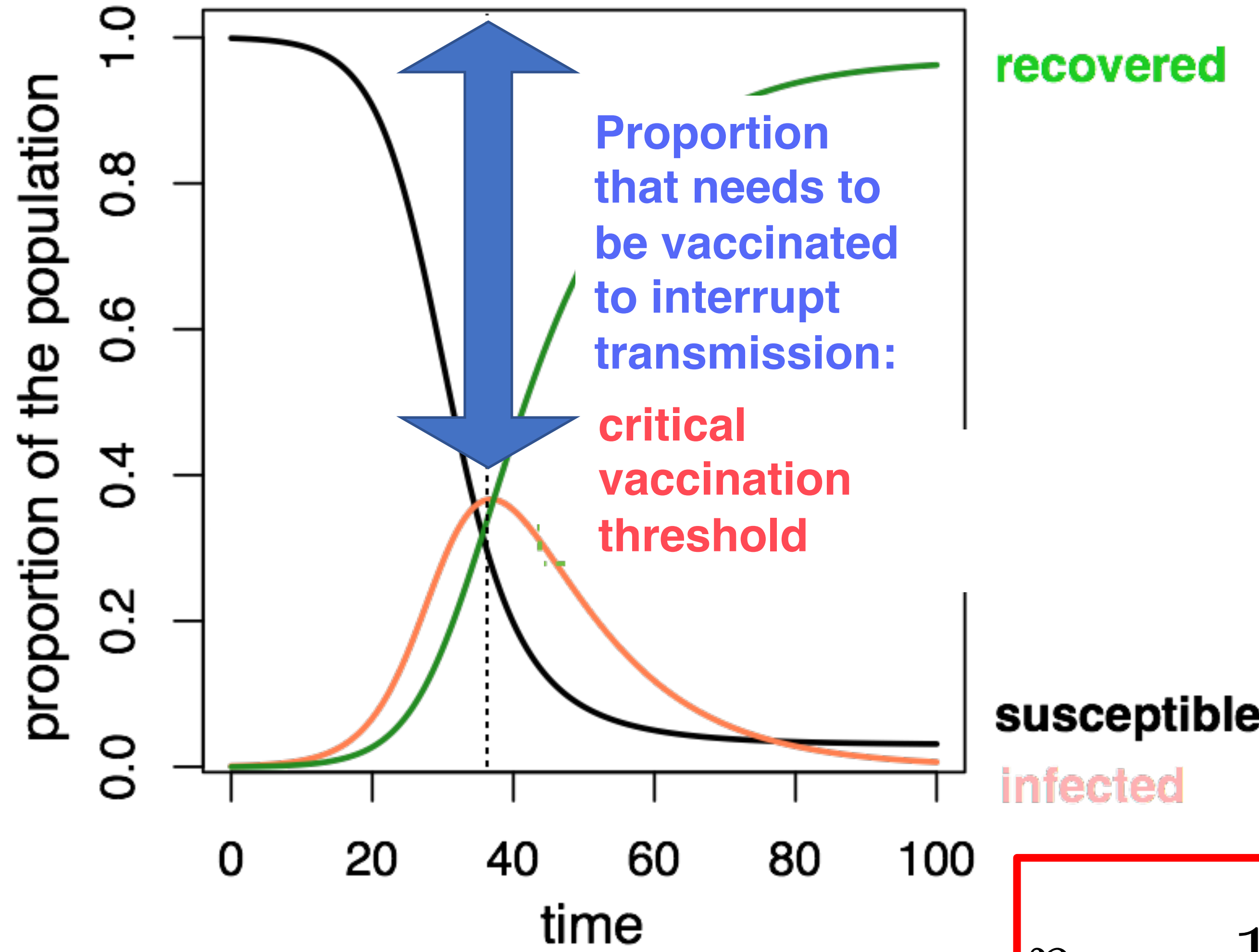


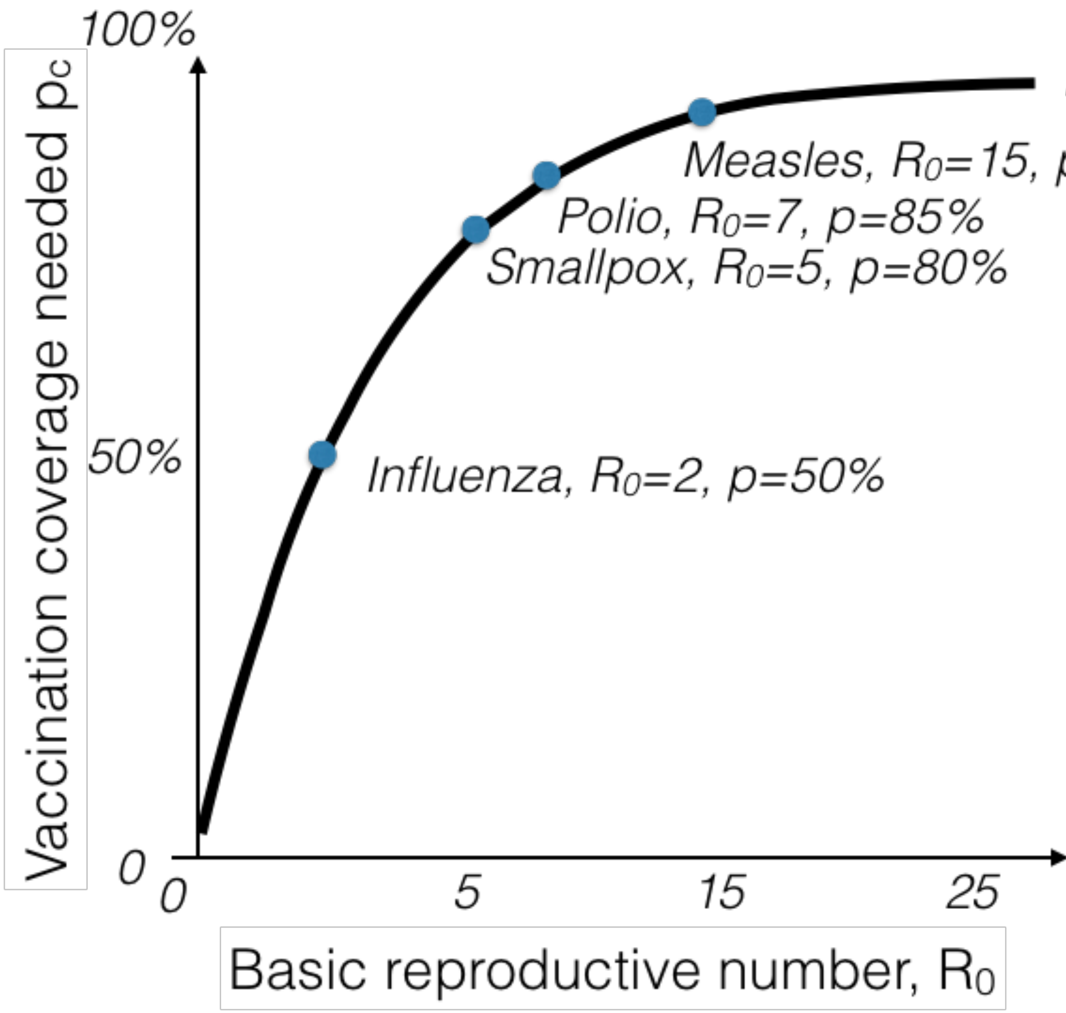
The SIR model : vaccination



$$\boxed{p_c = 1 - \frac{1}{R_0}}$$

The SIR model : eradication

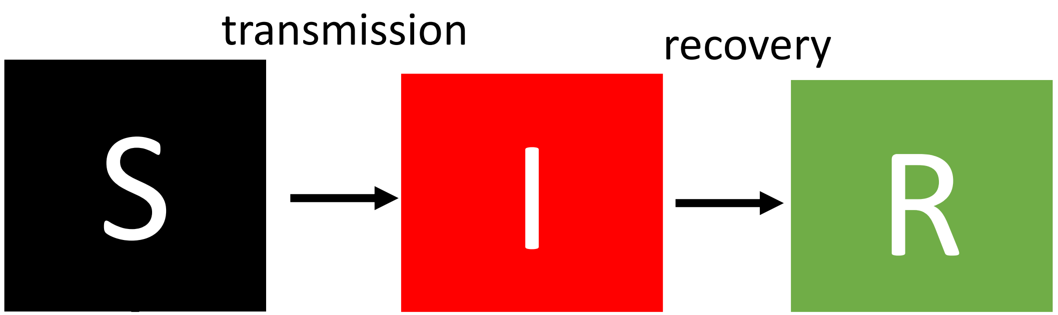
$$p_c = 1 - \frac{1}{R_0}$$



More transmissible diseases are harder to eradicate

The SIR model : extensions

Moving beyond a ‘closed’ population



The SIR model : extensions

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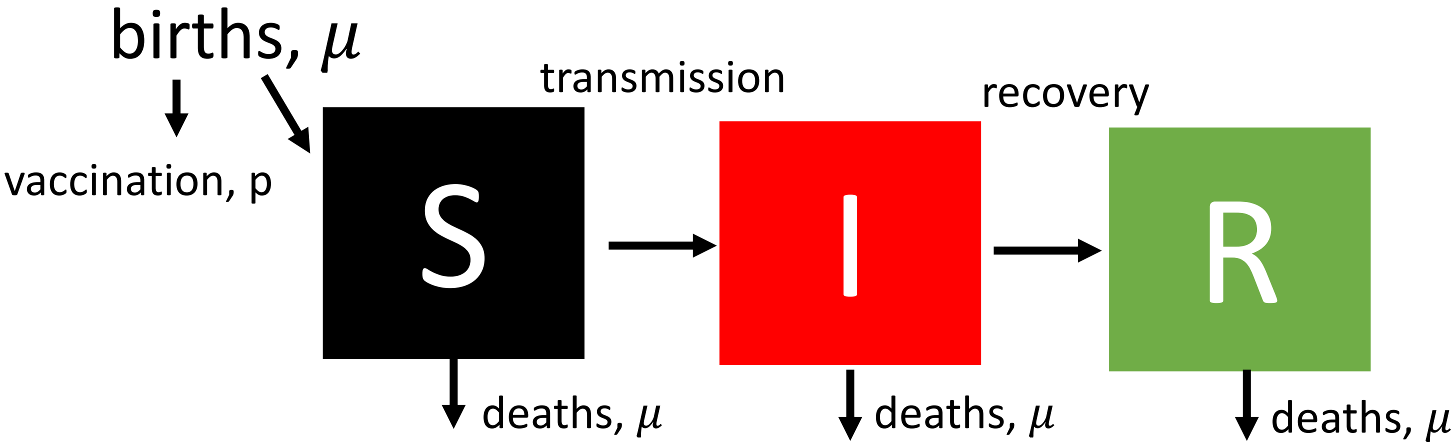

The SIR model : extensions

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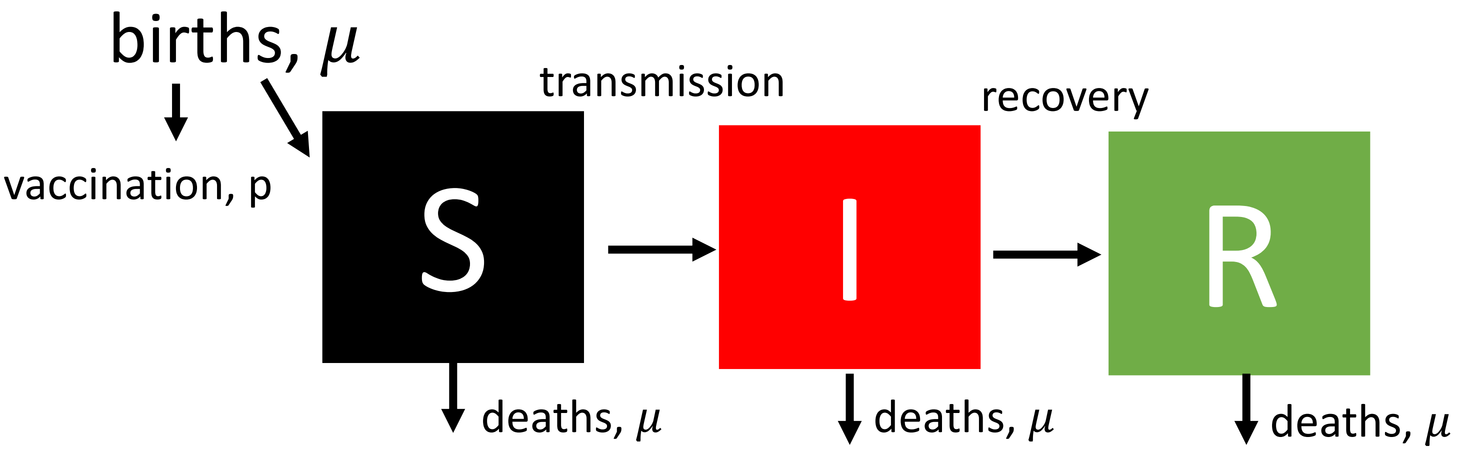

$$\frac{dS(t)}{dt} = \mu(1-p) - \beta S(t)I(t) - \mu S(t)$$

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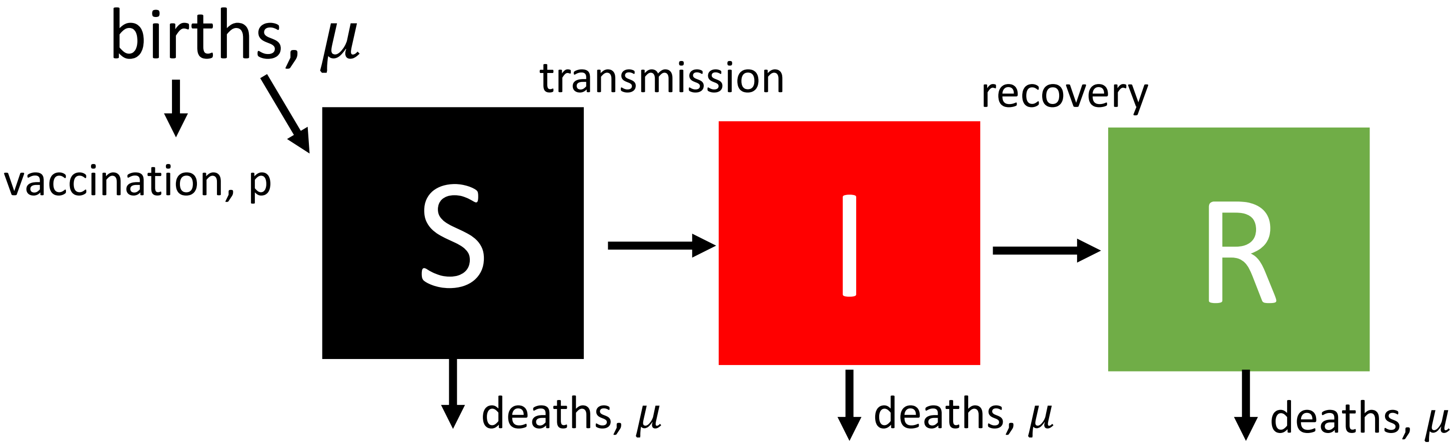
The SIR model : add births

Moving beyond a ‘closed’ population



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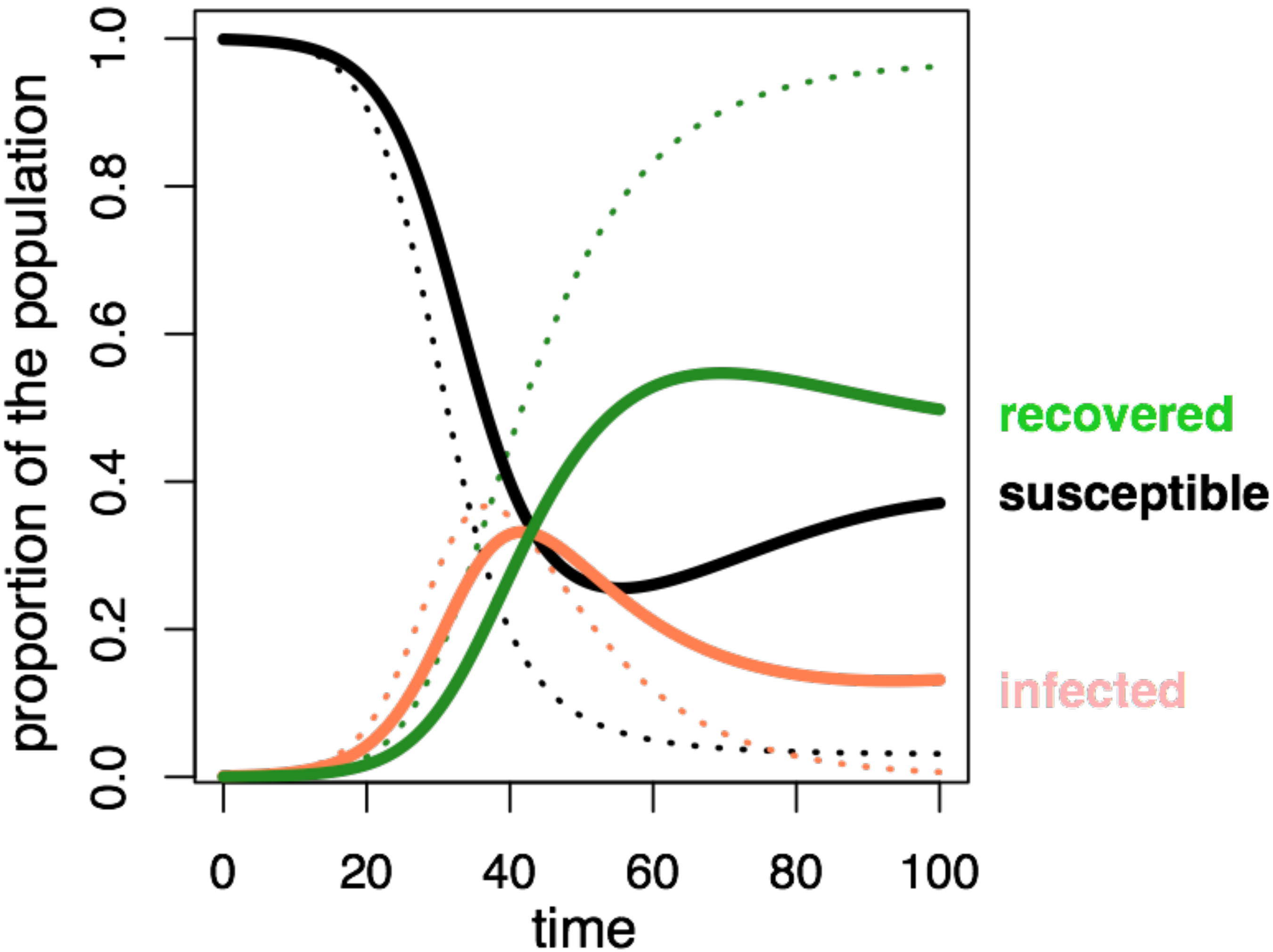
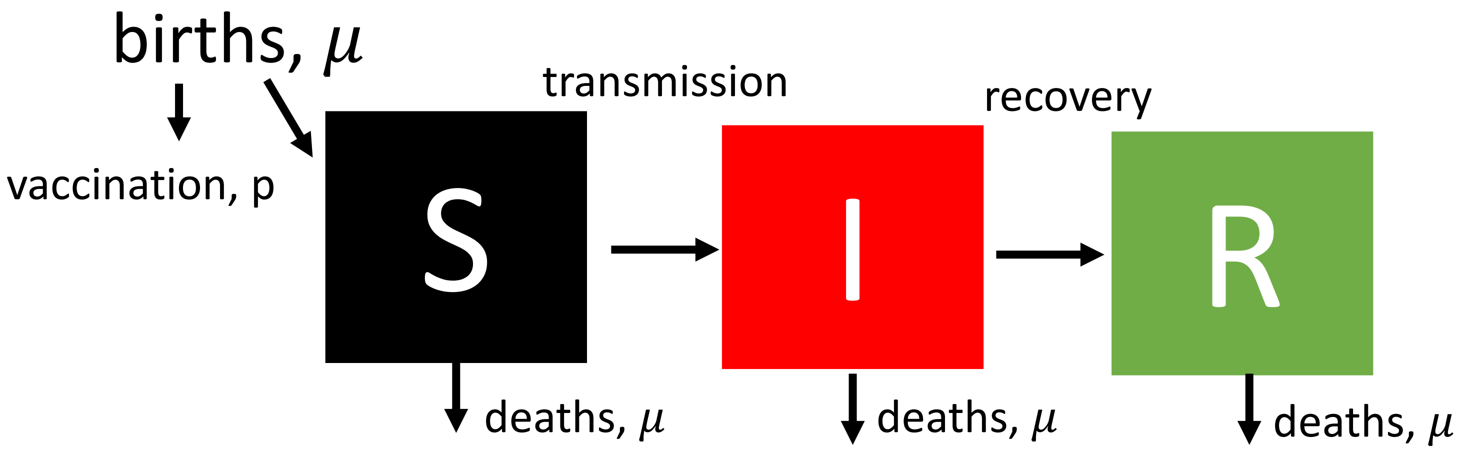
How will births impact dynamics?

The SIR model : add births

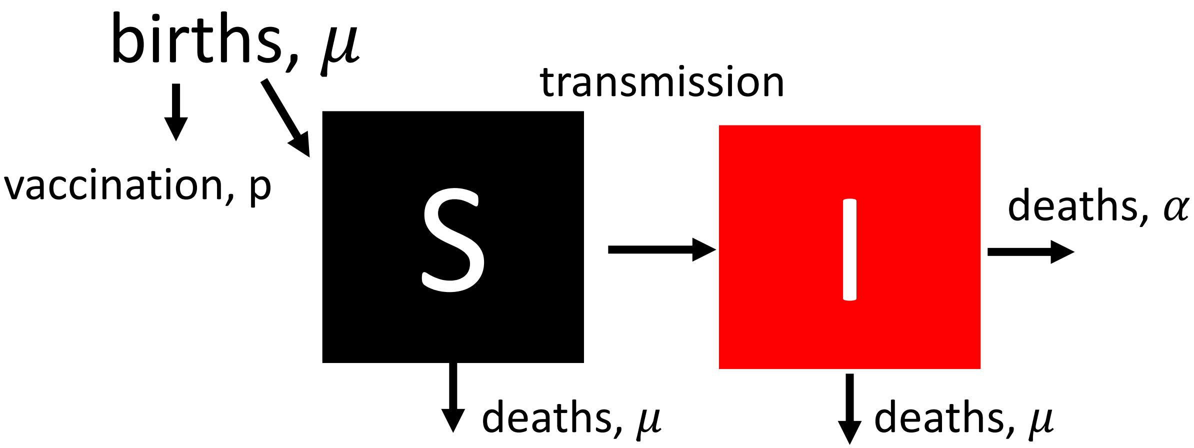


Beyond the SIR model



What do we change if infection is always FATAL?

Beyond the SIR model



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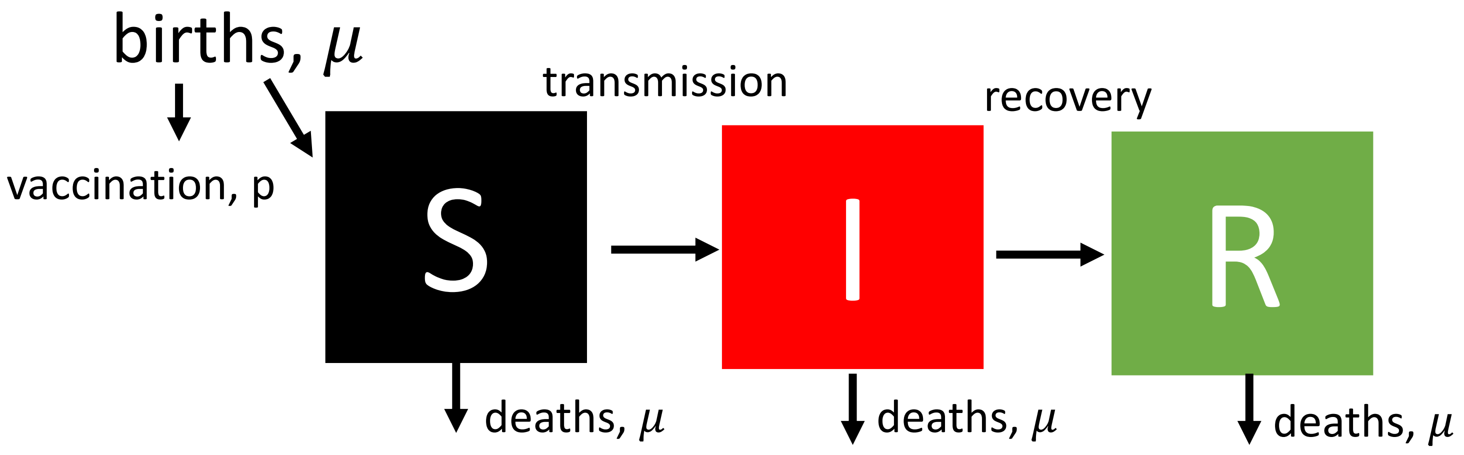
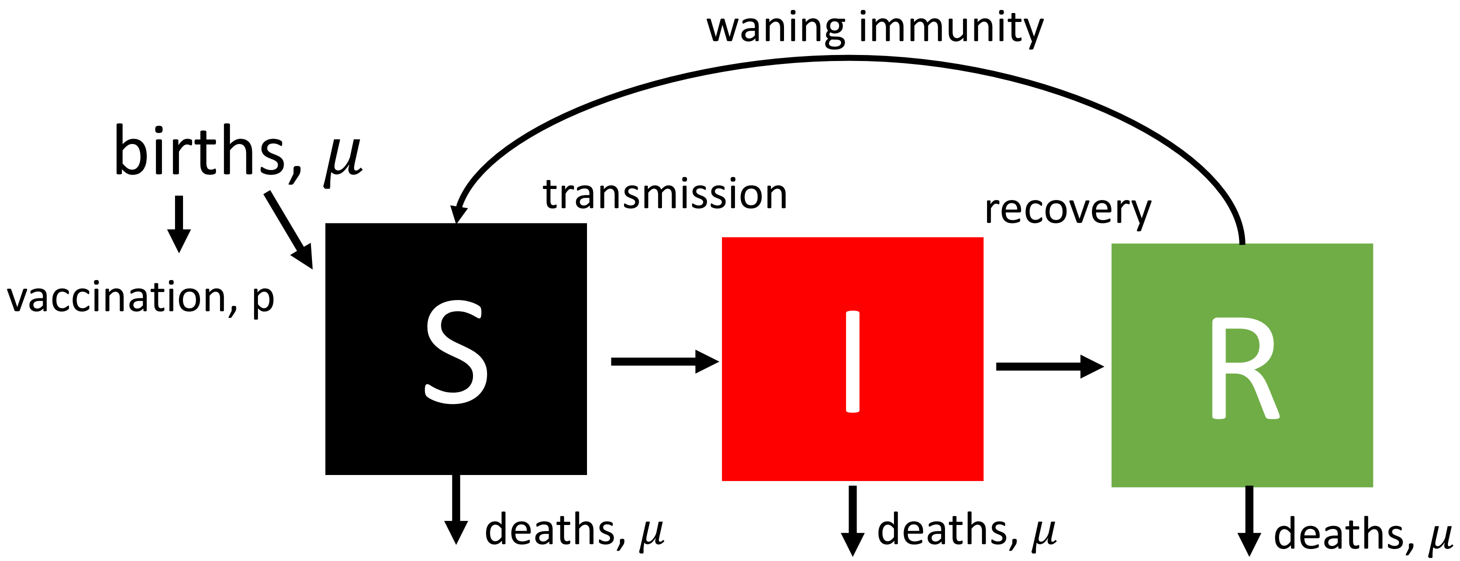


What if immunity wanes?

Beyond the SIR model



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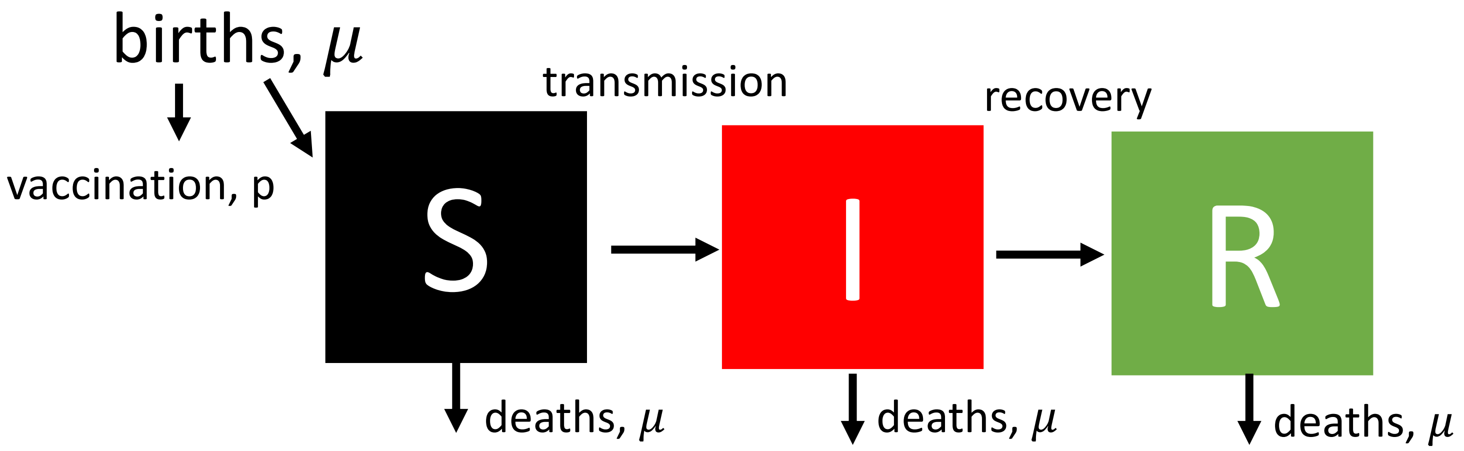
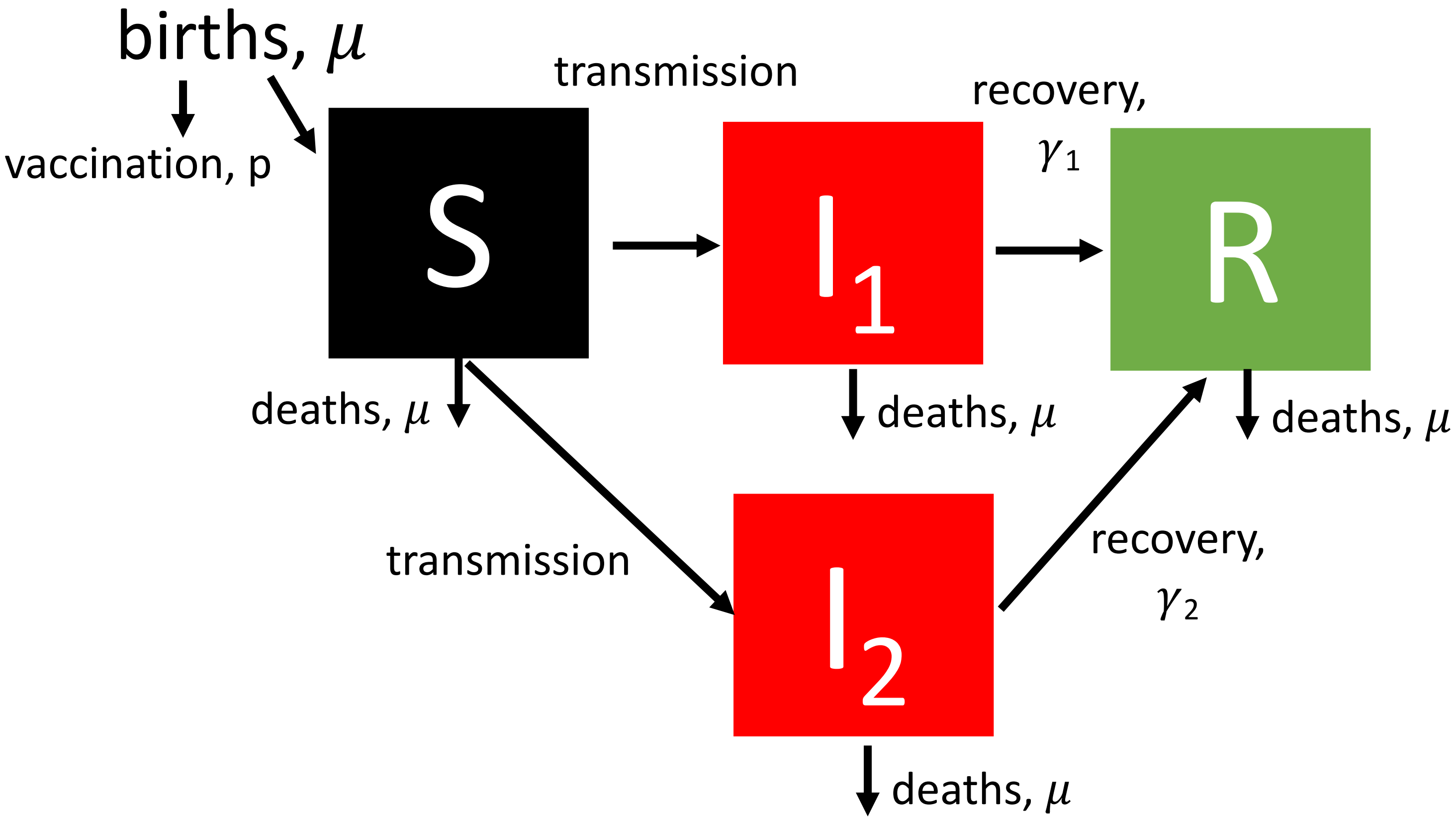


What if people recover at different rates?

Beyond the SIR model



What if people recover at different rates?

Key concepts

- Compartmental/mechanistic/mathematical models

  Modèles en compartiments

- Continuous vs. discrete models

  Modèles en temps continue vs. modèles en temps discrète

- Deterministic vs. stochastic models

  Modèles détérministique vs. stochastique

- Structured models

  Modèles structurés.

- Two population models

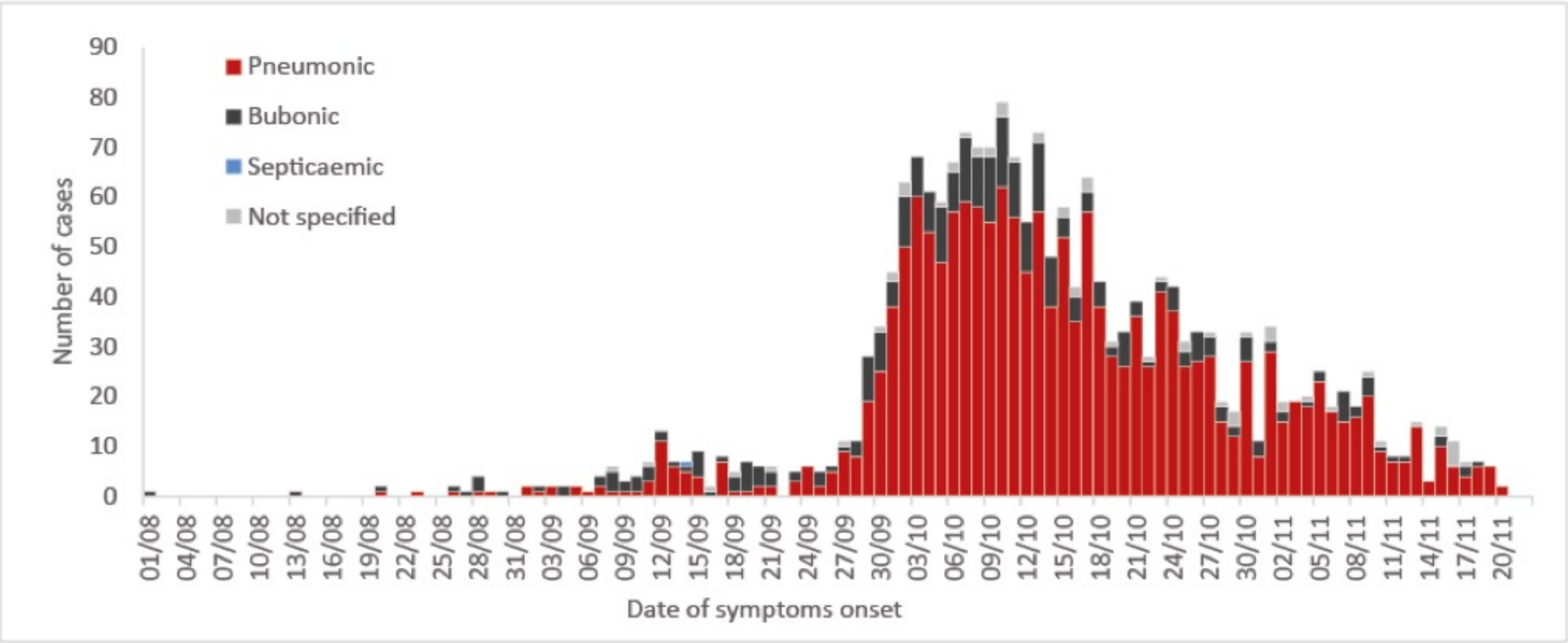
  Modèles des deux populations

- SIR models – and beyond!

  Modèles SIR – et au délà!

“All models are ______, some models are______.”

Which model?



http://www.who.int/csr/don/27-november-2017-plague-madagascar/en/