1. Population-level dynamics - R0

Follow Miller et al to find disease-free equilibrium and evaluate stability to get R0.
First solve assuming an SI pathogen with density-dependent regulation on the population

In[ ]:=
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Solve[{(S + Inf) * (b - q * (S + Inf)) - betar * S * Inf - mu * S == 0,
  betar * S * Inf - mu * Inf - alphar * Inf == 0}, {S, Inf}]
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Out[ ]=

$$\left\{\{S \to 0, Inf \to 0\}, \left\{S \to \frac{alphar + mu}{betar},\right.\right.$$
$$Inf \to \frac{1}{2\,betar^2\,q}\left(-alphar\,betar^2 + b\,betar^2 - betar^2\,mu - 2\,alphar\,betar\,q - 2\,betar\,mu\,q - betar^{3/2}\sqrt{\left(alphar^2\,betar - 2\,alphar\,b\,betar + b^2\,betar + 2\,alphar\,betar\,mu - 2\,b\,betar\,mu + betar\,mu^2 + 4\,alphar^2\,q + 4\,alphar\,mu\,q\right)}\right)\Big\}, \left\{S \to \frac{alphar + mu}{betar},\right.$$
$$Inf \to \frac{1}{2\,betar^2\,q}\left(-alphar\,betar^2 + b\,betar^2 - betar^2\,mu - 2\,alphar\,betar\,q - 2\,betar\,mu\,q + betar^{3/2}\sqrt{\left(alphar^2\,betar - 2\,alphar\,b\,betar + b^2\,betar + 2\,alphar\,betar\,mu - 2\,b\,betar\,mu + betar\,mu^2 + 4\,alphar^2\,q + 4\,alphar\,mu\,q\right)}\right)\Big\}, \left\{S \to \frac{b - mu}{q}, Inf \to 0\right\}\Big\}$$

We find both a disease-free equilibrium (DFE) and an endemic-infection equilibrium.
At DFE: S* = $\frac{b-mu}{q}$ and I* = 0.
At endemic infection: S*= $\frac{alphar+mu}{betar}$ and I* =

$$\frac{1}{2\,betar^2\,q}\left(-alphar\,betar^2 + b\,betar^2 - betar^2\,mu - 2\,alphar\,betar\,q - 2\,betar\,mu\,q + betar^{3/2}\sqrt{\left(alphar^2\,betar - 2\,alphar\,b\,betar + b^2\,betar + 2\,alphar\,betar\,mu - 2\,b\,betar\,mu + betar\,mu^2 + 4\,alphar^2\,q + 4\,alphar\,mu\,q\right)}\right)$$

Now, we evaluate the stability of the Jacobian matrix at DFE. This occurs when either (a) the trace of the Jacobian is positive or (b) the determinant of the Jacobian is negative. A little algebra will establish that both of these approaches yield the same equation, which we call R_0, representing the conditions under which a pathogen can invade the system.
Solving for that expression with this Jacobian yields the following equation for R_{0} in this system :

R0 = $\frac{betar\,(b-mu)}{b\,(alphar+mu)}$