

Introduction to Compartmental Models

Introduction aux modèles à compartiments

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Goals for this lecture

- Understand the difference between statistical and compartmental models
Comprendre la différence entre les modèles statistiques et les modèles à compartiments
- Understand the difference between parameters and state variables
Comprendre la différence entre les paramètres et les variables d'état
- Understand the difference between discrete-time and continuous-time models
Comprendre la différence entre les modèles à temps discret et les modèles à temps continu
- Understand how to formalize and conceptualize compartmental models
Comprendre comment formaliser et conceptualiser les modèles à compartiments

Compartmental/Mechanistic/Mathematical Models

1. Populations are divided into compartments
Les populations sont subdivisées en compartiments

Compartmental/Mechanistic/Mathematical Models

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Les individus d'un compartiment sont mélangés de manière homogène

Compartmental/Mechanistic/Mathematical Models

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Les compartiments et les taux de transition sont déterminés par les systèmes biologiques

Compartmental/Mechanistic/Mathematical Models

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Les compartiments et les taux de transition sont déterminés par les systèmes biologiques
4. Rates of transfer between compartments are expressed mathematically
Les taux de transition entre les compartiments sont exprimés mathématiquement

How are these different from statistical models?

En quoi sont-elles différentes des modèles statistiques?

How are these different from statistical models?

En quoi sont-elles différentes des modèles statistiques?

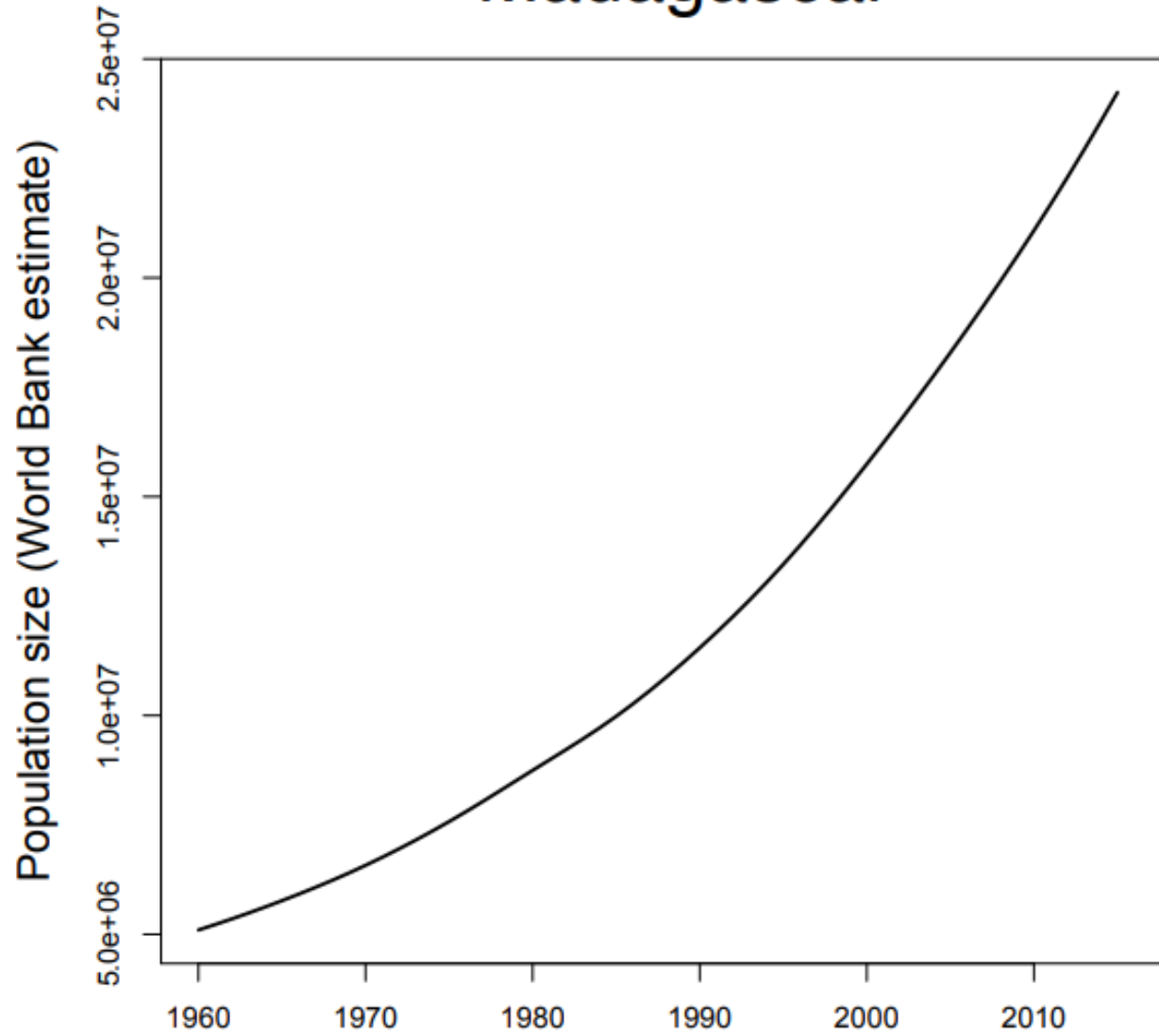
Compartmental models make explicit hypotheses about
biological mechanisms that drive dynamics
(may not be realistic, but still explicit)

Les modèles à compartements font des hypothèses explicites
sur les mécanismes biologiques qui régissent la dynamique
(peut ne pas être réalistes, mais toujours explicites)

1. Simple Population Models

Les modèles simples de population

Madagascar



<http://databank.worldbank.org>

What can we say
about the population
of Madagascar?

How would a model
help us? What kind of
model should we
use?

The basic population model

How does the population of Madagascar grow over time?

Comment est-ce que la population de Madagascar augmente avec le passage du temps ?

The basic population model

Compartmental models (mechanistic models)

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2. Individuals within a compartment are homogenously mixed
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Les modèles à compartiments (les modèles mécaniste)

1. Les populations sont subdivisées en compartiments
2. Les individus d'un compartiment sont mélangés de manière homogène
3. Les compartiments et les taux de transition sont déterminés par les systèmes biologiques
4. Les taux de transition entre les compartiments sont exprimés mathématiquement

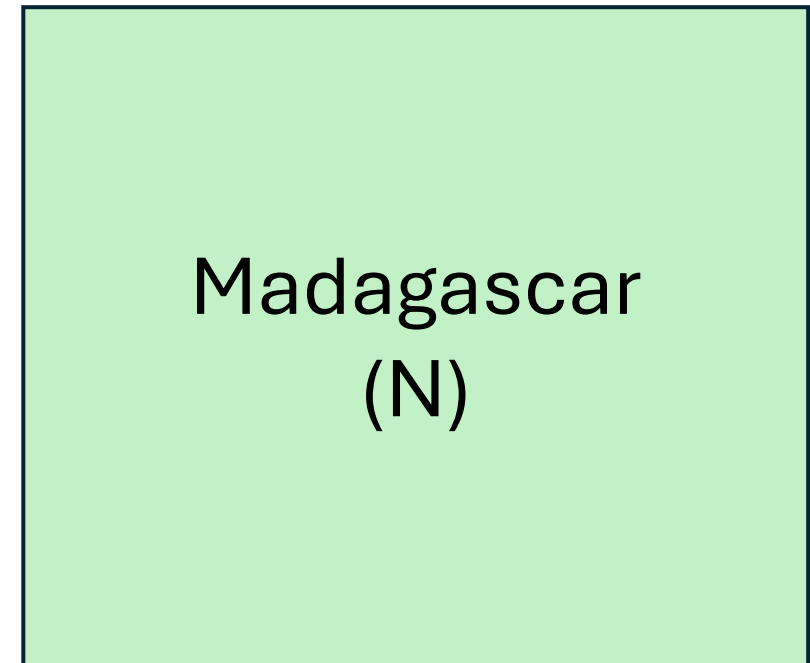
The basic population model

Compartmental models (mechanistic models)

1. *Populations are divided into compartments*
2. Individuals within a compartment are homogenously mixed
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4. Rates of transferring between compartments are expressed mathematically

N = state variable =
the data we want to explain

Square =
compartment



The basic population model

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Madagascar
(N)



The basic population model

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Madagascar
(N)

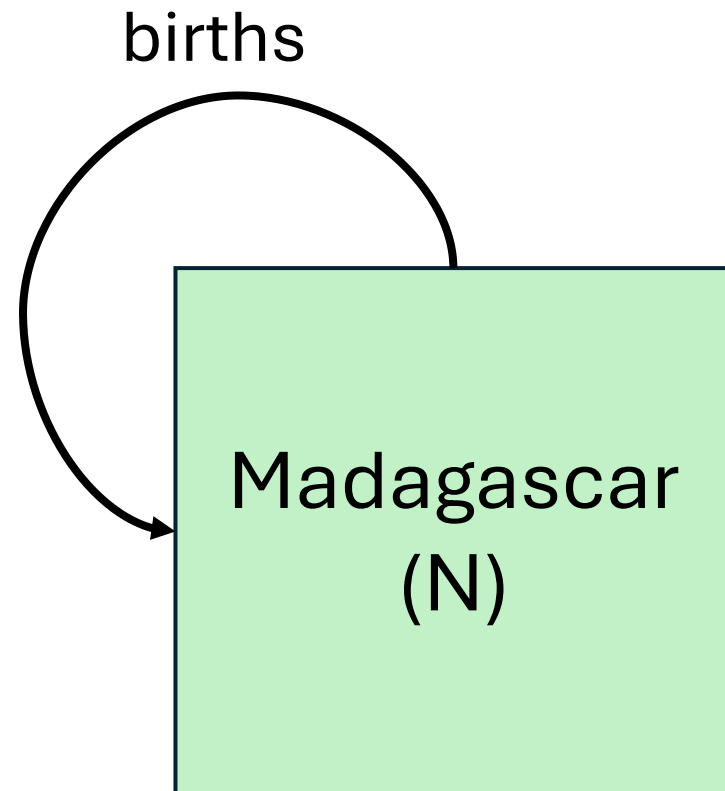
How does the population grow?

Comment est-ce que la population augmente?

The basic population model

Compartmental models (mechanistic models)


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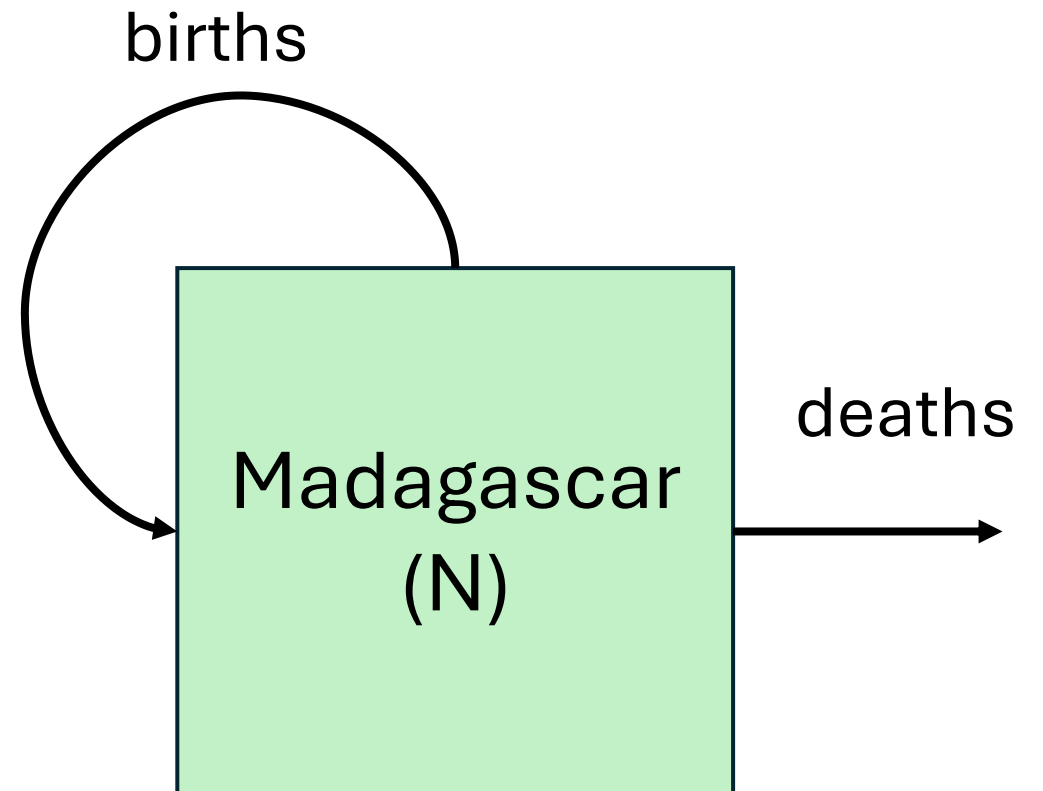
How does the population decrease?

Comment est-ce que la population diminue ?

The basic population model

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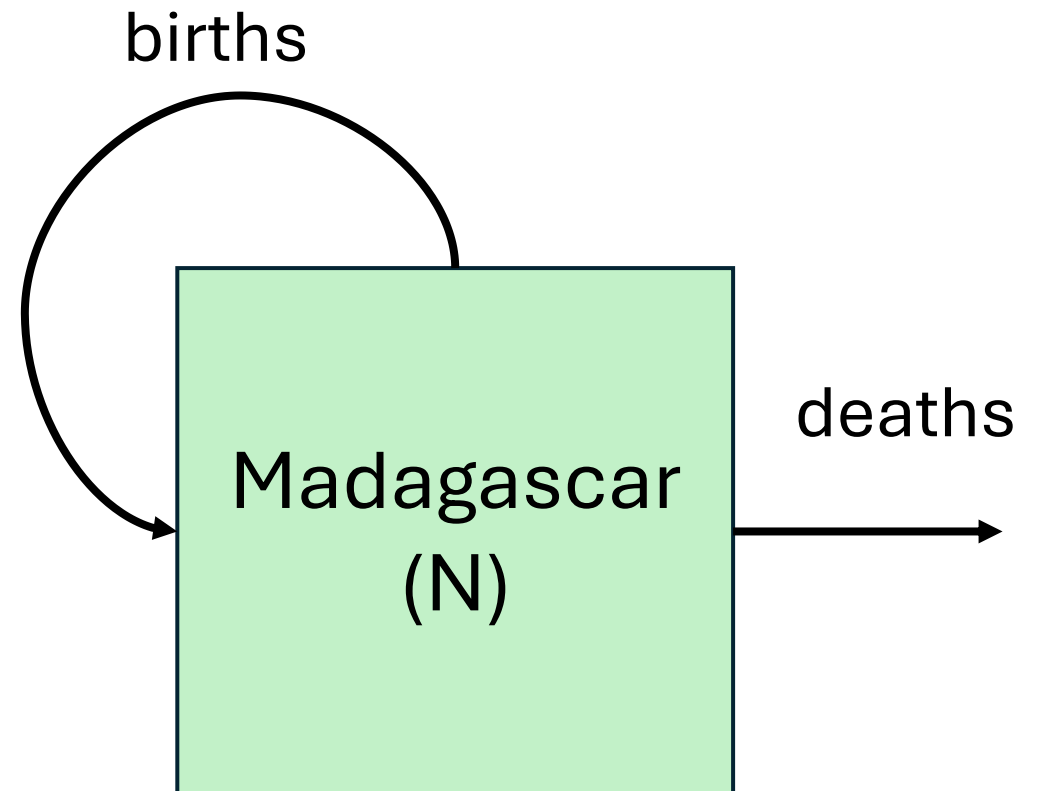
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What is a big assumption we are making here?

C'est quoi une hypothèse importante que nous faisons ici ?



The basic population model

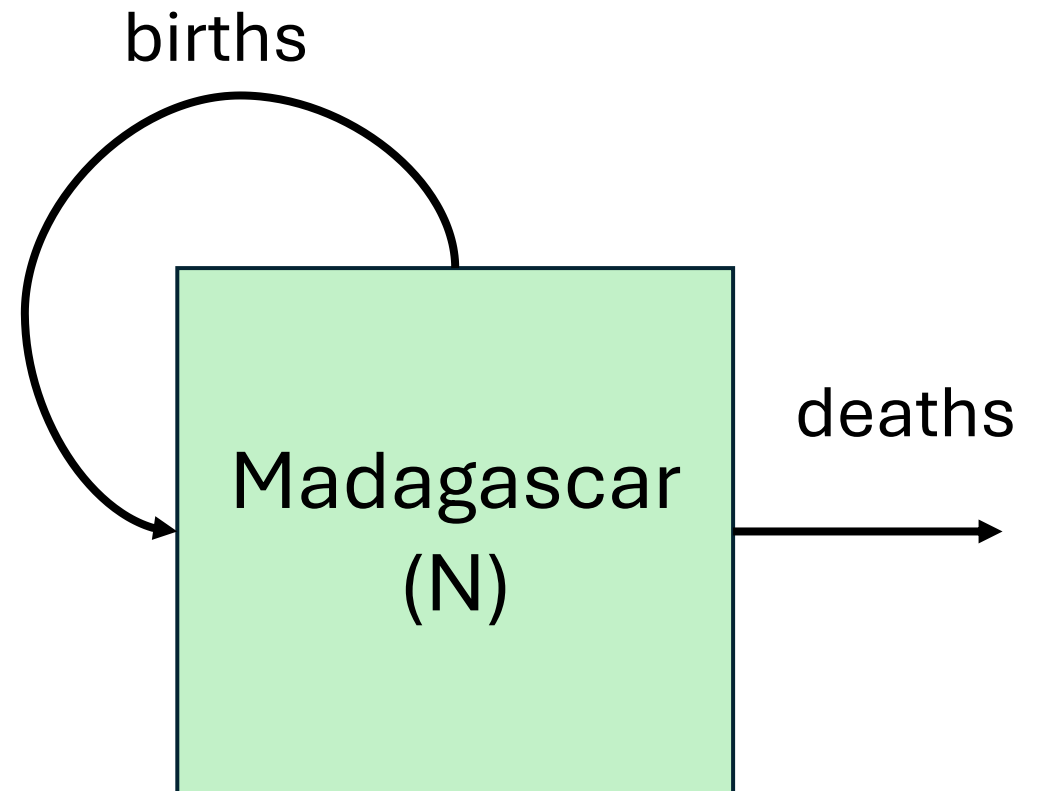
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Homogenous mixing
La mélange homogène

No immigration
Sans immigration

Same birth and death rate for each person
Les mêmes taux de natalité et de mortalité pour chaque personne

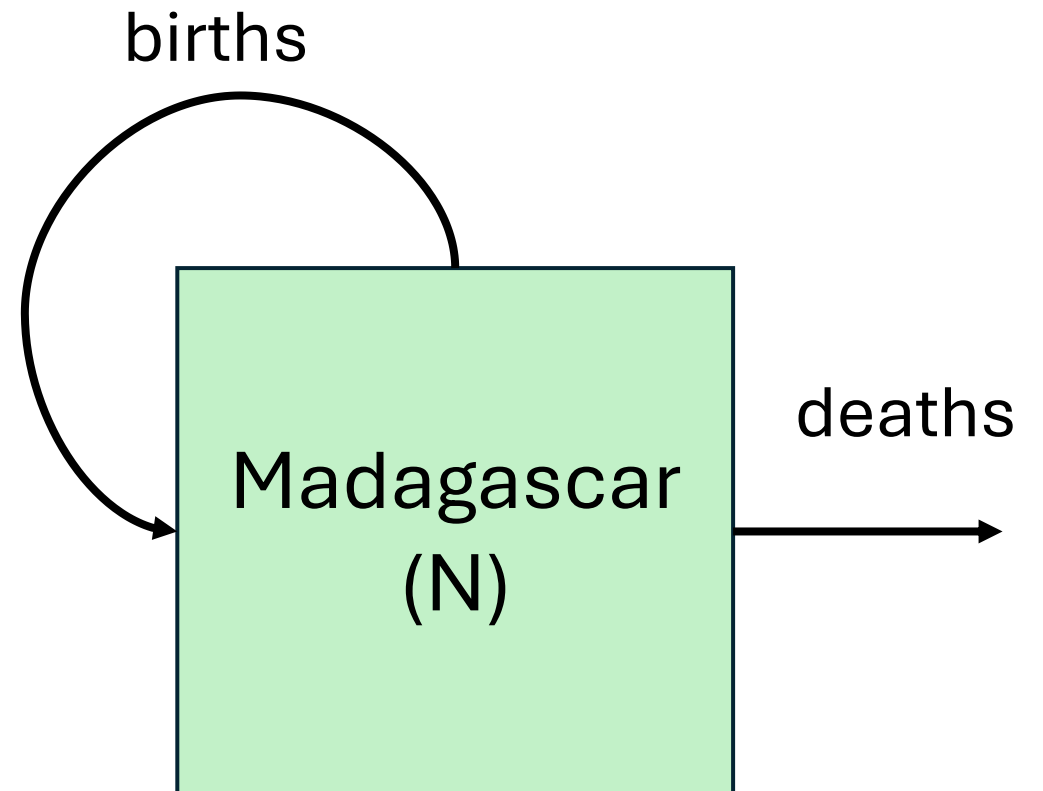


The basic population model

Compartmental models (mechanistic models)

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$$N_{t+1} =$$

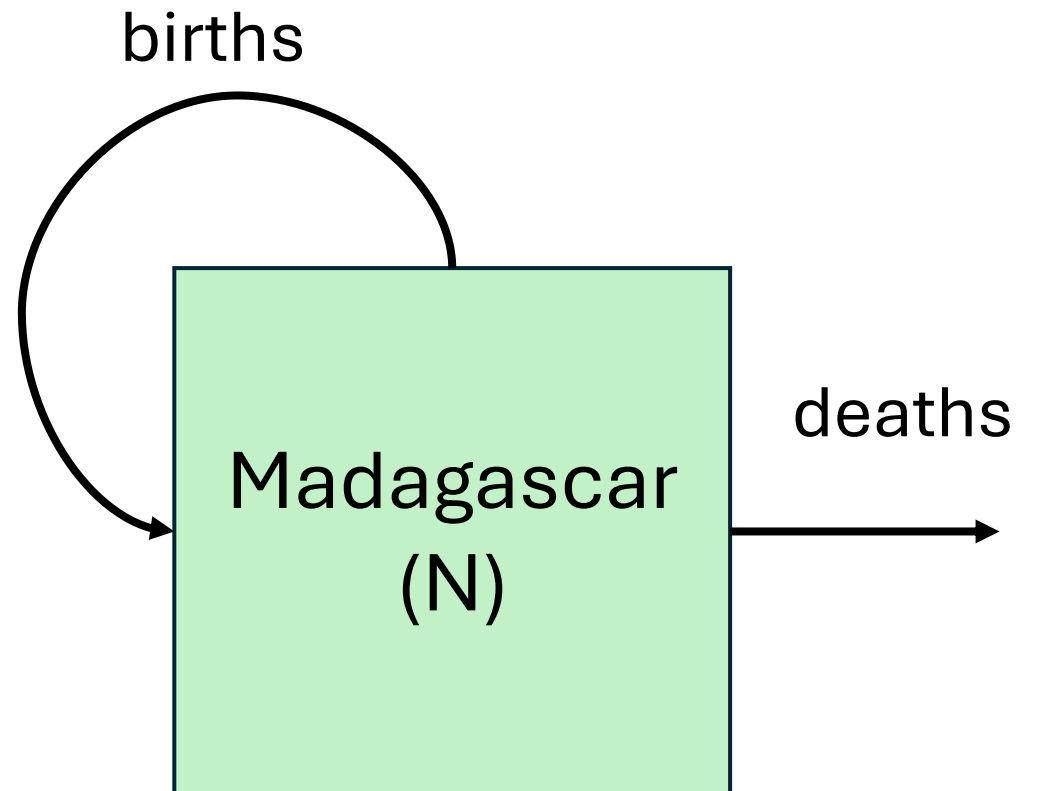


The basic population model

Compartmental models (mechanistic models)

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$$N_{t+1} = (\text{births}) * N_t$$

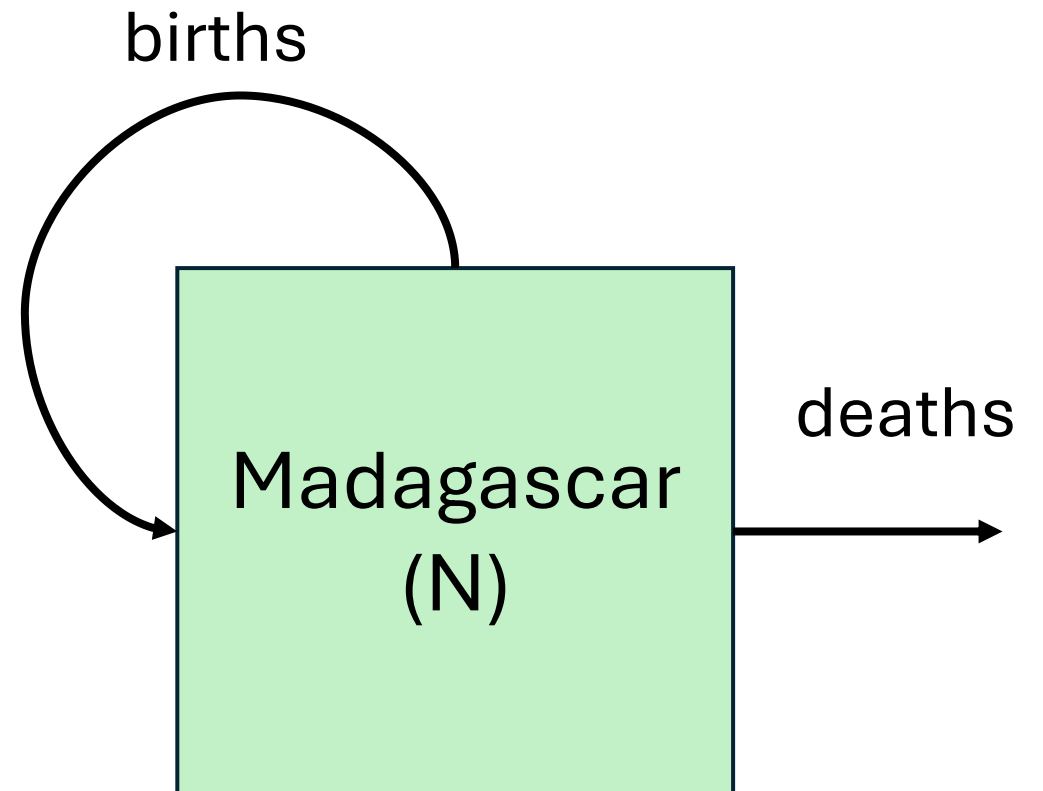


The basic population model

Compartmental models (mechanistic models)

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$$N_{t+1} = (\text{births}) * N_t - (\text{deaths}) * N_t$$



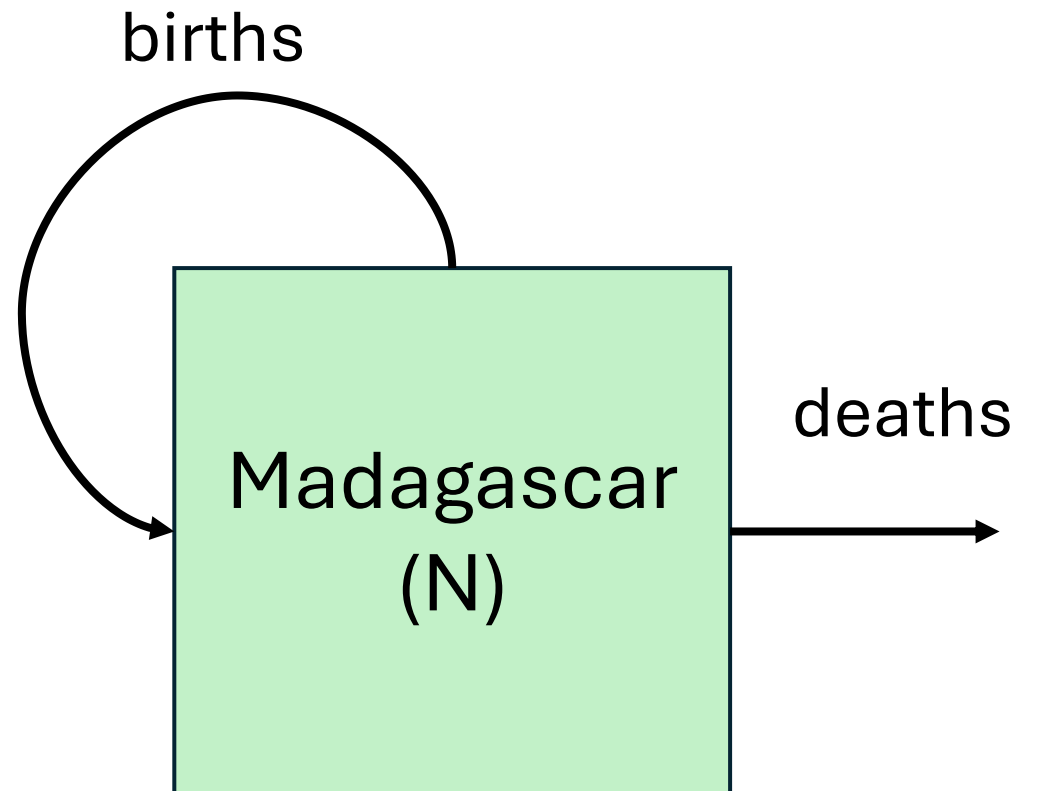
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The basic population model

Compartmental models (mechanistic models)

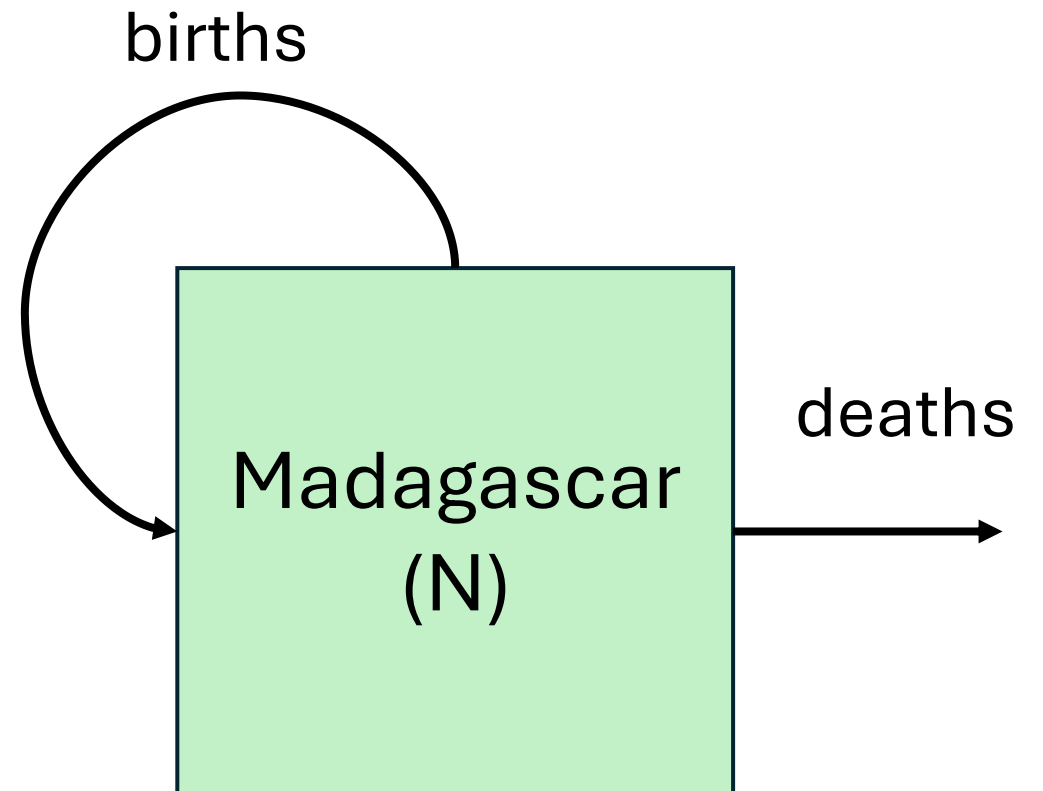
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$$N_{t+1} = (\text{births} - \text{deaths}) * N_t$$

$$N_{t+1} = \lambda * N_t$$

λ = pop intrinsic growth rate



Checking In

What are the main assumptions of a single population model?

Quelles sont les principales hypothèses d'un modèle d'une seule population?

What does lambda represent?

Que représente le lambda?



Checking In

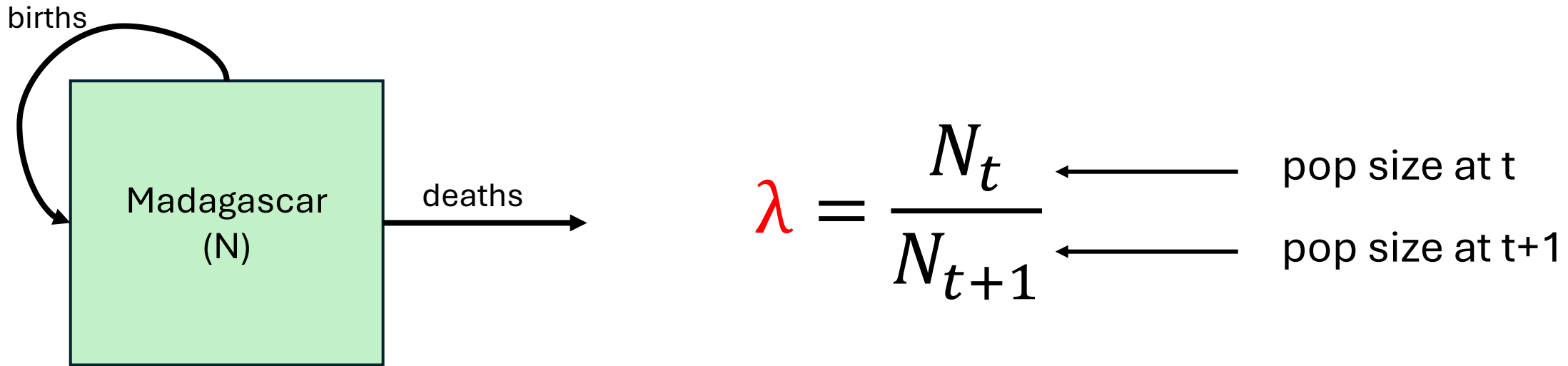
What are the main assumptions of a single population model?

- No immigration (*sans immigration*)
- Homogenous mixing (*mélange homogène*)
- Same birth and death rate for each person (*les mêmes taux de natalité et de mortalité pour chaque personne*)

What is lambda?

- Population intrinsic growth rate (*taux de croissance intrinsèque*)

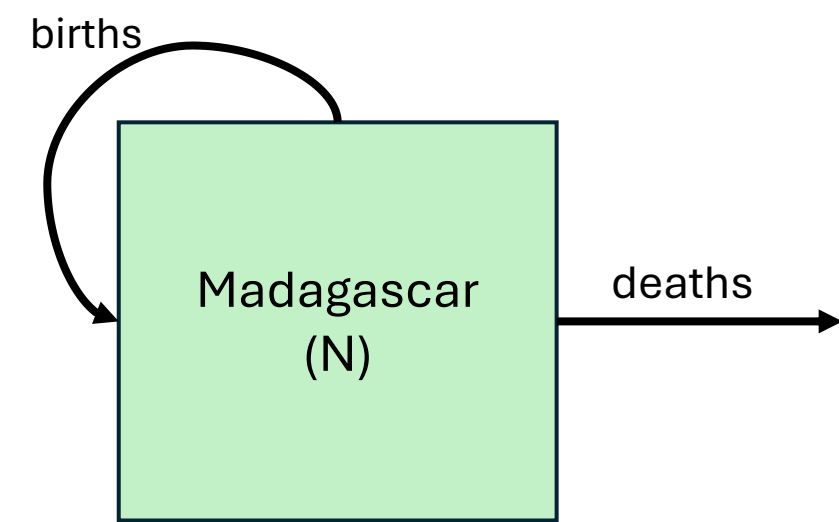
The basic population model



This is for one time step, how do we generalize this equation to work for all time steps?

C'est pour un pas de temps, comment généraliser cette équation pour tous les pas de temps?

The basic population model



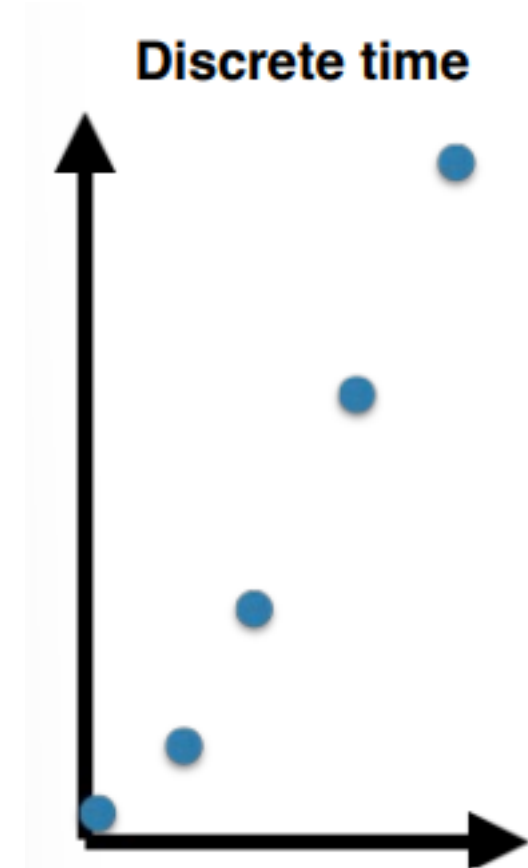
$$\lambda = \frac{N_t}{N_{t+1}}$$

$$N_1 =$$

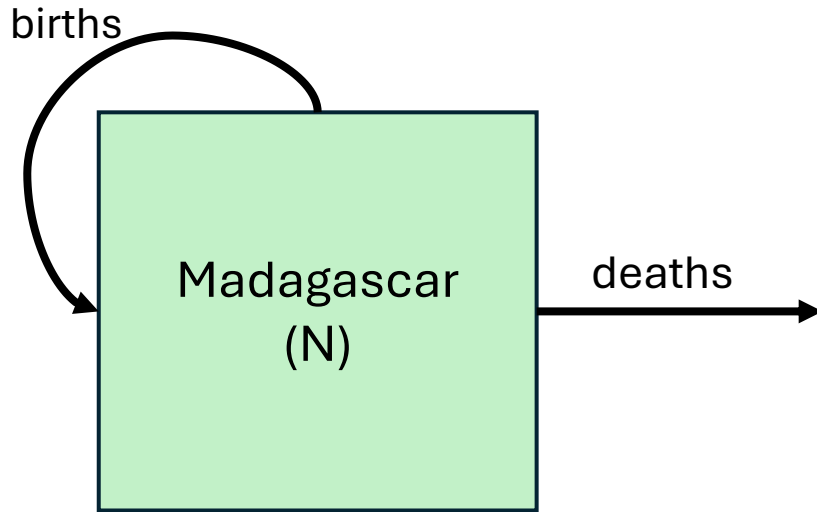
$$N_2 =$$

$$N_3 =$$

$$N_t =$$



The basic population model



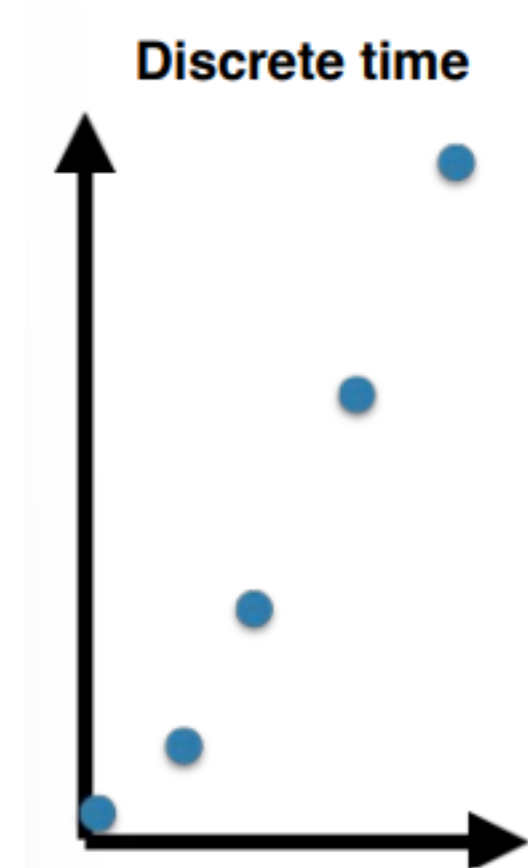
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$$N_1 = N_0 \lambda$$

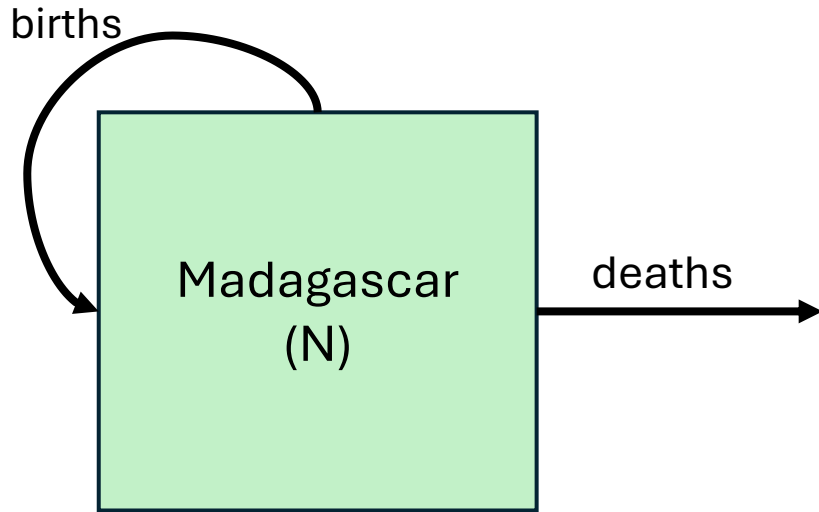
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The basic population model



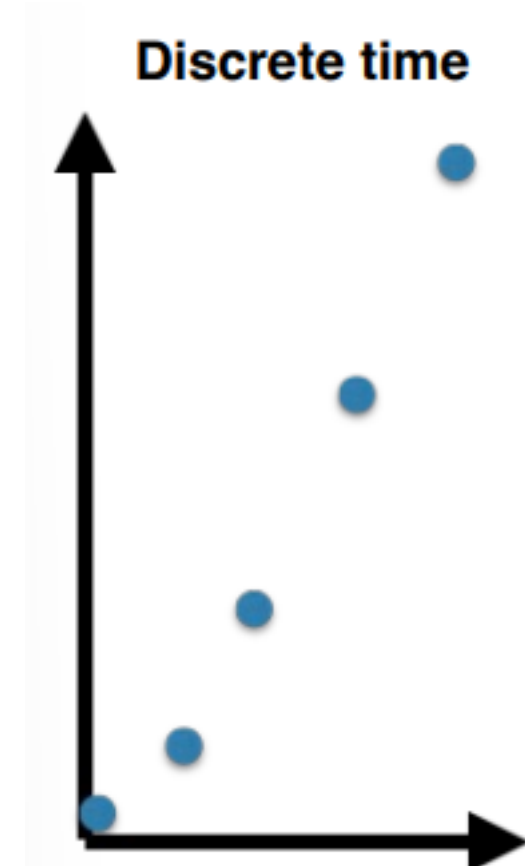
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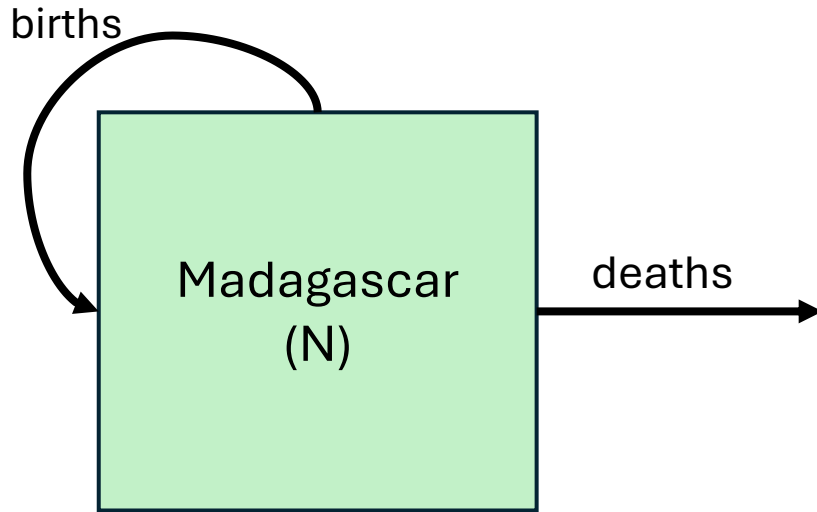
$$N_2 = N_1 \lambda = [N_0 \lambda] \lambda = \lambda^2 N_0$$

$$N_3 =$$

$$N_t =$$



The basic population model



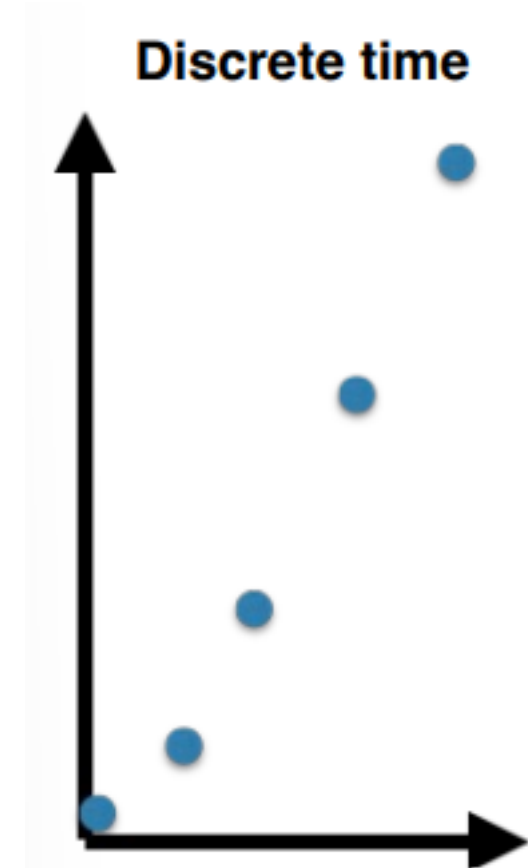
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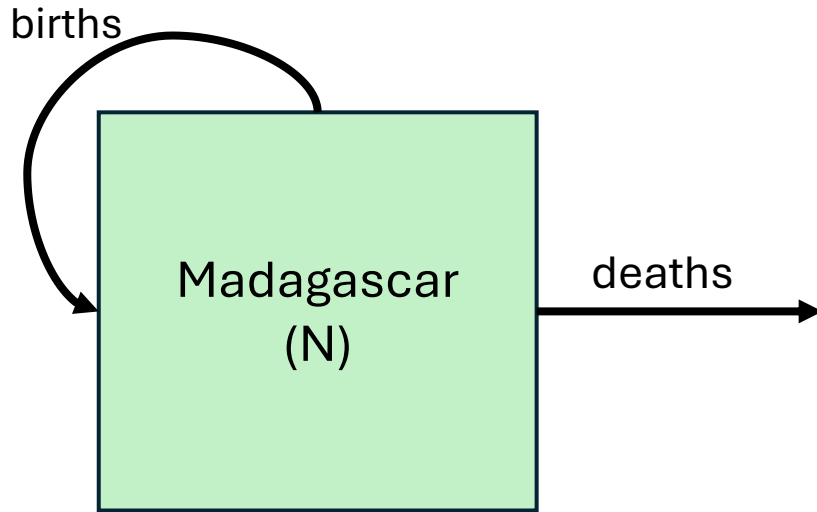
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The basic population model



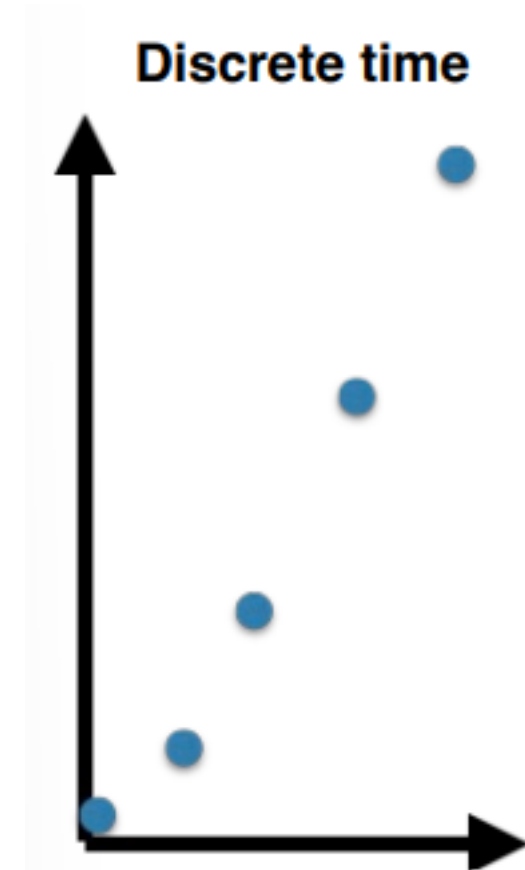
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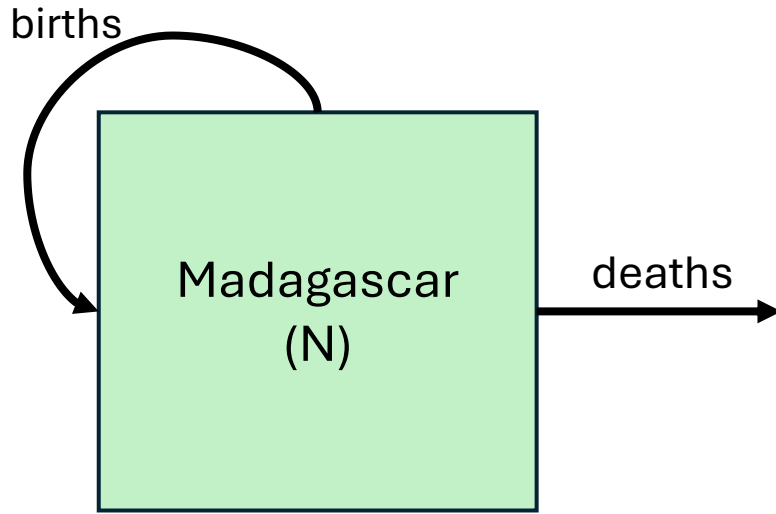
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$$N_3 = \lambda^3 N_0$$

$$N_t = \lambda^t N_0$$



The basic population model



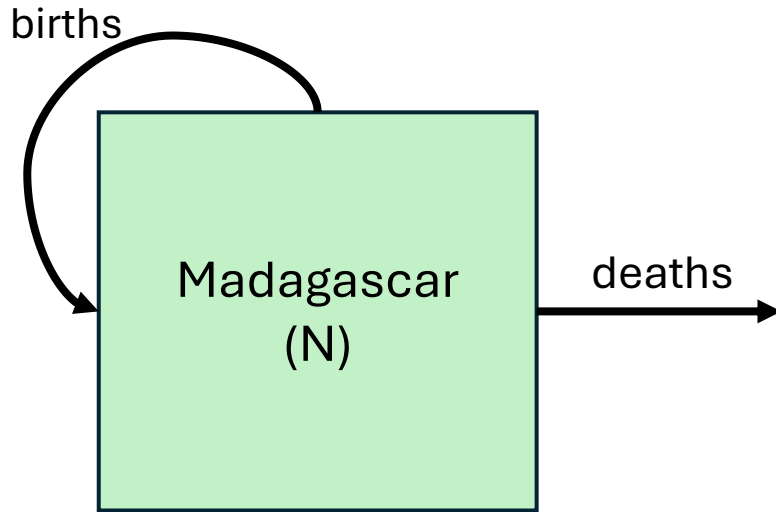
$$\lambda = \frac{N_t}{N_{t+1}}$$

$$N_t = \lambda^t N_0$$

What if we want to know
the population size for
any time t ?
Not just where we have
data?

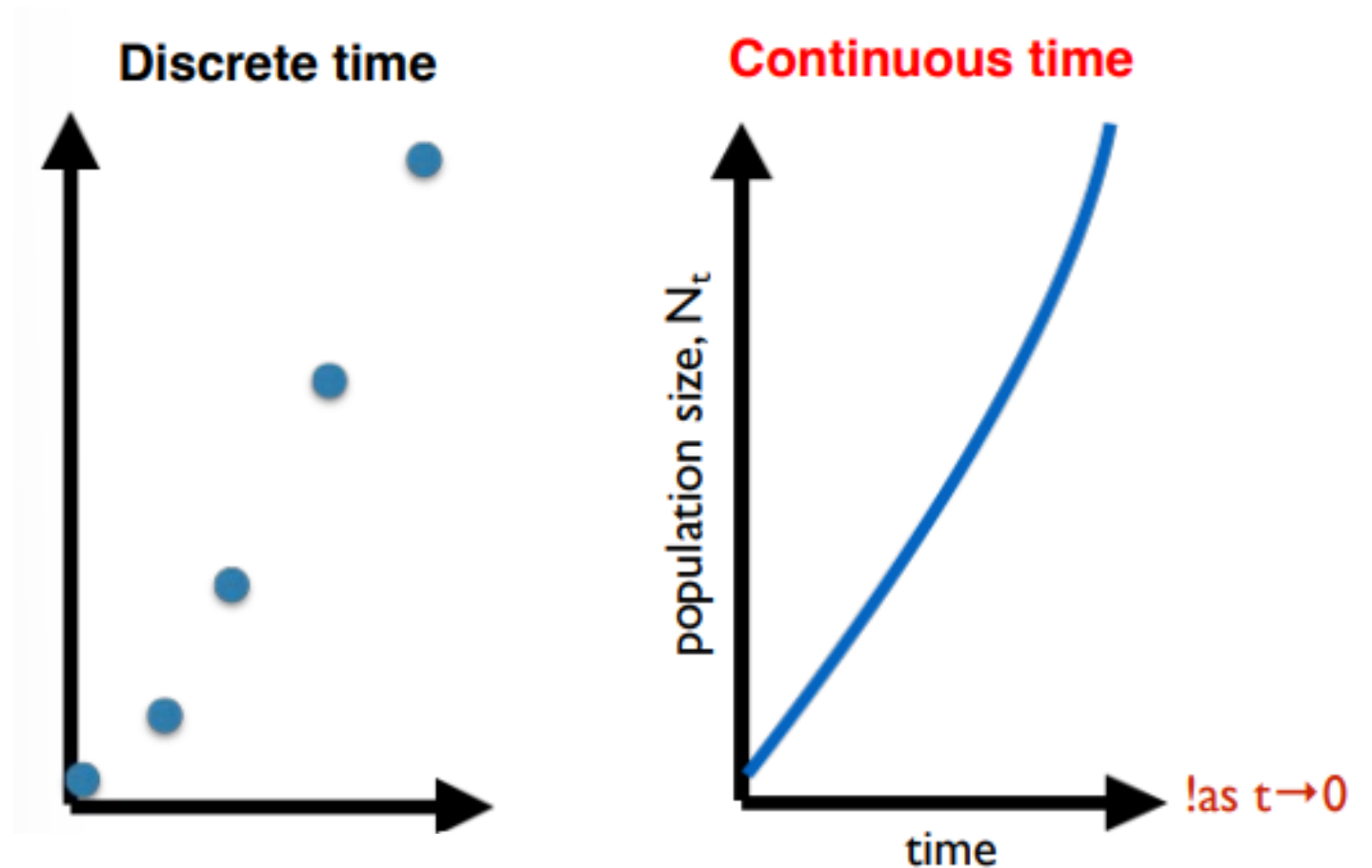


The basic population model

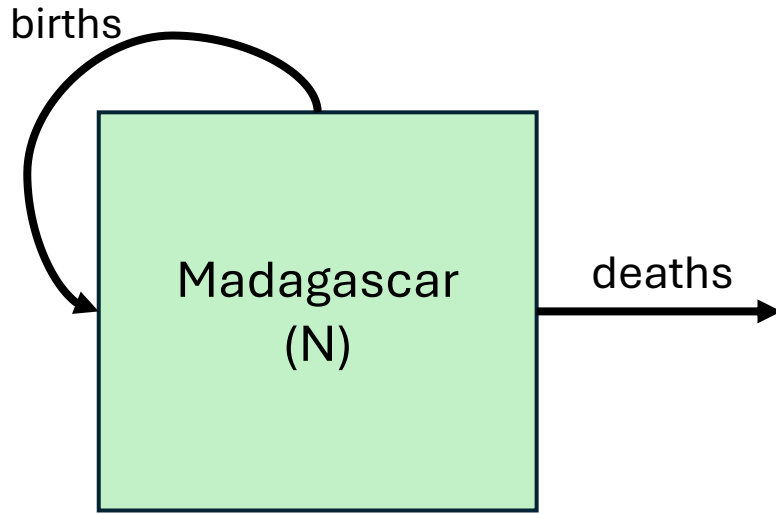


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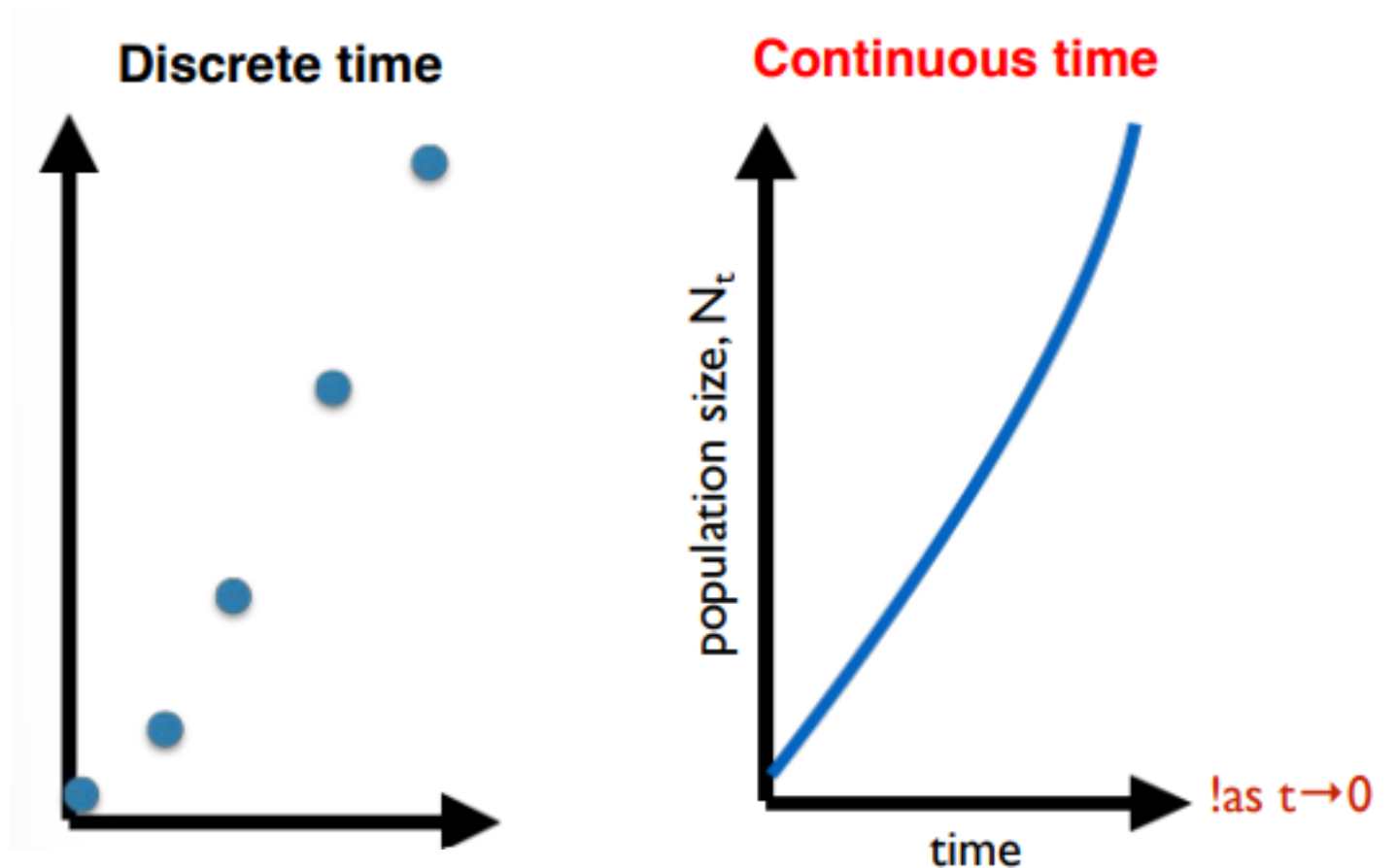


The basic population model



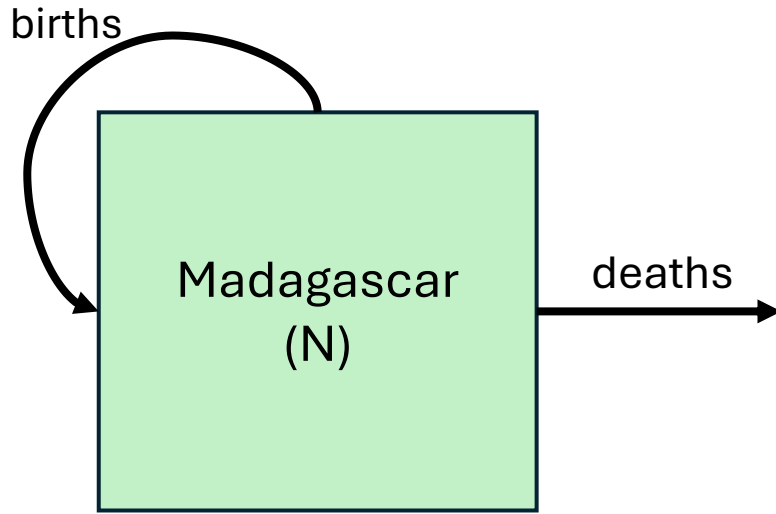
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$$N_t = \lambda^t N_0$$



How do we get the same type of equation for continuous time?

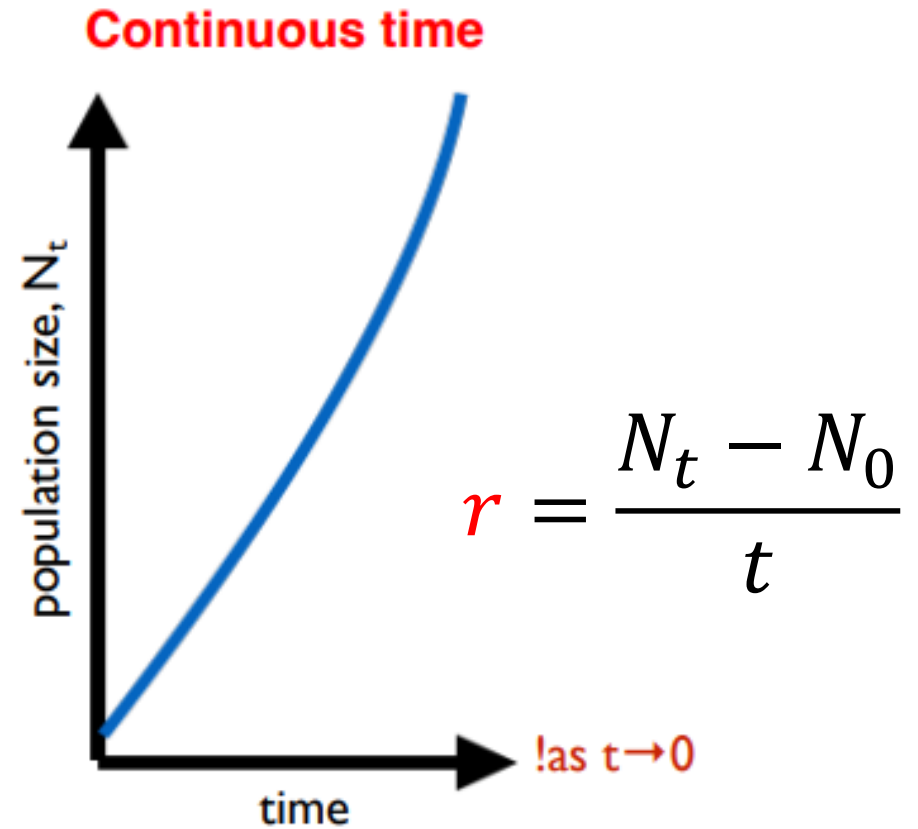
The basic population model



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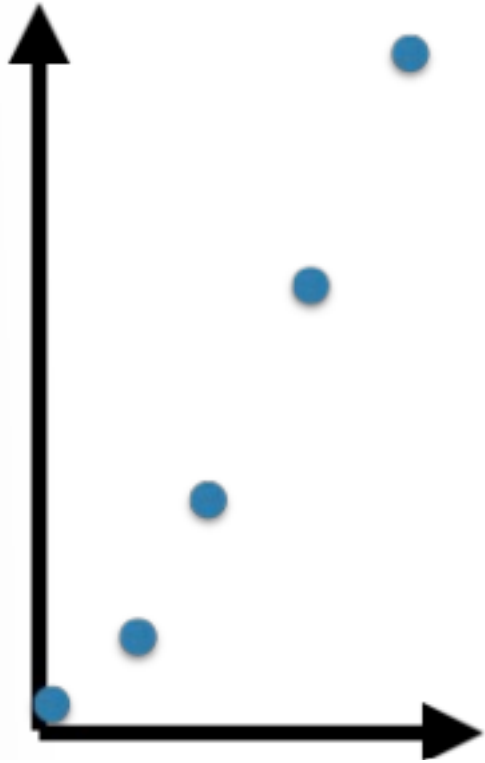


$$r = \frac{N_t - N_0}{t}$$

$$\frac{dN}{dt} = rN$$

The basic population model

Discrete time



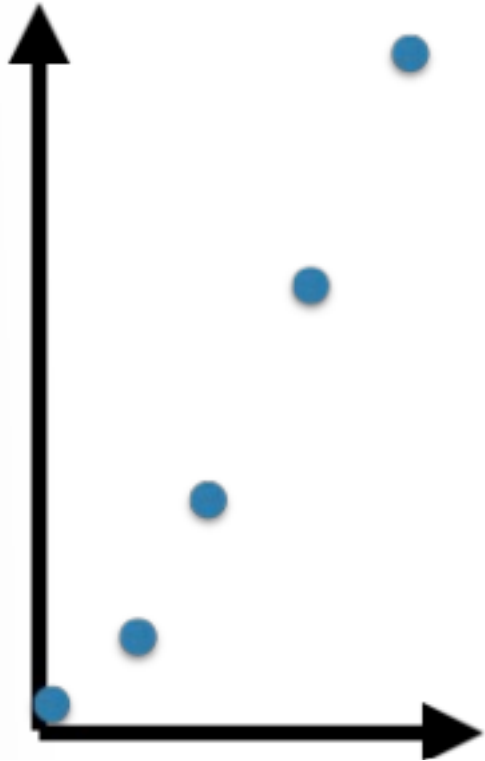
$$N_t = \lambda^t N_0$$

Continuous time

$$\frac{dN_t}{dt} = rN_t$$

The basic population model

Discrete time



$$N_t = \lambda^t N_0$$

Continuous time

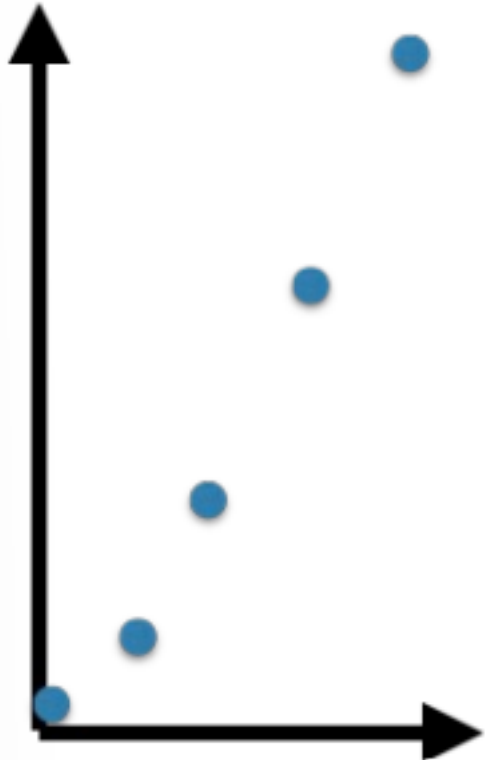
$$\frac{dN_t}{dt} = rN_t$$

Separation of variables

$$\frac{dN_t}{N_t} = r dt$$

The basic population model

Discrete time



$$N_t = \lambda^t N_0$$

Continuous time

$$\frac{dN_t}{dt} = rN_t$$

Separation of variables

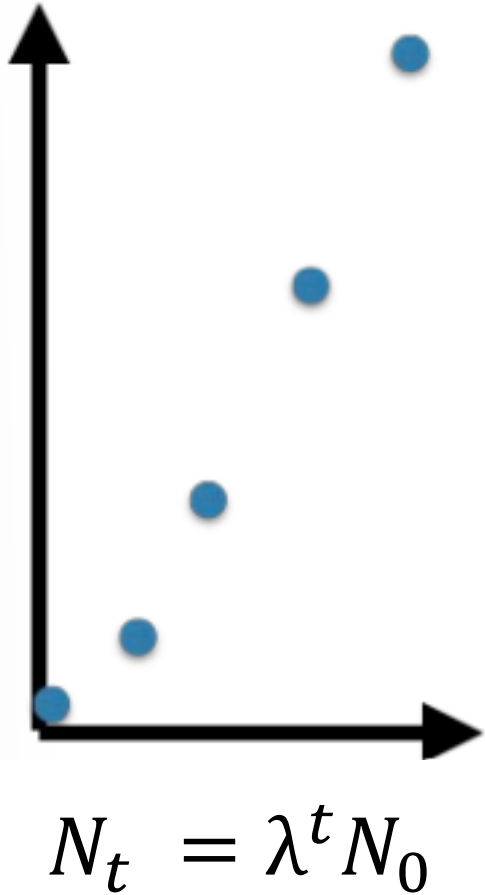
$$\frac{dN_t}{N_t} = r dt$$

Integrate both sides

$$\int_{N_0}^{N_t} \frac{dN_t}{N_t} = r \int_0^t dt$$

The basic population model

Discrete time



Continuous time

$$\frac{dN_t}{dt} = rN_t$$

Separation of variables

$$\frac{dN_t}{N_t} = r dt$$

Integrate both sides

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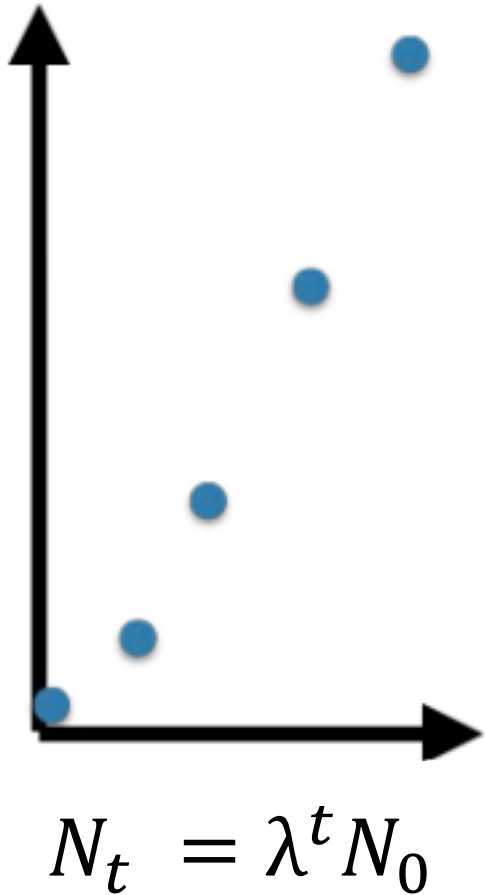
By definition

$$\ln N_t - \ln N_0 = rt$$

$$\ln \frac{N_t}{N_0} = rt$$

The basic population model

Discrete time



Continuous time

$$\frac{dN_t}{dt} = rN_t$$

Separation of variables

$$\frac{dN_t}{N_t} = r dt$$

Integrate both sides

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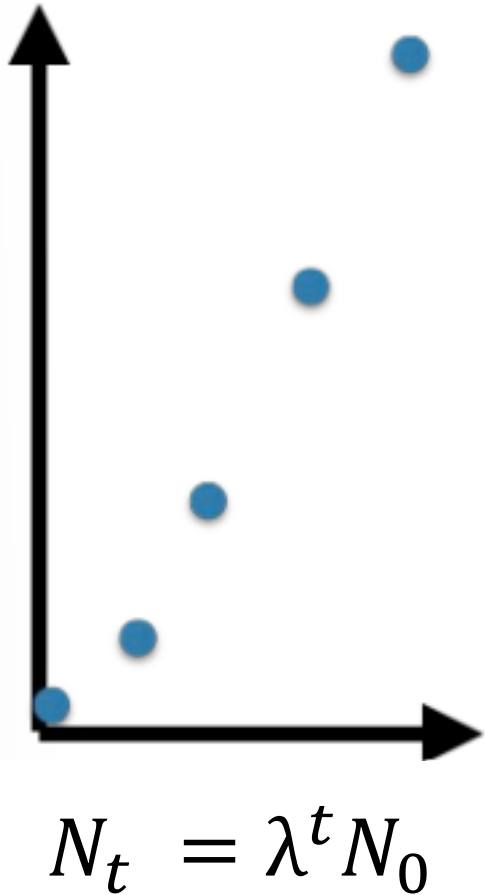
$$\ln \frac{N_t}{N_0} = rt$$

Exponentiate

$$\frac{N_t}{N_0} = e^{rt}$$

The basic population model

Discrete time



Continuous time

$$\frac{dN_t}{dt} = rN_t$$

Separation of variables

$$\frac{dN_t}{N_t} = r dt$$

Integrate both sides

$$\int_{N_0}^{N_t} \frac{dN_t}{N_t} = r \int_0^t dt$$

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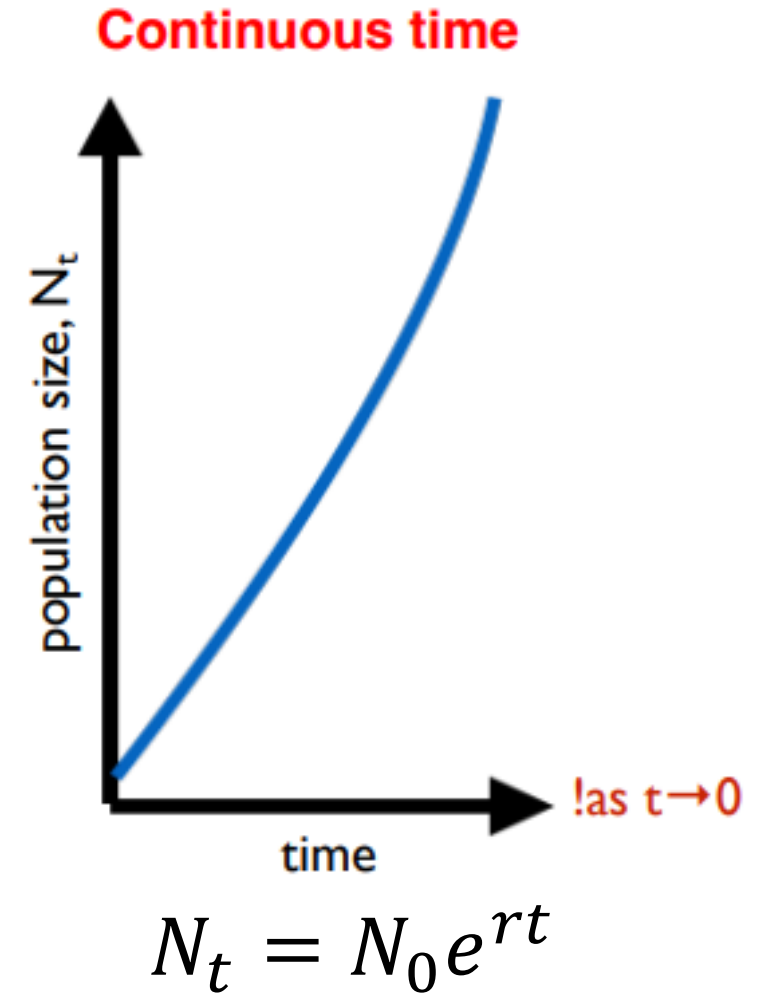
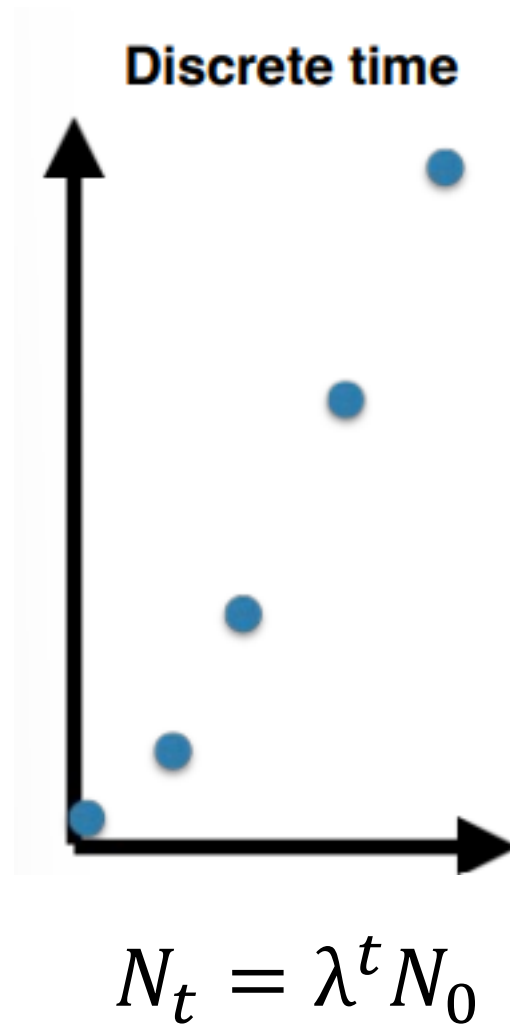
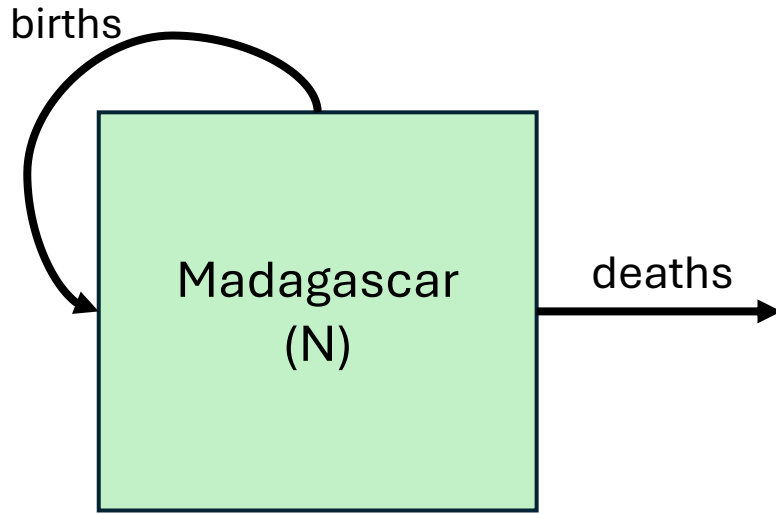
Exponentiate

$$\frac{N_t}{N_0} = e^{rt}$$

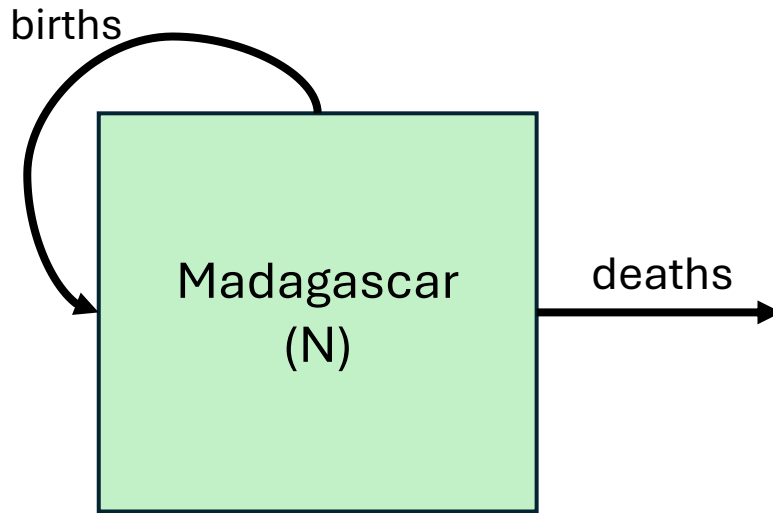
Solve for $N(t)$:

$$N_t = N_0 e^{rt}$$

The basic population model



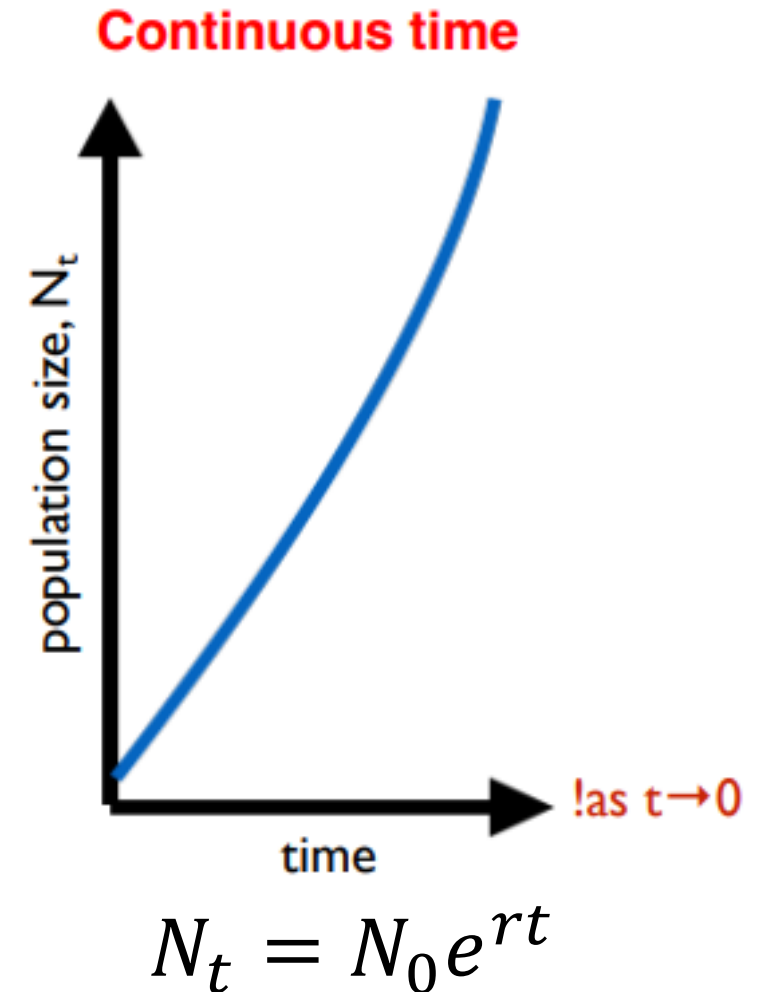
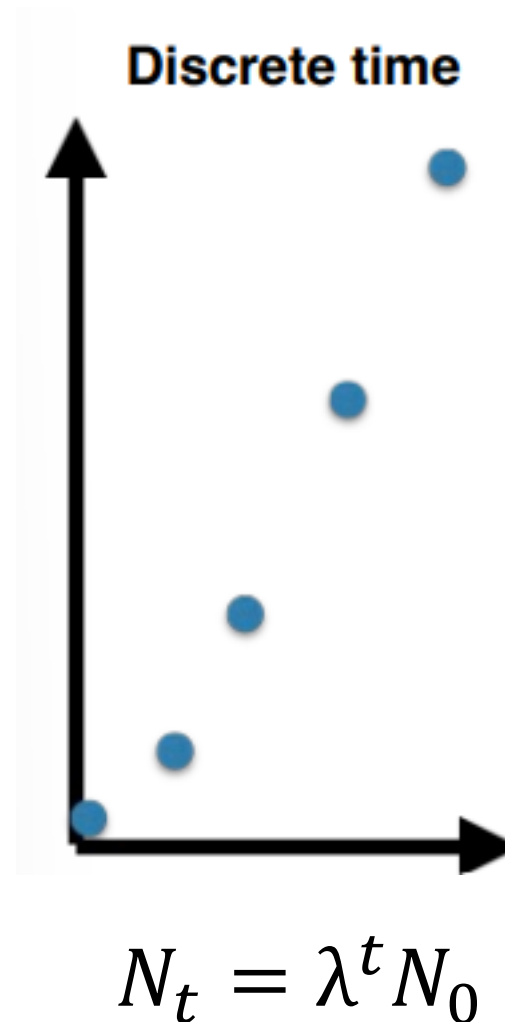
The basic population model



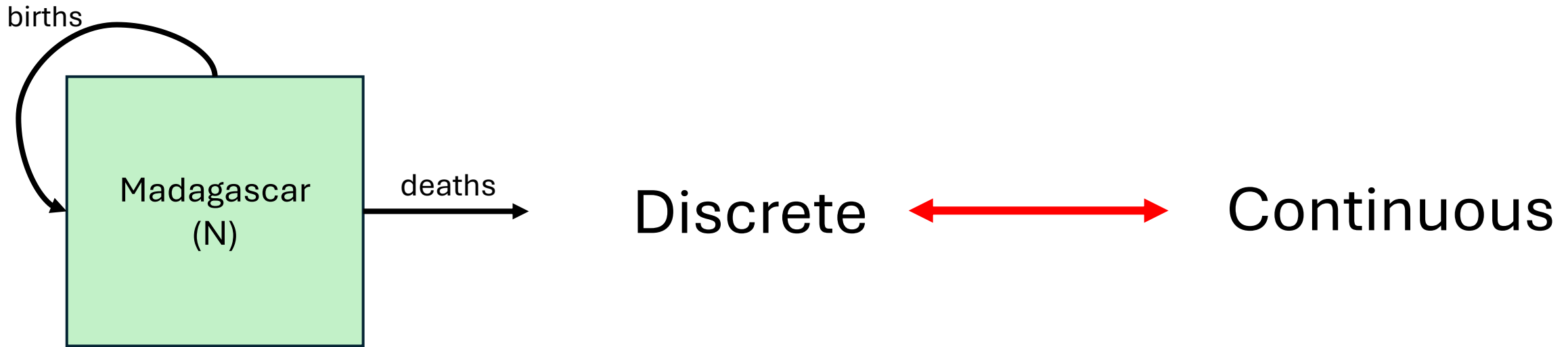
When does a population grow?

Discrete: $\lambda > 1$

Continuous: $r > 0$



The basic population model



Continuous models can be discretized; discrete models can be approximated by continuous ones. The appropriate choice may depend on the data/question.

Les modèles à temps continu peuvent être discrétisés; les modèles à temps discret peuvent être approximatés par ceux à temps continu. Le choix approprié peut dépendre des données/de la question.

Checking In

What is the difference between discrete and continuous models?

Quelle est la différence entre les modèles à temps discret et à temps continu?

What math is used in discrete population models?

Continuous population models?

Quel type de mathématique est utilisé dans les modèles à temps discret et à temps continu?



Checking In

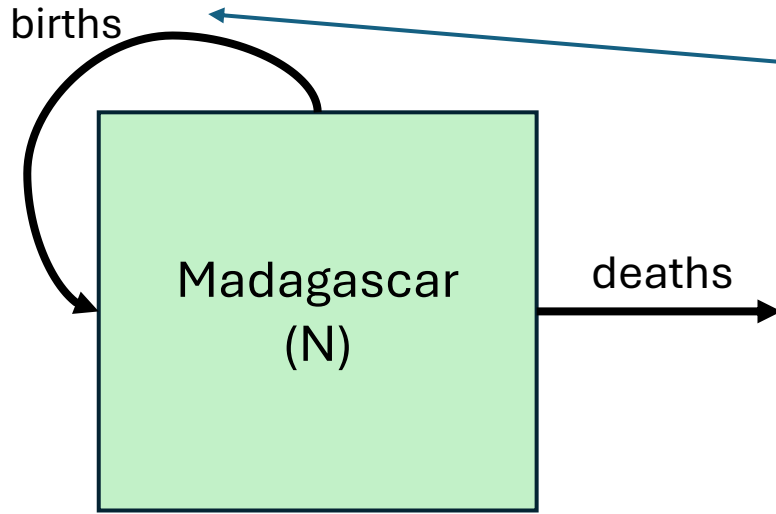
What is the difference between discrete and continuous models?

- Discrete: state variable only changes at distinct time steps
- Continuous: state variables change continuously (tiny tiny time steps)

What math is used in discrete population models? Continuous population models?

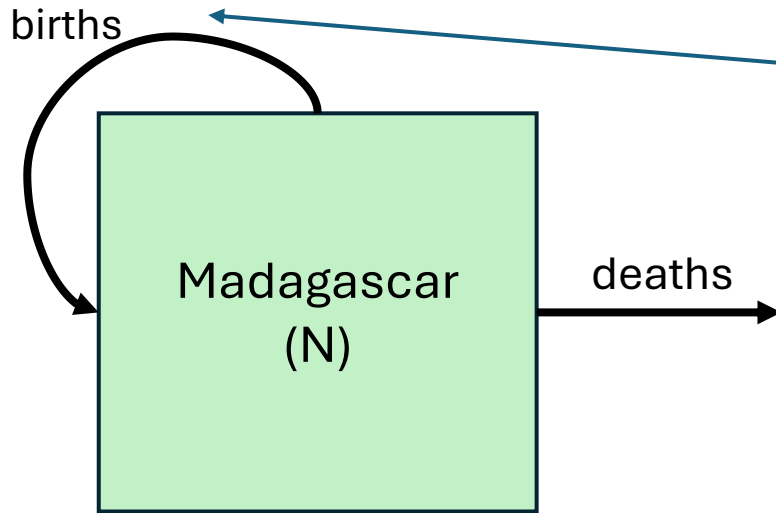
- Algebra, Calculus

The basic population model



What about these rates?
Are they the same every
year? And in every person?
Why might they be
different?

The basic population model



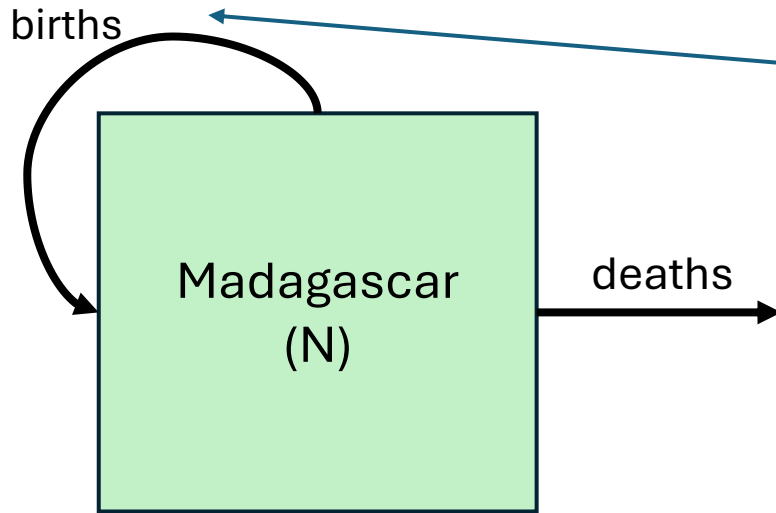
What about these rates?
Are they the same every year? And in every person?
Why might they be different?

Reproductive age (âge de procréer)

Death rate increases with age (le taux de mortalité s'accroît avec l'âge)

Diseases/other health factors (les maladies / d'autres facteurs de santé)

The basic population model



What about these rates?
Are they the same every year? And in every person?
Why might they be different?

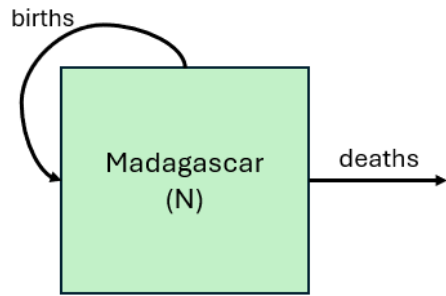
Reproductive age (âge de procréer)

Death rate increases with age (le taux de mortalité s'accroît avec l'âge)

Diseases/other health factors (les maladies / d'autres facteurs de santé)

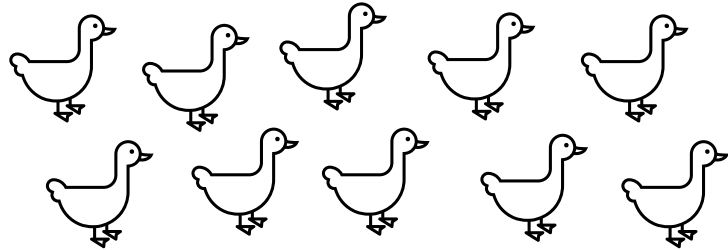
How do we incorporate random variation in these rates?

Comment intégrer la variation aléatoire de ces taux ?



The basic population model

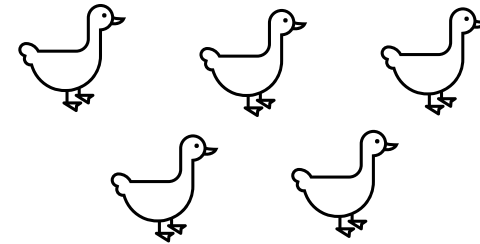
starting population

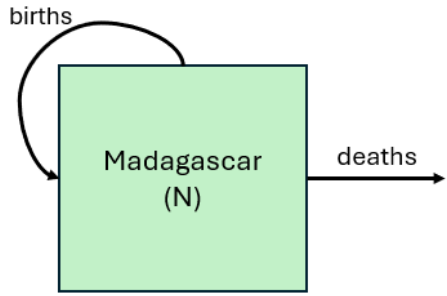


if **deterministic** “always the same”



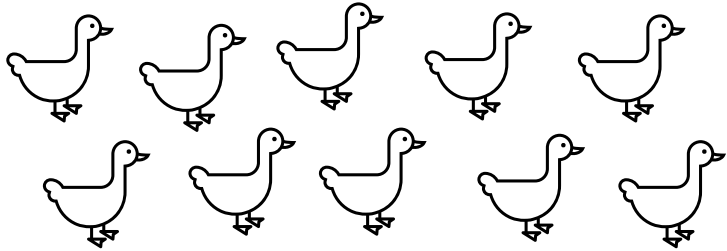
probability of death = 0.5





The basic population model

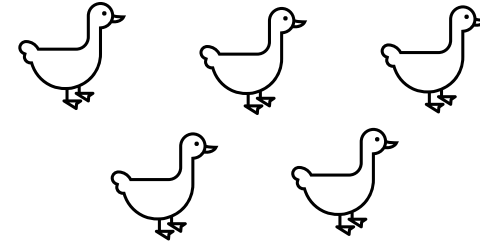
starting population



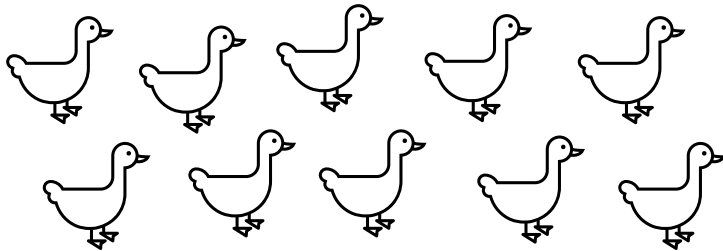
if **deterministic** “always the same”



probability of death = 0.5



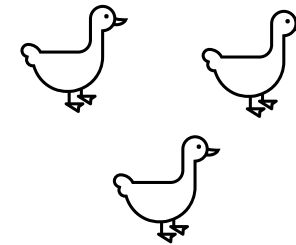
starting population

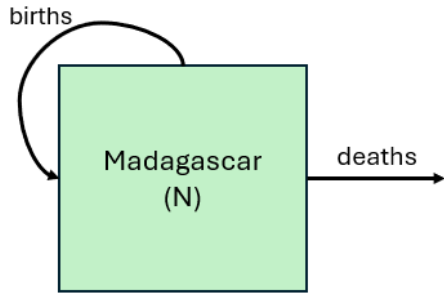


if **stochastic** “up to chance”



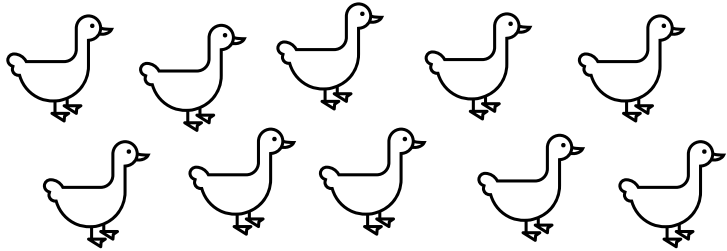
probability of death = 0.5





The basic population model

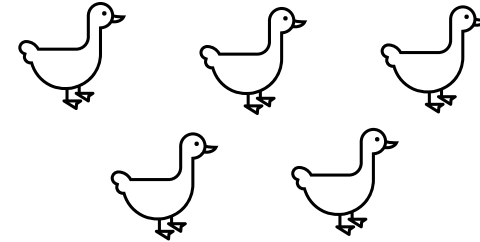
starting population



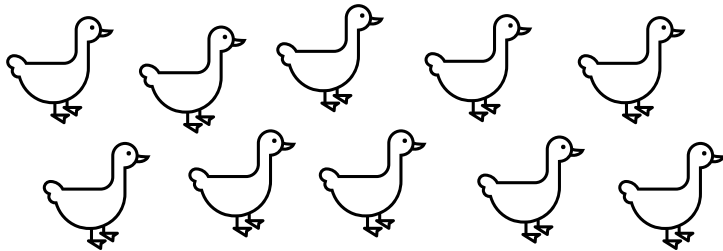
if **deterministic** “always the same”



probability of death = 0.5



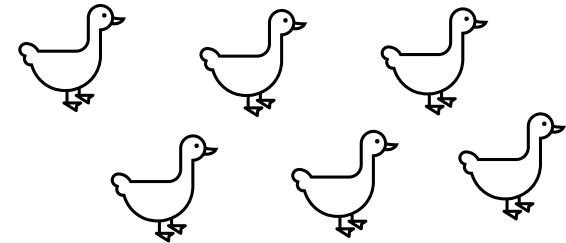
starting population

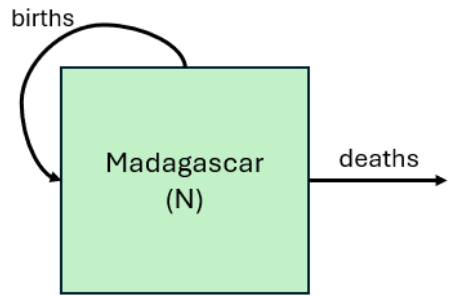


if **stochastic** “up to chance”



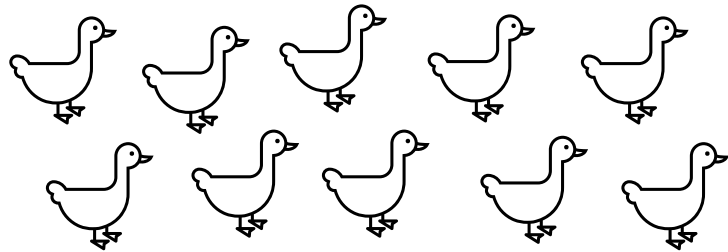
probability of death = 0.5





The basic population model

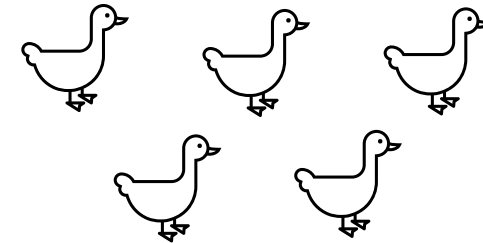
starting population



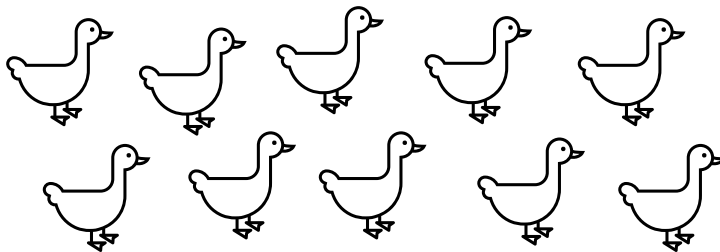
if **deterministic** “always the same”



probability of death = 0.5



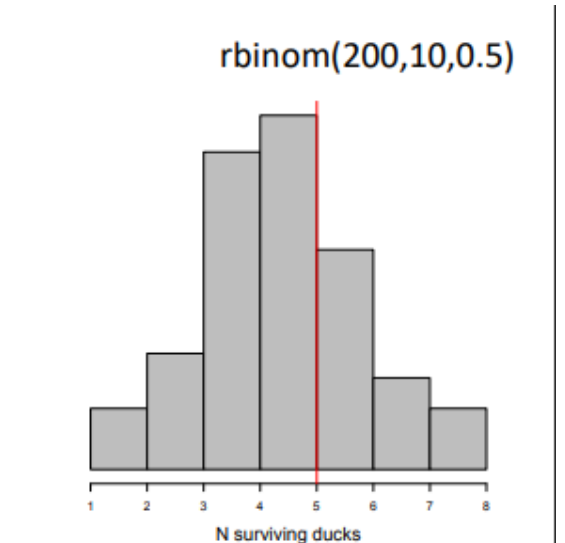
starting population

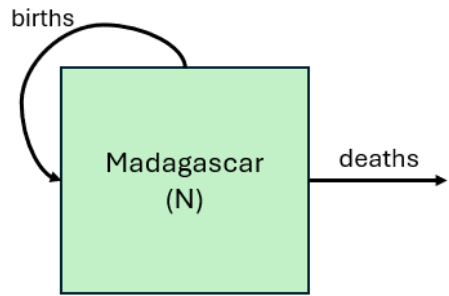


if **stochastic**



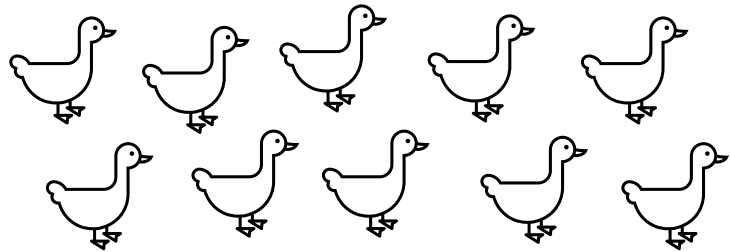
probability of death = 0.5





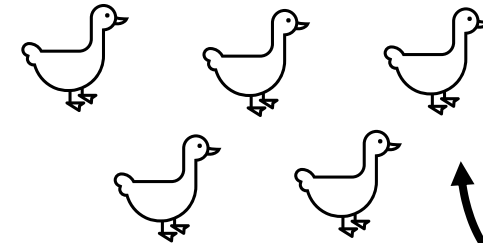
The basic population model

starting population



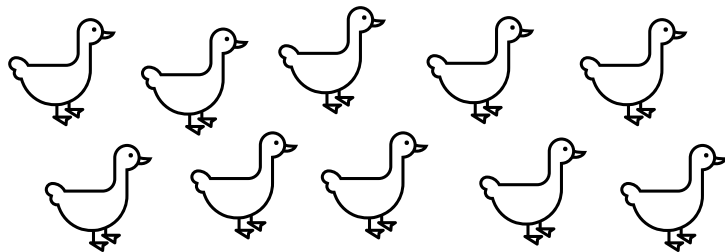
probability of death = 0.5

if **deterministic** “always the same”



If you test your 10 ducks many times, on average you get 5

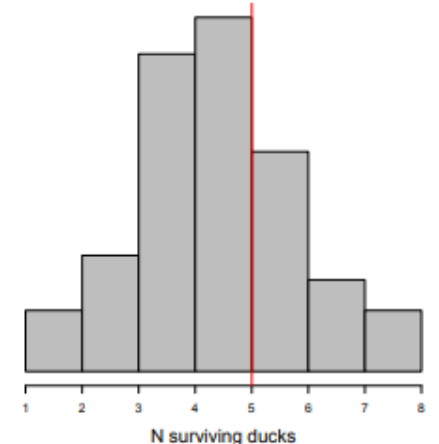
starting population



probability of death = 0.5

if **stochastic**

`rbinom(200,10,0.5)`



Checking In

What is the difference between deterministic and stochastic?
Quelle est la difference entre déterministe et stochastique?



Checking In

What is the difference between deterministic and stochastic?

- Deterministic = always the same
- Stochastic = up to chance

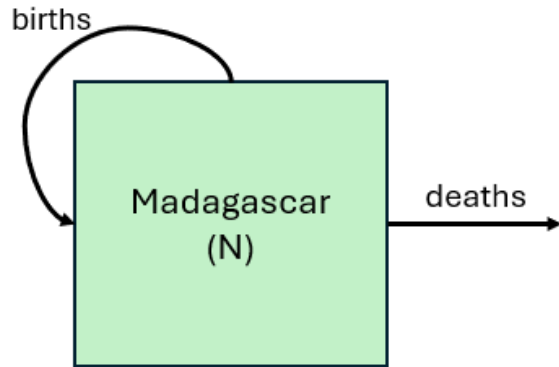
Key concepts

- Compartmental / mechanistic / mathematical models
Modèles à compartiments
- Continuous vs. discrete models
Les modèles à temps discret et les modèles à temps continu
- Deterministic vs. stochastic models
Modèles déterministe vs. stochastique

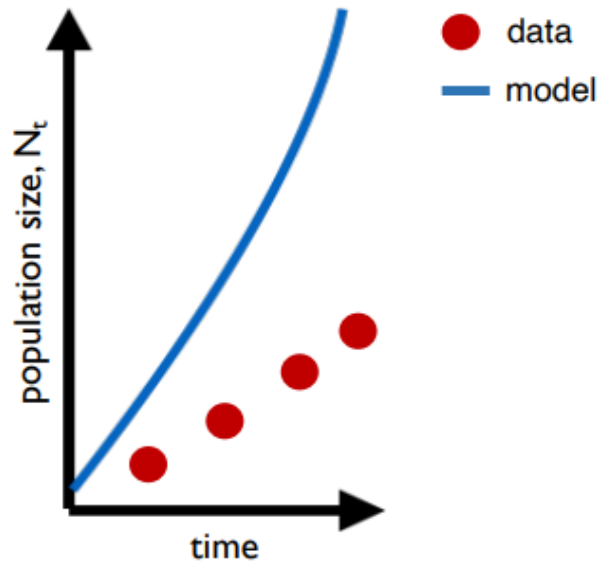
2. Structured Population Models

Les modèles de la population structurées

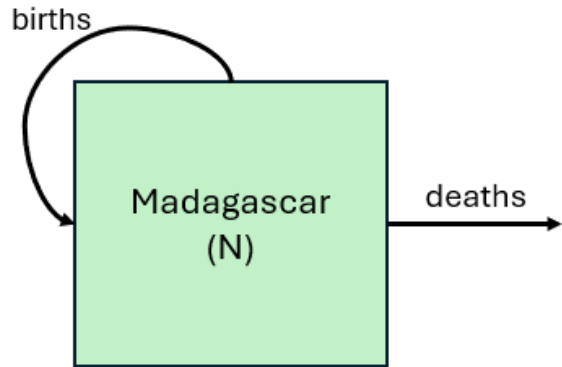
The structured population model



Why does the model perform poorly?

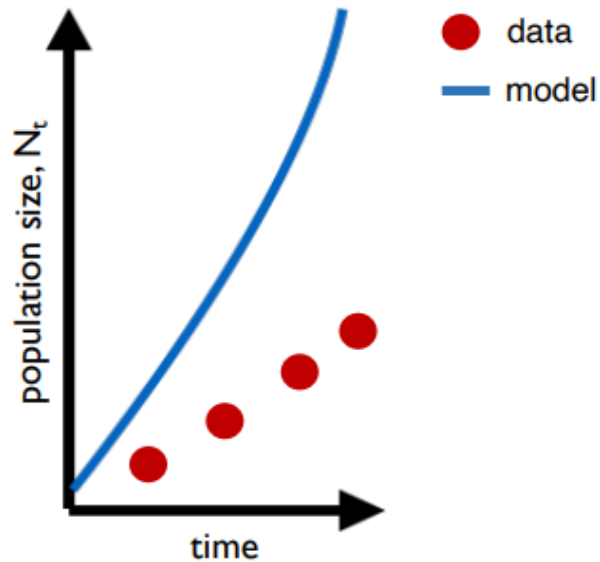


The structured population model



Why does the model perform poorly?

We need population structure!



That means distinguishing babies from adults.

The basic population model

Compartmental models (mechanistic models)

1. Populations are divided into compartments
2. Individuals within a compartment are homogenously mixed
3. Compartments and transition rates are determined by biological systems
4. Rates of transferring between compartments are expressed mathematically

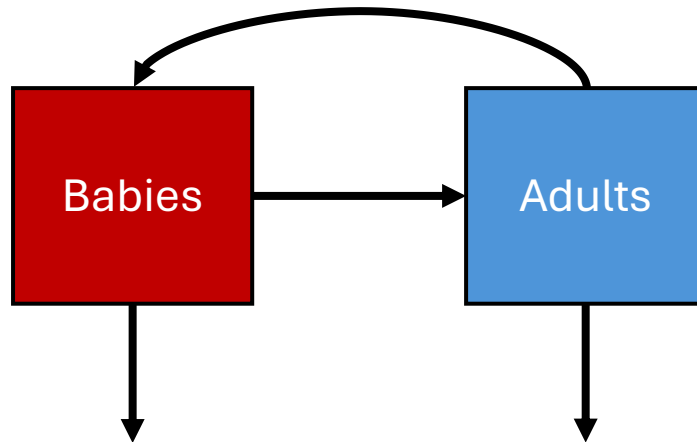
How does the population of Madagascar grow over time?

Comment est-ce que la population de Madagascar croît avec le passage du temps ?

The basic population model

Compartmental models (mechanistic models)

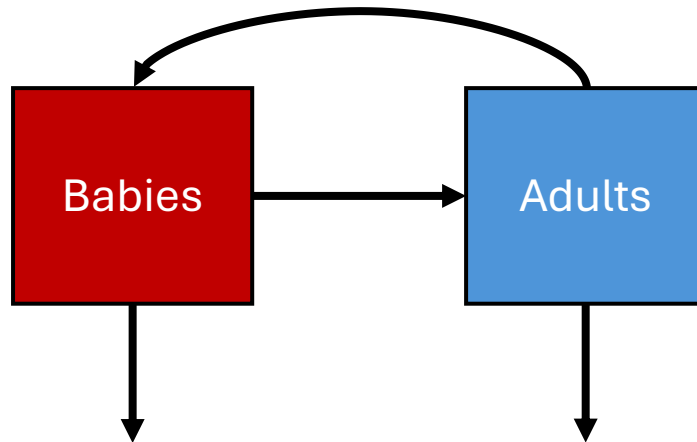
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The basic population model

Compartmental models (mechanistic models)

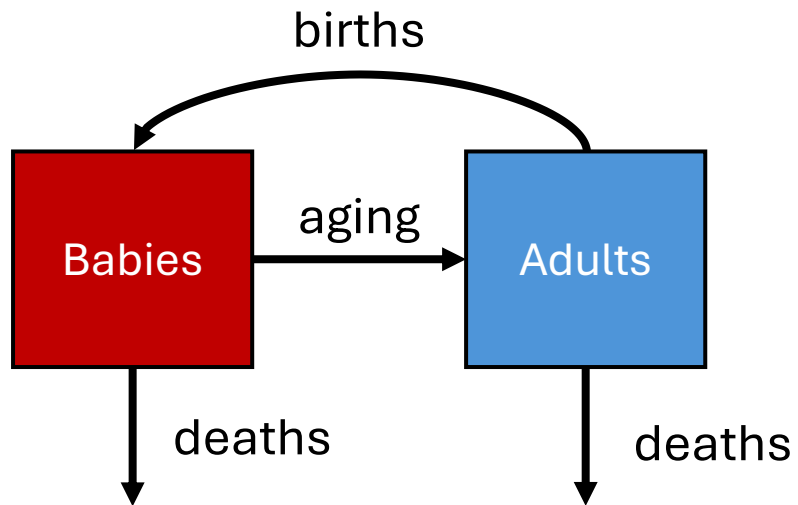
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The basic population model

Compartmental models (mechanistic models)

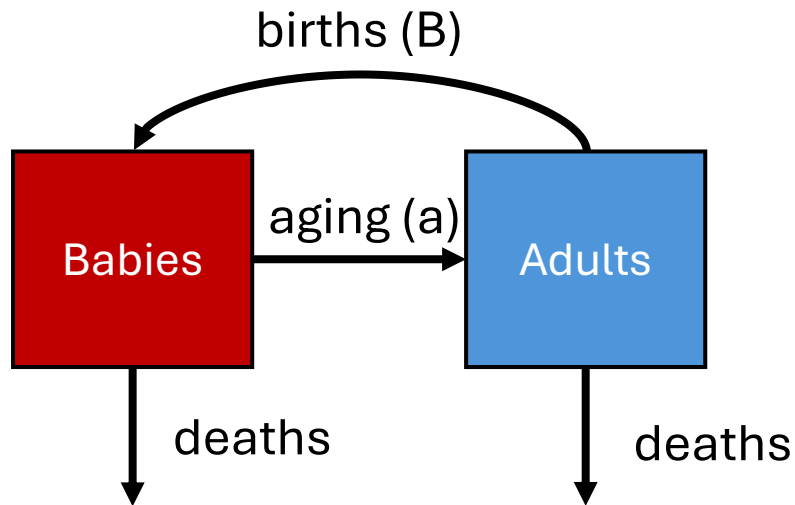
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The basic population model

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$$N_{t+1} = \lambda N_t$$

pop size at t+1 pop size at t

$$n_{t+1} = A n_t$$

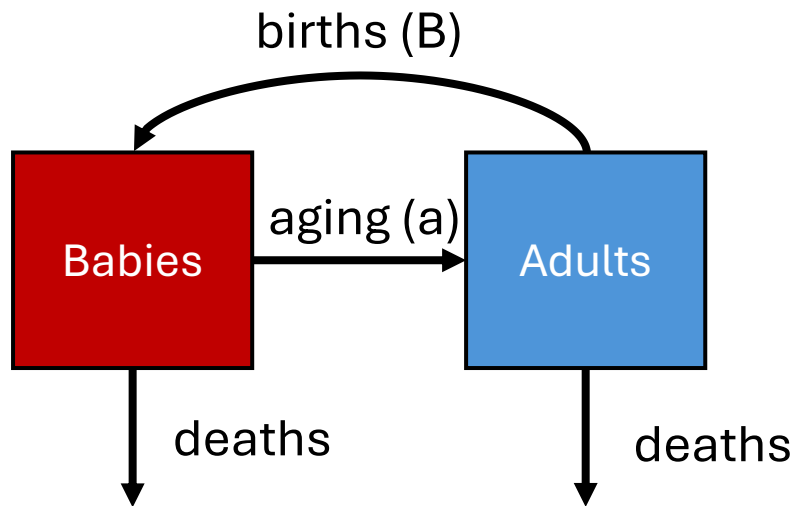
matrix of rates vector of population sizes

*discrete time

The basic population model

Compartmental models (mechanistic models)

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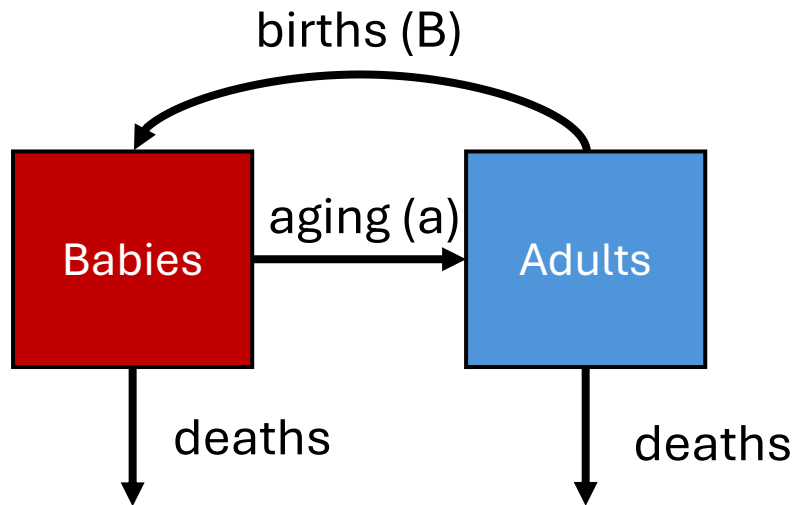
$$n_{t+1} = An_t$$

$$\begin{array}{c} A \\ \begin{array}{|c|c|} \hline s_b(1-a) & B \\ \hline S_b a & s_a \\ \hline \end{array} \end{array} \times \begin{array}{c} n_t \\ \begin{array}{|c|} \hline n_b \\ \hline n_a \\ \hline \end{array} \end{array} = \begin{array}{c} n_{t+1} \\ \begin{array}{|c|} \hline s_b(1-a)n_b + bn_a \\ \hline S_b an_b + s_a n_a \\ \hline \end{array} \end{array}$$

The basic population model

Compartmental models (mechanistic models)

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$$n_{t+1} = An_t$$

$$\begin{array}{c|c} & A \\ \hline n_t & \begin{array}{c|c} s_b(1-a) & B \\ \hline s_b a & s_a \end{array} \\ \hline n_{t+1} & \begin{array}{c|c} s_b(1-a)n_b + bn_a & \\ \hline s_b a n_b + s_a n_a & \end{array} \end{array}$$

Population growth will depend on population structure!

La croissance démographique dépendra de la structure de la population

Key concepts

- Compartmental / mechanistic / mathematical models
Modèles à compartiments
- Continuous vs. discrete models
Les modèles à temps discrets et les modèles à temps continu
- Deterministic vs. stochastic models
Modèles déterministe vs. stochastique
- Structured models
Modèles structurés

Checking In

How do we modify a basic population model to make it structured?
Comment modifier un modèle de population pour le structurer?



Checking In

How do we modify a basic population model to make it structured?

- Two compartments (adults and babies)
- Vector/matrix of values

3. Two-population model

Les modèles de deux populations

The predator-prey model

Compartmental models (mechanistic models)

1. Populations are divided into compartments
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3. Compartments and transition rates are determined by biological systems
4. Rates of transferring between compartments are expressed mathematically

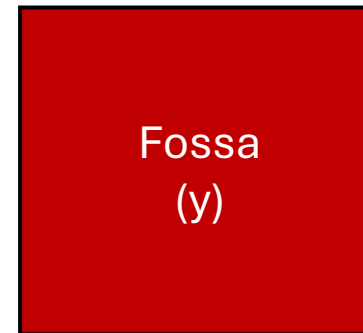
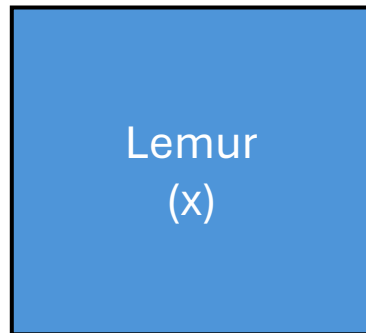
How does the population of fossa regulate the population of lemurs in Ranomafana?

Comment la population de fossa régule la population de lémuriens à Ranomafana?

The predator-prey model

Compartmental models (mechanistic models)

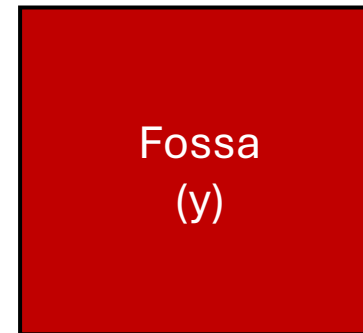
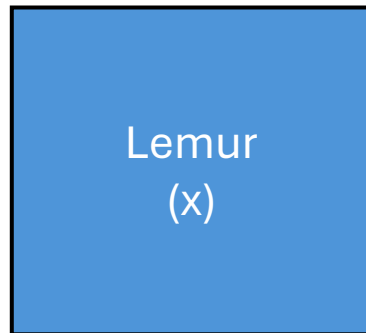
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The predator-prey model

Compartmental models (mechanistic models)

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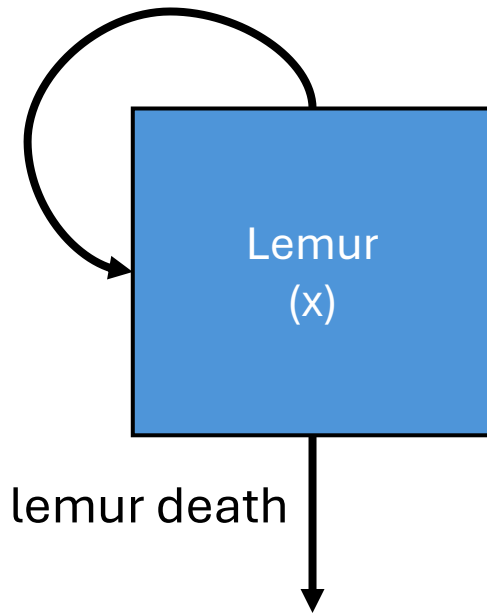


The predator-prey model

Compartmental models (mechanistic models)

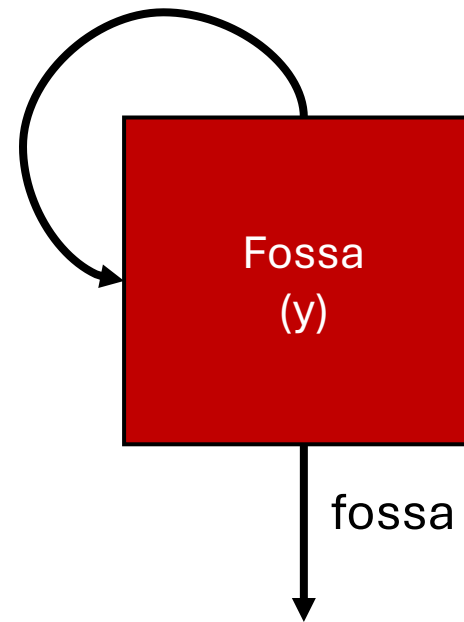
1. Populations are divided into compartments
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lemur reproduction



lemur death

fossa reproduction

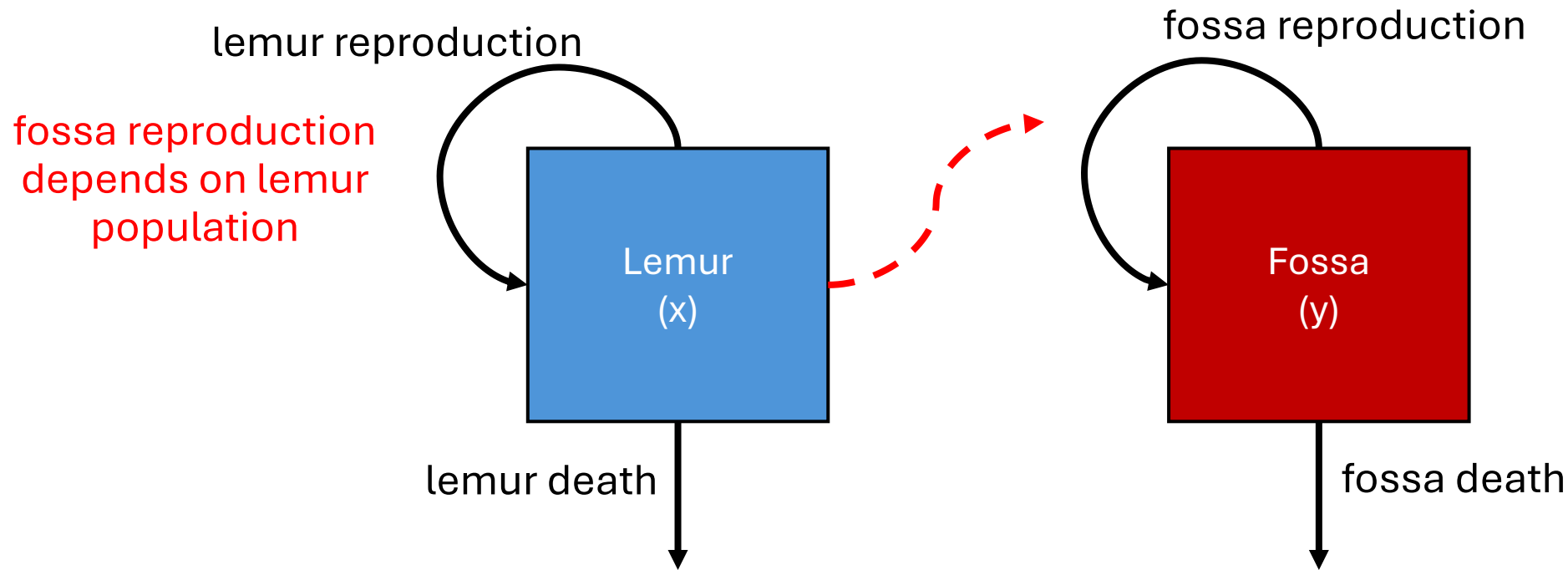


fossa death

The predator-prey model

Compartmental models (mechanistic models)

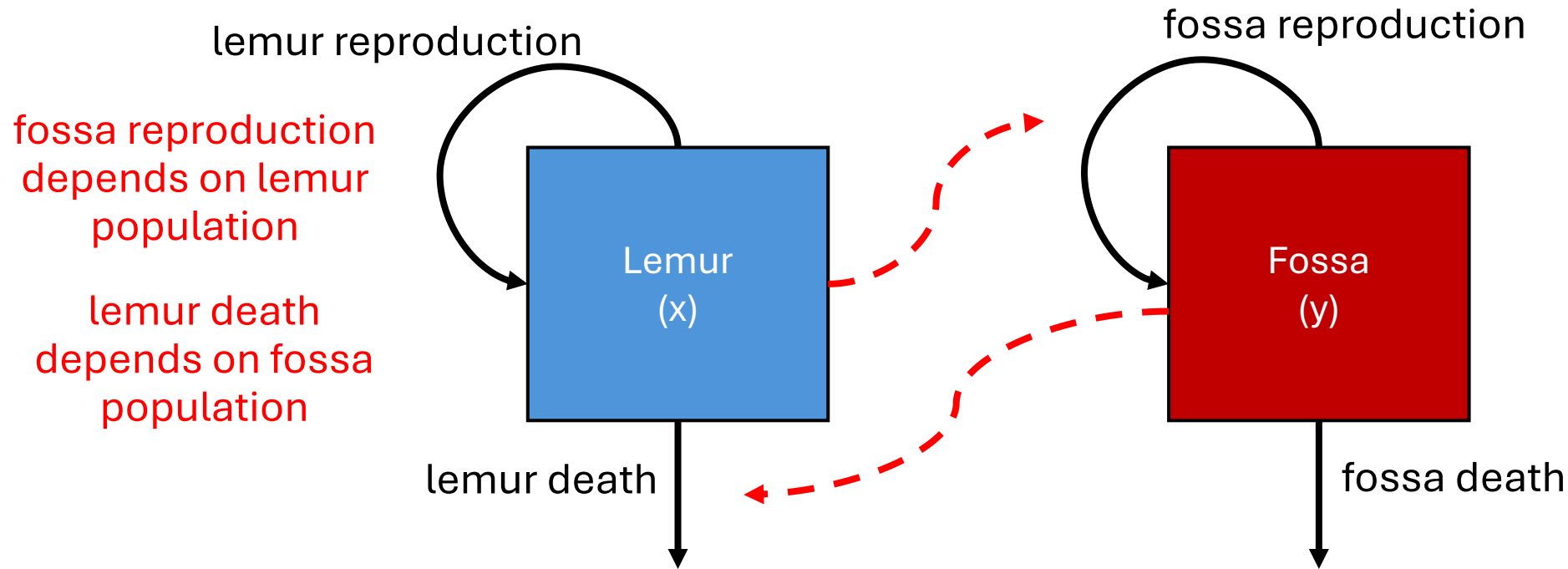
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The predator-prey model

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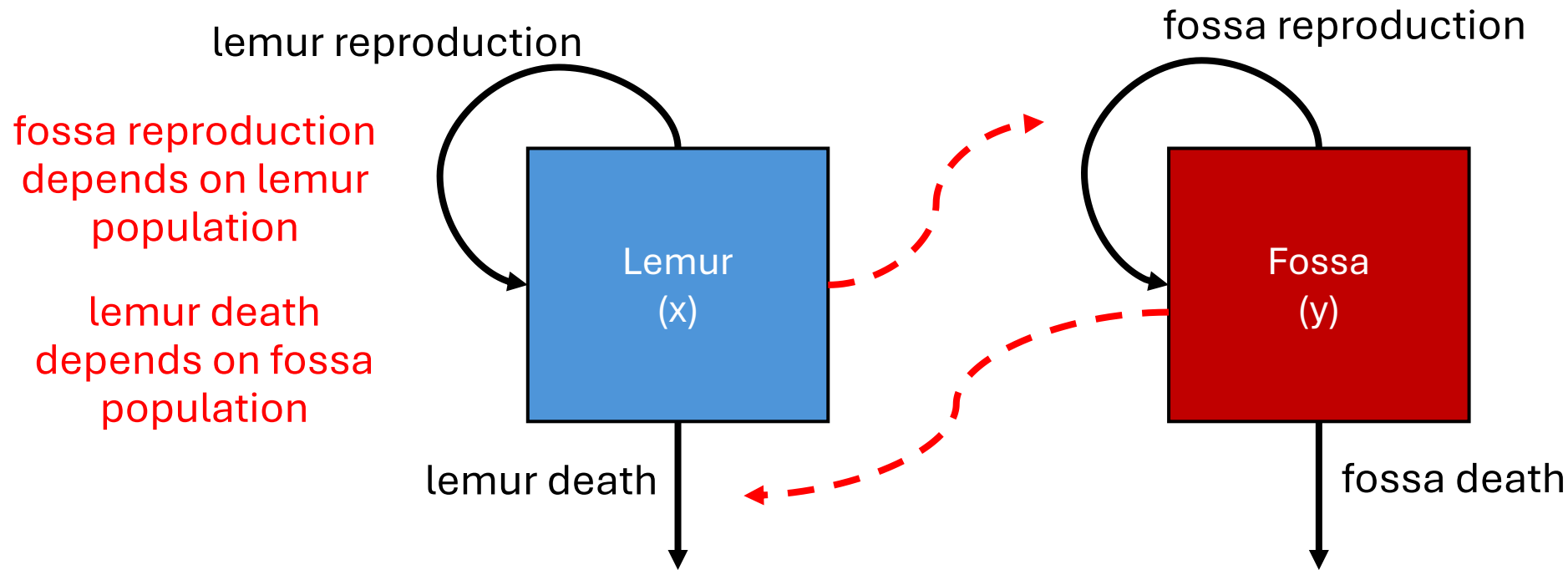
The predator-prey model

Compartmental models (mechanistic models)

1. Populations are divided into compartments
2. Individuals within a compartment are homogenously mixed
3. Compartments and transition rates are determined by biological systems
4. *Rates of transferring between compartments are expressed mathematically*

Parameters

lemur rep. rate
lemur death rate
fossa rep. rate
fossa death rate



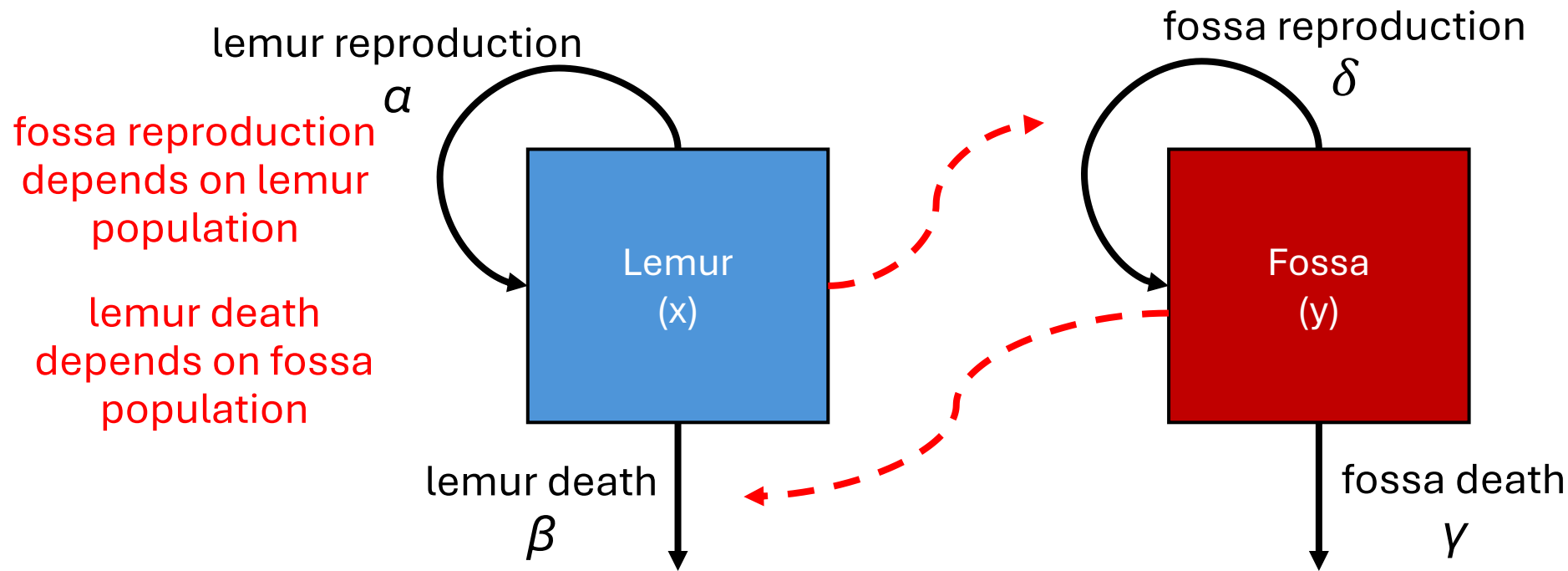
The predator-prey model

Compartmental models (mechanistic models)

1. Populations are divided into compartments
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4. *Rates of transferring between compartments are expressed mathematically*

Parameters

- α lemur rep. rate
 β lemur death rate
 δ fossa rep. rate
 γ fossa death rate



$$\frac{dx}{dt} = x(\alpha - \beta y)$$

$$\frac{dy}{dt} = y(\delta x - \gamma)$$

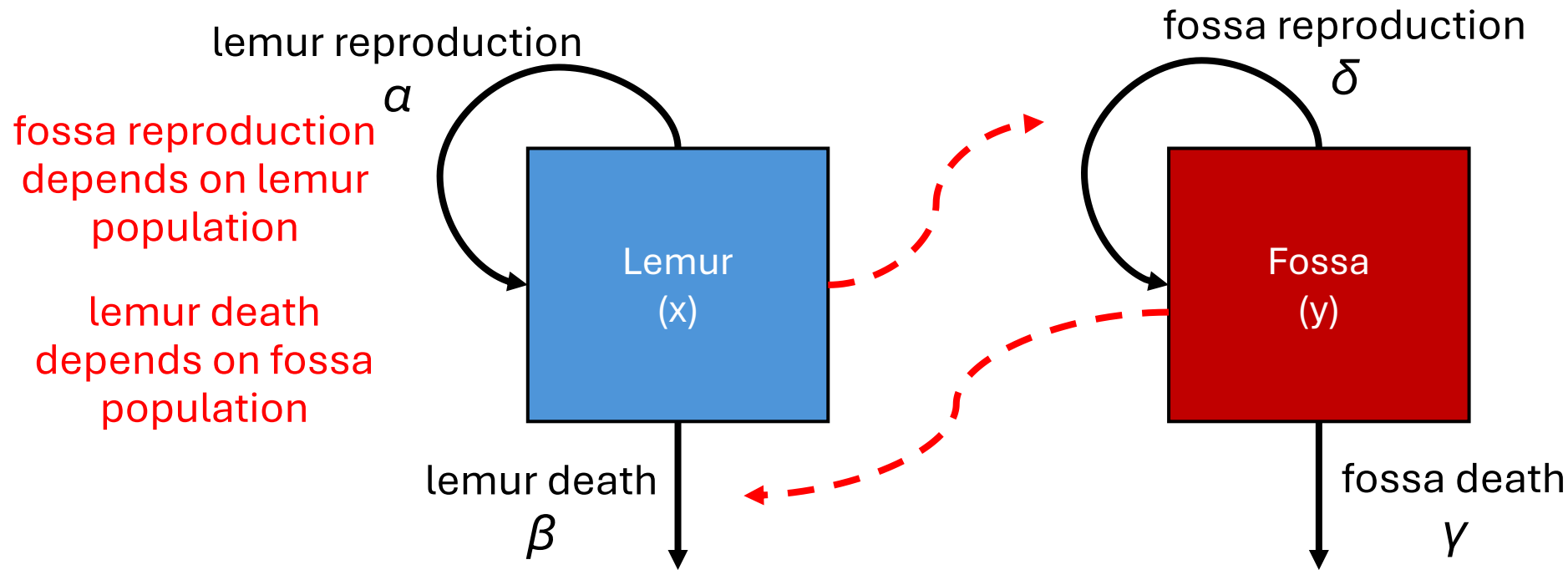
The predator-prey model

Compartmental models (mechanistic models)

1. Populations are divided into compartments
2. Individuals within a compartment are homogenously mixed
3. Compartments and transition rates are determined by biological systems
4. *Rates of transferring between compartments are expressed mathematically*

Some Assumptions:

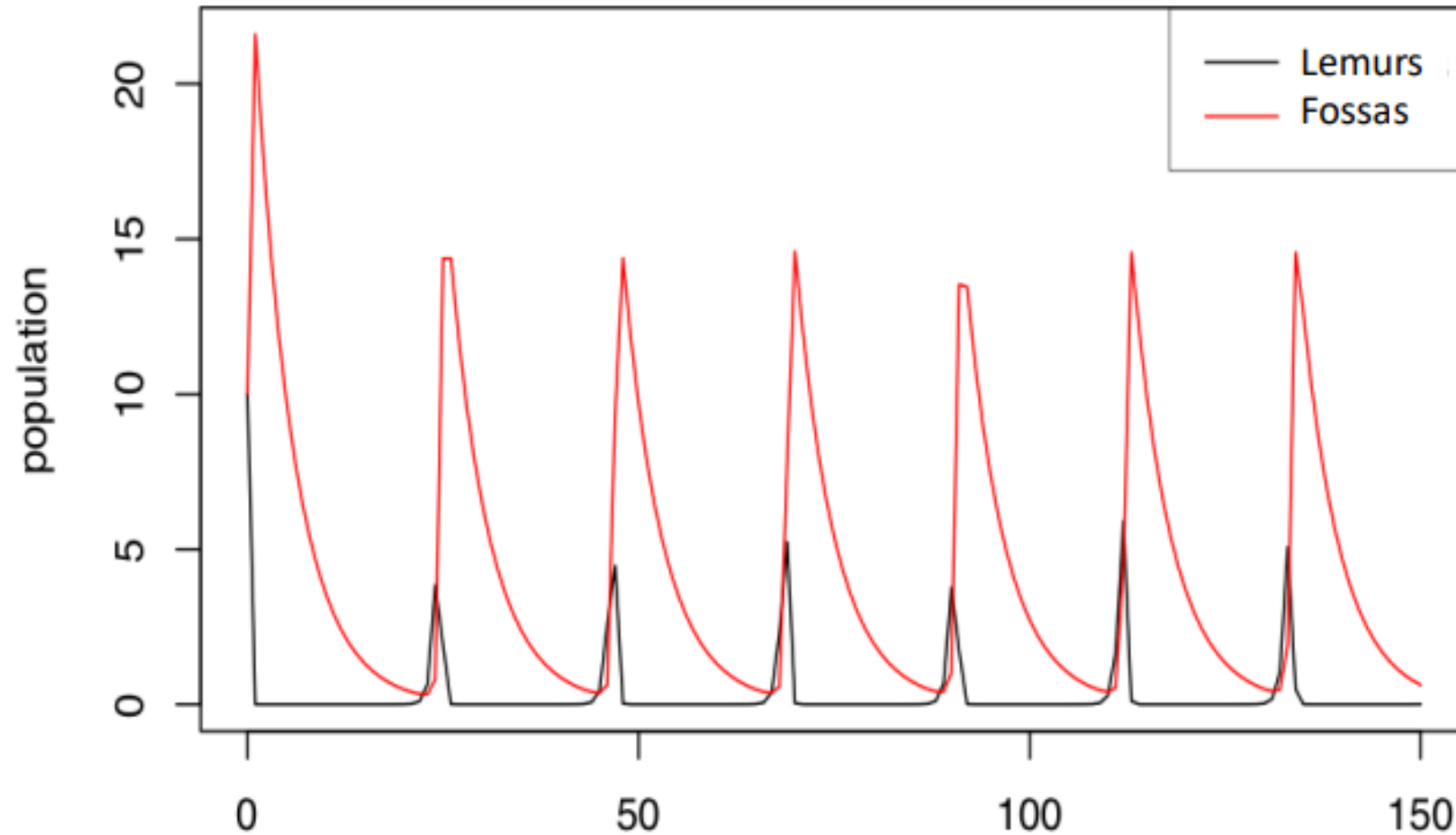
- The lemur has unlimited food supply
- The lemur only dies from being eaten by a fossa
- The fossa is totally dependent on a single prey species as its only food supply



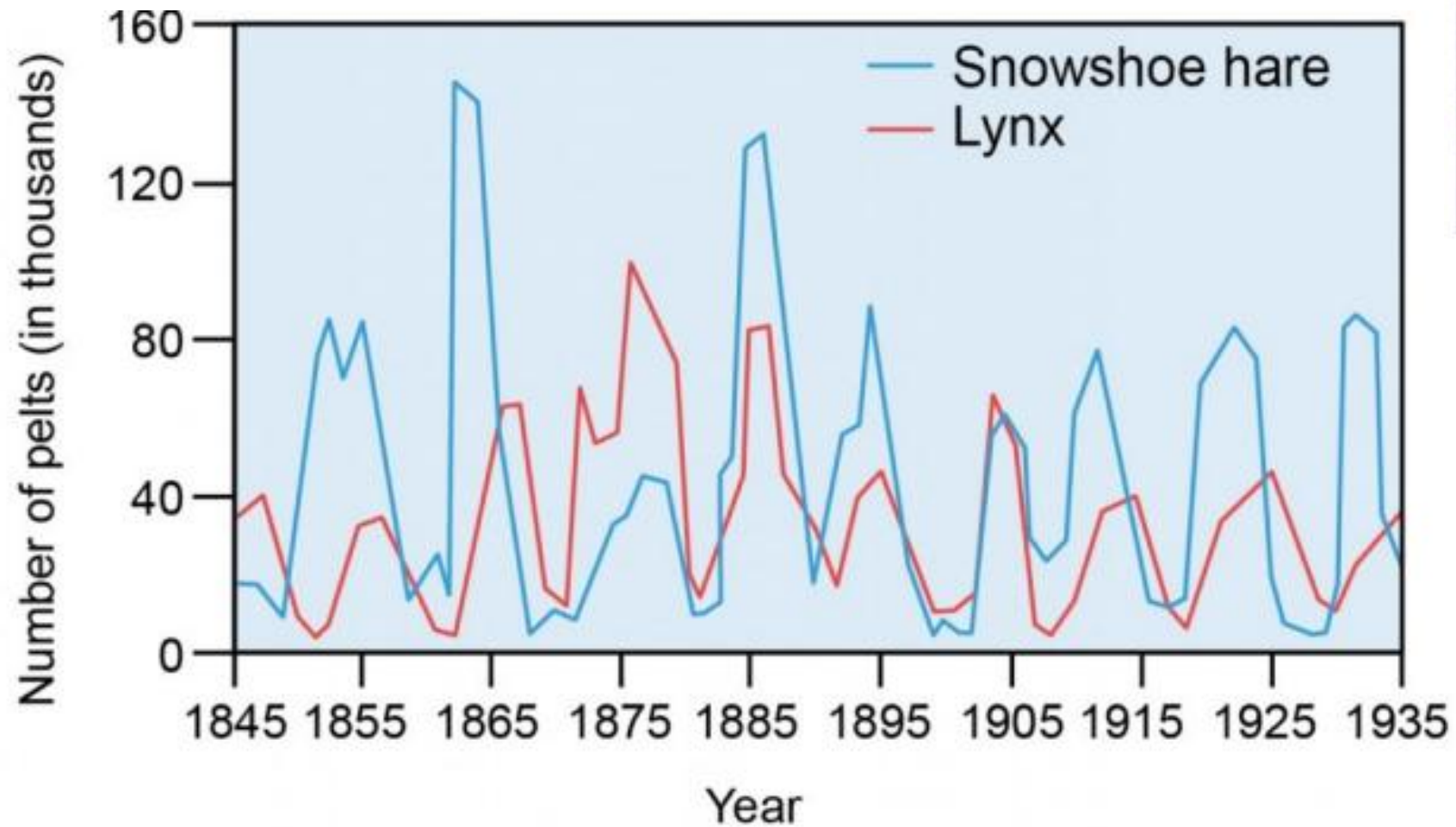
$$\frac{dx}{dt} = x(\alpha - \beta y)$$

$$\frac{dy}{dt} = y(\delta x - \gamma)$$

The predator-prey model



The predator-prey model



Key concepts

- Compartmental / mechanistic / mathematical models
Modèles à compartiments
- Continuous vs. discrete models
Les modèles à temps discrets et les modèles à temps continu
- Deterministic vs. stochastic models
Modèles déterministe vs. stochastique
- Structured models
Modèles structurés
- Two population models
Modèles des deux populations

Checking In

What pattern can we see in simple predator-prey relationships?

Quel motif est visible dans les relations simples prédateur-proie ?

What could we modify to make this model more complex/realistic?

Qu'est-ce qu'on peut modifier pour rendre ce modele plus complexe/réaliste ?



Checking In

What pattern can we see in simple predator-prey relationships?

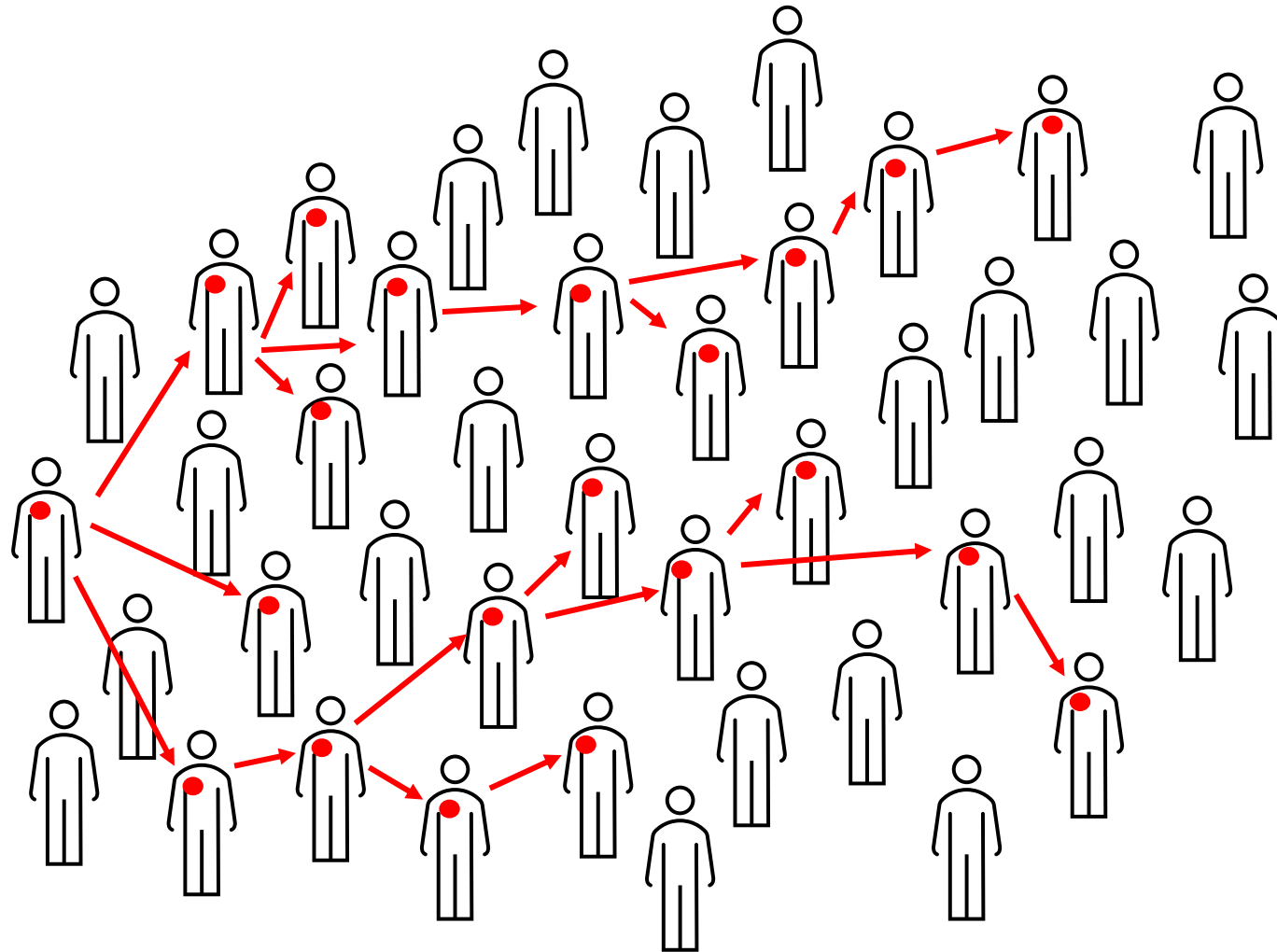
- Cycles / oscillations

What could we modify to make this model more complex/realistic?

- Lemurs can die of other causes
- Fossas can eat other things

4. SIR models

Les modèles SIR



The SIR model

Compartmental models (mechanistic models)

1. Populations are divided into compartments
2. Individuals within a compartment are homogenously mixed
3. Compartments and transition rates are determined by biological systems
4. Rates of transferring between compartments are expressed mathematically

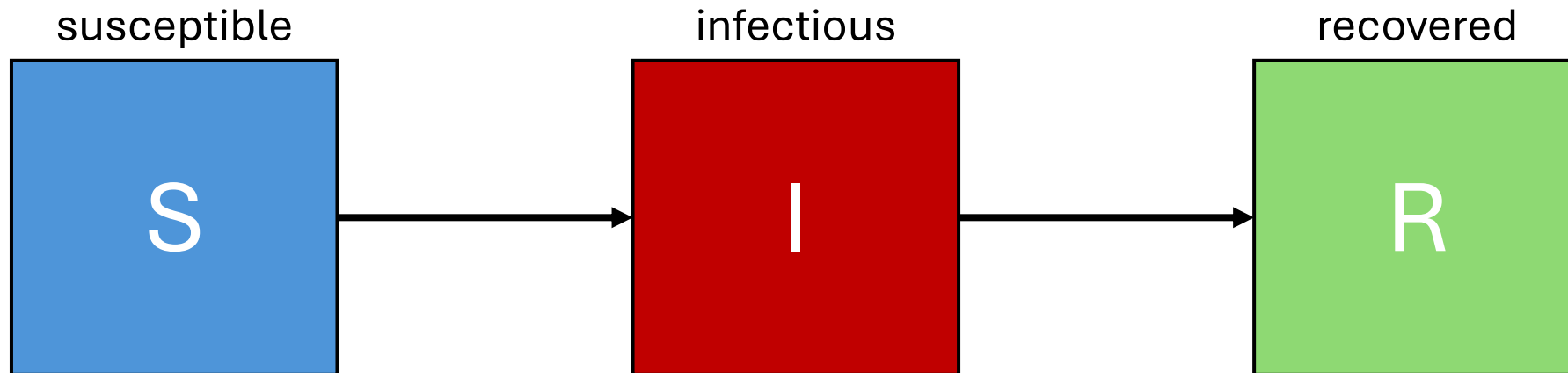
How does measles transmit through Antananarivo?

Comment la rougeole se transmet-elle à Antananarivo?

The SIR model

Compartmental models (mechanistic models)

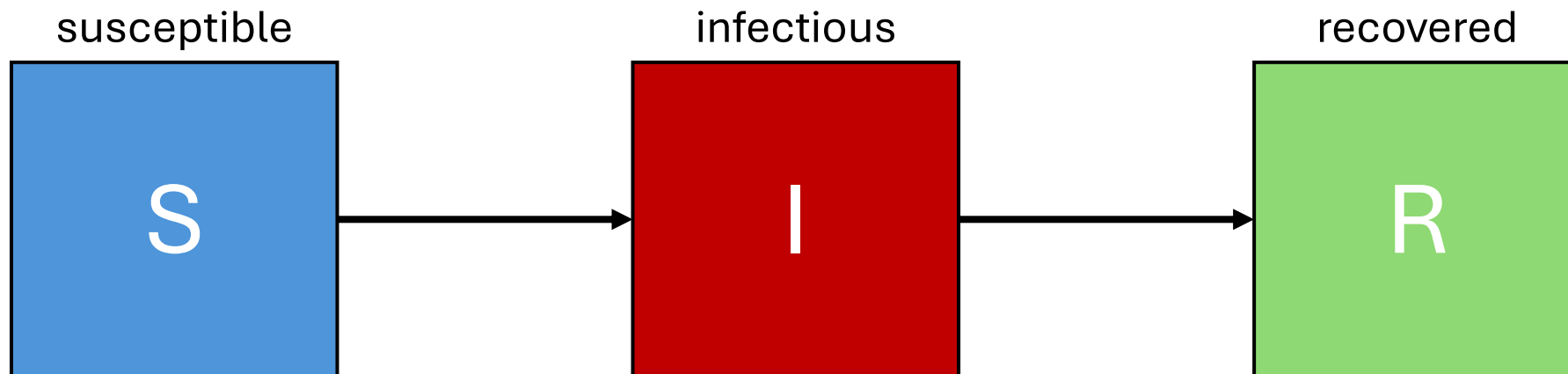
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The SIR model

Compartmental models (mechanistic models)

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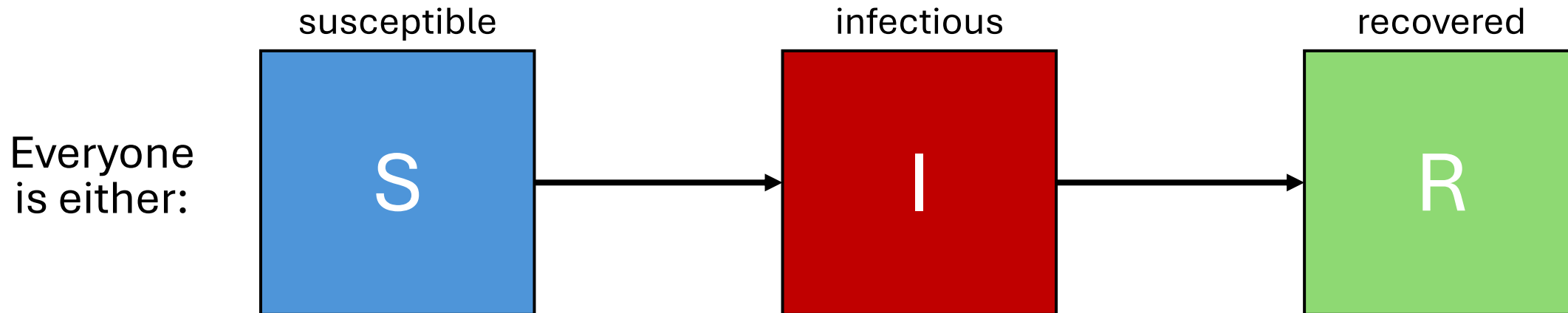
What are the big assumptions here?

Quelles sont les grandes hypothèses?

The SIR model

Compartmental models (mechanistic models)

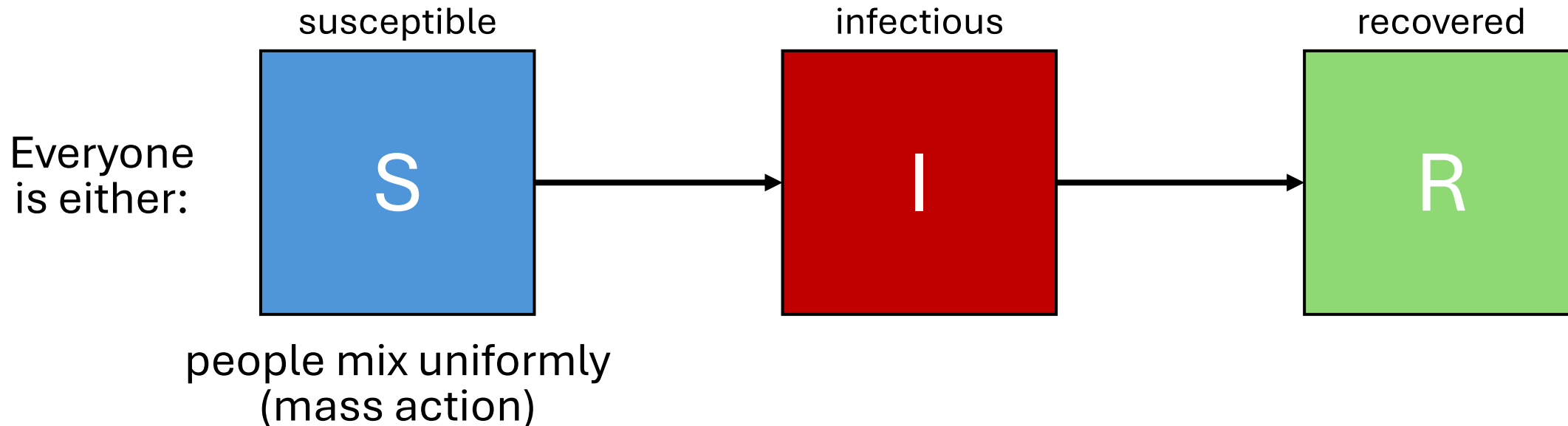
1. *Populations are divided into compartments*
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The SIR model

Compartmental models (mechanistic models)

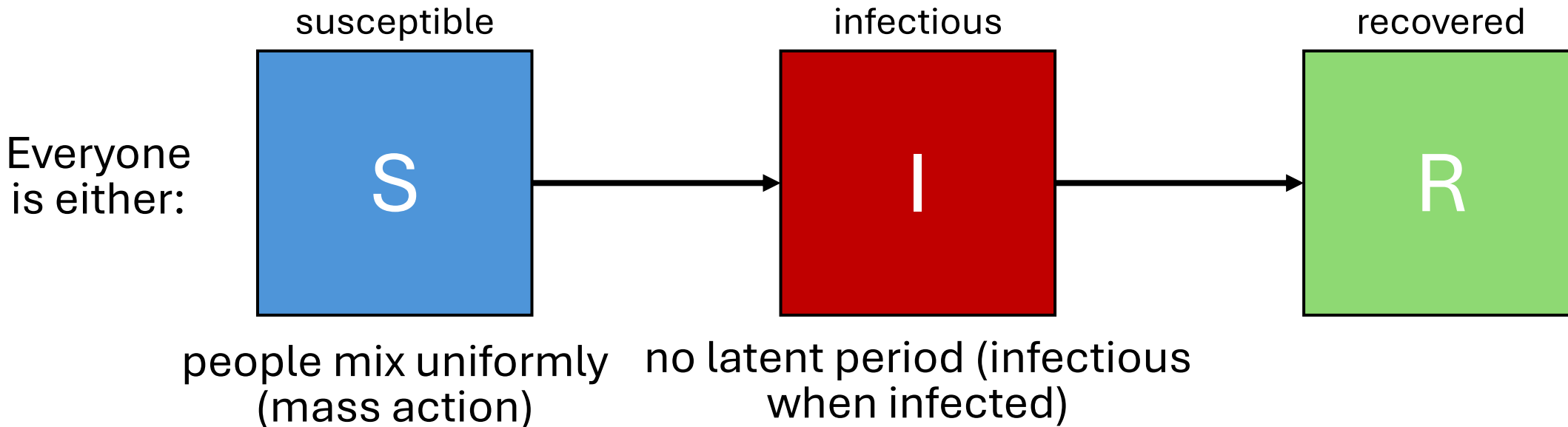
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The SIR model

Compartmental models (mechanistic models)

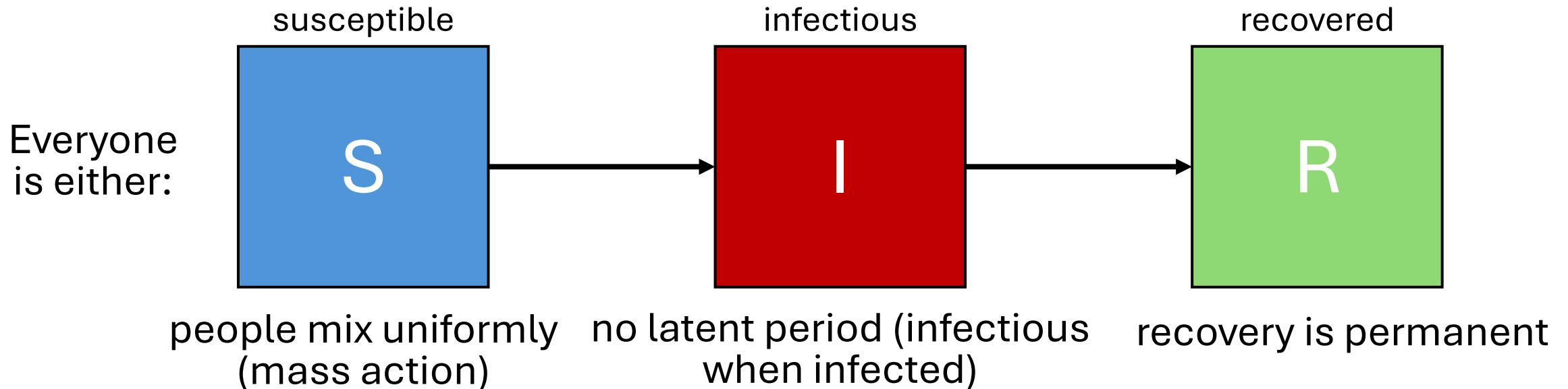
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The SIR model

Compartmental models (mechanistic models)

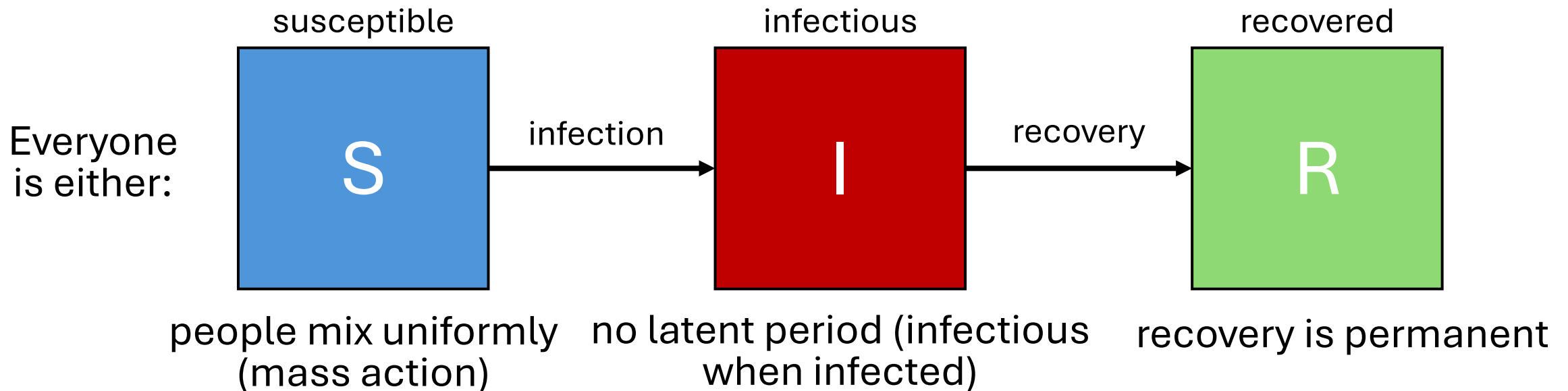
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The SIR model

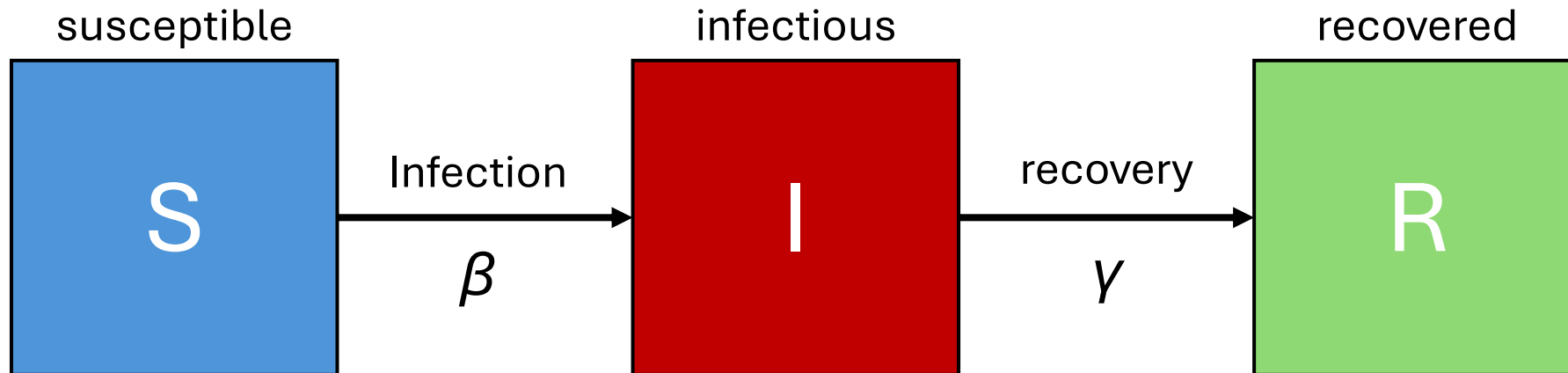
Compartmental models (mechanistic models)

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4. *Rates of transferring between compartments are expressed mathematically*

$$\frac{dS(t)}{dt} = -\beta S(t)I(t)$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t)$$

$$\frac{dR(t)}{dt} = \gamma I(t)$$



β = transmission rate
taux de transmission

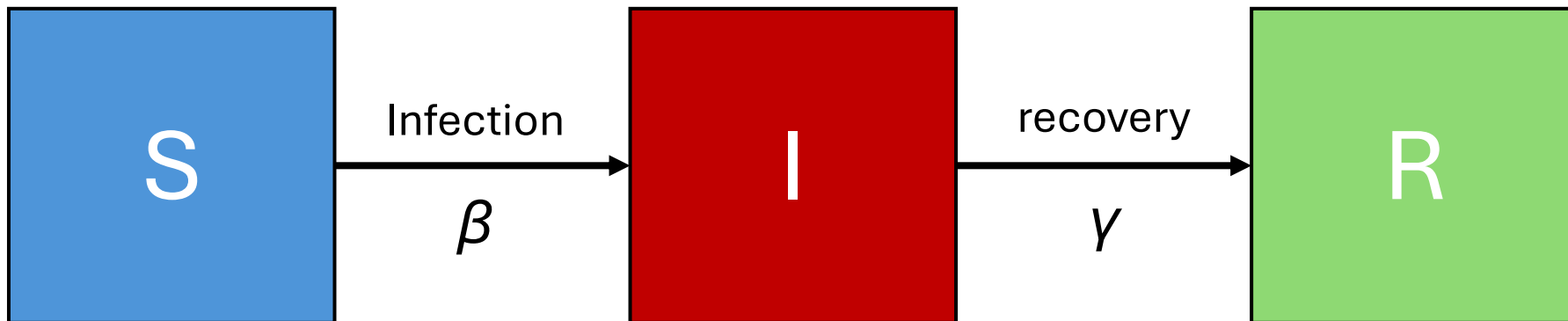
γ = recovery rate
taux de guérison

The SIR model

Compartmental models (mechanistic models)

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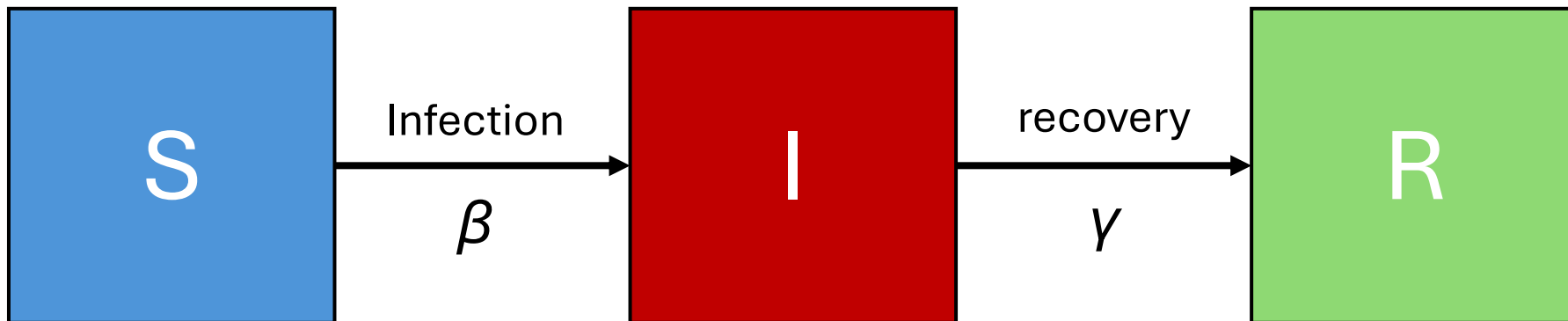
We keep track of the change in the number or proportion of **susceptible**, infected, and recovered individuals over time

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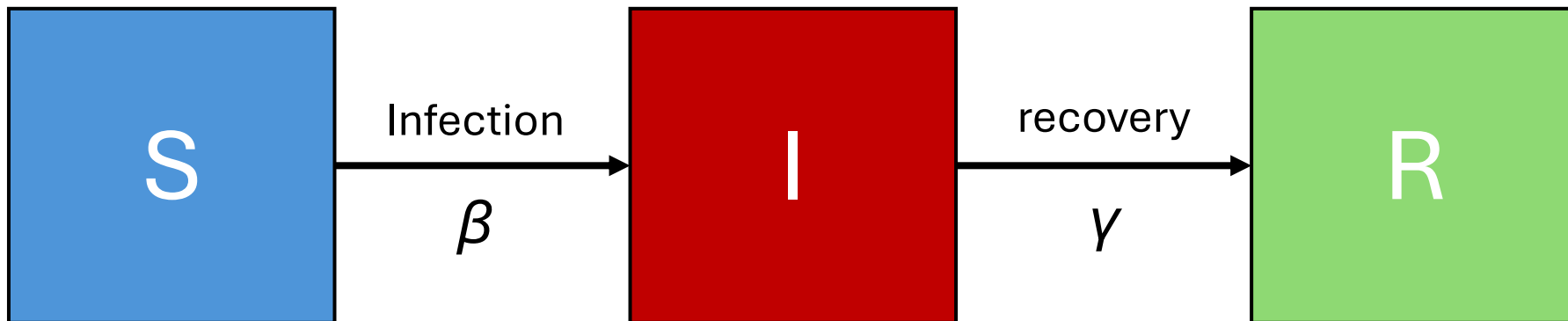
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The SIR model

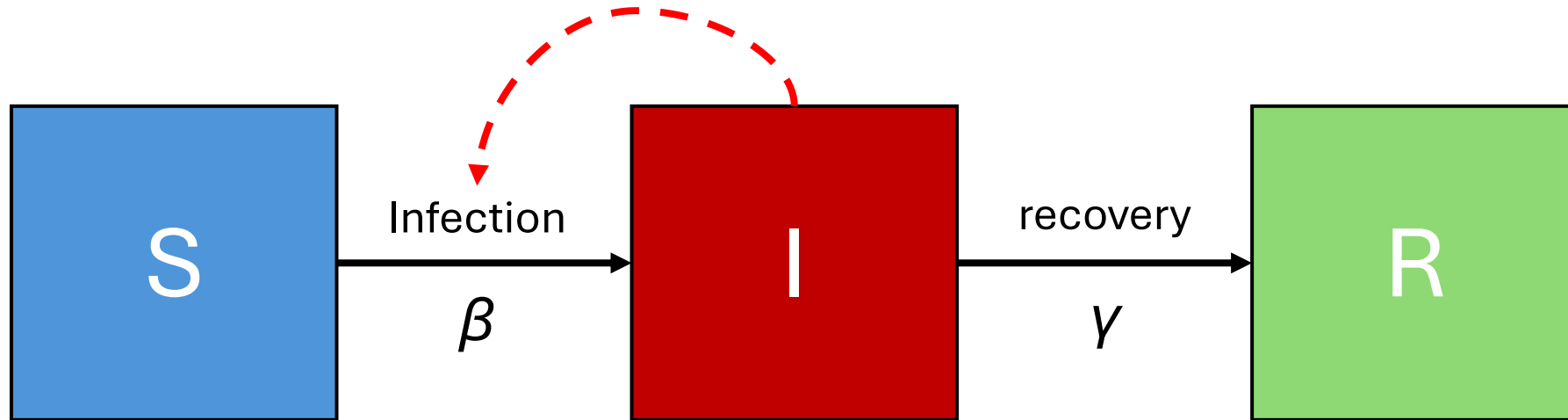
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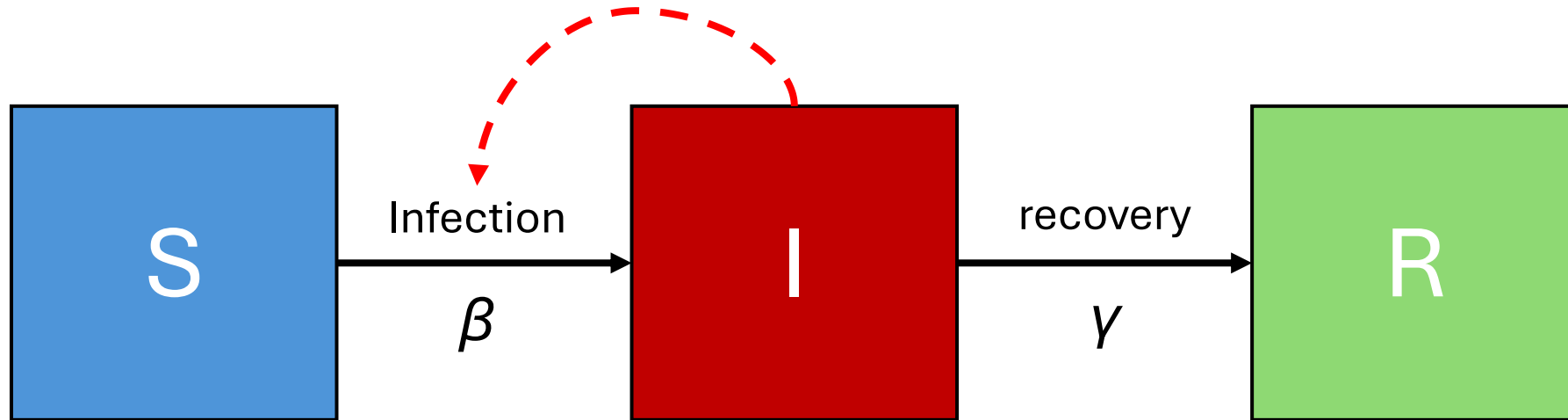
infected numbers influence the transmission rate le nombre d'infectés influence le taux de transmission

The SIR model

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All people are accounted for as they move through compartments.

The SIR model

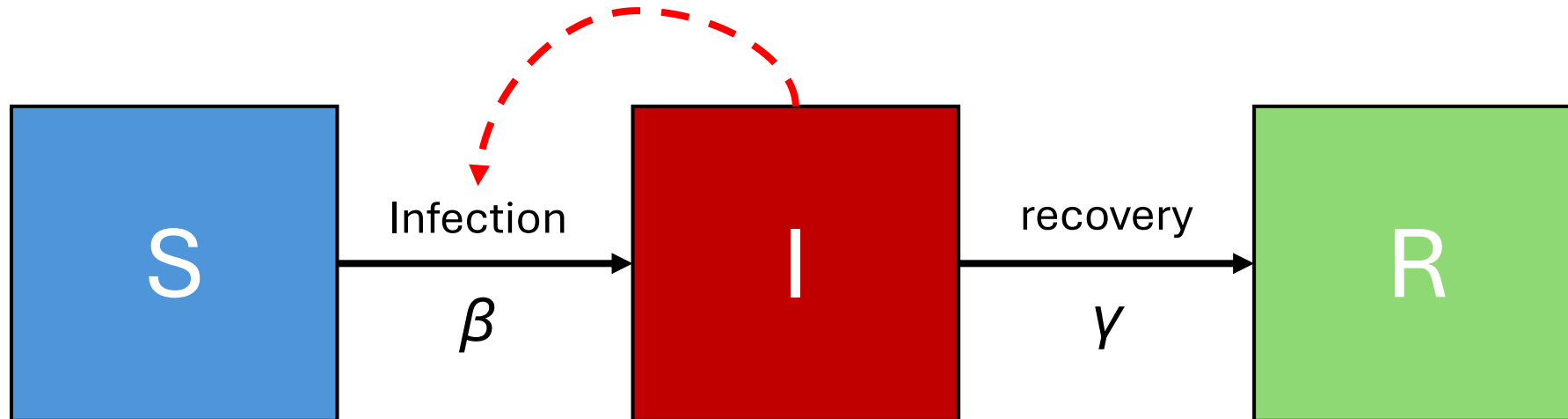
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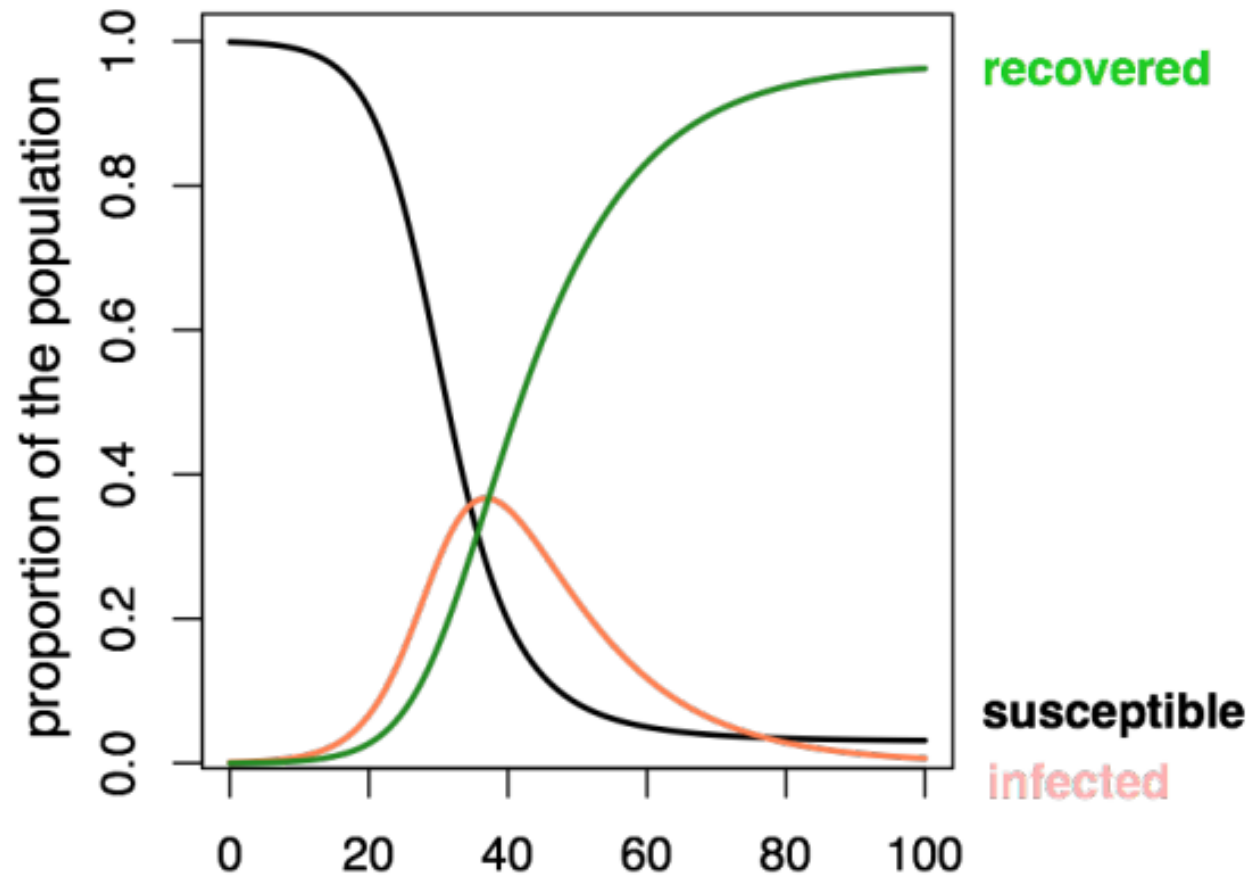
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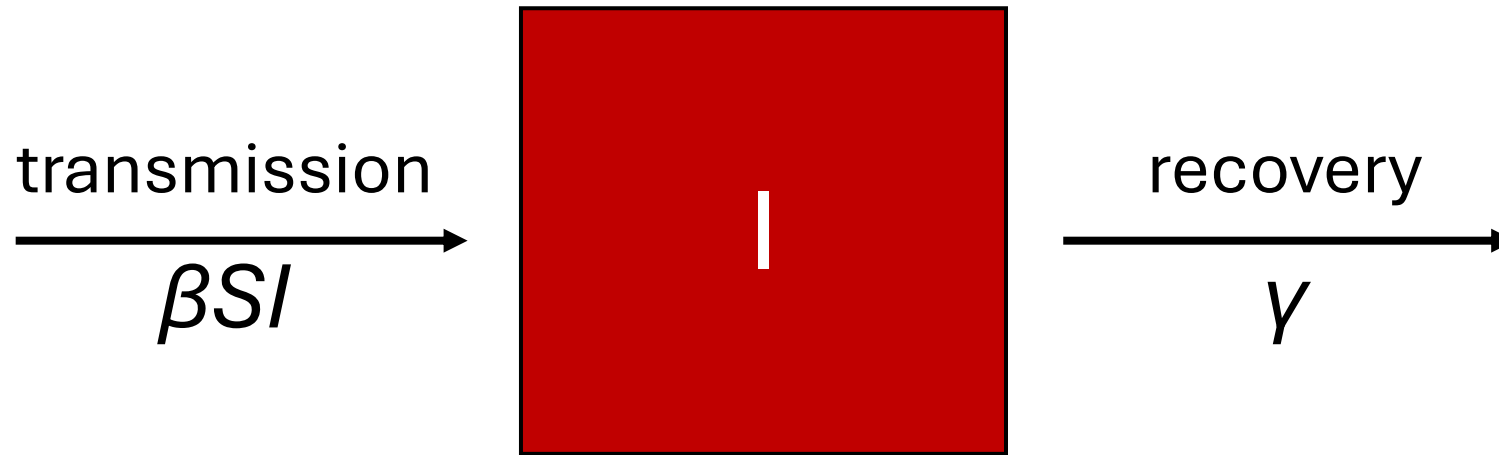
What will the dynamics look like?

À quoi ressemblera la dynamique?

The SIR model



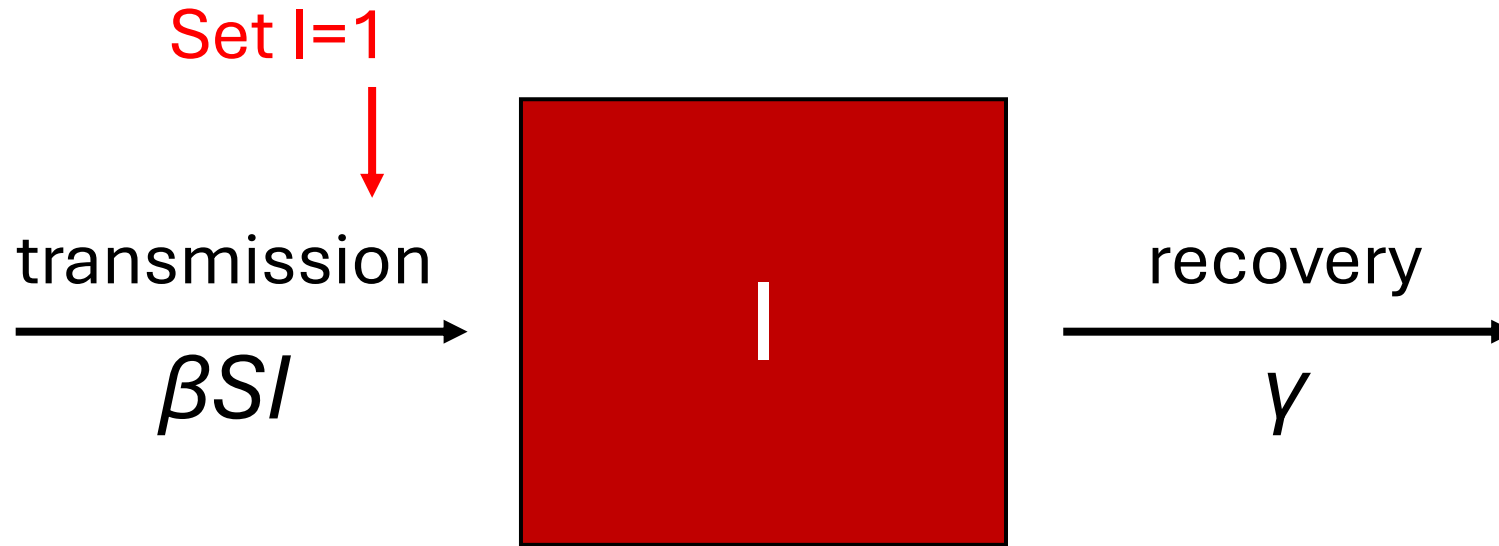
The SIR model



How do we describe how an infection moves through a population?

Comment décrire le moyen dont une infection se transmet dans une population ?

The SIR model



$$R_0 = \beta N / \gamma$$

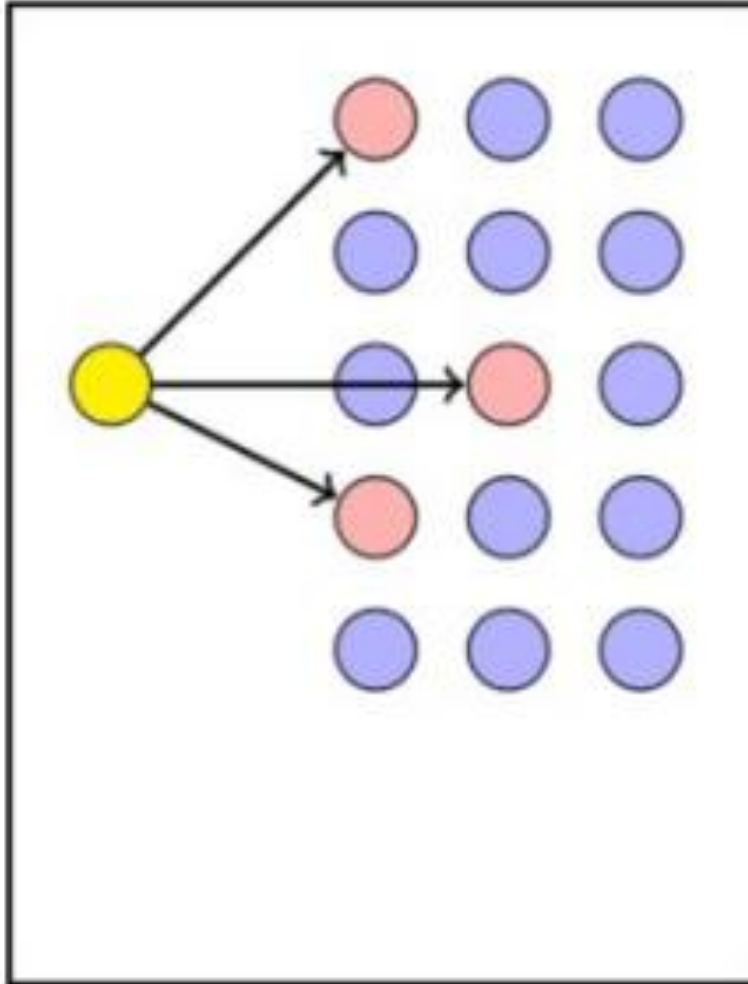
The average number of persons infected by an infectious individual when everyone is susceptible ($S=100\%$, or $S=1$, start of an epidemic)

Le nombre moyen de personnes infectés par un individu infectueux quand tout le monde est sensible ($S=100\%$, au début d'une épidémie)

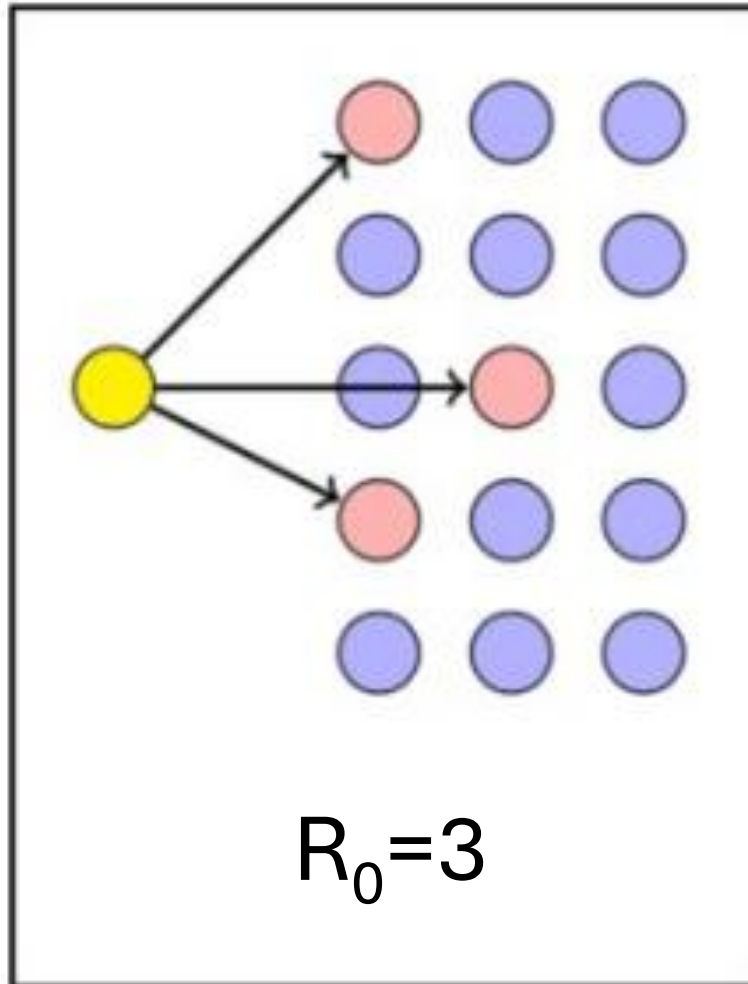
The SIR model

What is the value of R_0 ?

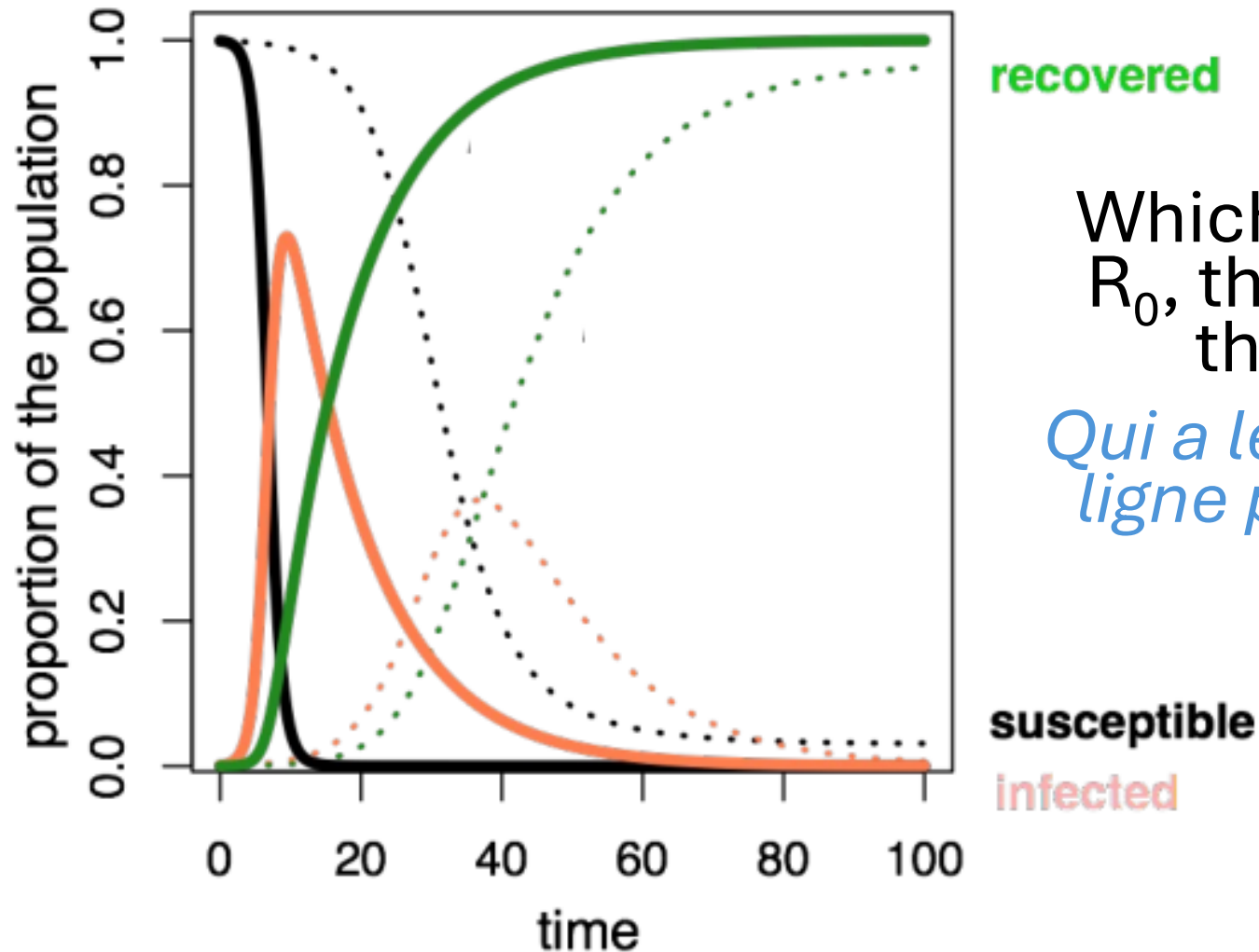
Quelle est la valeur de R_0 ?



The SIR model



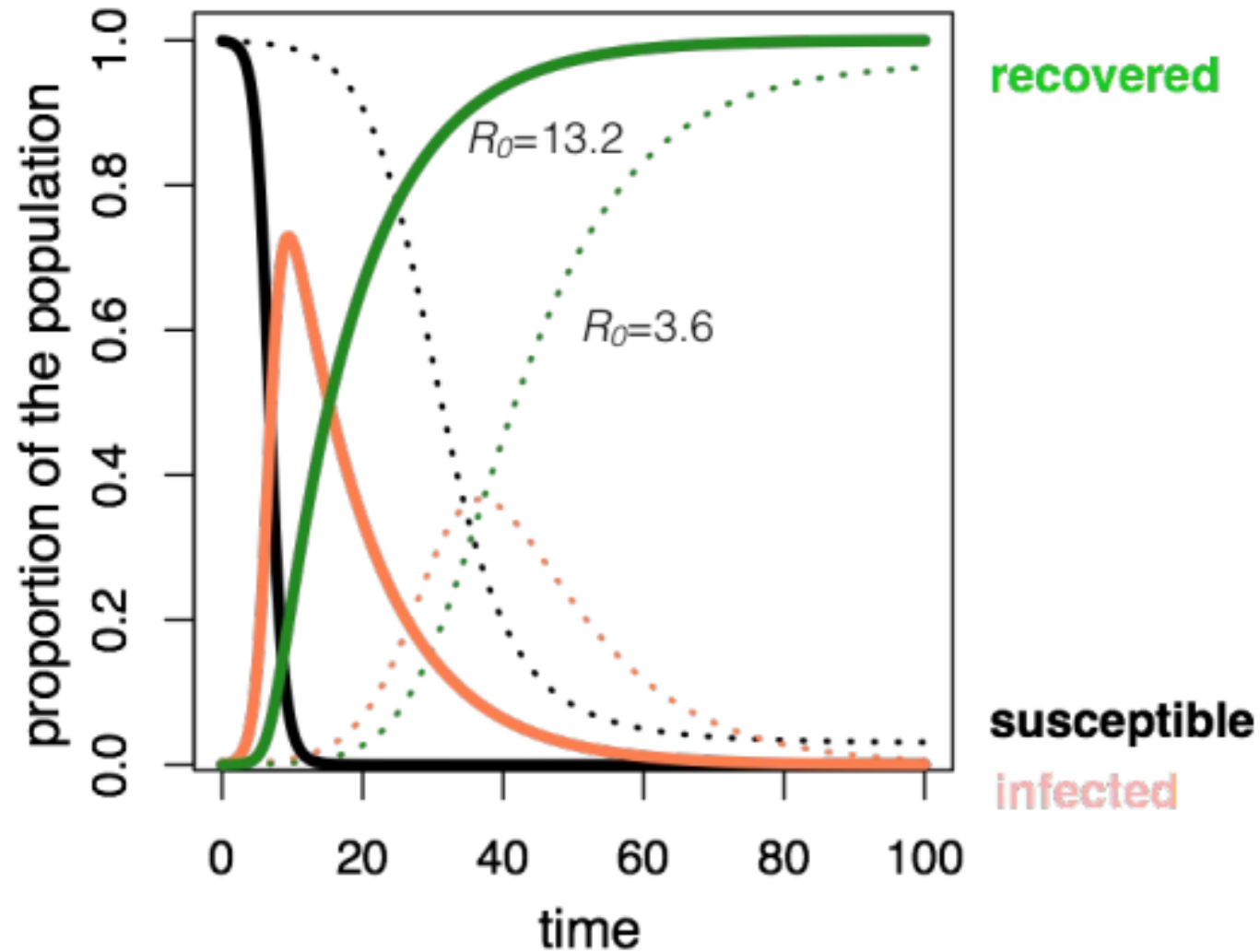
The SIR model



Which has the higher R_0 , the dotted line or the solid line?

Qui a le R_0 le plus élevé, ligne pointillé ou ligne continue?

The SIR model



Checking In

What is R_0 ?

C'est quoi, R_0 ?

How could you modify this simple SIR model to represent COVID-19?

Comment pourriez-vous modifier ce modèle SIR simple pour représenter COVID-19?



Checking In

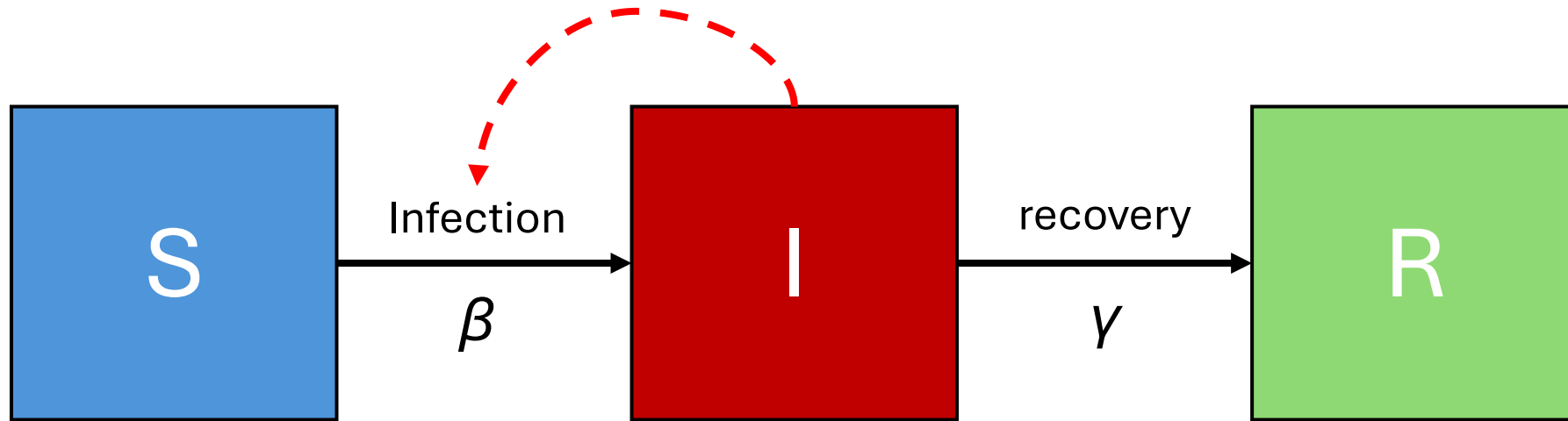
What is R_0 ?

- The average number of secondary infections from the first infections individual

How could you modify this simple SIR model to represent COVID-19?

- Re-infection, incubation period, social distancing, vaccination

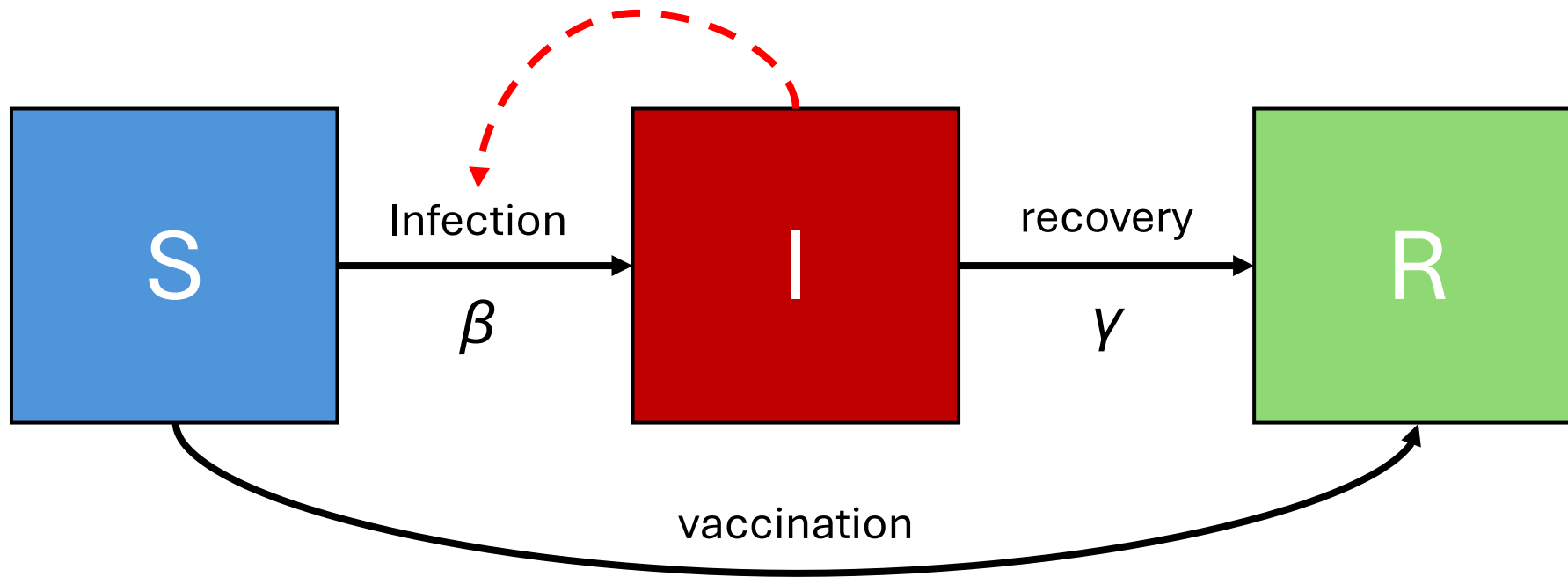
The SIR model : vaccination



Vaccination moves people out of susceptibles into the immune (recovered) class.

La vaccination éloigne les personnes sensibles de la maladie dans la classe immunitaire (rétablie).

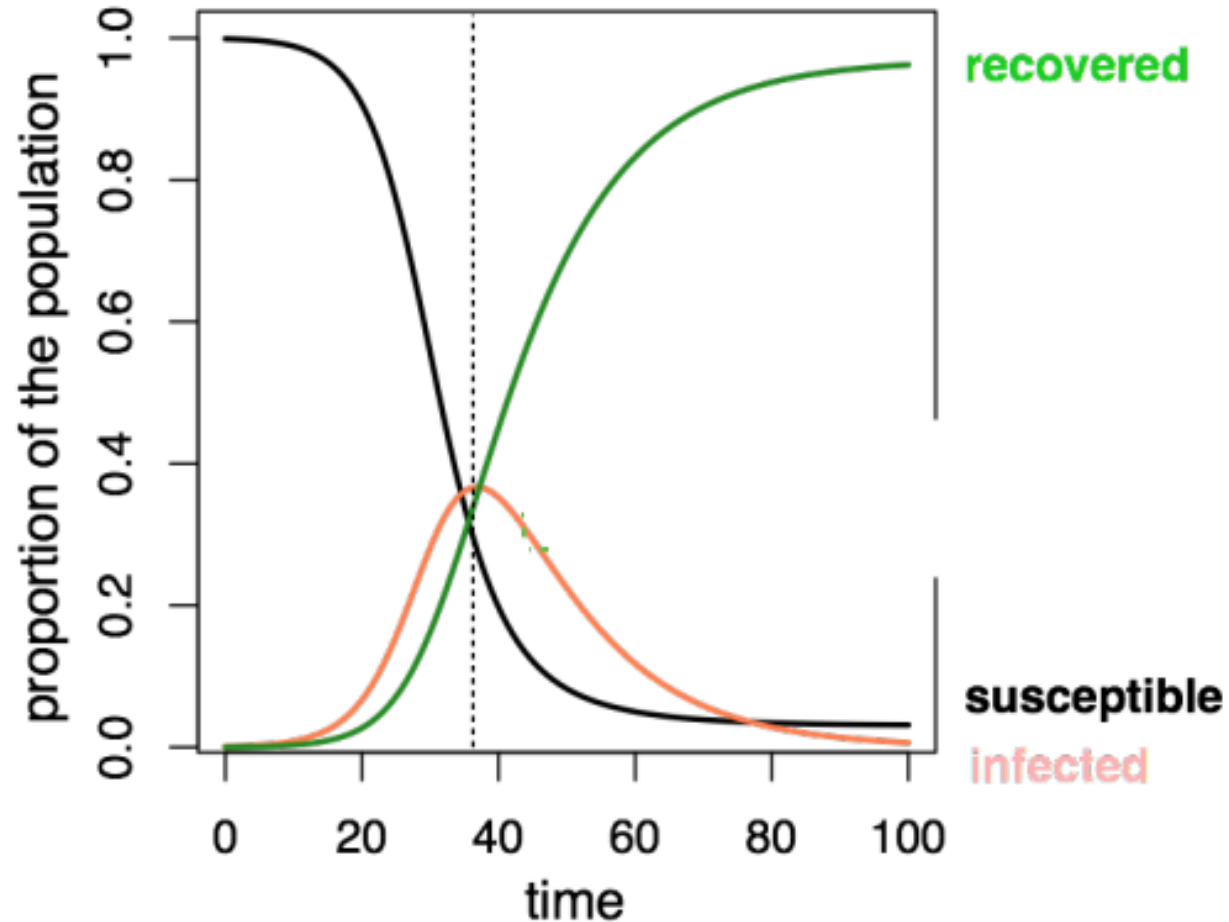
The SIR model : vaccination



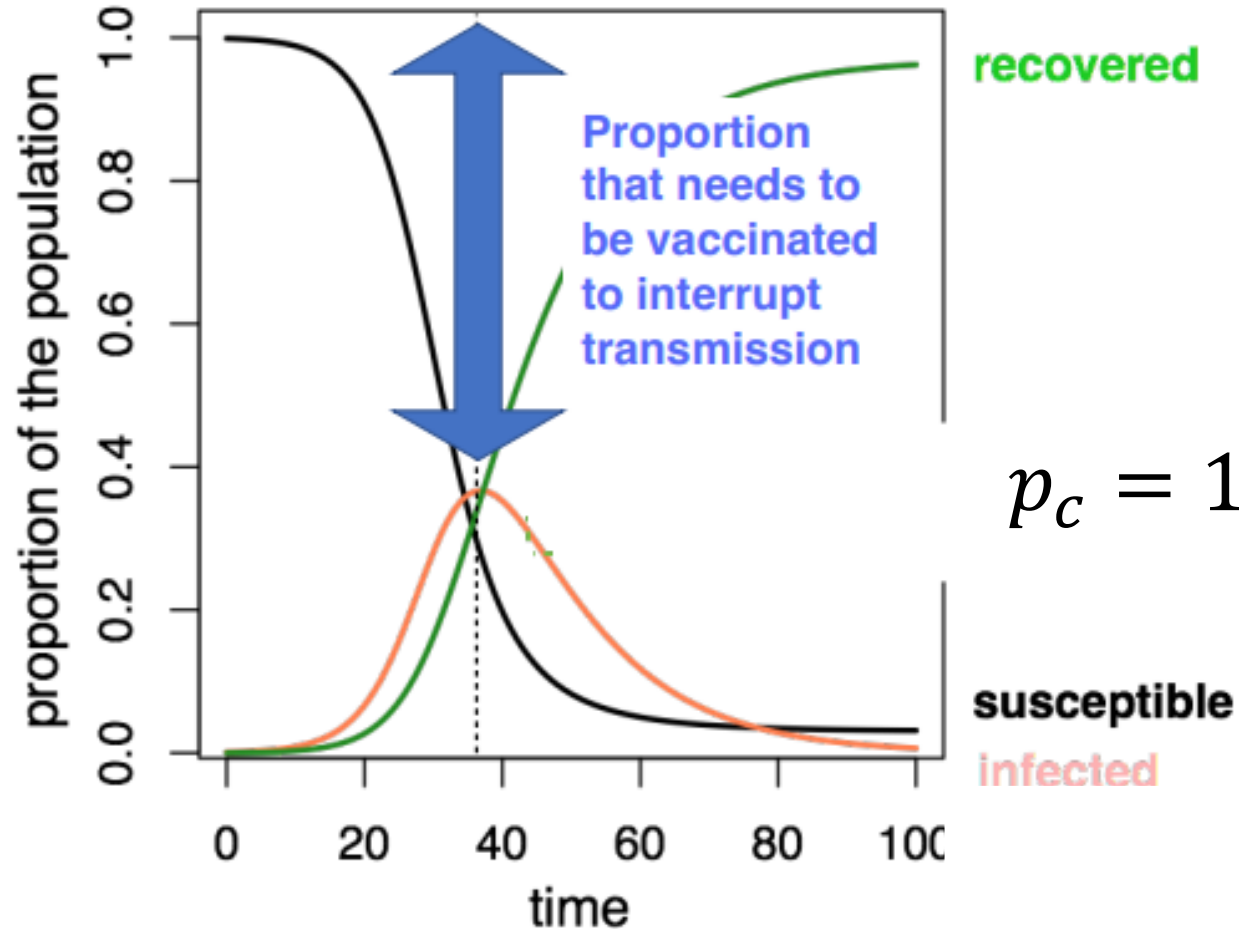
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The SIR model : vaccination

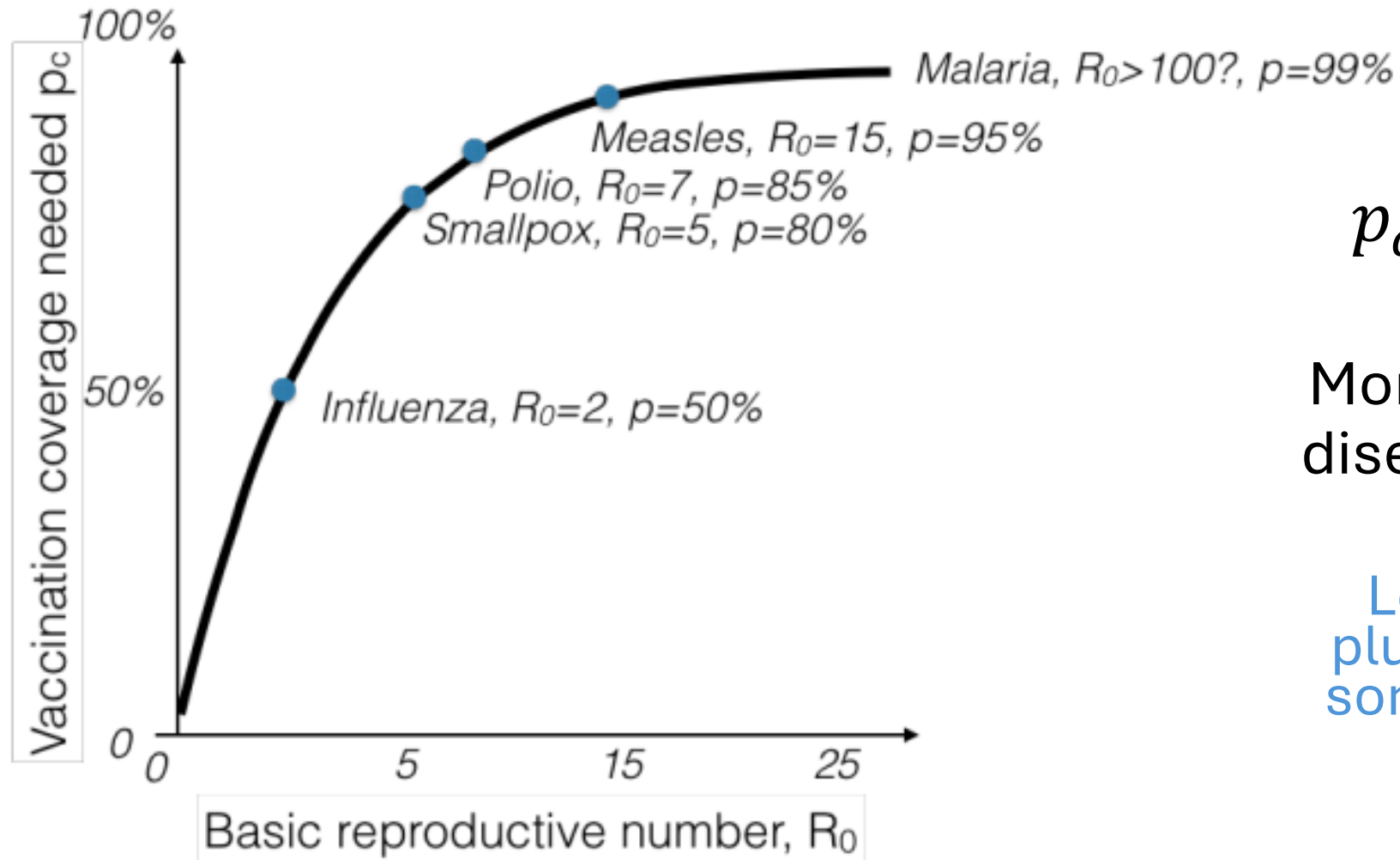


The SIR model : vaccination



$$p_c = 1 - \frac{1}{R_0}$$

The SIR model : vaccination

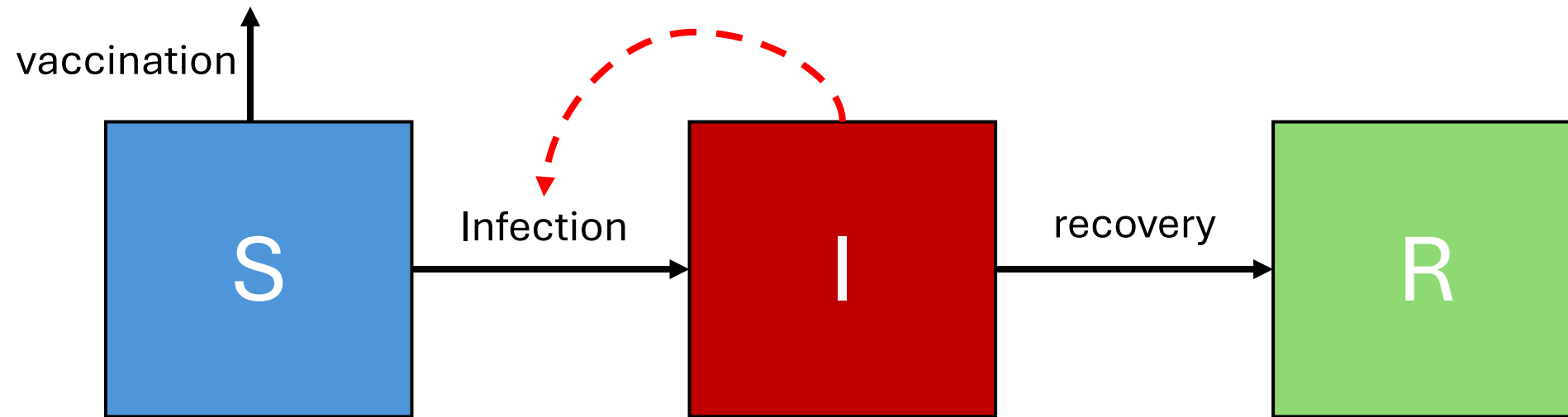


$$p_c = 1 - \frac{1}{R_0}$$

More transmissible diseases are harder to eradicate

Les maladies les plus transmissibles sont plus difficiles à éradiquer

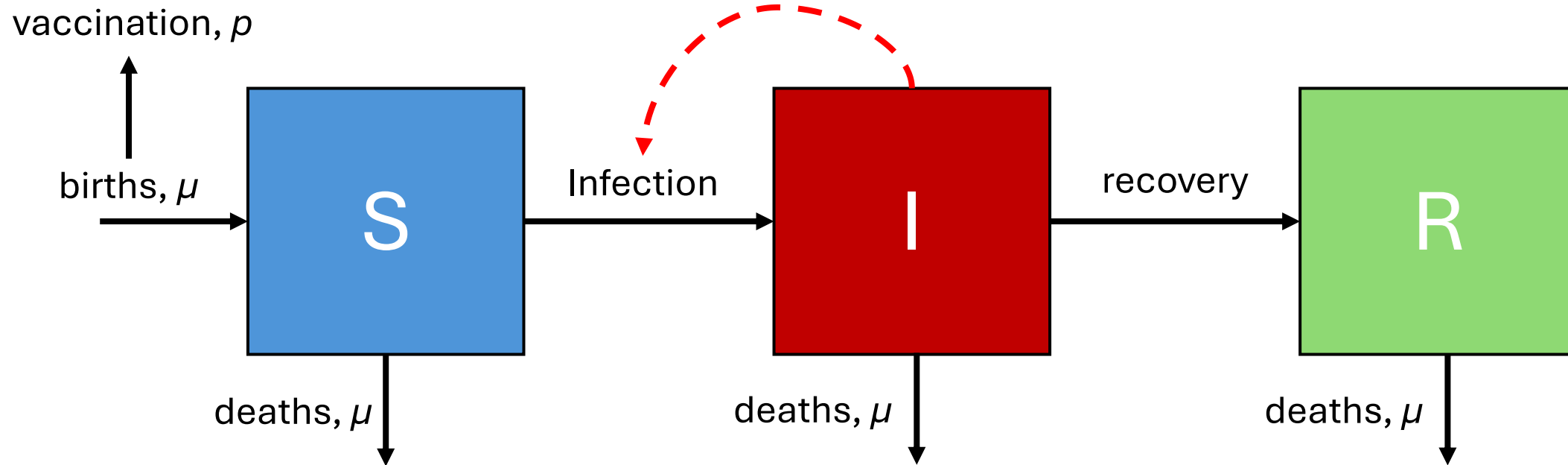
The SIR model : extensions



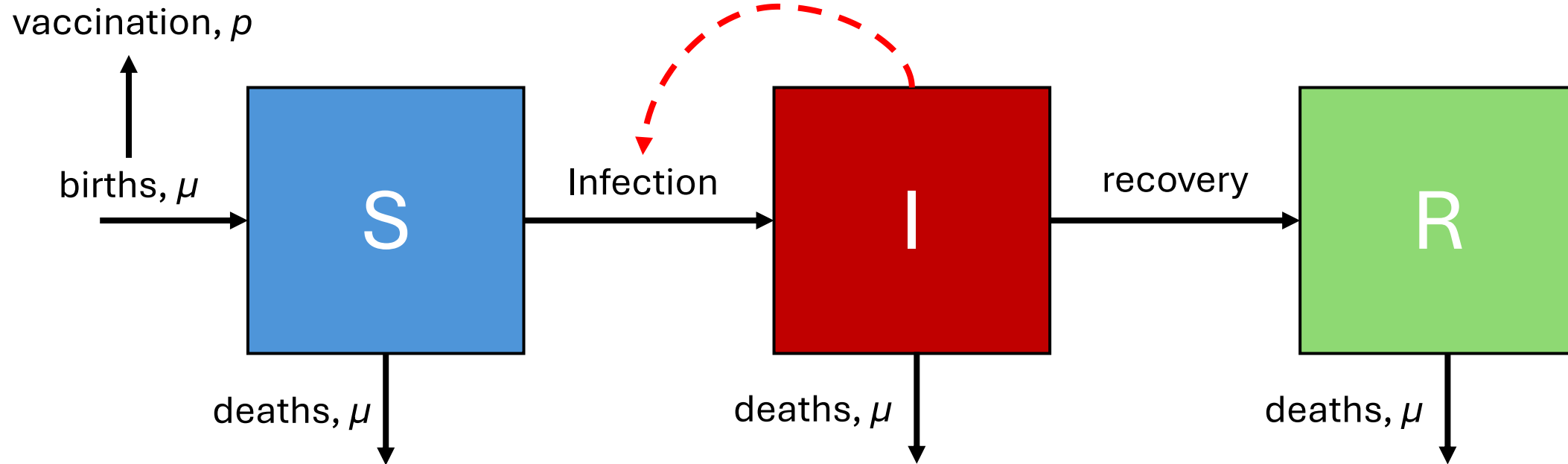
What do we change if we incorporate births and deaths?

Que change-t-on si on inclut des naissances et des décès?

The SIR model : extensions



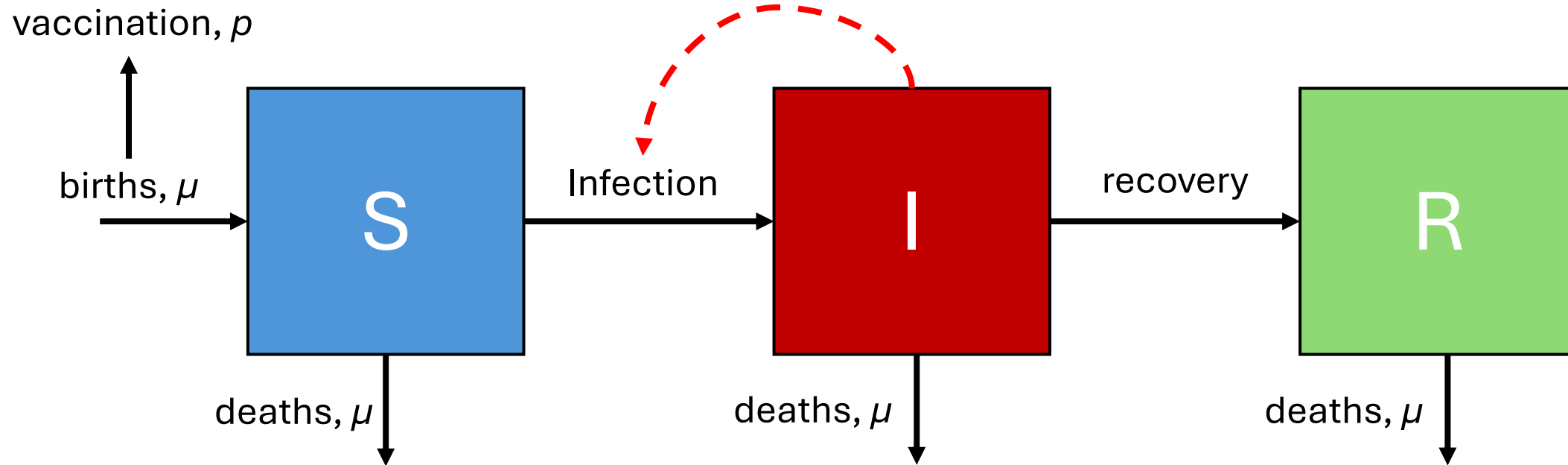
The SIR model : extensions



$$\frac{dS(t)}{dt} = \mu(1 - p) - \beta S(t)I(t) - \mu S(t)$$

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The SIR model : extensions

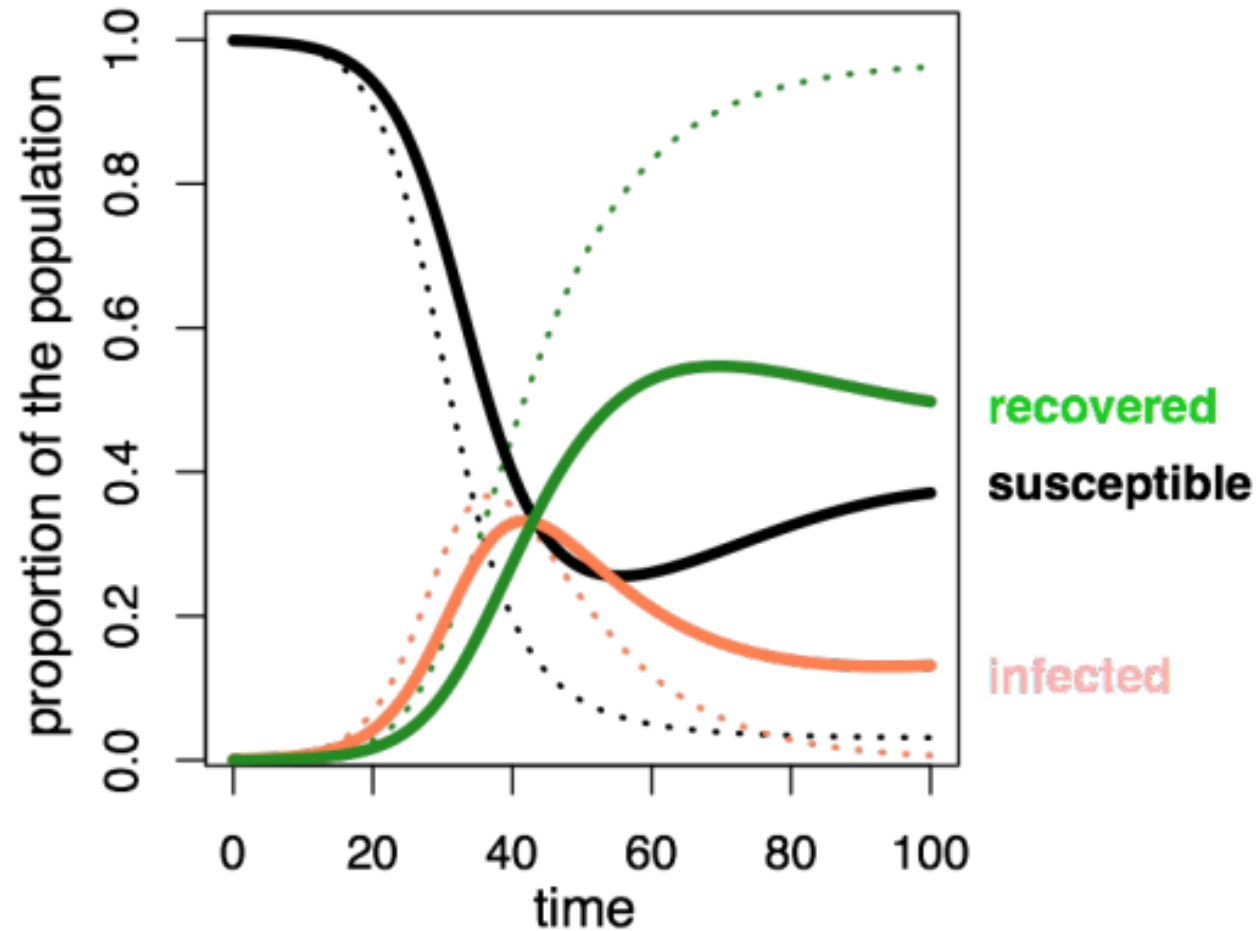


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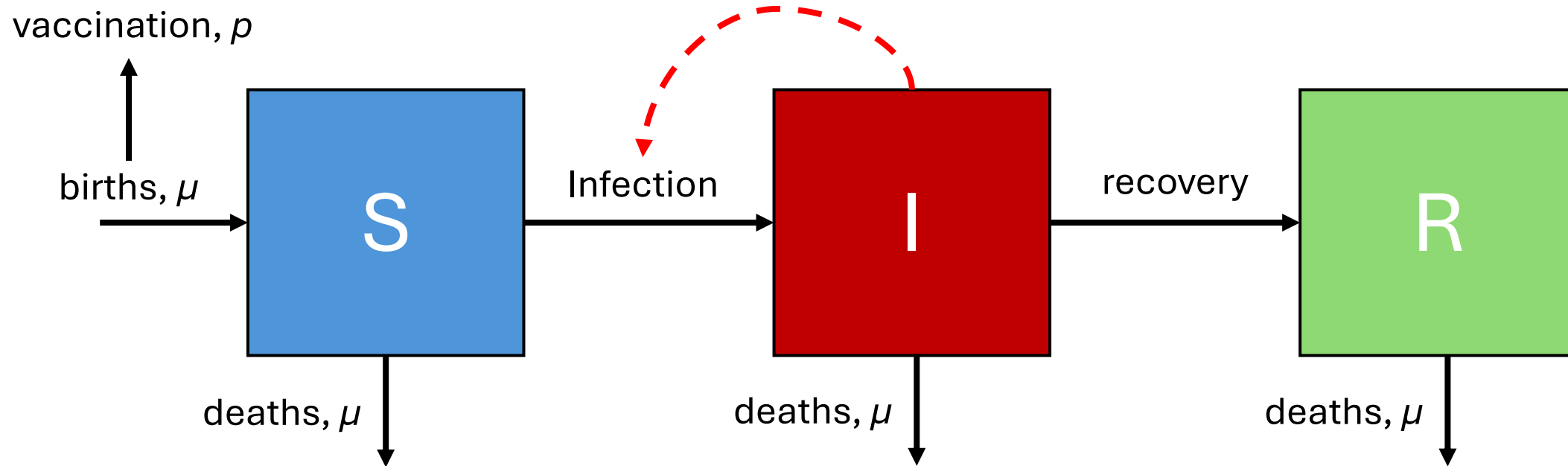
How will births
impact
dynamics?

The SIR model : extensions



SIR with births

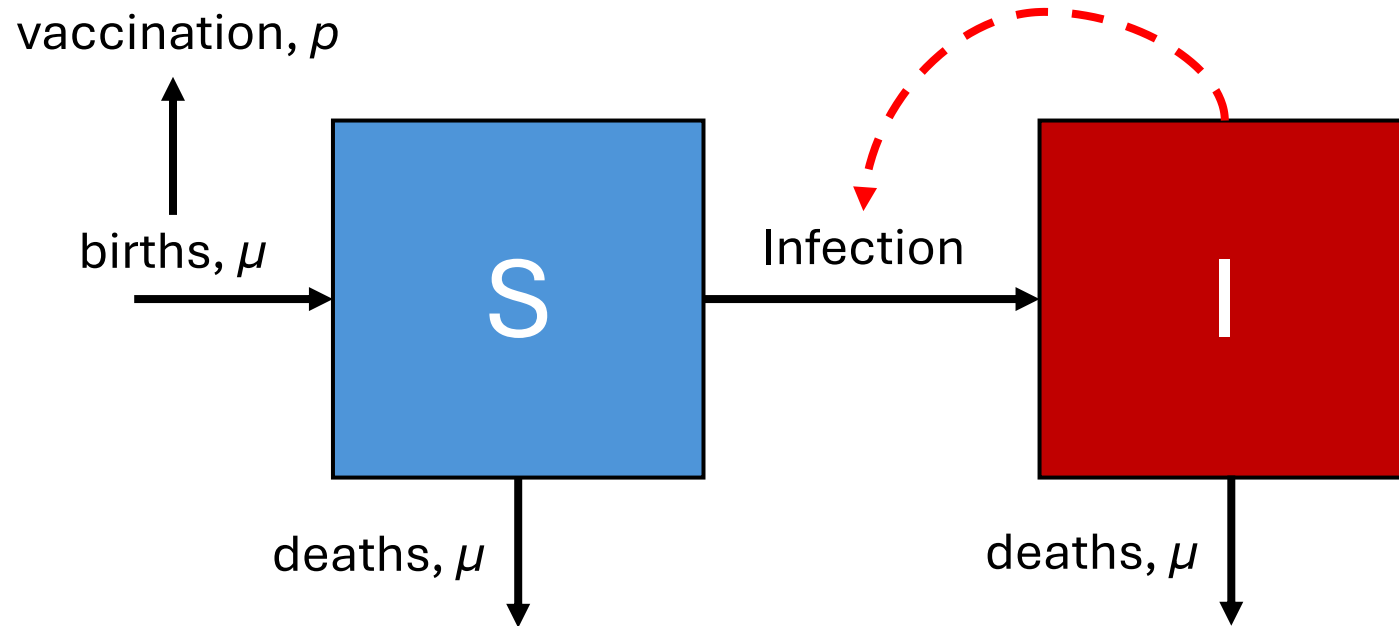
The SIR model : extensions



What do we change if infection is always fatal?

Que change-t-on si l'infection est toujours mortelle?

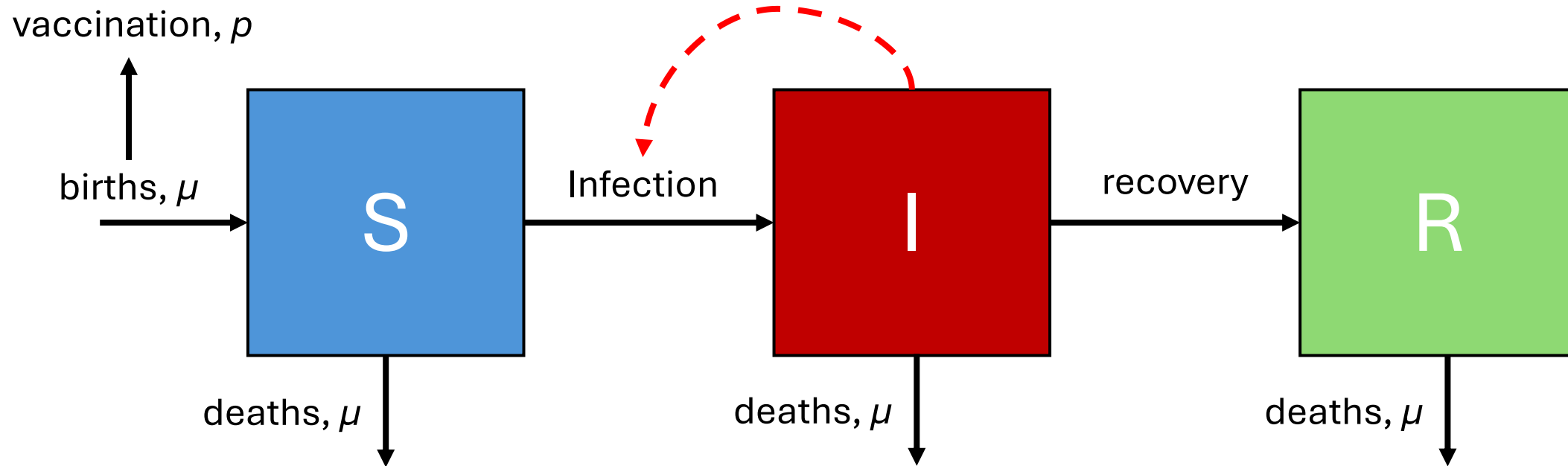
The SI model



What do we change if infection is always fatal? No recovered class

Que change-t-on si l'infection est toujours mortelle? Pas d'une classe récupérée

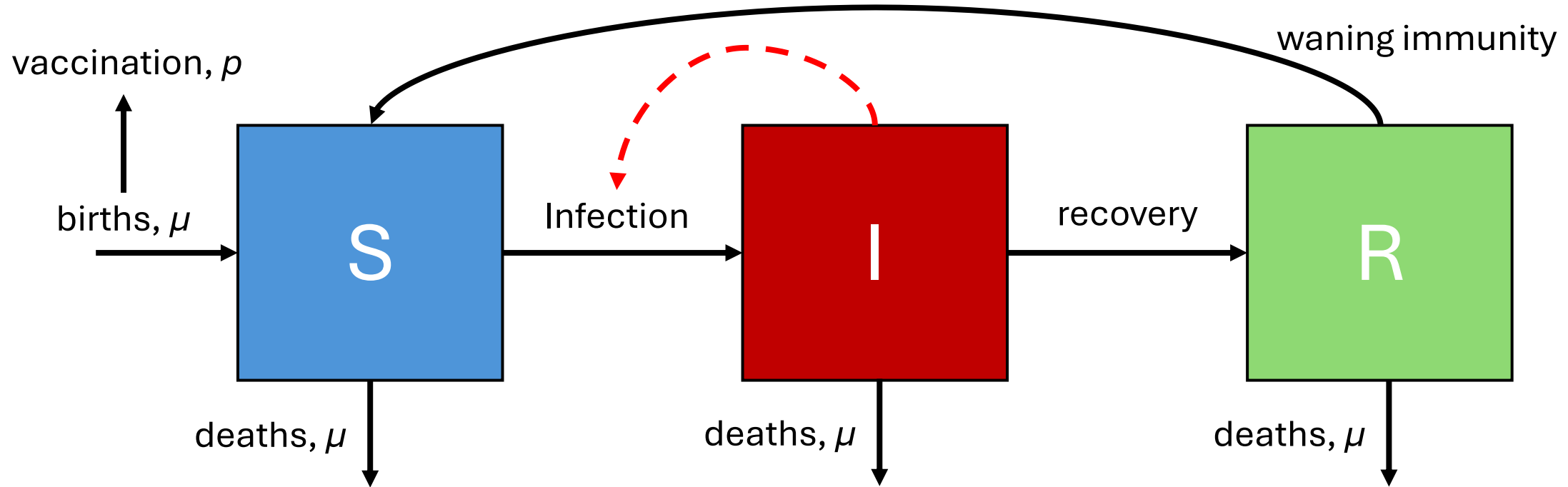
The SIR model : extensions



What do we change if immunity wanes?

Que change-t-on si l'immunité diminue?

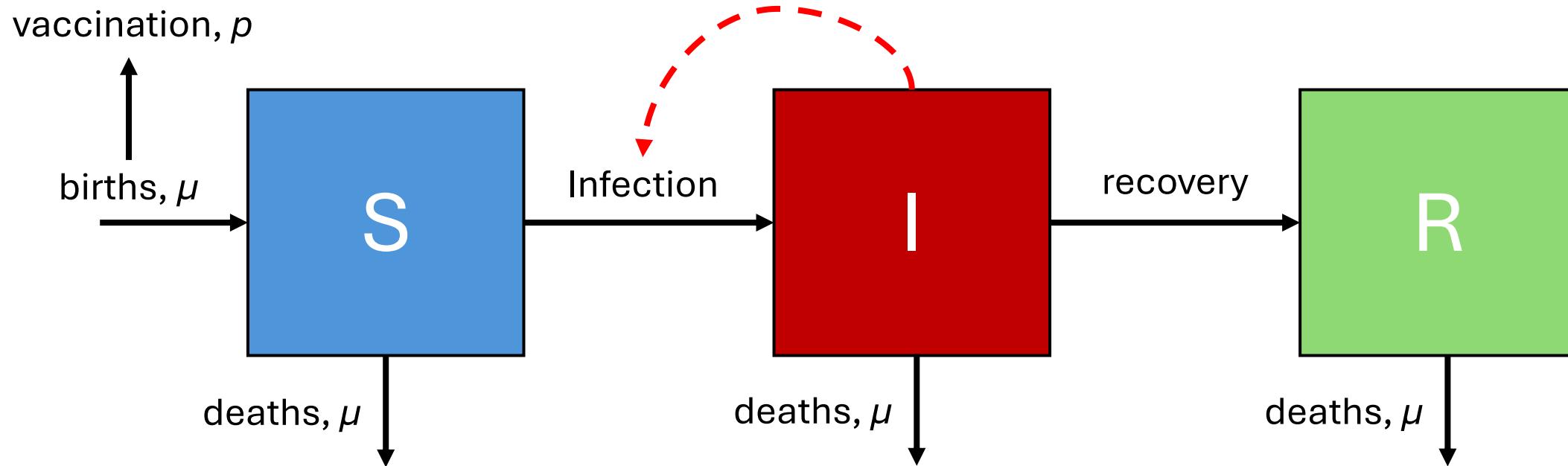
The SIRS model



What do we change if immunity wanes? Recovered individuals become susceptible

Que change-t-on si l'immunité diminue? Les individus récupérés deviennent sensibles

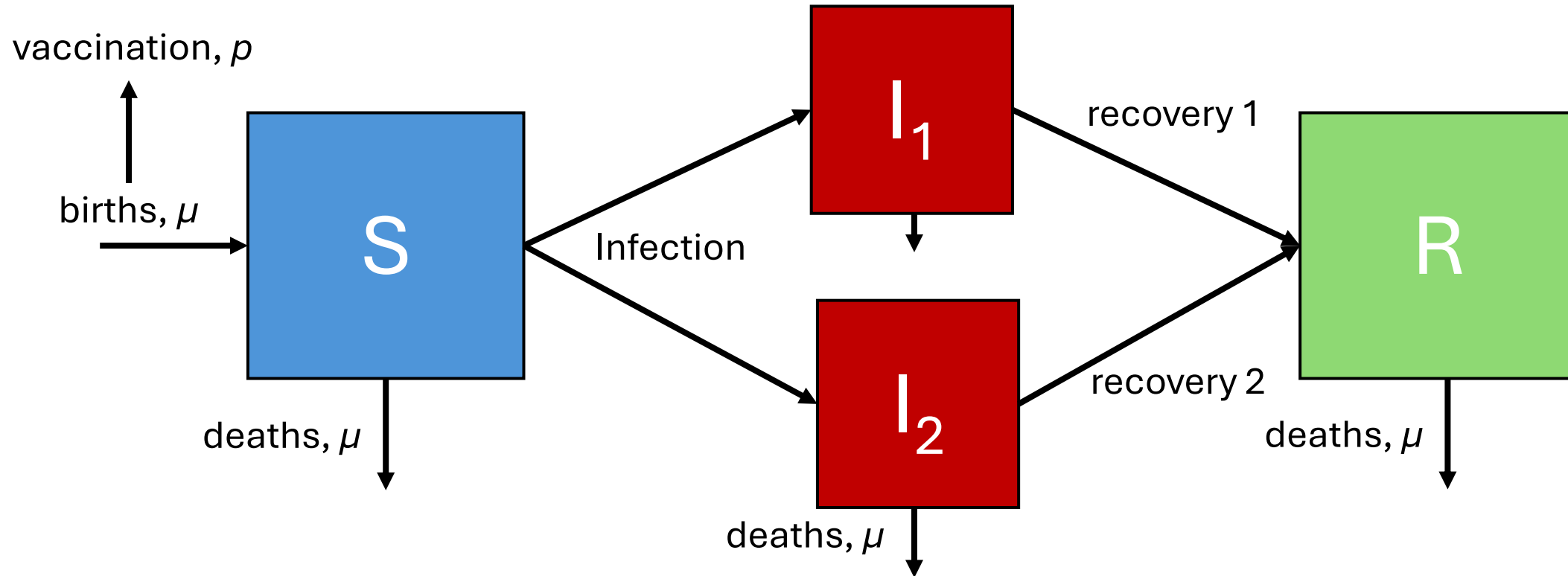
The SIR model : extensions



What do we change if people recover at different rates?

Que change-t-on si les taux de récupération diffèrent?

The SIR model : extensions



What do we change if people recover at different rates?

Que change-t-on si les taux de récupération diffèrent?

Key concepts

- Compartmental / mechanistic / mathematical models
Modèles à compartiments
- Continuous vs. discrete models
Les modèles à temps discrets et les modèles à temps continu
- Deterministic vs. stochastic models
Modèles déterministe vs. stochastique
- Structured models
Modèles structurés
- Two population models
Modèles des deux populations
- SIR models – and beyond!
Modèles SIR – et au delà !

R Tutorial

