Module 2. Laplace Transforms

INTRODUCTION: The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required for engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready tables of Laplace transforms reduce the problem of solving differential equations to mere algebraic manipulations.

Definition: Let $f(t)$ be a function of t (defined for all positive values of t). The Laplace transform of $f(t)$, denoted by $L[f(t)]$, is defined by the equation

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$$

provided the integral converges. In the definition, s is a parameter which may be a real or complex number. $L[f(t)]$ is denoted by $\bar{f}(s)$ or $F(s)$. The symbol L is called the Laplace transform operator.

Remarks

1. The Laplace transform (an integral transform) converts a function $f(t)$ into a function $\bar{f}(s)$.

2. The Laplace transform technique is very useful in solving linear differential equations with initial conditions. It is a powerful tool for solving electrical circuit and systems problems.

Linearity Property

If a, b and c are constants and f, g and h are functions of t, then

$$L[a\,f(t)+b\,g(t)-c\,h(t)] = a\,L[f(t)]+b\,L[g(t)]-c\,L[h(t)]$$

2.1 Laplace Transform of Elementary Functions

1. $L[1] = \frac{1}{s}$ & $L[k] = \frac{k}{s}$ **(where k is any constant)**

Examples: $L[2] = \frac{2}{s}$ $L[-5] = \frac{-5}{s}$

2. $L[e^{at}] = \frac{1}{s-a}$ & $L[e^{-at}] = \frac{1}{s+a}$

Examples: $L[e^{2t}] = \frac{1}{s-2}$ $L[e^{-3t}] = \frac{1}{s+3}$

3. $L[\sin at] = \frac{a}{s^2+a^2}$ & $L[\cos at] = \frac{s}{s^2+a^2}$

Examples: $L[\sin 2t] = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$ $L[\cos t] = \frac{s}{s^2+1^2} = \frac{s}{s^2+1}$

4. $L[\sinh at] = \frac{a}{s^2-a^2}$ & $L[\cosh at] = \frac{s}{s^2-a^2}$

Examples: $L[\sinh t] = \frac{1}{s^2-1^2} = \frac{1}{s^2-1}$ $L[\cosh 2t] = \frac{s}{s^2-2^2} = \frac{s}{s^2-4}$

5. $L[t^n] = \frac{n!}{s^{n+1}}$, **if n is a positive integer**

Examples: $L[t] = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$ $L[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

Note: $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$, **if n is real number .**

SOLVED PROBLEMS

1. Find the Laplace transform of $e^{2t} + 4t^3 - 2sin3t + 3cos3t$

Solution: Let $f(t) = e^{2t} + 4t^3 - 2sin3t + 3cos3t$

Take the Laplace transform of both sides

$$L[f(t)] = L[e^{2t} + 4t^3 - 2sin3t + 3cos3t]$$

$$= L[e^{2t}] + 4L[t^3] - 2L[sin3t] + 3L[cos3t] \qquad \text{(by linearity)}$$

$We\ have\ L[e^{at}] = \frac{1}{s-a}, L[t^n] = \frac{n!}{s^{n+1}}, L[\sin at] = \frac{a}{s^2+a^2}$ & $L[\cos at] = \frac{s}{s^2+a^2}$

$$\therefore L[f(t)] = \frac{1}{s-2} + 4\left(\frac{3!}{s^4}\right) - 2\left(\frac{3}{s^2+9}\right) + 3\left(\frac{s}{s^2+9}\right)$$

$$= \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}$$

Thus $\boldsymbol{L[f(t)] = \frac{1}{s-2} + \frac{24}{s^4} + \frac{3s-6}{s^2+9}}$.

2. Find the Laplace transform of $\sin 2t\, sin3t$

Solution: Let $f(t) = \sin 2t\, sin3t$

$$\text{We know that } \sin A\, sinB = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\therefore f(t) = sin2t\, sin3t = \frac{1}{2}[\cos(-t) - cos5t]$$

$$f(t) = \frac{1}{2}[\cos t - \cos 5t] \qquad [\cos(-\theta) = \cos\theta]$$

Take the Laplace transform of both sides

$$L[f(t)] = L\left[\frac{1}{2}[\cos t - \cos 5t]\right]$$

$$L[f(t)] = \frac{1}{2}[L(\cos t) - L(\cos 5t)] \qquad \text{(by linearity)}$$

$We\ have\ \ L[\cos at] = \dfrac{s}{s^2+a^2}$

$$\text{Thus } \boldsymbol{L[f(t)] = \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]}.$$

3. Find the Laplace transform of $cos^2 2t$

Solution: Let $f(t) = cos^2 2t$

$$\text{We know that } cos^2\theta = \frac{1}{2}(1 + cos2\theta)$$

$$\therefore f(t) = \frac{1}{2}(1 + \cos 4t)$$

Take the Laplace transform of both sides

$$L[f(t)] = L\left[\frac{1}{2}(1 + \cos 4t)\right]$$

$$L[f(t)] = \frac{1}{2}[L(1) + L(\cos 4t)] \qquad \text{(by linearity)}$$

$$We\ have\ \ L[1] = \frac{1}{s} \quad \& \quad L[\cos at] = \frac{s}{s^2 + a^2}$$

$$\text{Thus } \boldsymbol{L[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 16}\right]}.$$

4. Find the Laplace transform of $(sint - cost)^2$

Solution: Let $f(t) = (sint - cost)^2$

$$= \sin^2 t + \cos^2 t - 2\sin t \cos t$$

We know that $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\ cos\ \theta = \sin 2\theta$

$$\therefore f(t) = 1 - \sin 2t$$

Take the Laplace transform of both sides

$$L[f(t)] = L[1 - \sin 2t]$$

$$L[f(t)] = L(1) - L(\sin 2t) \qquad \text{(by linearity)}$$

$$We\ have\ \ L[1] = \frac{1}{s} \quad \& \quad L[\sin at] = \frac{a}{s^2 + a^2}$$

$$\text{Thus } \boldsymbol{L[f(t)] = \frac{1}{s} - \frac{2}{s^2 + 4}}.$$

5. Find the Laplace transform of $\cos(at + b)$

Solution: Let $f(t) = \cos(at + b)$

We know that $\cos(A + B) = cosA\ cosB - sinA \sin B$

$$\therefore f(t) = cosat\ cosb - sinat \sin b$$

Take the Laplace transform of both sides

$$L[f(t)] = L[cosat\ cosb - sinat \sin b]$$

$$L[f(t)] = \cos b\, L(\cos at) - \sin b\, L(\sin at) \qquad \text{(by linearity)}$$

$$We\ have\ \ L[\cos at] = \frac{s}{s^2 + a^2} \quad \& \quad L[\sin at] = \frac{a}{s^2 + a^2}$$

$$\therefore L[f(t)] = cosb\left(\frac{s}{s^2 + a^2}\right) - sinb\left(\frac{a}{s^2 + a^2}\right)$$

$$= \frac{s\cos b}{s^2 + a^2} - \frac{a \sin b}{s^2 + a^2}$$

$$\text{Thus } \boldsymbol{L[f(t)] = \frac{s\ cosb - a\ sinb}{s^2 + a^2}}.$$

EXERCISE PROBLEMS

Find the Laplace transform of the following functions:

1. $\sin(a+bt)$ **Answer:** $\dfrac{s\sin a + b\cos a}{s^2+b^2}$

2. $\sin 3t \cos 2t$ **Answer:** $\dfrac{1}{2}\left\{\dfrac{5}{s^2+25}+\dfrac{1}{s^2+1}\right\}$

3. $\cos 3t \cos 2t$ **Answer:** $\dfrac{1}{2}\left\{\dfrac{s}{s^2+25}+\dfrac{s}{s^2+1}\right\}$

4. $\sin^2 2t$ **Answer:** $\dfrac{1}{2}\left\{\dfrac{1}{s}-\dfrac{s}{s^2+16}\right\}$

5. $\sin 2t \cos 2t$ **Answer:** $\dfrac{2}{s^2+16}$

6. $4t-3$ **Answer:** $\dfrac{4}{s^2}-\dfrac{3}{s}$

7. $3e^{-2t}$ **Answer:** $\dfrac{3}{s+2}$

2.2 Properties of Laplace Transforms

Property 1 [First Shifting Property]

If $L[f(t)] = \bar{f}(s)$, then $L[e^{at}f(t)] = \bar{f}(s-a)$.

Or

If $L[f(t)] = \bar{f}(s)$, then $L[e^{at}f(t)] = \left[\bar{f}(s)\right]_{s \to s-a}$ (s is replaced by $s-a$)

Or

$$L[e^{at}f(t)] = [L\{f(t)\}]_{s\to s-a}$$

SOLVED PROBLEMS

1. Find the Laplace transform of $e^{-3t}(2cos5t - 3sin5t)$

Solution: Let $f(t) = 2cos5t - 3sin5t$

Take the Laplace transform of both sides

$$L[f(t)] = L[2\cos 5t - 3\sin 5t]$$

$$= 2L[\cos 5t] - 3L[\sin 5t] \qquad \text{(by linearity)}$$

$We\ have\ \ L[\cos at] = \dfrac{s}{s^2+a^2}$ & $L[\sin at] = \dfrac{a}{s^2+a^2}$

$$\therefore L[f(t)] = 2\cdot\frac{s}{s^2+5^2} - 3\cdot\frac{5}{s^2+5^2}$$

$$= \frac{2s}{s^2+25} - \frac{15}{s^2+25}$$

$$\therefore L[f(t)] = \frac{2s-15}{s^2+25} = \bar{f}(s)$$

By shifting property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[e^{at}f(t)] = \left[\bar{f}(s)\right]_{s\to s-a}$

$$\therefore L[e^{-3t}f(t)] = \left[\bar{f}(s)\right]_{s\to s+3}$$

$$= \left[\frac{2s-15}{s^2+25}\right]_{s\to s+3}$$

$$= \frac{2(s+3)-15}{(s+3)^2+25}$$

$$\text{Thus } \boldsymbol{L[e^{-3t}(2\cos 5t - 3\sin 5t)] = \frac{2s-9}{s^2+6s+34}}$$

2. Find the Laplace transform of $e^{2t}\,cos^2t$

Solution: Let $f(t) = cos^2t$

$$f(t) = \frac{1}{2}(1 + \cos 2t)$$

Take the Laplace transform of both sides

$$L[f(t)] = L\left[\frac{1}{2}(1 + \cos 2t)\right]$$

$$= \frac{1}{2}[L[1] + L[cos2t]] \qquad \text{(by linearity)}$$

$$We\ have\ \ L[1] = \frac{1}{s} \quad \& \quad L[\cos at] = \frac{s}{s^2 + a^2}$$

$$\therefore\ L[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 4}\right] = \bar{f}(s)$$

By shifting property, we have

$$\text{If } L[f(t)] = \bar{f}(s), \text{ then } L[e^{at}f(t)] = [\bar{f}(s)]_{s \to s-a}$$

$$\therefore L[e^{2t}f(t)] = [\bar{f}(s)]_{s \to s-2}$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 4}\right]_{s \to s-2}$$

$$\text{Thus } \boldsymbol{L[e^{2t}cos^2t] = \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4}\right]}$$

3. Find the Laplace transform of t^2e^{-2t}

Solution: Let $f(t) = t^2$

Take the Laplace transform of both sides

$$L[f(t)] = L[t^2]$$

$$We\ have\ \ L[t^n] = \frac{n!}{s^{n+1}}$$

$$\therefore\ L[f(t)] = \frac{2!}{s^{2+1}} = \frac{2}{s^3} = \bar{f}(s)$$

By shifting property, we have

$$\text{If } L[f(t)] = \bar{f}(s), \text{ then } L[e^{at}f(t)] = [\bar{f}(s)]_{s \to s-a}$$

$$\therefore L[e^{-2t}f(t)] = [\bar{f}(s)]_{s \to s+2}$$

$$= \left[\frac{2}{s^3}\right]_{s \to s+2}$$

$$\text{Thus } \boldsymbol{L}[\boldsymbol{e}^{-2t}\boldsymbol{t}^2] = \frac{\mathbf{2}}{(\boldsymbol{s}+\mathbf{2})^3}$$

EXERCISE PROBLEMS

Find the Laplace transform of the following functions:

1. $e^{2t}(3t^2 - cos4t)$ **Answer**: $\frac{6}{(s-2)^3} - \frac{(s-2)}{s^2-4s+20}$

2. $e^{-t}sin^2 3t$ **Answer**: $\frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+36}\right]$

3. $e^{3t}\,(2t+5)^2$ **Answer**: $\frac{8}{(s-3)^3} + \frac{25}{(s-3)} + \frac{20}{(s-3)^2}$

<u>Property 2</u> [Multiplication by t^n property]

If $L[f(t)] = \bar{f}(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$, where $n = 1, 2, 3 \ldots$

Or

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}], \quad \text{where } n = 1, 2, 3 \ldots$$

SOLVED PROBLEMS

1. Find the Laplace transform of $t^4 e^{-3t}$

Solution: Let $f(t) = e^{-3t}$

Take the Laplace transform of both sides

$$L[f(t)] = L[e^{-3t}] = \frac{1}{s+3} = \bar{f}(s)$$

By multiplication by t^n property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$, where $n = 1, 2, 3 \ldots$

$$\therefore L[t^4 f(t)] = (-1)^4 \frac{d^4}{ds^4}[\bar{f}(s)]$$

$$= \frac{d^4}{ds^4}\left[\frac{1}{s+3}\right] \qquad use \ \frac{d}{dx}\left[\frac{1}{x^n}\right] = -\frac{n}{x^{n+1}}$$

$$= \frac{d^3}{ds^3}\left[-\frac{1}{(s+3)^2}\right] = -\frac{d^3}{ds^3}\left[\frac{1}{(s+3)^2}\right]$$

$$= -\frac{d^2}{ds^2}\left[-\frac{2}{(s+3)^3}\right] = 2\frac{d^2}{ds^2}\left[\frac{1}{(s+3)^3}\right]$$

$$= 2\frac{d}{ds}\left[-\frac{3}{(s+3)^4}\right] = -6\frac{d}{ds}\left[\frac{1}{(s+3)^4}\right]$$

$$= -6\left[-\frac{4}{(s+3)^5}\right]$$

Thus $\boldsymbol{L[t^4 e^{-3t}] = \frac{24}{(s+3)^5}}$

Alternative Method: Rewrite the given as $e^{-3t} t^4$

By shifting property, we have $L[e^{at} f(t)] = [L\{f(t)\}]_{s \to s-a}$

$$\therefore L[e^{-3t} t^4] = [L(t^4)]_{s \to s+3}$$

$$= \left[\frac{4!}{s^{4+1}}\right]_{s \to s+3} = \left[\frac{24}{s^5}\right]_{s \to s+3}$$

$$\text{Thus } \boldsymbol{L[e^{-3t}\, t^4] = \frac{24}{(s+3)^5}}$$

2. Find the Laplace transform of $\boldsymbol{t\, cosat}$

Solution: Let $f(t) = \cos at$

Take the Laplace transform of both sides

$$L[f(t)] = L[\cos at] = \frac{s}{s^2+a^2} = \bar{f}(s)$$

By multiplication by t^n property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$, where $n = 1, 2, 3 \ldots$

$$\therefore L[tf(t)] = (-1)^1 \frac{d^1}{ds^1}[\bar{f}(s)]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\left[\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right] \qquad \text{(by quotient rule)}$$

$$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right] = -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$\text{Thus } \boldsymbol{L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}}$$

3. Find the Laplace transform of $\boldsymbol{t\, e^{-t} sin3t}$

Solution: Rewrite the given as $e^{-t}\, t\, sin3t$

First let us find $L[t\, sin3t]$

By multiplication by t^n property, we have $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[L[f(t)]]$

$$\therefore L[t\, sin3t] = (-1)^1 \frac{d^1}{ds^1}[L[\sin 3t]] = -\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$

$$= -\left[-\frac{3}{(s^2+9)^2}(2s)\right]$$

$$L[t\, sin3t] = \frac{6s}{(s^2+9)^2}$$

By shifting property, we have $L[e^{at} f(t)] = [L\{f(t)\}]_{s\to s-a}$

$$\therefore L[e^{-t}\,(t\, sin3t)] = [L(t\sin 3t)]_{s\to s+1}$$

$$= \left[\frac{6s}{(s^2+9)^2}\right]_{s \to s+1}$$

$$= \left[\frac{6(s+1)}{[(s+1)^2+9]^2}\right]$$

Thus $\boldsymbol{L[e^{-t}\, t\, sin3t] = \frac{6(s+1)}{(s^2+2s+10)^2}}$

4. Find the Laplace transform of $t^2\ sinat$

Solution: Let $f(t) = \sin at$

Take the Laplace transform of both sides

$$L[f(t)] = L[\sin at] = \frac{a}{s^2+a^2} = \bar{f}(s)$$

By multiplication by t^n property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$, where $n = 1, 2, 3 \ldots$

$$\therefore L[t^2\, f(t)] = (-1)^2 \frac{d^2}{ds^2}[\bar{f}(s)]$$

$$= \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{d}{ds}\left[-\frac{a}{(s^2+a^2)^2}(2s)\right] \qquad by\ \frac{d}{dx}\left[\frac{1}{x^n}\right] = -\frac{n}{x^{n+1}}$$

$$= -2a\frac{d}{ds}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$= -2a\left[\frac{(s^2+a^2)^2(1) - s[2(s^2+a^2)(2s)]}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{(s^2+a^2)\{(s^2+a^2) - 4s^2\}}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{(a^2-3s^2)}{(s^2+a^2)^3}\right]$$

Thus $\boldsymbol{L[t^2 \sin at] = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}}$

EXERCISE PROBLEMS

Find the Laplace transform of the following functions:

1. $t\sin^2 t$ **Answer**: $\dfrac{2(3s^2+4)}{s^2(s^2+4)^2}$

2. $t^2\cos at$ **Answer**: $\dfrac{2s^3-6a^2s}{(s^2+a^2)^3}$

3. $t\,e^{-2t}\sin 4t$ **Answer**: $\dfrac{8(s+2)}{s^2+4s+20}$

<u>Property 3</u> [Division by *t* property]

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \bar{f}(s)ds$ provided the integral exists

Or

$$L\left[\frac{1}{t}f(t)\right] = \int_s^\infty [L\{f(t)\}]ds$$

SOLVED PROBLEMS

1. Find the Laplace transform of $\frac{1-e^t}{t}$

Solution: Let $f(t) = 1 - e^t$

Take the Laplace transform of both sides

$$L[f(t)] = L[1 - e^t] = \frac{1}{s} - \frac{1}{s-1} = \bar{f}(s)$$

By division by t property, we have

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \bar{f}(s)ds$ provided the integral exists

$$\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds = [\log s - \log(s-1)]_s^\infty$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty = \left[\log\left\{\frac{s}{s\left(1-\frac{1}{s}\right)}\right\}\right]_s^\infty$$

$$= \left[\log\left(\frac{1}{1-\frac{1}{s}}\right)\right]_s^\infty = \log\left(\frac{1}{1-\frac{1}{\infty}}\right) - \log\left(\frac{1}{1-\frac{1}{s}}\right)$$

$$= \log\left(\frac{1}{1-0}\right) - \log\left(\frac{s}{s-1}\right) \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \log 1 - \log\left(\frac{s}{s-1}\right)$$

$$= 0 - \log\left(\frac{s}{s-1}\right) \qquad [\log 1 = 0]$$

$$= \log\left(\frac{s-1}{s}\right) \qquad \left[\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)\right]$$

Thus $\boldsymbol{L\left[\frac{1-e^t}{t}\right] = \log\left(\frac{s-1}{s}\right)}$ ***or*** $\boldsymbol{L\left[\frac{1-e^t}{t}\right] = \log\left(1 - \frac{1}{s}\right)}$

2. Find the Laplace transform of $\dfrac{\boldsymbol{cosat - cosbt}}{\boldsymbol{t}}$

Solution: Let $f(t) = \cos at - \cos bt$

Take the Laplace transform of both sides

$$L[f(t)] = L[\cos at - \cos bt] = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} = \bar{f}(s)$$

By division by t property, we have

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \bar{f}(s)ds$ provided the integral exists

$$\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2 + a^2) - \frac{1}{2}\log(s^2 + b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left\{\frac{s^2\left(1 + \frac{a^2}{s^2}\right)}{s^2\left(1 + \frac{b^2}{s^2}\right)}\right\}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1 + \frac{a^2}{s^2}}{1 + \frac{b^2}{s^2}}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{1 + \frac{a^2}{\infty}}{1 + \frac{b^2}{\infty}}\right) - \log\left(\frac{1 + \frac{a^2}{s^2}}{1 + \frac{b^2}{s^2}}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1 + 0}{1 + 0}\right) - \log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)\right] \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)\right] \qquad [\log 1 = 0]$$

$$= \frac{1}{2}\log\left(\frac{s^2 + b^2}{s^2 + a^2}\right) \qquad \left[\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)\right]$$

$$= \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)^{\frac{1}{2}} \qquad [\log m^n = n\log m]$$

Thus $\boldsymbol{L}\left[\dfrac{\boldsymbol{cosat - cosbt}}{\boldsymbol{t}}\right] = \boldsymbol{log}\left(\sqrt{\dfrac{s^2 + b^2}{s^2 + a^2}}\right)$

3. Find the Laplace transform of $\frac{e^{-3t}-e^{-4t}}{t}$

Solution: Let $f(t) = e^{-3t} - e^{-4t}$

Take the Laplace transform of both sides

$$L[f(t)] = L[e^{-3t} - e^{-4t}] = \frac{1}{s+3} - \frac{1}{s+4} = \bar{f}(s)$$

By division by t property, we have

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \bar{f}(s)ds$ provided the integral exists

$$\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right)ds$$

$$= [\log(s+3) - \log(s+4)]_s^\infty$$

$$= \left[\log\left(\frac{s+3}{s+4}\right)\right]_s^\infty = \left[\log\left\{\frac{s(1+\frac{3}{s})}{s\left(1+\frac{4}{s}\right)}\right\}\right]_s^\infty$$

$$= \left[\log\left(\frac{1+\frac{3}{s}}{1+\frac{4}{s}}\right)\right]_s^\infty = \log\left(\frac{1+\frac{3}{\infty}}{1+\frac{4}{\infty}}\right) - \log\left(\frac{1+\frac{3}{s}}{1+\frac{4}{s}}\right)$$

$$= \log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{s+3}{s+4}\right) \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \log 1 - \log\left(\frac{s+3}{s+4}\right)$$

$$= 0 - \log\left(\frac{s+3}{s+4}\right) \qquad [\log 1 = 0]$$

$$= \log\left(\frac{s+4}{s+3}\right) \qquad \left[\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)\right]$$

Thus $\boldsymbol{L\left[\frac{e^{-3t}-e^{-4t}}{t}\right] = log\left(\frac{s+4}{s+3}\right)}$

EXERCISE PROBLEMS

Find the Laplace transform of the following functions:

1. $\dfrac{sint}{t}$ **Answer**: $cot^{-1}s$

2. $\dfrac{e^{at} - cosbt}{t}$ **Answer**: $log\left(\dfrac{\sqrt{s^2 + b^2}}{(s - a)}\right)$

3. $\dfrac{e^{-at} - e^{-bt}}{t}$ **Answer**: $log\left(\dfrac{s + b}{s + a}\right)$

2.3 Inverse Laplace Transforms

Definition: If $L[f(t)] = \bar{f}(s)$, then $f(t)$ is called the **inverse Laplace transform** of $\bar{f}(s)$ and we write symbolically $L^{-1}[\bar{f}(s)] = f(t)$ where L^{-1} is called the inverse Laplace transformation operator.

Linearity Property

If $\bar{f}_1(s)$ and $\bar{f}_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$L^{-1}[c_1\bar{f}_1(s) + c_2\bar{f}_2(s)] = c_1L^{-1}[\bar{f}_1(s)] + c_2L^{-1}[\bar{f}_2(s)] = c_1f_1(t) + c_2f_2(t)$$

where c_1 and c_2 are any constants.

Inverse Laplace Transform of Standard Functions

1. $L^{-1}\left[\frac{1}{s}\right] = 1$ & $L^{-1}\left[\frac{k}{s}\right] = k$ (where k is any constant)

Examples: $L^{-1}\left[\frac{3}{s}\right] = 3$ $\quad L^{-1}\left[\frac{-7}{s}\right] = -7$

2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$ & $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

Examples: $L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$ $\quad L^{-1}\left[\frac{1}{s+5}\right] = e^{-5t}$

3. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}sinat$ & $L^{-1}\left[\frac{s}{s^2+a^2}\right] = cosat$

Examples:

$$L^{-1}\left[\frac{1}{s^2+9}\right] = L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3}\sin 3t$$

$$L^{-1}\left[\frac{1}{s^2+5}\right] = L^{-1}\left[\frac{1}{s^2+(\sqrt{5})^2}\right] = \frac{1}{\sqrt{5}}\sin\sqrt{5}t$$

$$L^{-1}\left[\frac{s}{s^2+4}\right] = L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t$$

$$L^{-1}\left[\frac{s}{s^2+3}\right] = L^{-1}\left[\frac{s}{s^2+(\sqrt{3})^2}\right] = \cos\sqrt{3}t$$

4. $\mathbf{L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{1}{a}\,sinhat}$ & $\mathbf{L^{-1}\left[\frac{s}{s^2-a^2}\right]=coshat}$

Examples: $\mathrm{L}^{-1}\left[\frac{1}{s^2-4}\right]=\mathrm{L}^{-1}\left[\frac{1}{s^2-2^2}\right]=\frac{1}{2}\sinh 2t$

$$\mathrm{L}^{-1}\left[\frac{1}{s^2-3}\right]=\mathrm{L}^{-1}\left[\frac{1}{s^2-(\sqrt{3})^2}\right]=\frac{1}{\sqrt{3}}\sinh\sqrt{3}t$$

$$\mathrm{L}^{-1}\left[\frac{s}{s^2-9}\right]=\mathrm{L}^{-1}\left[\frac{s}{s^2-3^2}\right]=\cosh 3t$$

$$\mathrm{L}^{-1}\left[\frac{s}{s^2-5}\right]=\mathrm{L}^{-1}\left[\frac{s}{s^2-(\sqrt{5})^2}\right]=\cosh\sqrt{5}t$$

5. $\mathbf{L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}}$, **if n is a positive integer**

Examples: $\mathrm{L}^{-1}\left[\frac{1}{s^2}\right]=\frac{t^{2-1}}{(2-1)!}=\frac{t}{1}=t.$ $\quad \mathrm{L}^{-1}\left[\frac{1}{s^4}\right]=\frac{t^{4-1}}{(4-1)!}=\frac{t^3}{3!}=\frac{t^3}{6}$

2.4 Methods of Finding Inverse Laplace Transforms

METHOD 1: Direct Method

SOLVED PROBLEMS

1. Find the inverse Laplace transform of $\mathbf{\frac{2}{s+3}+\frac{5s}{s^2+9}}$

Solution: Let $\bar{f}(s)=\frac{2}{s+3}+\frac{5s}{s^2+9}$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)]=L^{-1}\left[\frac{2}{s+3}+\frac{5s}{s^2+9}\right]$$

$$=2L^{-1}\left[\frac{1}{s+3}\right]+5L^{-1}\left[\frac{s}{s^2+3^2}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}}$ & $\mathbf{L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at}$

$$\therefore L^{-1}[\bar{f}(s)]=2e^{-3t}+5\cos 3t$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)] = f(t) = 2e^{-3t} + 5\cos 3t}$.

2. Find the inverse Laplace transform of $\frac{s^2 - 3s + 4}{s^3}$

Solution: Let $\bar{f}(s) = \frac{s^2 - 3s + 4}{s^3}$

The above can be written as

$$\bar{f}(s) = \frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3} = \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 4L^{-1}\left[\frac{1}{s^3}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s}\right] = 1}$ & $\mathbf{L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}}$

$$\therefore L^{-1}[\bar{f}(s)] = 1 - 3\left(\frac{t^{2-1}}{(2-1)!}\right) + 4\left(\frac{t^{3-1}}{(3-1)!}\right)$$

$$= 1 - 3\left(\frac{t}{1!}\right) + 4\left(\frac{t^2}{2!}\right)$$

$$= 1 - 3\frac{t}{1} + 4\frac{t^2}{2}$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)] = f(t) = 1 - 3t + 2t^2}$.

3. Find the inverse Laplace transform of $\frac{2s + 3}{s^2 - 8}$

Solution: Let $\bar{f}(s) = \frac{2s + 3}{s^2 - 8}$

The above can be written as

$$\bar{f}(s) = \frac{2s}{s^2 - 8} + \frac{3}{s^2 - 8} = \frac{2s}{s^2 - (\sqrt{8})^2} + \frac{3}{s^2 - (\sqrt{8})^2}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{2s}{s^2 - (\sqrt{8})^2} + \frac{3}{s^2 - (\sqrt{8})^2}\right]$$

$$= 2L^{-1}\left[\frac{s}{s^2-(\sqrt{8})^2}\right] + 3L^{-1}\left[\frac{1}{s^2-(\sqrt{8})^2}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at}$ **&** $\mathbf{L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at}$

$$\therefore L^{-1}[\bar{f}(s)] = 2\cosh\sqrt{8}t + 3\frac{1}{\sqrt{8}}\sinh\sqrt{8}t$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)] = f(t) = 2\cosh\sqrt{8}t + \frac{3}{\sqrt{8}}\sinh\sqrt{8}t}$

4. Find the inverse Laplace transform of $\frac{5}{2s-3} + \frac{4s}{9-s^2}$

Solution: Let $\bar{f}(s) = \frac{5}{2s-3} + \frac{4s}{9-s^2}$

The above can be written as

$$\bar{f}(s) = \frac{5}{2(s-3/2)} - \frac{4s}{s^2-9}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{5}{2(s-3/2)} - \frac{4s}{s^2-9}\right]$$

$$= \frac{5}{2}L^{-1}\left[\frac{1}{s-3/2}\right] - 4L^{-1}\left[\frac{s}{s^2-3^2}\right] \quad \text{(by linearity)}$$

We have $\boldsymbol{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}}$ **&** $\boldsymbol{L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at}$

$$\therefore L^{-1}[\bar{f}(s)] = \frac{5}{2}e^{\frac{3}{2}t} - 4\cosh 3t$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)] = f(t) = \frac{5}{2}e^{\frac{3}{2}t} - 4\cosh 3t}$.

5. Find the inverse Laplace transform of $\frac{2s-5}{4s^2+25}$

Solution: Let $\bar{f}(s) = \frac{2s-5}{4s^2+25}$

The above can be written as

$$\bar{f}(s) = \frac{2s-5}{4\left(s^2+\frac{25}{4}\right)} = \frac{2s}{4[s^2+(5/2)^2]} - \frac{5}{4[s^2+(5/2)^2]}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{s}{2\{s^2+(5/2)^2\}} - \frac{5}{4\{s^2+(5/2)^2\}}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right] - \frac{5}{4}L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at}$ **&** $\mathbf{L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at}$

$$\therefore L^{-1}[\bar{f}(s)] = \frac{1}{2}\cos\left(\frac{5}{2}t\right) - \frac{5}{4}\cdot\frac{1}{5/2}\sin\left(\frac{5}{2}t\right)$$

$$= \frac{1}{2}\cos\left(\frac{5}{2}t\right) - \frac{5}{4}\cdot\frac{2}{5}\sin\left(\frac{5}{2}t\right)$$

$$= \frac{1}{2}\cos\left(\frac{5}{2}t\right) - \frac{1}{2}\sin\left(\frac{5}{2}t\right)$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)] = f(t) = \frac{1}{2}\left[\cos\left(\frac{5}{2}t\right) - \sin\left(\frac{5}{2}t\right)\right]}$.

EXERCISE PROBLEMS

Find the inverse Laplace transform of the following functions:

1. $\frac{2s+5}{s^2+4}$ **Answer**: $2\cos 2t + \frac{5}{2}\sin 2t$
2. $\frac{3s^2+4}{s^5}$ **Answer**: $\frac{3t^2}{2} + \frac{t^4}{6}$
3. $\frac{1}{3s^2+16}$ **Answer**: $\frac{1}{4\sqrt{3}}\sin\left(\frac{4}{\sqrt{3}}t\right)$
4. $\frac{1}{s+2} + \frac{3}{2s+5} - \frac{4}{3s-2}$ **Answer**: $e^{-2t} + \frac{3}{2}e^{-\frac{5}{2}t} - \frac{4}{3}e^{\frac{2}{3}t}$

METHOD 2: Shifting Property

If $L^{-1}[\bar{f}(s)] = f(t)$, then $L^{-1}[\bar{f}(s-a)] = e^{at}L^{-1}[\bar{f}(s)] = e^{at}f(t)$

Or

If $L^{-1}[\bar{f}(s)] = f(t)$, then $L^{-1}[\bar{f}(s+a)] = e^{-at}L^{-1}[\bar{f}(s)] = e^{-at}f(t)$

SOLVED PROBLEMS

1. Find the inverse Laplace transform of $\frac{1}{(s+2)^2}$

Solution: By shifting property, we have

$$L^{-1}[\bar{f}(s+a)] = e^{-at}L^{-1}[\bar{f}(s)]$$

$$\therefore \mathrm{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]$$

We have $\mathbf{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

$$\therefore \mathrm{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}\left(\frac{t^{2-1}}{(2-1)!}\right) = e^{-2t}\left(\frac{t}{1}\right)$$

$$\text{Thus } \mathbf{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-at}f(t) = e^{-2t}t$$

2. Find the Inverse Laplace transform of $\frac{s}{(s-3)^5}$

Solution: $\frac{s}{(s-3)^5} = \frac{s+3-3}{(s-3)^5} = \frac{(s-3)+3}{(s-3)^5}$

By shifting property, we have

$$L^{-1}[\bar{f}(s-a)] = e^{at}L^{-1}[\bar{f}(s)]$$

$$\therefore \mathrm{L}^{-1}\left[\frac{s}{(s-3)^5}\right] = \mathrm{L}^{-1}\left[\frac{(s-3)+3}{(s-3)^5}\right] = e^{3t}L^{-1}\left[\frac{s+3}{s^5}\right]$$

$$= e^{3t}L^{-1}\left[\frac{s}{s^5} + \frac{3}{s^5}\right] = e^{3t}L^{-1}\left[\frac{1}{s^4} + \frac{3}{s^5}\right]$$

$$= e^{3t}\left[L^{-1}\left\{\frac{1}{s^4}\right\} + 3L^{-1}\left\{\frac{1}{s^5}\right\}\right] \qquad \text{by linearity}$$

$$\textbf{We have } L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

$$\therefore L^{-1}\left[\frac{s}{(s-3)^5}\right] = e^{3t}\left[\left\{\frac{t^{4-1}}{(4-1)!}\right\} + 3\left\{\frac{t^{5-1}}{(5-1)!}\right\}\right]$$

$$= e^{3t}\left[\left\{\frac{t^3}{3!}\right\} + 3\left\{\frac{t^4}{4!}\right\}\right]$$

$$= e^{3t}\left[\left\{\frac{t^3}{6}\right\} + 3\left\{\frac{t^4}{24}\right\}\right]$$

Thus $\boldsymbol{L^{-1}\left[\frac{s}{(s-3)^5}\right] = e^{at}f(t) = e^{3t}\left(\frac{t^3}{6} + \frac{t^4}{8}\right)}$

3. Find the Inverse Laplace transform of $\frac{3s+1}{(s+1)^4}$

Solution: $\frac{3s+1}{(s+1)^4} = \frac{3s+1+3-3}{(s+1)^4} = \frac{3s+3-2}{(s+1)^4} = \frac{3(s+1)-2}{(s+1)^4}$

By shifting property, we have

$$L^{-1}[\bar{f}(s+a)] = e^{-at}L^{-1}[\bar{f}(s)]$$

$$\therefore L^{-1}\left[\frac{3s+1}{(s+1)^4}\right] = L^{-1}\left[\frac{3(s+1)-2}{(s+1)^4}\right] = e^{-(1)t}L^{-1}\left[\frac{3s-2}{s^4}\right]$$

$$= e^{-t}L^{-1}\left[\frac{3s}{s^4} - \frac{2}{s^4}\right] = e^{-t}L^{-1}\left[\frac{3}{s^3} - \frac{2}{s^4}\right]$$

$$= e^{-t}\left[3L^{-1}\left\{\frac{1}{s^3}\right\} - 2L^{-1}\left\{\frac{1}{s^4}\right\}\right] \qquad \text{by linearity}$$

$$\textbf{We have } L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

$$\therefore L^{-1}\left[\frac{3s+1}{(s+1)^4}\right] = e^{-t}\left[3\left\{\frac{t^{3-1}}{(3-1)!}\right\} - 2\left\{\frac{t^{4-1}}{(4-1)!}\right\}\right]$$

$$= e^{-t}\left[3\left\{\frac{t^2}{2!}\right\} - 2\left\{\frac{t^3}{3!}\right\}\right]$$

$$= e^{-t}\left[3\left\{\frac{t^2}{2}\right\} - 2\left\{\frac{t^3}{6}\right\}\right]$$

Thus $\boldsymbol{L^{-1}\left[\frac{3s+1}{(s+1)^4}\right] = e^{-at}f(t) = e^{-t}\left(\frac{3t^2}{2} - \frac{t^3}{3}\right)}$

4. Find the inverse Laplace transform of $\dfrac{s+3}{s^2-4s+13}$

Solution: $s^2-4s+13 = s^2-4s+13+2^2-2^2$

$$= [s^2-(2)(2)s+2^2]+13-2^2$$

$$= (s-2)^2+9$$

Now $\dfrac{s+3}{s^2-4s+13} = \dfrac{s+3}{(s-2)^2+9} = \dfrac{s+3+2-2}{(s-2)^2+9} = \dfrac{(s-2)+5}{(s-2)^2+9}$

By shifting property, we have

$$L^{-1}[\bar{f}(s-a)] = e^{at}L^{-1}[\bar{f}(s)]$$

$$\therefore \mathrm{L}^{-1}\left[\frac{s+3}{s^2-4s+13}\right] = \mathrm{L}^{-1}\left[\frac{(s-2)+5}{(s-2)^2+9}\right] = e^{(2)t}L^{-1}\left[\frac{s+5}{s^2+9}\right]$$

$$= e^{2t}\left[L^{-1}\left\{\frac{s}{s^2+3^2}+\frac{5}{s^2+3^2}\right\}\right]$$

$$= e^{2t}\left[L^{-1}\left\{\frac{s}{s^2+3^2}\right\}+5L^{-1}\left\{\frac{1}{s^2+3^2}\right\}\right] \qquad \text{by linearity}$$

We have $\mathbf{L^{-1}}\left[\dfrac{\boldsymbol{s}}{\boldsymbol{s^2+a^2}}\right] = \mathbf{cos}\,\boldsymbol{at}$ & $\mathbf{L^{-1}}\left[\dfrac{\mathbf{1}}{\boldsymbol{s^2+a^2}}\right] = \dfrac{\mathbf{1}}{\boldsymbol{a}}\mathbf{sin}\,\boldsymbol{at}$

$$\therefore \mathrm{L}^{-1}\left[\frac{s+3}{s^2-4s+13}\right] = e^{2t}\left[\cos 3t + 5\frac{1}{3}\sin 3t\right]$$

$$\text{Thus } \boldsymbol{L^{-1}}\left[\frac{\boldsymbol{s+3}}{\boldsymbol{s^2-4s+13}}\right] = \boldsymbol{e^{at}f(t)} = e^{2t}\left(\cos 3t + \frac{5}{3}\sin 3t\right)$$

EXERCISE PROBLEMS

Find the inverse Laplace transform of the following functions:

1. $\dfrac{s}{(s+2)^2}$ **Answer**: $e^{-2t}(1-2t)$

2. $\dfrac{s}{s^2+4s+5}$ **Answer**: $e^{-2t}(\cos t - 2\sin t)$

3. $\dfrac{s+2}{s^2-2s+5}$ **Answer**: $e^{t}(\cos 2t + \frac{3}{2}\sin 2t)$

4. $\dfrac{2s}{s^2+2s+5}$ **Answer**: $e^{-t}(2\cos 2t - \sin 2t)$

METHOD 3: Partial Fraction Method

By partial fraction, we have

1. $$\frac{1}{(s+a)(s+b)} = \frac{A}{(s+a)} + \frac{B}{(s+b)}$$

2. $$\frac{1}{(s+a)(s+b)^2} = \frac{A}{(s+a)} + \frac{B}{(s+b)} + \frac{C}{(s+b)^2}$$

3. $$\frac{1}{(s+a)(s^2+bs+c)} = \frac{A}{(s+a)} + \frac{Bs+C}{(s^2+bs+c)}$$

SOLVED PROBLEMS

1. Find the Inverse Laplace transform of $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$

Solution: By partial fraction we can write the given as

$$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) ----(1)$$

Put $s = 1$ in (1) $2-6+5 = A(1-2)(1-3)$ $1 = A(-1)(-2)$ $1 = 2A$ $\therefore \boldsymbol{A = \frac{1}{2}}$	Put $s = 2$ in (1) $2(4)-6(2)+5 = B(2-1)(2-3)$ $8-12+5 = B(1)(-1)$ $1 = -B$ $\therefore \boldsymbol{B = -1}$

Put $s = 3$ in (1) $2(9)-6(3)+5 = C(3-1)(3-2)$ $18-18+5 = C(1)(2)$ $5 = 2C \quad \therefore \boldsymbol{C = \frac{5}{2}}$

$$\therefore \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{1/2}{s-1} + \frac{(-1)}{s-2} + \frac{5/2}{s-3}$$

Take the inverse Laplace transform of both sides

$$L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right] = L^{-1}\left[\frac{1/2}{s-1}+\frac{(-1)}{s-2}+\frac{5/2}{s-3}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s-2}\right] + \frac{5}{2}L^{-1}\left[\frac{1}{s-3}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}}$

$$\therefore\ L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right] = \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$$

$$\text{Thus } \boldsymbol{L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right] = f(t) = \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}}.$$

2. Find the Inverse Laplace transform of $\boldsymbol{\frac{s-1}{s^2+3s+2}}$

Solution: Here $\frac{s-1}{s^2+3s+2} = \frac{s-1}{(s+1)(s+2)}$ by Factorization

By partial fraction we can write the above as

$$\frac{s-1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$s-1 = A(s+2) + B(s+1) ----(1)$$

Put $s = -2$ in (1) $-2-1 = A(0) + B(-2+1)$ $-3 = -B$ $\therefore \boldsymbol{B = 3}$	Put $s = -1$ in (1) $-1-1 = A(-1+2) + B(0)$ $-2 = A$ $\therefore \boldsymbol{A = -2}$

$$\therefore \frac{s-1}{(s+1)(s+2)} = \frac{-2}{(s+1)} + \frac{3}{(s+2)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}\left[\frac{s-1}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{-2}{(s+1)} + \frac{3}{(s+2)}\right]$$

$$= -2L^{-1}\left[\frac{1}{s+1}\right] + 3L^{-1}\left[\frac{1}{s+2}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}}$

$$\therefore\ L^{-1}\left[\frac{s-1}{(s+1)(s+2)}\right] = -2e^{-t} + 3e^{-2t}$$

Thus $\boldsymbol{L^{-1}\left[\frac{s-1}{s^2+3s+2}\right] = f(t) \; = 3e^{-2t} - 2e^{-t}}$.

3. Find the Inverse Laplace transform of $\frac{4s+5}{(s+2)(s-1)^2}$

Solution: By partial fraction we can write the given as

$$\frac{4s+5}{(s+2)(s-1)^2} = \frac{A}{(s+2)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

$$4s+5 = A(s-1)^2 + B(s+2)(s-1) + C(s+2) - - - -(1)$$

Put $s = -2$ in (1) $-8+5 = A(-2-1)^2 + B(0) + C(0)$ $-3 = A(9)$ $\therefore \boldsymbol{A = \frac{-1}{3}}$	Put $s = 1$ in (1) $4+5 = A(0) + B(0) + C(1+2)$ $9 = C(3)$ $\therefore \boldsymbol{C = 3}$

Put $s = 0$ in (1) $5 = A - 2B + 2C$ $5 = -\frac{1}{3} - 2B + 2(3) = \frac{17}{3} - 2B$ $2B = \frac{17}{3} - 5$ $2B = \frac{2}{3} \quad \therefore \boldsymbol{B = \frac{1}{3}}$

$$\therefore \frac{4s+5}{(s+2)(s-1)^2} = \frac{-\frac{1}{3}}{(s+2)} + \frac{\frac{1}{3}}{(s-1)} + \frac{3}{(s-1)^2}$$

Take the inverse Laplace transform of both sides

$$L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right] = L^{-1}\left[\frac{-\frac{1}{3}}{(s+2)} + \frac{\frac{1}{3}}{(s-1)} + \frac{3}{(s-1)^2}\right]$$

$$= -\frac{1}{3}L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right] + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] \qquad \text{(by linearity)}$$

We have $\mathbf{L^{-1}}\left[\frac{1}{s+a}\right] = e^{-at}, \mathbf{L^{-1}}\left[\frac{1}{s-a}\right] = e^{at}$ & $L^{-1}[\bar{f}(s-a)] = e^{at}L^{-1}[\bar{f}(s)]$

$$\therefore \; L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right] = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} + 3\, e^{(1)t}L^{-1}\left[\frac{1}{s^2}\right]$$

We have $\mathbf{L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}}$

$$\therefore\; L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right] = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} + 3\,e^{t}\,\frac{t^{2-1}}{(2-1)!}$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} + 3\,e^{t}\,\frac{t}{1!}$$

Thus $\boldsymbol{L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right] = f(t) = \left(\frac{1}{3} + 3t\right)e^{t} - \frac{1}{3}e^{-2t}}$.

4. Find the Inverse Laplace transform of $\frac{s}{(s-3)(s^2+4)}$

Solution: By partial fraction we can write the given as

$$\frac{s}{(s-3)(s^2+4)} = \frac{A}{(s-3)} + \frac{Bs+C}{(s^2+4)}$$

$$s = A(s^2+4) + (Bs+C)(s-3) ----(1)$$

Put s = 3 in (1) $3 = A(9+4) + (Bs+C)(0)$ $3 = A(13)$ $\therefore \boldsymbol{A = \frac{3}{13}}$	Put s = 0 in (1) $0 = 4A - 3C$ $0 = 4\left(\frac{3}{13}\right) - 3C =$ $3C = \frac{12}{13} \quad \therefore \boldsymbol{C = \frac{4}{13}}$

$$(1) \Longrightarrow s = As^2 + 4A + Bs^2 - 3Bs + Cs - 3C ----(2)$$

Comparing the coefficients of s^2 in (2), we get

$$0 = A + B$$

$$B = -A = -\left(\frac{3}{13}\right)$$

$$\therefore \boldsymbol{B = -\frac{3}{13}}$$

$$\therefore \frac{s}{(s-3)(s^2+4)} = \frac{\frac{3}{13}}{(s-3)} + \frac{\left(\frac{-3}{13}\right)s + \frac{4}{13}}{(s^2+4)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right] = L^{-1}\left[\frac{\frac{3}{13}}{(s-3)} + \frac{\left(\frac{-3}{13}\right)s + \frac{4}{13}}{(s^2+4)}\right]$$

$$= L^{-1}\left[\frac{\frac{3}{13}}{(s-3)} + \frac{\left(\frac{-3}{13}\right)s}{(s^2+4)} + \frac{\frac{4}{13}}{(s^2+4)}\right]$$

$$= \frac{3}{13}L^{-1}\left[\frac{1}{s-3}\right] - \frac{3}{13}L^{-1}\left[\frac{s}{s^2+2^2}\right] + \frac{4}{13}L^{-1}\left[\frac{1}{s^2+2^2}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}, L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at}$ **&** $\mathbf{L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at}$

$$\therefore\ L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right] = \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{4}{13}\left(\frac{1}{2}\sin 2t\right)$$

$$= \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t$$

Thus $\boldsymbol{L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right] = f(t) = \frac{1}{13}[3e^{3t} - 3\cos 2t + 2\sin 2t]}$

EXERCISE PROBLEMS

Find the inverse Laplace transform of the following functions:

1. $\frac{s}{(s+2)(s+3)}$ **Answer**: $-2e^{-2t} + 3e^{-3t}$
2. $\frac{1-7s}{(s-3)(s-1)(s+2)}$ **Answer**: $-2e^{-2t} + 3e^{-3t}$
3. $\frac{s^2+s-2}{s(s+3)(s-2)}$ **Answer**: $\frac{7}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$
4. $\frac{1}{(s+2)(s+1)^2}$ **Answer**: $e^{-2t} - e^{-t} + te^{-t}$
5. $\frac{1}{(s-1)(s^2+1)}$ **Answer**: $\frac{1}{2}[e^t - \cos t - \sin t]$

2.5 Convolution

Definition: The convolution of two functions $f(t)$ & $g(t)$ usually denoted by $f(t) * g(t)$ is defined in the form of an integral as follows

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

Note: Convolution operation '$*$' is commutative i.e., $f(t) * g(t) = g(t) * f(t)$.

Convolution Theorem for Laplace Transform: If $L[f(t)] = \bar{f}(s)$ and $L[g(t)] = \bar{g}(s)$ then

$$L[f(t) * g(t)] = \bar{f}(s)\bar{g}(s)$$

where, $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

Convolution Theorem for Inverse Laplace Transform: If $L^{-1}[\bar{f}(s)] = f(t)$ & $L^{-1}[\bar{g}(s)] = g(t)$, then

$$L^{-1}[\bar{f}(s)\,\bar{g}(s)] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$$

or

$$L^{-1}[\bar{f}(s)\,\bar{g}(s)] = \int_0^t f(t-u)g(u)du = f(t) * g(t)$$

SOLVED PROBLEMS

1) Apply convolution theorem to find inverse Laplace transform of $\frac{1}{s(s+a)}$.

Solution: First we express the given function as a product of two functions

$$\frac{1}{s(s+a)} = \frac{1}{s}\frac{1}{(s+a)} = \bar{f}(s)\,\bar{g}(s)$$

$$\text{Let } \bar{f}(s) = \frac{1}{s} \quad \& \quad \bar{g}(s) = \frac{1}{(s+a)}$$

$$f(t) = L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s}\right] = 1$$

$$g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$\therefore\ \boldsymbol{f(t) = 1} \qquad \& \qquad \boldsymbol{g(t) = e^{-at}}$$

By Convolution theorem, we have

$$L^{-1}[\bar{f}(s)\,\bar{g}(s)] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$$

$$\therefore L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t (1)\left[e^{-a(t-u)}\right]du$$

$$= \int_0^t e^{-at}e^{au}du = e^{-at}\int_0^t e^{au}\,du$$

$$= e^{-at}\left[\frac{e^{au}}{a}\right]_0^t$$

$$= \frac{e^{-at}}{a}\left[e^{at} - 1\right]$$

$$\therefore\ \boldsymbol{L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1}{a}\left[1 - e^{-at}\right]}$$

2) Use convolution theorem to find inverse Laplace transform of $\boldsymbol{\frac{1}{(s-1)(s+3)}}$.

Solution: First we express the given function as a product of two functions

$$\frac{1}{(s-1)(s+3)} = \frac{1}{(s-1)}\frac{1}{(s+3)} = \bar{f}(s)\,\bar{g}(s)$$

$$\text{Let } \bar{f}(s) = \frac{1}{(s-1)} \qquad \& \qquad \bar{g}(s) = \frac{1}{(s+3)}$$

$$f(t) = L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s-1}\right] = e^{t}$$

$$g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$$

$$\therefore\ \boldsymbol{f(t) = e^{t}} \qquad \& \qquad \boldsymbol{g(t) = e^{-3t}}$$

By Convolution theorem, we have

$$L^{-1}[\bar{f}(s)\,\bar{g}(s)] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$$

$$\therefore L^{-1}\left[\frac{s}{(s-1)(s+3)}\right]=\int_0^t [e^u][e^{-3(t-u)}]du$$

$$=\int_0^t e^{u-3t+3u}\,du$$

$$=e^{-3t}\int_0^t e^{4u}\,du$$

$$=e^{-3t}\left[\frac{e^{4u}}{4}\right]_0^t=\frac{e^{-3t}}{4}[e^{4t}-e^0]$$

$$\therefore \boldsymbol{L^{-1}\left[\frac{1}{(s-1)(s+3)}\right]=\frac{1}{4}(e^t-e^{-3t})}$$

3) Using convolution theorem find inverse Laplace transform of $\boldsymbol{\frac{1}{(s+1)(s^2+4)}}$.

Solution: First we express the given function as a product of two functions

$$\frac{1}{(s+1)(s^2+4)}=\frac{1}{(s+1)}\frac{1}{(s^2+4)}=\bar{f}(s)\,\bar{g}(s)$$

$$\text{Let } \bar{f}(s)=\frac{1}{(s+1)} \quad \& \quad \bar{g}(s)=\frac{1}{(s^2+4)}$$

$$f(t)=L^{-1}[\bar{f}(s)]=L^{-1}\left[\frac{1}{s+1}\right]=e^{-t}$$

$$g(t)=L^{-1}[\bar{g}(s)]=L^{-1}\left[\frac{1}{s^2+4}\right]=L^{-1}\left[\frac{1}{s^2+2^2}\right]=\frac{1}{2}\sin 2t$$

$$\therefore \boldsymbol{f(t)=e^{-t} \qquad \& \qquad g(t)=\frac{1}{2}\sin 2t}$$

By Convolution theorem, we have

$$L^{-1}[\bar{f}(s)\,\bar{g}(s)]=\int_0^t f(t-u)g(u)du=f(t)*g(t)$$

$$\therefore L^{-1}\left[\frac{1}{(s+1)(s^2+4)}\right]=\int_0^t [e^{-(t-u)}]\left[\frac{1}{2}\sin 2u)\right]du$$

$$=\frac{1}{2}\int_0^t e^{-t+u}\sin 2u\,du=\frac{1}{2}\int_0^t e^{-t}e^u\sin 2u\,du$$

$$=\frac{e^{-t}}{2}\int_0^t e^u\sin 2u\,du$$

We know that $\int e^{au} \sin bu \, du = \frac{e^{au}}{a^2+b^2}(a \sin bu - b \cos bu)$

$$\therefore L^{-1}\left[\frac{1}{(s+1)(s^2+4)}\right] = \frac{e^{-t}}{2}\left[\frac{e^{u}}{1^2+2^2}(\sin 2u - 2\cos 2u)\right]_0^t$$

$$= \frac{e^{-t}}{2}\left[\frac{e^{u}}{5}(\sin 2u - 2\cos 2u)\right]_0^t$$

$$= \frac{e^{-2t}}{10}[e^{2u}(\sin 2u - 2\cos 2u)]_0^t$$

$$= \frac{e^{-2t}}{13}[\{e^{2t}(2\cos 3t + 3\sin 3t)\} - \{e^{0}(2\cos 0 + 3\sin 0)\}]$$

$$= \frac{e^{-2t}}{13}[e^{2t}(2\cos 3t + 3\sin 3t) - (1)\{2(1) + 3(0)\}]$$

$$= \frac{e^{-2t}}{13}(2e^{2t}\cos 3t + 3e^{2t}\sin 3t - 2)$$

$$= \frac{1}{13}(2\cos 3t + 3\sin 3t - 2e^{-2t})$$

Thus $\boldsymbol{L^{-1}\left[\frac{s}{(s+2)(s^2+9)}\right] = \frac{1}{13}(2\cos 3t + 3\sin 3t - 2e^{-2t})}$

EXERCISE PROBLEMS

Find the inverse Laplace transform of the following functions using convolution theorem:

1. $\frac{1}{s(s^2+a^2)}$ **Answer**: $\frac{1}{a^2}(1-\cos at)$
2. $\frac{s}{(s^2+4)^2}$ **Answer**: $\frac{t \sin 2t}{4}$
3. $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ **Answer**: $\frac{a \sin at - b \sin bt}{a^2 - b^2}$
4. $\frac{1}{(s-1)(s^2+1)}$ **Answer**: $\frac{1}{2}(e^{t} - \sin t - \cos t)$
5. $\frac{2}{s^2(s^2+4)}$ **Answer**: $\frac{1}{4}(2t - \sin 2t)$

2.6 Application of Laplace Transform to Differential Equations

Laplace transform method of solving differential equations yields particular solution without the necessity of first finding the general solution and then evaluating the arbitrary constants. In general, this method is shorter than our earlier methods and is especially useful for solving linear differential equations with constant coefficients.

Working Procedure to solve linear differential equation with constant coefficient by Laplace transform method.

❖ Take the Laplace transform of both sides of the differential equation and use the following formulae then the given initial conditions

$$L[y'(t)] = s\bar{y}(s) - y(0)$$

$$L[y''(t)] = s^2\bar{y}(s) - sy(0) - y'(0)$$

$$L[y'''(t)] = s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0)$$

$$\text{In general,}\quad L\left[y^{(n)}(t)\right] = s^n\bar{y}(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$$

$$where\ \ \bar{y}(s) = L[y(t)]$$

❖ Transpose the terms with minus sign to the RHS.

❖ Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.

❖ Take the inverse transform of both sides and use appropriate method to find inverse Laplace transform. This gives y as a function of t which is desired solution satisfying the given conditions.

SOLVED PROBLEMS

1) Use Laplace transform method to solve $\frac{dy}{dt} + y = te^{-t}$ with $y(0) = 2$.

Solution: Given $y'(t) + y(t) = te^{-t}$

Taking the Laplace transform of both the sides

$$L[y'(t)] + L[y(t)] = L[te^{-t}]$$

Using $L[y'(t)] = s\bar{y}(s) - y(0),\ \ L[y(t)] = \bar{y}(s)$

$$\& \ \ L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}],\ \text{we get}$$

$$[s\bar{y}(s) - y(0)] + \bar{y}(s) = -\frac{d}{ds}[L\{e^{-t}\}]$$

Using the given condition $y(0) = 2$, the above reduces to

$$s\bar{y}(s) - 2 + \bar{y}(s) = -\frac{d}{ds}\left[\frac{1}{s+1}\right]$$

$$(s+1)\bar{y}(s) - 2 = -\left[-\frac{1}{(s+1)^2}\right]$$

$$(s+1)\bar{y}(s) = \frac{1}{(s+1)^2} + 2$$

$$(s+1)\bar{y}(s) = \frac{1 + 2(s+1)^2}{(s+1)^2}$$

$$\therefore \bar{y}(s) = \frac{1 + 2(s+1)^2}{(s+1)^3}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{y}(s)] = y(t) = L^{-1}\left[\frac{1 + 2(s+1)^2}{(s+1)^3}\right]$$

$$\boldsymbol{We\ have\ L^{-1}[\bar{f}(s+a)] = e^{-at}L^{-1}[\bar{f}(s)]}$$

$$\therefore y(t) = e^{-t}L^{-1}\left[\frac{1+2s^2}{s^3}\right] = e^{-t}L^{-1}\left[\frac{1}{s^3} + \frac{2s^2}{s^3}\right] = e^{-t}L^{-1}\left[\frac{1}{s^3} + \frac{2}{s}\right]$$

$$= e^{-t}\left[L^{-1}\left(\frac{1}{s^3}\right) + 2L^{-1}\left(\frac{1}{s}\right)\right] \qquad \text{by linearity}$$

$$\textbf{We have } \mathbf{L^{-1}}\left[\frac{\mathbf{1}}{\boldsymbol{s}}\right] = \mathbf{1} \quad \& \quad \mathbf{L^{-1}}\left[\frac{\mathbf{1}}{\boldsymbol{s^n}}\right] = \frac{\boldsymbol{t^{n-1}}}{(\boldsymbol{n}-\mathbf{1})!}$$

$$= e^{-t}\left[\frac{t^{3-1}}{(3-1)!} + 2(1)\right] = e^{-t}\left[\frac{t^2}{2!} + 2(1)\right]$$

$$\text{Thus } \boldsymbol{y(t) = e^{-t}\left(2 + \frac{t^2}{2}\right)}$$

2) Use Laplace transform technique to solve $\boldsymbol{\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-t}}$

$\boldsymbol{with\ y(0) = y'(0) = 0.}$

Solution: Given $y''(t) + 4y'(t) + 3y(t) = e^{-t}$

Taking the Laplace transform of both the sides

$$L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = L[e^{-t}]$$

Using $L[y''(t)] = s^2\bar{y}(s) - sy(0) - y'(0)$, $L[y'(t)] = s\bar{y}(s) - y(0)$

$L[y(t)] = \bar{y}(s)$ and $L[e^{-at}] = \dfrac{1}{s+a}$ we get

$$[s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 3\bar{y}(s) = \frac{1}{s+1}$$

Using the initial conditions $y(0) = 0$ & $y'(0) = 0$, the above reduces to

$$s^2\bar{y}(s) + 4s\bar{y}(s) + 3\bar{y}(s) = \frac{1}{s+1}$$

$$(s^2 + 4s + 3)\bar{y}(s) = \frac{1}{s+1}$$

$$(s+1)(s+3)\bar{y}(s) = \frac{1}{(s+1)}$$

$$\therefore \bar{y}(s) = \frac{1}{(s+1)^2(s+3)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{y}(s)] = y(t) = L^{-1}\left[\frac{1}{(s+1)^2(s+3)}\right] - - - - - - (1)$$

$$\text{Let } \frac{1}{(s+1)^2(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s+3} \quad \text{by partial fraction}$$

$$\therefore 1 = A(s+1)(s+3) + B(s+3) + C(s+1)^2 - - - -(2)$$

Put $s = -1$ in (2) $1 = A(0) + B(-1+3) + C(0)$ $1 = 2B$ $\therefore \boldsymbol{B = \frac{1}{2}}$	Put $s = -3$ in (2) $1 = A(0) + B(0) + C(-3+1)^2$ $1 = C(4)$ $\therefore \boldsymbol{C = \frac{1}{4}}$

Put $s = 0$ in (2) $1 = 3A + 3B + C$ $1 = 3A + 3\left(\frac{1}{2}\right) + \frac{1}{4} = 3A + \frac{7}{4}$ $3A = 1 - \frac{7}{4} = -\frac{3}{4} \qquad \therefore \boldsymbol{A = -\frac{1}{4}}$

$$\therefore \frac{1}{(s+1)^2(s+3)} = \frac{-1/4}{(s+1)} + \frac{1/2}{(s+1)^2} + \frac{1/4}{s+3}$$

Using the above in (1), we get

$$y(t) = L^{-1}\left[\frac{1}{(s+1)^2(s+3)}\right] = L^{-1}\left[\frac{-1/4}{(s+1)} + \frac{1/2}{(s+1)^2} + \frac{1/4}{s+3}\right]$$

$$y(t) = -\frac{1}{4}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{4}L^{-1}\left[\frac{1}{s+3}\right] \qquad \text{by linearity}$$

We have $\mathbf{L^{-1}}\left[\frac{\mathbf{1}}{\boldsymbol{s+a}}\right] = \boldsymbol{e^{-at}}$ & $\boldsymbol{L^{-1}[\bar{f}(s+a)] = e^{-at}L^{-1}[\bar{f}(s)]}$

$$\therefore y(t) = -\frac{1}{4}e^{-(1)t} + \frac{1}{2}e^{-(1)t}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{4}e^{-3t}$$

We have $\mathbf{L^{-1}}\left[\frac{\mathbf{1}}{\boldsymbol{s^n}}\right] = \frac{\boldsymbol{t^{n-1}}}{\boldsymbol{(n-1)!}}$

$$\therefore y(t) = -\frac{1}{4}e^{-t} + \frac{1}{2}e^{-t}\left[\frac{t^{2-1}}{(2-1)!}\right] + \frac{1}{4}e^{-3t}$$

$$= -\frac{1}{4}e^{-t} + \frac{1}{2}e^{-t}\left(\frac{t}{1}\right) + \frac{1}{4}e^{-3t}$$

Thus $\boldsymbol{y(t) = \left(\frac{t}{2} - \frac{1}{4}\right)e^{-t} + \frac{1}{4}e^{-3t}}$

3) Apply Laplace transform method to solve $\boldsymbol{\frac{d^2y}{dt^2} - y = t \;\; with \; y(0) = 0}$ **and** $\boldsymbol{y'(0) = 0}$.

Solution: Given $\quad y''(t) - y(t) = t$

Taking the Laplace transform of both the sides

$$L[y''(t)] - L[y(t)] = L[t]$$

Using $\boldsymbol{L[y''(t)] = s^2\bar{y}(s) - sy(0) - y'(0)}$, $\boldsymbol{L[y(t)] = \bar{y}(s)}$

and $\boldsymbol{L[t] = \frac{1}{s^2}}$ **we get**

$$[s^2\bar{y}(s) - sy(0) - y'(0)] - \bar{y}(s) = \frac{1}{s^2}$$

Using the initial conditions $y(0) = 0$ & $y'(0) = 0$, the above reduces to

$$s^2\bar{y}(s) - \bar{y}(s) = \frac{1}{s^2}$$

$$(s^2 - 1)\bar{y}(s) = \frac{1}{s^2}$$

$$\therefore \bar{y}(s) = \frac{1}{s^2(s^2-1)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{y}(s)] = y(t) = L^{-1}\left[\frac{1}{s^2(s^2-1)}\right] - - - - - (1)$$

$$\text{Let } \frac{1}{s^2(s^2-1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2-1} \quad \text{by partial fraction}$$

$$1 = (As+B)(s^2-1) + (Cs+D)s^2$$

$$\therefore 1 = As^3 + Bs^2 - As - B + Cs^3 + Ds^2 - - - -(2)$$

Put $s = 0$ in (2), we get

$$1 = -B \quad \therefore \boldsymbol{B = -1}$$

Comparing the coefficients of s in (2), we get

$$0 = -A \quad \therefore \boldsymbol{A = 0}$$

Comparing the coefficients of s^2 in (2), we get

$$0 = B + D$$

$$D = -B = -(-1) = 1 \quad \therefore \boldsymbol{D = 1}$$

Comparing the coefficients of s^3 in (2), we get

$$0 = A + C$$

$$C = -A = -(0) = 0 \quad \therefore \boldsymbol{C = 0}$$

$$\therefore \frac{1}{s^2(s^2-1)} = \frac{(0)s+(-1)}{s^2} + \frac{(0)s+1}{s^2-1} = \frac{-1}{s^2} + \frac{1}{s^2-1}$$

Using the above in (1), we get

$$y(t) = L^{-1}\left[\frac{1}{s^2(s^2-1)}\right] = L^{-1}\left[\frac{-1}{s^2} + \frac{1}{s^2-1}\right]$$

$$y(t) = -L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s^2-1^2}\right] \quad \text{by linearity}$$

We have $\mathbf{L}^{-1}\left[\frac{\boldsymbol{1}}{\boldsymbol{s^n}}\right] = \frac{\boldsymbol{t^{n-1}}}{\boldsymbol{(n-1)!}} \quad \mathbf{L}^{-1}\left[\frac{\boldsymbol{1}}{\boldsymbol{s^2-a^2}}\right] = \frac{\boldsymbol{1}}{\boldsymbol{a}}\boldsymbol{\sinh at}$

$$\therefore y(t) = -\frac{t^{2-1}}{(2-1)!} + \frac{1}{1}\sinh 1t$$

$$\therefore y(t) = -\left(\frac{t}{1!}\right) + \sinh t = -t + \sinh t$$

$$\text{Thus } \boldsymbol{y(t) = \sinh t - t}$$

Self-Study:

5) An altenating e. m. f. $\boldsymbol{E\sin\omega t}$ is applied to an inductance $\boldsymbol{L}$ and capacitance $\boldsymbol{C}$ in series, show that by using Laplace transform method that the current in the circuit is $\boldsymbol{\frac{E\omega}{L(p^2-\omega^2)}(\cos\omega t-\cos pt)}$ where $\boldsymbol{p^2=\frac{1}{LC}}$.

Solution: Let i be a current and q be the charge in the circuit, then its differential equation is

$$\mathbf{L}\frac{di}{dt}+\frac{q}{\boldsymbol{C}}=\boldsymbol{E}\sin\omega t$$

$$\mathbf{L}\,\mathrm{i}'(\mathrm{t})+\frac{q}{\boldsymbol{C}}=\boldsymbol{E}\sin\omega t$$

Taking the Laplace transform of both the sides

$$\mathbf{L}\,L[i'(t)]+\frac{1}{\boldsymbol{C}}L[q]=\boldsymbol{E}L[\sin\omega t]$$

Using $\boldsymbol{L[i'(t)]=s\,\bar{i}(s)-i(0)}$ and $\boldsymbol{L[i(t)]=\bar{i}(s)}$, we get

$$\mathbf{L}[s\,\bar{i}(s)-i(0)]+\frac{1}{\boldsymbol{C}}L[q]=\boldsymbol{E}\left[\frac{\omega}{s^2+\omega^2}\right]$$

Since $i=0\ \&\ q=0\ \ at\ t=0\ [intially]$, the above reduces to

$$\mathbf{L}\,s\,\bar{i}(s)+\frac{1}{\boldsymbol{C}}L[q]=\frac{\boldsymbol{E}\omega}{s^2+\omega^2}----(1)$$

Also, we have $i=\frac{dq}{dt}\Longrightarrow L[i(t)]=L[q'(t)]\Longrightarrow\bar{i}(s)=sL[q]-q(0)$

$$\Longrightarrow\ \bar{i}(s)=sL[q]-0\Longrightarrow\boldsymbol{L[q]=\frac{\bar{i}(s)}{s}}$$

Now equation (1) become

$$\mathbf{L}\,s\,\bar{i}(s)+\frac{1}{\boldsymbol{C}}\frac{\bar{i}(s)}{s}=\frac{\boldsymbol{E}\omega}{s^2+\omega^2}\Longrightarrow\left[\mathbf{L}\,s\ +\frac{1}{\boldsymbol{C}s}\right]\bar{i}(s)=\frac{\boldsymbol{E}\omega}{s^2+\omega^2}$$

$$\frac{\mathbf{L}}{\mathrm{s}}\left[s^2\ +\frac{1}{\mathbf{LC}}\right]\bar{i}(s)=\frac{\boldsymbol{E}\omega}{s^2+\omega^2}$$

$$\frac{\mathbf{L}}{\mathrm{s}}[s^2\ +p^2]\bar{i}(s)=\frac{\boldsymbol{E}\omega}{s^2+\omega^2}\qquad where\ p^2=\frac{1}{\mathbf{LC}}$$

$$\therefore \bar{i}(s) = \frac{\boldsymbol{E}\omega s}{\mathbf{L}(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}$$

$$\bar{i}(s) = \frac{\boldsymbol{E}\omega}{\mathbf{L}}\,\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{i}(s)] = i(t) = L^{-1}\left[\frac{\boldsymbol{E}\omega}{\mathbf{L}}\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}\right]$$

$$i(t) = \frac{\boldsymbol{E}\omega}{\mathbf{L}}L^{-1}\left[\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}\right] ----(2)$$

$$\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)} = \frac{As+B}{s^2+p^2}+\frac{Cs+D}{s^2+\omega^2} \quad \text{by partial fraction}$$

$$\therefore s = (As+B)(s^2+\omega^2)+(Cs+D)(s^2+p^2)$$

$$s = As^3+Bs^2+\omega^2As+B\omega^2+Cs^3+Ds^2+p^2Cs+Dp^2 ----(3)$$

Comparing the coefficients of s^3 in (3), we get

$$0 = A + C \quad \Longrightarrow \quad C = -A$$

Comparing the coefficients of s^2 in (3), we get

$$0 = B + D \quad \Longrightarrow \quad D = -B$$

Comparing the coefficients of s in (3), we get

$$1 = \omega^2A + p^2C \Longrightarrow 1 = \omega^2A - p^2A$$

$$\Longrightarrow 1 = (\omega^2-p^2)A \Longrightarrow \boldsymbol{A} = \frac{\mathbf{1}}{\boldsymbol{\omega^2-p^2}}$$

$$\text{Since } C = -A \Longrightarrow \boldsymbol{C} = -\frac{\mathbf{1}}{\boldsymbol{\omega^2-p^2}}$$

Comparing the constants in (3), we get **or** Put $s = 0$ in (2), we get

$$0 = \omega^2B + p^2D \Longrightarrow 0 = \omega^2B - p^2B$$

$$\Longrightarrow 0 = (\omega^2-p^2)B \Longrightarrow B = \frac{0}{\omega^2-p^2} \Longrightarrow \boldsymbol{B} = \mathbf{0}$$

$$\text{Since } D = -B \Longrightarrow \boldsymbol{D} = \mathbf{0}$$

$$\therefore \frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)} = \frac{\left(\frac{1}{\omega^2-p^2}\right)s+(0)}{s^2+p^2}+\frac{\left(-\frac{1}{\omega^2-p^2}\right)s+(0)}{s^2+\omega^2}$$

$$\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)} = \left(\frac{1}{\omega^2-p^2}\right)\frac{s}{s^2+p^2}-\left(\frac{1}{\omega^2-p^2}\right)\frac{s}{s^2+\omega^2}$$

Take the inverse Laplace transform of both sides

$$\therefore L^{-1}\left[\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}\right]=L^{-1}\left[\left(\frac{1}{\omega^2-p^2}\right)\frac{s}{s^2+p^2}-\left(\frac{1}{\omega^2-p^2}\right)\frac{s}{s^2+\omega^2}\right]$$

$$L^{-1}\left[\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}\right]=\left(\frac{1}{\omega^2-p^2}\right)L^{-1}\left[\frac{s}{s^2+p^2}\right]-\left(\frac{1}{\omega^2-p^2}\right)L^{-1}\left[\frac{s}{s^2+\omega^2}\right]$$

We have $\mathbf{L^{-1}}\left[\frac{\boldsymbol{s}}{\boldsymbol{s^2+a^2}}\right]=\boldsymbol{\cos at}$

$$\therefore L^{-1}\left[\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}\right]=\left(\frac{1}{\omega^2-p^2}\right)\cos pt-\left(\frac{1}{\omega^2-p^2}\right)\cos\omega t$$

$$=\left(\frac{1}{\omega^2-p^2}\right)[\cos pt-\cos\omega t]$$

$$=\frac{-(\cos\omega t-\cos pt)}{-(p^2-\omega^2)}$$

$$\therefore L^{-1}\left[\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}\right]=\frac{\cos\omega t-\cos pt}{p^2-\omega^2}$$

Using above result in (2), we get

$$i(t)=\frac{\boldsymbol{E}\omega}{\mathbf{L}}\left[\frac{\cos\omega t-\cos pt}{p^2-\omega^2}\right]$$

$$\boldsymbol{Thus}\quad \boldsymbol{i(t)}=\frac{\boldsymbol{E\omega}}{\mathbf{L}(\boldsymbol{p^2-\omega^2})}(\mathbf{cos}\,\boldsymbol{\omega t}-\mathbf{cos}\,\boldsymbol{pt})$$

EXERCISE PROBLEMS

1) Apply Laplace transform method to solve $\frac{dy}{dt} + y = sin\ t\ \ with\ y(0) = 0.$

Answer: $y(t) = \frac{1}{2}[e^{-t} + \sin t - \cos t]$

2) Use Laplace transform method to solve $\frac{d^2y}{dt^2} + y = t\ \ with\ y(0) = y'(0) = 0$.

Answer: $y(t) = t - \sin t + \cos t$

3) Apply Laplace transform technique to solve $y'' - 3y' - 4y = 2\,e^{-t}\ \ with$

$y(0) = 1\ and\ y'(0) = 1$.

Answer: $y(t) = \frac{12}{25}e^{4t} + \frac{13}{25}e^{-t} - \frac{2}{5}te^{-t}$

4) Use Laplace transform technique to solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 10 \sin t\ \ e^{-t}\ \ with$

$y(0) = 0\ and\ y'(0) = 0$.

Answer: $y(t) = \sin t - 2\cos t + \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$

5) Use Laplace transform method to solve $y''' + 2y'' - y' - 2y = 0\ \ given$

$y(0) = y'(0) = 0\ and\ y''(0) = 6.$

Answer: $y(t) = e^{t} - 3e^{-t} + 2e^{-2t}$

2.7 Fourier Transforms

INTRODUCTION: The **Fourier Transform** provides a frequency domain representation of the original signal. It is expansion of **Fourier Series** to the non-periodic signals. The Fourier transform of a function of time is a complex-valued function of the frequency, whose magnitude (absolute value) represents the amount of that frequency present in the original function. The Fourier transform is not limited to functions of time, but the domain of the original function is commonly referred to as the time domain. Fourier Transform is useful in the study of solution of partial differential equations associated with initial boundary value problems. Fourier Transform methods have long been proved to extremely useful in all fields of science and technology, such as signal analysis, image processing, radio-astronomy, seismology, spectroscopy and crystallography. There is also an **inverse Fourier transform** that mathematically synthesizes the original function from its frequency domain representation.

2.8 Fourier & Inverse Fourier Transforms

Definition: The (infinite) Fourier transform of a real valued function $f(x)$ is defined as

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

provided the integral exists. On integration, we obtain a function of s which is usually denoted by $\bar{f}(s)$.

$$i.e., \qquad F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{isx} dx = \bar{f}(s).$$

The inverse Fourier transform of $\bar{f}(s)$ is defined as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{-isx} ds.$$

Linearity Property

If $\bar{f}(s)$ & $\bar{g}(s)$ are Fourier transforms of $f(x)$ & $g(x)$ respectively, then

$$F[af(x) \pm bg(x)] = aF[f(x)] \pm bF[g(x)] = a\bar{f}(s) \pm b\bar{g}(s)$$

where a & b are constants.

Note: $e^{i\theta} = \cos\theta + i\sin\theta$ & $e^{-i\theta} = \cos\theta - i\sin\theta$

$|x| < a \Rightarrow -a < x < a$ & $|x| \leq a \Rightarrow -a \leq x \leq a$

SOLVED PROBLEMS

1) Find the Fourier transform of $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$.

Solution: Given $f(x) = \begin{cases} x^2 & for -a < x < a \\ 0, & otherwise \end{cases}$.

The Fourier transform is given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{isx} dx = \bar{f}(s)$$

$$\therefore F[f(x)] = \int_{-a}^{a} f(x)\, e^{isx} dx = \int_{-a}^{a} x^2\, e^{isx} dx$$

$$= \left[(x^2)\left(\frac{e^{isx}}{is}\right) - (2x)\left(\frac{e^{isx}}{i^2s^2}\right) + (2)\left(\frac{e^{isx}}{i^3s^3}\right)\right]_{-a}^{a}$$

$$= \frac{1}{is}[x^2e^{isx}]_{-a}^{a} + \frac{2}{s^2}[xe^{isx}]_{-a}^{a} - \frac{2}{is^3}[e^{isx}]_{-a}^{a} \qquad \because \frac{1}{i^3} = -\frac{1}{i}$$

$$= \frac{1}{is}\left[(a)^2e^{is(a)} - (-a)^2e^{is(-a)}\right] + \frac{2}{s^2}\left[(a)e^{is(a)} - (-a)e^{is(-a)}\right] - \frac{2}{is^3}\left[e^{is(a)} - e^{is(-a)}\right]$$

$$= \frac{a^2}{is}[e^{ias} - e^{-ias}] + \frac{2a}{s^2}[e^{ias} + e^{-ias}] - \frac{2}{is^3}[e^{ias} - e^{-ias}]$$

$$= \frac{a^2}{is}[(\cos as + i\sin as) - (\cos as - i\sin as)]$$

$$+ \frac{2a}{s^2}[(\cos as + i\sin as) + (\cos as - i\sin as)]$$

$$- \frac{2}{is^3}[(\cos as + i\sin as) - (\cos as - i\sin as)]$$

$$= \frac{a^2}{is}[2i\sin as] + \frac{2a}{s^2}[2\cos as] - \frac{2}{is^3}[2i\sin as]$$

$$= \frac{2a^2\sin as}{s} + \frac{4a\cos as}{s^2} - \frac{4\sin as}{s^3}$$

$$= \frac{2a^2s^2\sin as + 4as\cos as - 4\sin as}{s^3}$$

Thus $\boldsymbol{F[f(x)] = \bar{f}(s) = \frac{(2a^2s^2-4)\sin as + 4as\cos as}{s^3}}$.

2) Find the Fourier transform of $f(x) = \begin{cases} 1 & for\ |x| < 1 \\ 0 & for\ |x| > 1 \end{cases}$. Hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$

Solution: Given $f(x) = \begin{cases} 1 & for -1 < x < 1 \\ 0, & otherwise \end{cases}$.

The Fourier transform is given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{isx} dx = \bar{f}(s)$$

$$\therefore F[f(x)] = \int_{-1}^{1} f(x)\, e^{isx} dx = \int_{-1}^{1} 1\, e^{isx} dx$$

$$= \int_{-1}^{1} e^{isx} dx = \left[\frac{e^{isx}}{is}\right]_{-1}^{1} = \frac{1}{is}[e^{isx}]_{-1}^{1}$$

$$= \frac{1}{is}\left[e^{is(1)} - e^{is(-1)}\right] = \frac{1}{is}[e^{is} - e^{-is}]$$

$$= \frac{1}{is}[\{\cos s + i\sin s\} - \{\cos s - i\sin s\}]$$

$$= \frac{1}{is}[\cos s + i\sin s - \cos s + i\sin s] = \frac{1}{is}[2i\sin s] = \frac{2\sin s}{s} = \bar{f}(s).$$

$$\textbf{Thus}\quad F[f(x)] = \bar{f}(s) = \frac{2\sin s}{s}.$$

Now by inverse Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{-isx} ds$$

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin s}{s}\, e^{-isx} ds$$

$$\text{Thus } f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}\, e^{-isx} ds$$

Put $x = 0$, we get

$$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}\, e^{-is(0)} ds = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}\, e^{0} ds = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds$$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds \qquad \text{since f(0)} = 1$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi$$

But we know that if f(x) is even, then $\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$

$$\text{Since } \frac{\sin \mathrm{s}}{\mathrm{s}} \text{ is even } \left[\frac{\sin s}{s} = \frac{\sin(-s)}{(-s)} = -\frac{\sin s}{-s} = \frac{\sin s}{s}\right]$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi \Rightarrow 2\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \pi$$

$$Thus \int_{0}^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

Changing variable s to x, we get

$$\boldsymbol{\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}}.$$

EXERCISE PROBLEMS

1) Find the Fourier transform of $f(x) = \begin{cases} x, & if\ |x| \le a \\ 0\ , & if\ |x| > a \end{cases}$.

Answer: $F[f(x)] = \bar{f}(s) = \frac{2i}{s^2}[sin\ as - as\ cos\ as]$

2) Find the Fourier transform of $f(x) = \begin{cases} 1, & if\ |x| \le a \\ 0\ , & if\ |x| > a \end{cases}$

Hence evaluate $\int_{-\infty}^{\infty} \frac{\sin ax}{x} dx$

Answer: $F[f(x)] = \bar{f}(s) = \frac{2\ sin\ as}{s} \quad and \quad \int_{-\infty}^{\infty} \frac{\sin ax}{x} dx = \pi$

3) Find the Fourier transform of $f(x) = e^{-ax}$ when $x > 0$.

Answer: $F[f(x)] = \bar{f}(s) = \frac{a + is}{a^2 + s^2}$

2.9 Fourier Sine and Cosine Transforms

Definition: If $f(x)$ is defined for all positive values of x then the Fourier sine and cosine transforms are given by

$$F_s[f(x)] = \int_0^\infty f(x)\sin sx\,dx = \bar{f}_s(s) ----(1)$$

$$F_c[f(x)] = \int_0^\infty f(x)\cos sx\,dx = \bar{f}_c(s) ----(2)$$

The inverse transforms are defined as

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(s)\sin sx\,ds \quad - \textbf{Inverse Fourier sine transform}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}_c(s)\cos sx\,ds \quad - \textbf{Inverse Fourier cosine transform}$$

Linearity Property

i) If $\bar{f}_s(s)$ & $\bar{g}_s(s)$are Fourier sine transforms of $f(x)$ & $g(x)$ respectively, then $F_s[af(x) \pm bg(x)] = aF_s[f(x)] \pm bF_s[g(x)] = a\bar{f}_s(s) \pm b\bar{g}_s(s)$ where a & b are constants.

ii) If $\bar{f}_c(s)$& $\bar{g}_c(s)$ are Fourier cosine transforms of $f(x)$ & $g(x)$ respectively, then $F_c[af(x) \pm bg(x)] = aF_c[f(x)] \pm bF_c[g(x)] = a\bar{f}_c(s) \pm b\bar{g}_c(s)$ where a & b are constants.

SOLVED PROBLEMS

1) Find the Fourier sine and cosine transforms of $f(x) = \begin{cases} 1, & 0 \le x < 2 \\ 0, & x \ge 2 \end{cases}$.

Solution: Given $f(x) = \begin{cases} 1, & 0 \le x < 2 \\ 0, & x \ge 2 \end{cases}$.

i) The Fourier sine transform is given by

$$F_s[f(x)] = \int_0^\infty f(x)\sin sx\,dx = \bar{f}_s(s)$$

$$\therefore F_s[f(x)] = \int_0^2 f(x)\sin sx\,dx + \int_2^\infty f(x)\sin sx\,dx$$

$$= \int_0^2 1\sin sx\,dx + \int_2^\infty (0)\sin sx\,dx$$

$$= \left[-\frac{\cos sx}{s}\right]_0^2 = -\frac{1}{s}[\cos sx]_0^2$$

$$= -\frac{1}{s}[\cos 2s - \cos 0] = -\frac{1}{s}[\cos 2s - 1] = \frac{1}{s}[1 - \cos 2s]$$

Thus $\boldsymbol{F_s[f(x)] = \bar{f}_s(s) = \dfrac{1 - \cos 2s}{s}}$.

ii) The Fourier cosine transform is given by

$$F_c[f(x)] = \int_0^\infty f(x)\cos sx\,dx = \bar{f}_c(s)$$

$$\therefore F_c[f(x)] = \int_0^2 f(x)\cos sx\,dx + \int_2^\infty f(x)\cos sx\,dx$$

$$= \int_0^2 1\cos sx\,dx + + \int_2^\infty (0)\cos sx\,dx$$

$$= \left[\frac{\sin sx}{s}\right]_0^2 = \frac{1}{s}[\sin sx]_0^2$$

$$= \frac{1}{s}[\sin 2s - \sin 0] = \frac{1}{s}[\sin 2s - 0] = \frac{\sin 2s}{s}$$

Thus $\boldsymbol{F_c[f(x)] = \bar{f}_c(s) = \dfrac{\sin 2s}{s}}$.

2. Find the Fourier sine transform of $f(x) = e^{-|x|}$. Hence show that

$$\int_0^\infty \frac{x\sin mx}{1+x^2}dx = \frac{\pi e^{-m}}{2}.$$

Solution: x being positive in $(0,\infty)$, $e^{-|x|} = e^{-x}$.

The Fourier sine transform is given by

$$F_s[f(x)] = \int_0^\infty f(x)\sin sx\,dx = \bar{f}_s(s)$$

$$\therefore \bar{f}_s(s) = \int_0^\infty e^{-x}\sin sx\,dx$$

$$\textit{We know that}\ \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$\therefore\ \bar{f}_s(s) = \left[\frac{e^{-x}}{(-1)^2+s^2}\{-1\sin sx - s\cos sx\}\right]_0^\infty$$

$$= \frac{1}{1+s^2}[e^{-x}\{-\sin sx - s\cos sx\}]_0^\infty$$

$$= \frac{1}{1+s^2}[e^{-\infty}\{-\sin\infty - s\cos\infty\} - e^0\{-\sin 0 - s\cos 0\}] \qquad But\ e^{-\infty} = 0$$

$$= \frac{1}{1+s^2}[0 - 1\{-(0) - s(1)\}] = \frac{s}{1+s^2}$$

$$\therefore\ \bar{f}_s(s) = \boldsymbol{\frac{s}{1+s^2}}$$

Now by inverse Fourier sine transform, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(s)\sin sx\,ds$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sx\,ds$$

Changing x to m, we get

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{s\sin sm}{1+s^2}ds$$

$$\int_0^\infty \frac{s\sin sm}{1+s^2}ds = \frac{\pi e^{-m}}{2}$$

Now changing s to x, we get

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}$$

3) Find the Fourier cosine transform of e^{-ax}. Hence evaluate $\int_0^\infty \frac{\cos \lambda x}{x^2+a^2} dx$

Solution: Given $f(x) = e^{-ax}$.

The Fourier cosine transform is given by

$$F_c[f(x)] = \int_0^\infty f(x) \cos sx \, dx = \bar{f}_c(s)$$

$$\therefore \bar{f}_c(s) = \int_0^\infty e^{-ax} \cos sx \, dx$$

We know that $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$

$$\therefore \bar{f}_c(s) = \left[\frac{e^{-ax}}{(-a)^2+s^2}\{-a \cos sx + s \sin sx\}\right]_0^\infty$$

$$= \frac{1}{a^2+s^2}[e^{-ax}\{-a \cos sx + s \sin sx\}]_0^\infty$$

$$= \frac{1}{a^2+s^2}[e^{-\infty}\{-a \cos \infty + s \sin \infty\} - e^0\{-a \cos 0 + s \sin 0\}] \qquad But\ e^{-\infty} = 0$$

$$= \frac{1}{a^2+s^2}[0 - 1\{-a(1) + s(0)\}] = \frac{a}{a^2+s^2}$$

$$\therefore \bar{f}_c(s) = \frac{\boldsymbol{a}}{\boldsymbol{a^2+s^2}}$$

Now by inverse Fourier cosine transform, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(s) \cos sx \, ds$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2} \cos sx \, ds$$

Changing x to λ, we get

$$e^{-a\lambda} = \frac{2}{\pi}\int_0^\infty \frac{a\cos\lambda s}{a^2+s^2}\,ds$$

$$\int_0^\infty \frac{\cos\lambda s}{a^2+s^2}\,ds = \frac{\pi e^{-a\lambda}}{2a}$$

Now changing s to x, we get

$$\boldsymbol{\int_0^\infty \frac{\cos\lambda x}{a^2+x^2}\,dx = \frac{\pi e^{-a\lambda}}{2a}}.$$

EXERCISE PROBLEMS

1) Find the Fourier sine and cosine transforms of $f(x) = 2x \;\; for\; 0 < x < 4$.

Answer: $F_s[f(x)] = \bar{f}_s(s) = \dfrac{2\, sin\, 4s - 8s\, cos\, 4s}{s^2}$

$$F_c[f(x)] = \bar{f}_c(s) = \frac{8s\, sin\, 4s + 2\, cos\, 4s - 2}{s^2}$$

2) Obtain the Fourier cosine transform of $f(x) = \begin{cases} x, & 0 < x < 2 \\ 0, & otherwise \end{cases}$.

Answer: $F_c[f(x)] = \bar{f}_c(s) = \dfrac{2s\, sin\, 2s + cos\, 2s - 1}{s^2}$

3) Find the Fourier cosine transform of $f(x) = e^{-x}$.

Answer: $F_c[f(x)] = \bar{f}_c(s) = \dfrac{1}{1+s^2}$

4) Find the Fourier sine transform of $f(x) = e^{-ax}$. Hence show that

$$\int_0^\infty \frac{\mathrm{x}\sin mx}{a^2+x^2}\,dx = \frac{\pi e^{-am}}{2}$$

Answer: $F_s[f(x)] = \bar{f}_s(s) = \dfrac{s}{a^2+s^2}$

5) Find the Fourier cosine transform of $f(x) = \dfrac{e^{-ax}}{x}$.

Answer: $F_c[f(x)] = \bar{f}_c(s) = -\dfrac{1}{2}\log(s^2 + a^2)$