Using Julia to compute Feigenbaum's constant α

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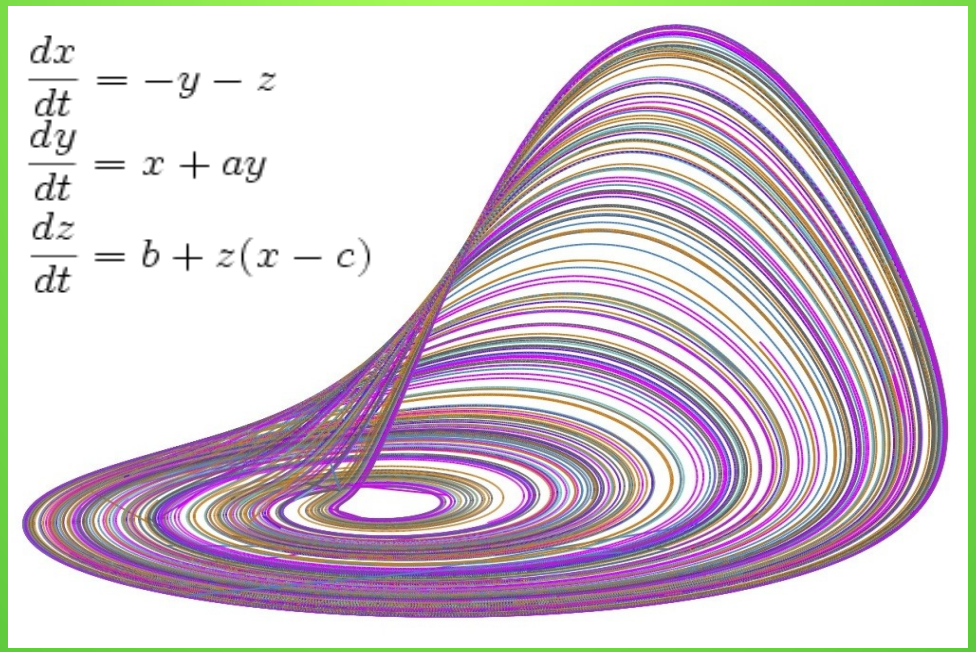
Cambridge Area Julia Users Network
Tuesday, Oct 17th, 2017



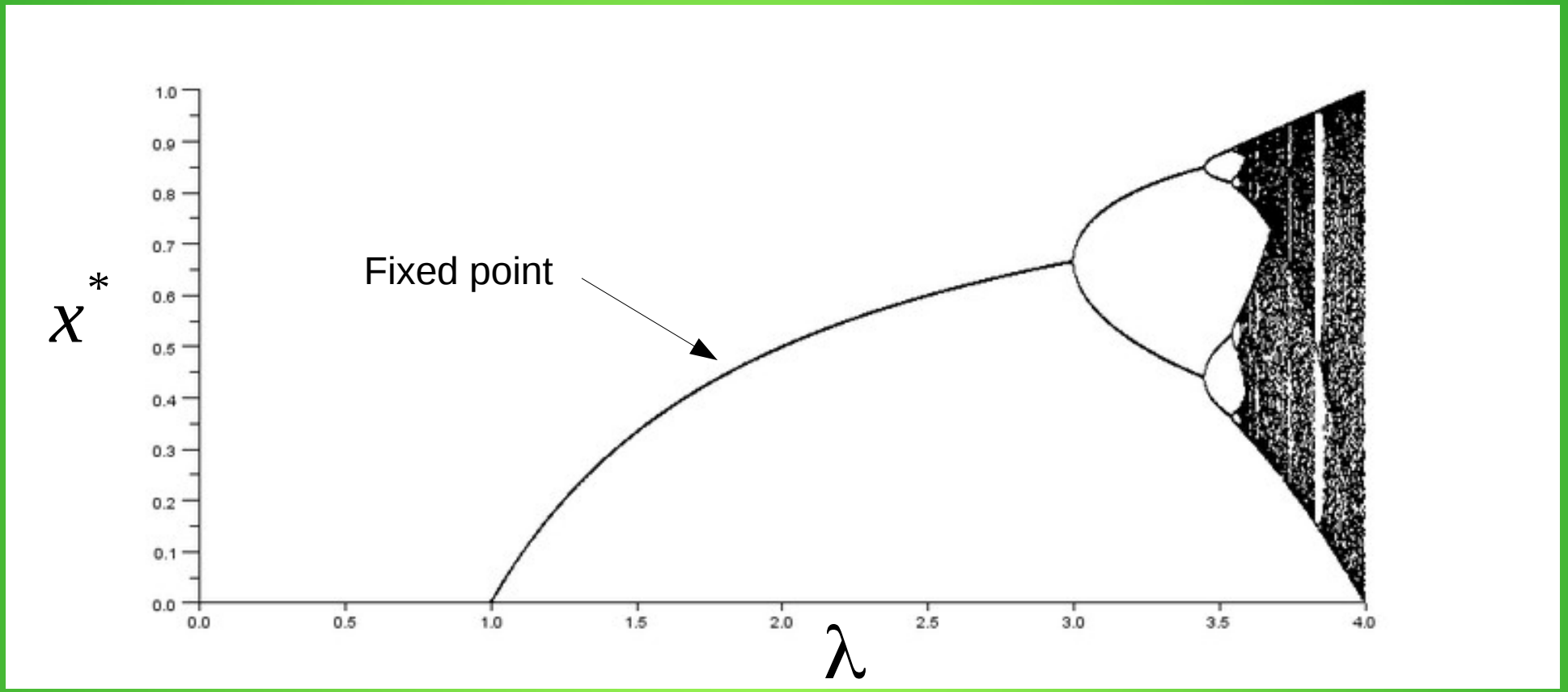
The iterated logistic map as a dynamical system

- Logistic map is a simple system which displays very interesting behavior.

$$x_{n+1} = \lambda x_n (1 - x_n)$$

- Upon repeated iteration, x often tends to a limiting value x^* which depends upon control parameter λ
- For some values of λ , there is no limiting value – *x* wanders chaotically.
- Popularized in a paper by biologist Robert May in Nature (1976).

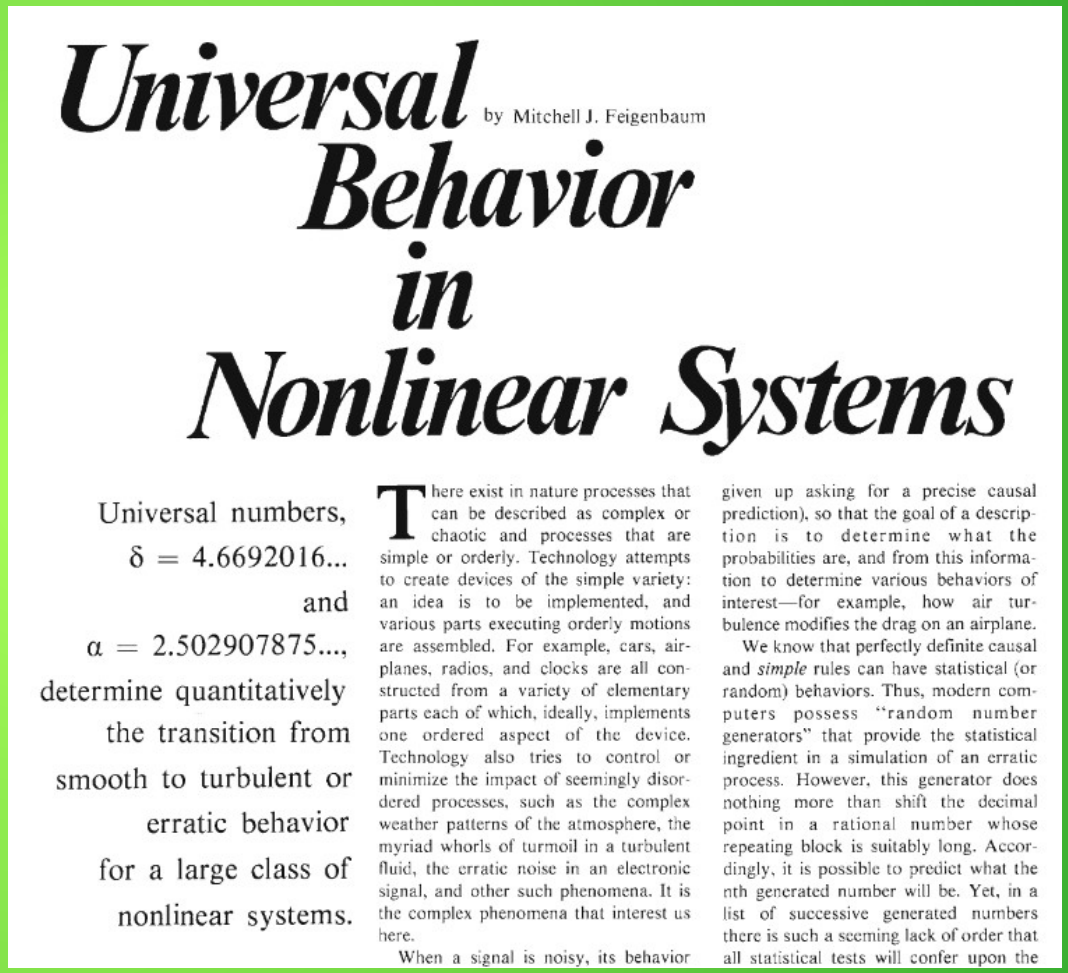
Bifurcation diagram



- Behavior of fixed point changes with increasing λ
- Note “period doubling” starting at $\lambda = 3$
- This observation was very exciting since it suggested a new approach to understanding turbulence and chaos.

Feigenbaum universality

- Feigenbaum observed that similar behavior occurred for many one-hump maps. ==> behavior is universal.
- He also found numerical constants which were related to the period-doubling behavior.
- He used renormalization group techniques to derive these constants.



Universal Behavior in Nonlinear Systems

by Mitchell J. Feigenbaum

Universal numbers, $\delta = 4.6692016...$ and $\alpha = 2.502907875...$, determine quantitatively the transition from smooth to turbulent or erratic behavior for a large class of nonlinear systems.

There exist in nature processes that can be described as complex or chaotic and processes that are simple or orderly. Technology attempts to create devices of the simple variety: an idea is to be implemented, and various parts executing orderly motions are assembled. For example, cars, airplanes, radios, and clocks are all constructed from a variety of elementary parts each of which, ideally, implements one ordered aspect of the device. Technology also tries to control or minimize the impact of seemingly disordered processes, such as the complex weather patterns of the atmosphere, the myriad whorls of turmoil in a turbulent fluid, the erratic noise in an electronic signal, and other such phenomena. It is the complex phenomena that interest us here.

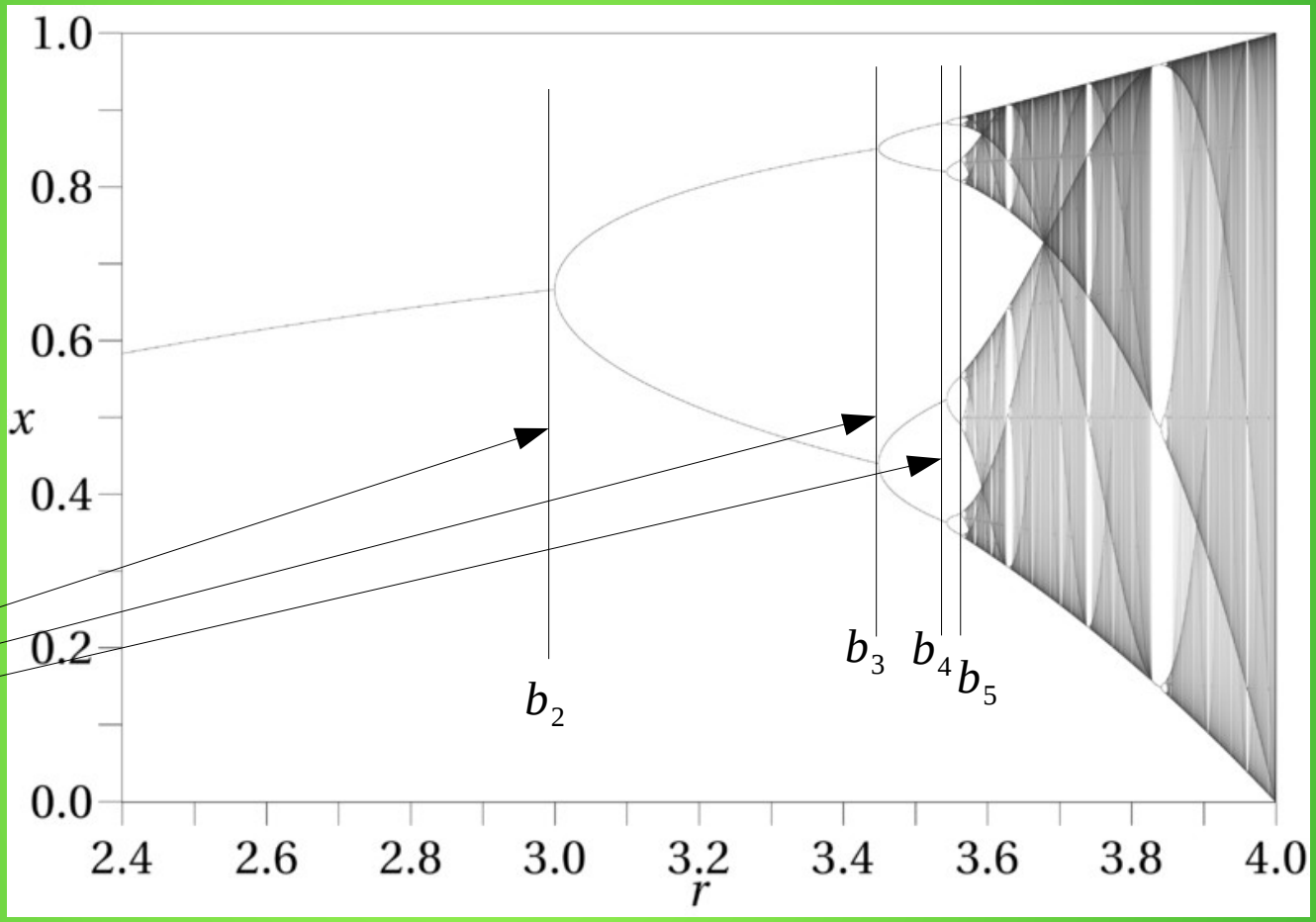
When a signal is noisy, its behavior given up asking for a precise causal prediction), so that the goal of a description is to determine what the probabilities are, and from this information to determine various behaviors of interest—for example, how air turbulence modifies the drag on an airplane.

We know that perfectly definite causal and *simple* rules can have statistical (or random) behaviors. Thus, modern computers possess "random number generators" that provide the statistical ingredient in a simulation of an erratic process. However, this generator does nothing more than shift the decimal point in a rational number whose repeating block is suitably long. Accordingly, it is possible to predict what the nth generated number will be. Yet, in a list of successive generated numbers there is such a seeming lack of order that all statistical tests will confer upon the

Exciting because renormalization techniques are are powerful in describing phase transitions. Very popular research topic in the 1970s and 1980s.

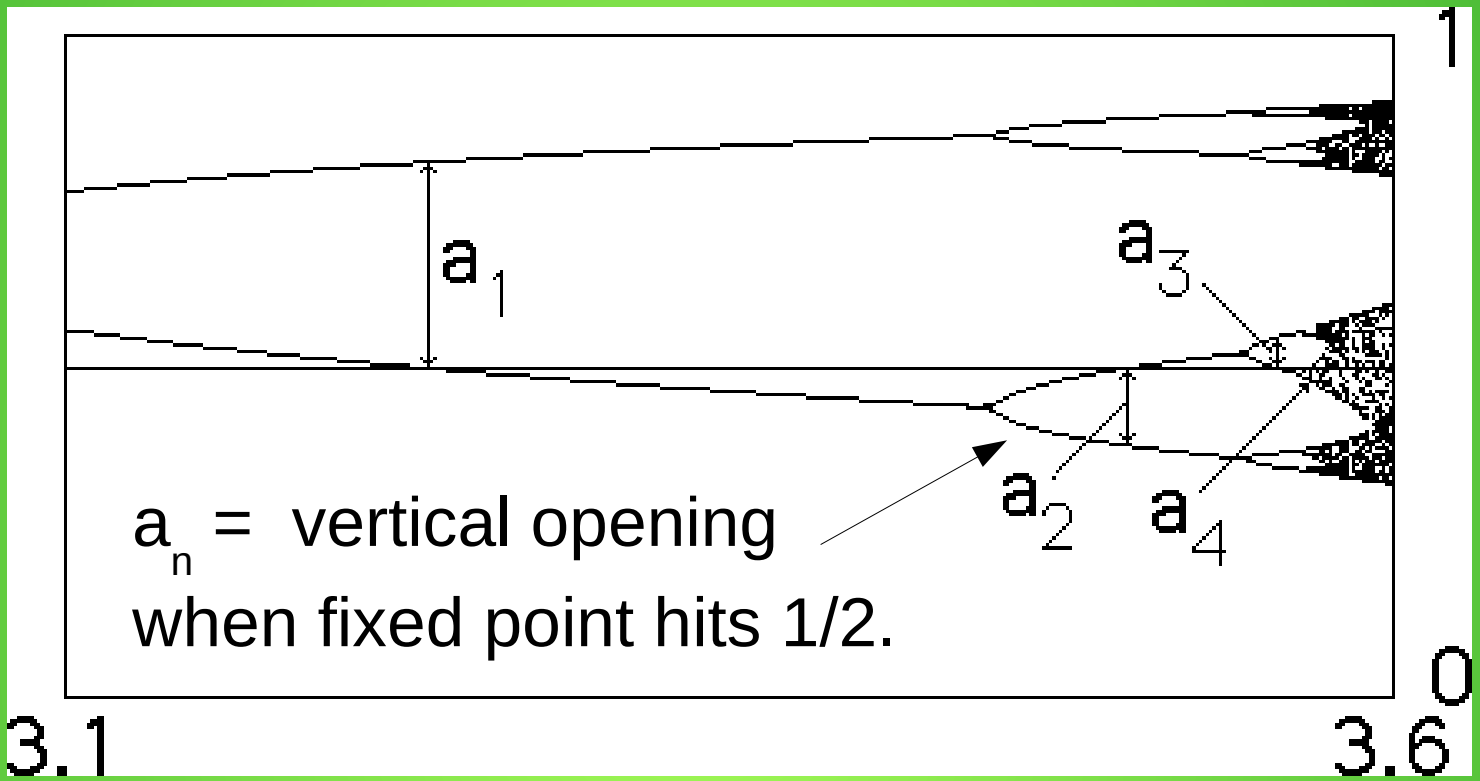
Feigenbaum's δ

Ratio of distances between bifurcations



$$\delta = \lim_{n \to \infty} \frac{b_n - b_{n-1}}{b_{n+1} - b_n} \approx 4.6692016\ldots$$

Feigenbaum's α



- α is ratio of successive a_n values:

$$\alpha = \lim_{n \to \infty} \frac{a_{n-1}}{a_n} \approx 2.502907876\ldots$$

How to compute α?

- Using a renormalization group argument, Feigenbaum proposed that the stretching behavior of the map was described by a universal function *g*(*z*) which obeyed this equation in the limit:

$$g(z) = \alpha\, g(g(z/\alpha))$$

In some loose sense, this equation says that applying two iterations of the map g(z) along with stretching and rescaling gives the original map g(z) back.

- It turns out that $\alpha = 1/g(1)$

My goal: Compute α to high precision

- Keith Briggs computed 576 decimal places (of which 344 were correct) digits in his PhD thesis (University of Melbourne, 1997).
- Current public record is 1018 digits computed by David Broadhurst in 1999 (referenced at the OEIS).
- Apparently, no new computations since then.

Feigenbaum constants to 1018 decimal places

From: David Broadhurst 22-Mar-1999
To: Simon Plouffe
CC: Keith Briggs, David Bailey, Steven Finch
Subj: Feigenbaum constants to 1018 decimal places

Keith Briggs' thesis gives values of alpha and delta to 576 decimal places.
I found that his values are good to 346 and 344 places, respectively.
The 1018 places below took 3 days, using 400MB on a 433MHz DECalpha,
with a collocation of N=700 points, checked in 3.5 days at N=720.
The Feigenbaum zero nearest to the origin was located to 1018 places.
The nearest complex singularity was located to 868 places.

Let g(x) and f(x) be even functions, with g(0)=f(0)=1

g(alpha*x)/alpha = g(g(x))
delta*f(alpha*x)/alpha = g'(g(x))*f(x) + f(g(x))

and delta as large as possible. Let {b,c,d} be positive numbers with

g(b) = 0 = 1/g(c+i*d)

and {b,c^2+d^2} as small as possible. Then

alpha = -2.
50290787509589282228390287321821578638127137672714997733619205677923546317959020670329964974643384129595231869995854723942182377785445179272863314993372578112163594879503744781260997380598671239711737328927665404401030669831383460009413932236449065788995122058431725078733774630878534242853519885875000423582469187408204281700901714823051821621619413199856066129382742649709844084470100805454967793676088812644640688518155270932400754250649715704704754199328317836453325624153786939571250970663879794926546231376745918909813116752434221110130913127837160951158341230841503716499702022468121964408121668652745804302624578256106715013852182164495325433498734874133527958153510165836054557635132765018107811948369459574850237398235452625632779475397269902012891516645793942019892024880339405169968655149447739653387697974123235406178198961124940959903531289977336118498473779461084288332938339039509008914086351525626803381414669279913310743349705143545201344643426475200162138461072992264199433277291897776905380259685...

delta = 4.
66920160910299067185320382046620161725818557747576863274565134300413433021131473713868974402394801381716598485518981513440862714

http://www.plouffe.fr/simon/constants/feigenbaum.txt

Why Julia?

- Built-in BigFloat type.
  - Required for high-precision computation.
- ForwardDiff.jl.
  - Automatic differentiation instead of hand-calculating differentials.
  - Used to compute Jacobian in Newton's Method.
- Runtime function definition (self-writing code).
  - Not used in this particular computation, but used in earlier attempts employing different approaches.
  - May be useful again if I use a Chebyshev expansion to represent $g(x)$.
- Personal interest

Computational strategy

- Expand $g(z)$ using even power series to order 2N:

$$g(z)=1+a_1 z^2+a_2 z^4+\cdots+a_N z^{2N}$$

- Define $f(z)$ using $g(z)$. We want $f(z) = 0$ for $-1 <= z <= 1$

$$f(z)=g(z)-\alpha\, g(g(z/\alpha))=0$$

- Evaluate $f(z)$ at N discrete points z_i, $i = 1 .. N$. This gives N equations in N unknowns. (Unknowns are coefficients a_i)

Computational strategy....

- Use Newton's method to solve system

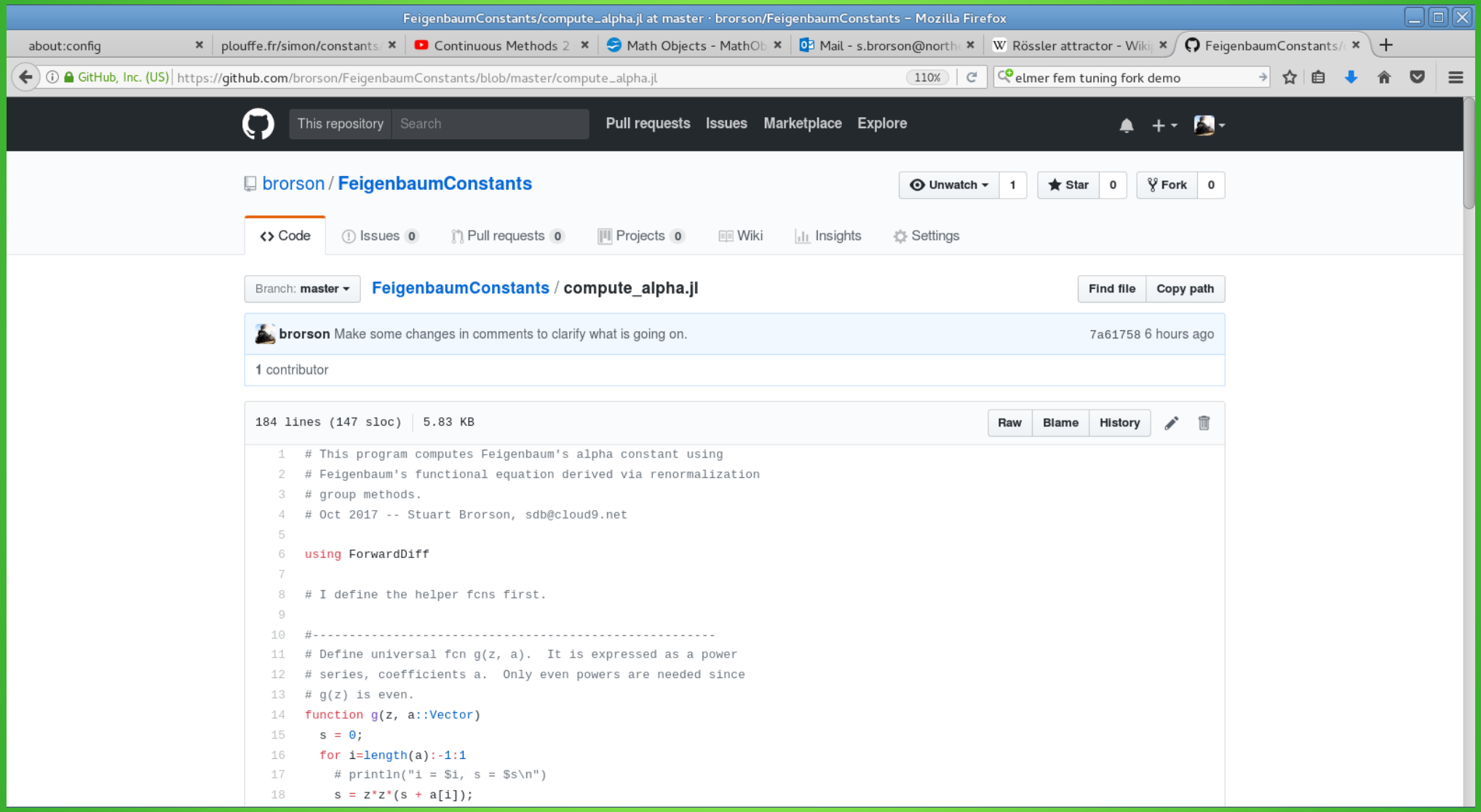
$$f\left(a_{1,}\, a_{2,}\, a_{3,} \ldots | z_1\right) = 0$$
$$f\left(a_{1,}\, a_{2,}\, a_{3,} \ldots | z_2\right) = 0$$
$$f\left(a_{1,}\, a_{2,}\, a_{3,} \ldots | z_3\right) = 0$$

etc .....

N equations,
N unknowns.

- Compute Jacobian using ForwardDiff.jl
- Call Newton's method from outer loop which walks up the number of digits computed, and uses results from last time to seed current computation.

Code



- Code available on github.com/brorson

My current best result – 1040 digits

2.50290787509589282228390287321821578638127137672714997733619205677923546317959020670329964974643383412959523186999585472394218237778544517927286331499337257811216359487950374478126099738059867123971173732892766540440103066983138346000941393223644906578899512205843172507873377463087853424285351988587500042358246918740820428170090171482305182162161941319985606612938274264970984408447010080545496779367608881264464068851815527093240075425064971570470475419932831783645332562415378693957125097066387979492654623137674591890981311675243422111013091312783716095115834123084150371649970202246812196440812166865274580430262457825610671501385218216449532543349873487413352795815351016583605455763513276501810781194836945957485023739823545262563277947539726990201289151664579394201989202488033940516996865514944773965338769797412323540617819896112494095990353128997733611849847377946108428833293833903950900891408635152562680338141466927991331074334970514354520134464342647520016213846107299226419943327729189777690538025968518850841613864279936834741390

- Verify result using Python program which does string comparisons.
- My result agrees with Broadhurst to 1020 digits (i.e. all digits in Broadhurst).
- Comparing two different runs together with different BigFloat precisions – first 1040 digits in common.

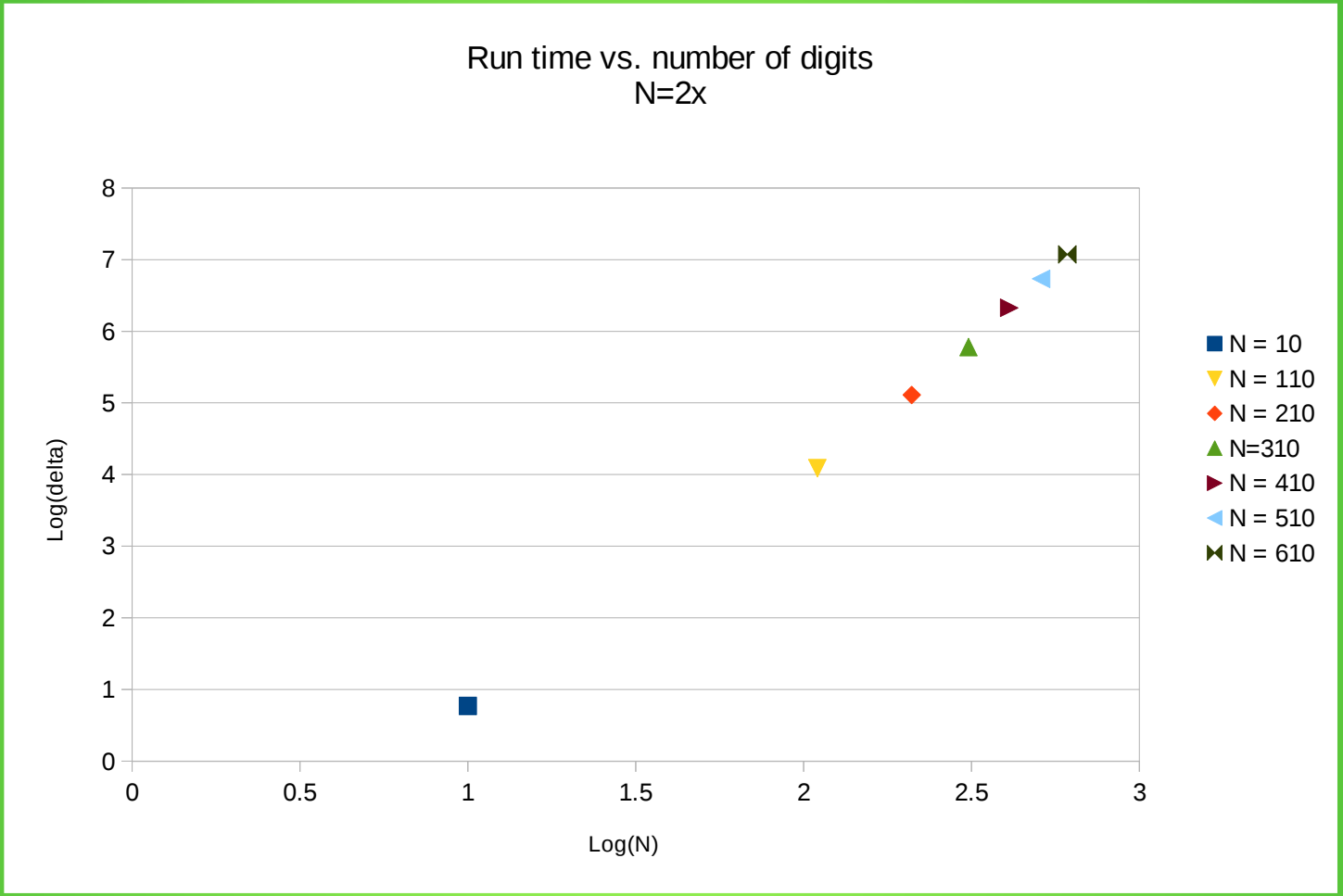
Future work

- Try to get 2000 digits.
- Compute δ.
- Use Chebyshev expansion instead of power series for accelerated convergence.

Thank you for your attention!

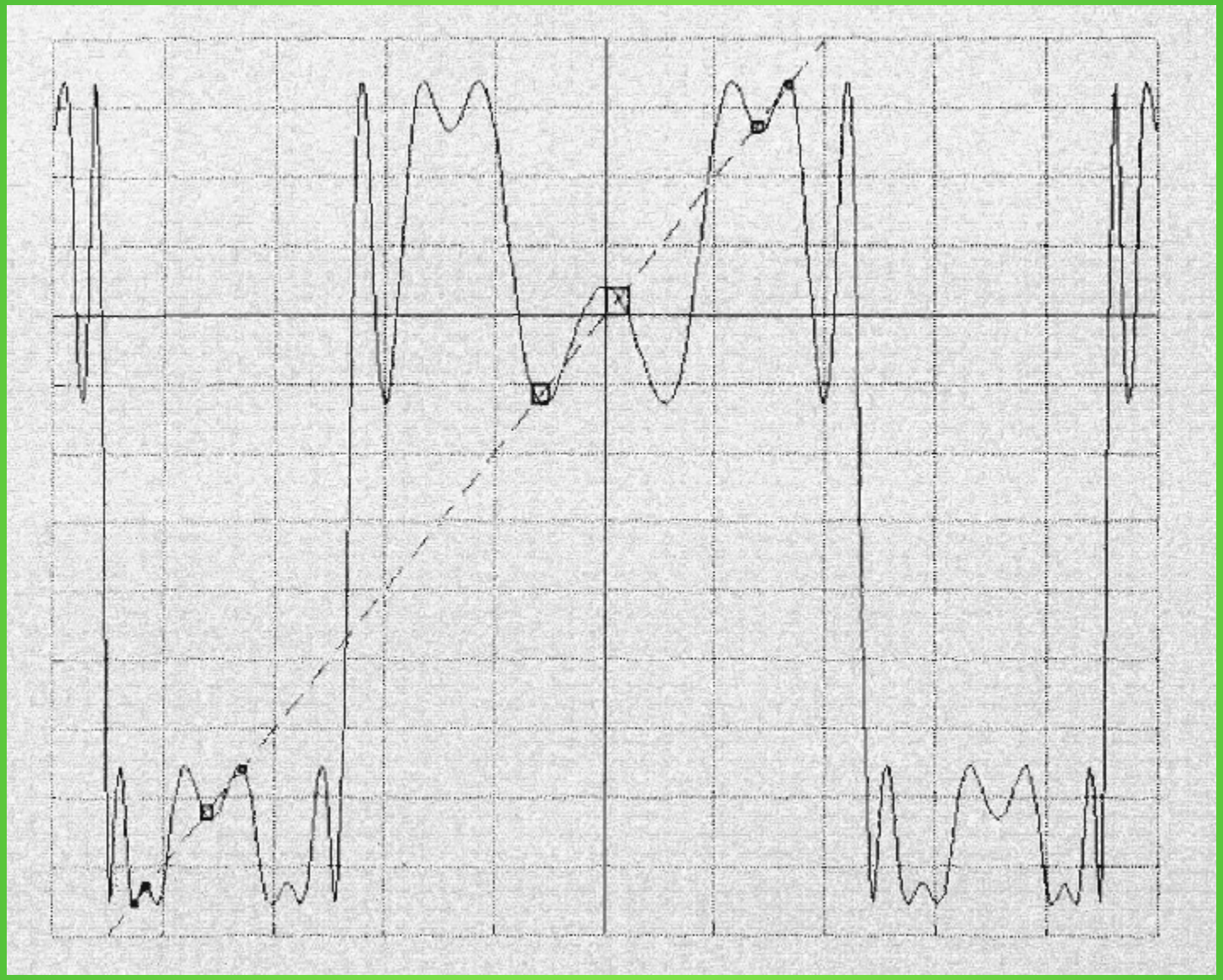
Supplementary slides

Scaling behavior for number of digits N



- Experimentally observed $O(N^{3.5})$ for up to 610 digits.
- I would guess $O(N^4)$ is asymptotic time complexity -- $O(N^3)$ for Gauss elimination, O(N) for MPFR.

The function g(z)



- Feigenbaum M.J. Universal behavior in nonlinear systems. Los Alamos Sci., 1 (1), p.4-27, 1980.

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Self-replicating attractor of a driven semiconductor oscillator

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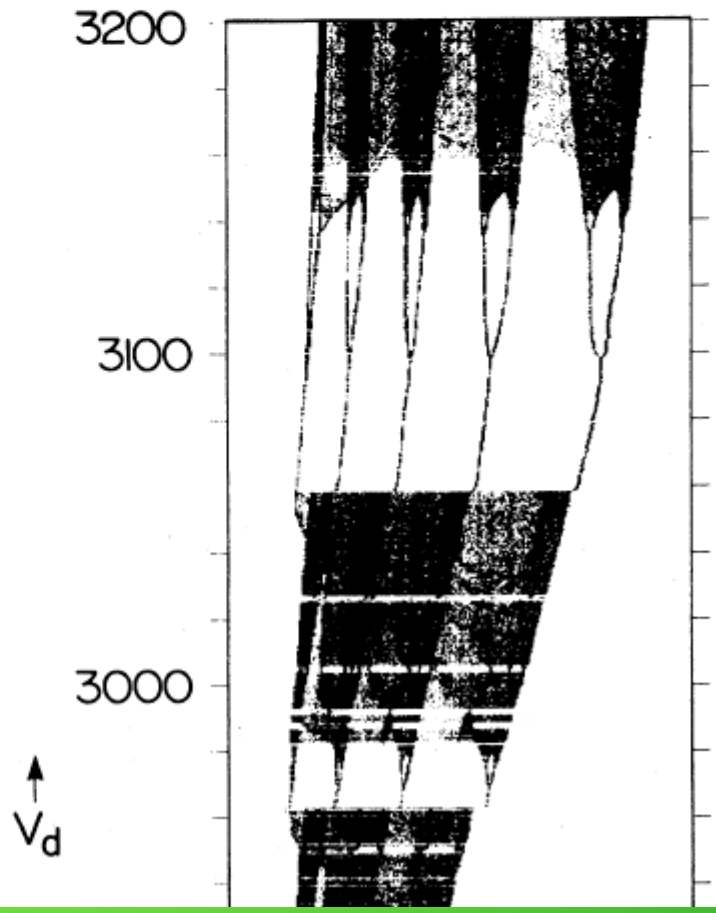
(Received 27 January 1983)

The measured attractor of a driven *R-L*-varactor system is found to be self-replicating and inconsistent with a simple logistic map. A calculation based on a standard model of a varactor diode is in qualitative agreement with the data.

It is well established that a driven series *RLC* circuit in which the capacitor is replaced by a varactor diode can exhibit period doubling, chaotic bands, and periodic windows at subharmonics of the fundamental drive frequency.[1,2] If the system is highly dissipative — that is, for large resistance — the measured period-doubling routes to chaos are consistent with the theory of Feigenbaum[3] and the order of appearance of the periodic windows agrees with the predictions of Metropolis, Stein, and Stein.[4] This would suggest that the underlying attractor of this system is quite similar to the logistic equation $x_{n+1}=4ax_n(1-x_n)$. However, for systems with smaller values of series resistance the true attractor is more complex although quite regular in its structure.

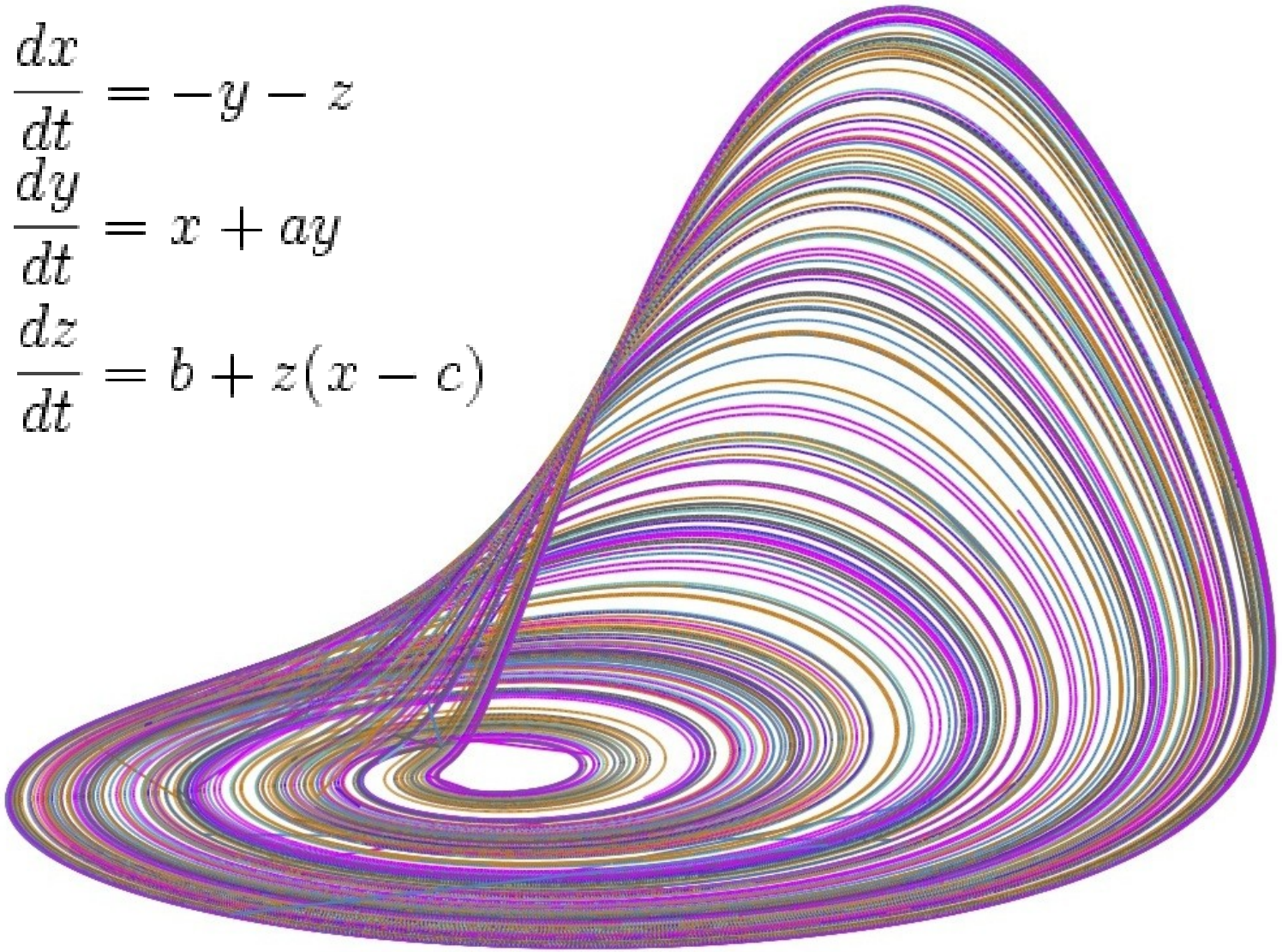
The system studied is a series combination of a sine wave voltage source, a resistor, an inductor, and a varactor diode. Circuit parameters are given in Table I and defined below. The amplitude of the voltage source, V_d, is digitally controlled: The digital range of 2750 to 3150 corresponds to a drive amplitude range of 1 to 10 V. Figure 1 is a bifurcation diagram of the current flowing through the circuit as a function of V_d: For each value of drive amplitude the current is sampled once per cycle for many cycles at the negative-slope zero crossing of the driving sine wave.

The structure is quite complex but exhibits the usual features of period doubling, chaotic bands, and periodic



Chaos

$$\frac{dx}{dt} = -y - z$$
$$\frac{dy}{dt} = x + ay$$
$$\frac{dz}{dt} = b + z(x - c)$$



- Phase space plot of Rössler system.