Rotating Wave Approximation for QOC

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1 Rotating Frame Transformation

1.1 Single rotation

We follow the template laid out in [CITE FIscher] to perform the rotating wave approximation for an N-level system with one control field. For starters, we take the laboratory frame Hamiltonian [CITE FISCHER]

$$H_{lab} = H_0 + \sum_{m=1}^{M} \Omega_m \cos(\omega_m t + \phi_m) \sum_{n'>n} g_{nn'} \sigma^x_{nn'} \tag{1}$$

This generalized hamiltonian maps perfectly to our systems, with H_0 representing all of our time independent elements. g_{nn} represents the prefactors on various drive terms (represented in our printed hamiltonians as Ω_n).

Sometimes we can ignore certain transitions because they are off-resonance. I would think this would apply to the transitions to the 2 state for our systems, but need to DOUBLE CHECK.

The first replacement is the basic RWA, replacing the driving terms with the terms from the RWA (FISCHER 3.25). also let $S = (n, n')$ which are on resonance and allowed.

$$H_{lab} \approx H_0 + \frac{\Omega}{2} \sum_{S} g_{nn'} \cos(\omega t + \phi) \sigma^x_{nn'} - \sin(\omega t + \phi) \sigma^y_{nn'}) \tag{2}$$

Our goal here is to get rid of the oscillation ωt. Again from fischer the desired transformation is:

$$|\psi\rangle_{rot} = e^{-iR} |\psi\rangle_{lab} \tag{3}$$

We take the derivative and obtain an equation for the transformed Hamiltonian:

$$H_{rot} = e^{iR} H_{lab} e^{-iR} + \frac{\mathrm{d}R}{\mathrm{d}t} \tag{4}$$

From this equation, our goal is to find R. The core of the RWA is the following transformation:

$$e^{-iR} \{\cos(\omega t + \phi)\sigma^x_{nn'} - \sin(\omega t + \phi)\sigma^y_{nn'}\} e^{iR} = \cos(\phi)\sigma^x_{nn'} - sin(\phi)\sigma^y_{nn'} \tag{5}$$

This transformation is applied over the set S in the sum in (2). The final step to the transformation is to find R such that these transformations hold. With that R in hand, we can see that (4) evaluates to

$$H_{rot} = H_0 + \frac{\mathrm{d}R}{\mathrm{d}t} + \frac{1}{2}\Omega \sum_S g_{nn'} \{\cos\phi\sigma^x_{nn'} - sin\phi\sigma^y_{nn'}\} \tag{6}$$

This is the rotating frame Hamiltonian, the key difference being the lack of oscillation on ω (carrier frequency), which is moved into the "generalised detuning term" $H_0 + \frac{\mathrm{d}R}{\mathrm{d}t}$. Note that in the two-level case this term reduces to $\frac{1}{2}\delta\omega\sigma^z$

What does equation (6) look like for IBM Q devices? All variables can be fillled in except for R, which can be solved through another process which will be shown after.

$$H_0 = H_\omega + H_{coupling} + H_{occupation_operator} \tag{7}$$
$$\Omega = 1 \tag{8}$$
$$g_{01} = \Omega_{d_0} \qquad \text{only one drive} \tag{9}$$

$g_{nn'}$ is technically contained in Ω because we are off resonance so only concerned with 0 to 1 transition

From 2.1 we have a value for R, so now we plug it all in and get a transformed hamiltonian

$$H_{rot} = H_0 + \omega\,[\sigma^z_{01} + \sigma^z_{12}] + \frac{\Omega_{d_0}}{2}[\cos\phi\sigma^x_{01} - \sin\phi\sigma^y_{01}] \tag{10}$$

In the single q athens, let's plug in H_0. Notice the ω in (10) is unnamed, this is the ω that we are driving our pulse with.

$$\begin{aligned} \mathcal{H}/\hbar = \frac{\omega_{q,0}}{2}(\mathbb{I} - \sigma^z_0) + J_{0,1}(\sigma^+_0\sigma^-_1 + \sigma^-_0\sigma^+_1) \\ + \omega\,[\sigma^z_{01} + \sigma^z_{12}] + \frac{\Omega_{d_0}}{2}[\cos\phi\sigma^x_{01} - \sin\phi\sigma^y_{01}] \end{aligned} \tag{11}$$

(11) is our final rotated hamiltonian. The next key question is how to get this into the format of QOC. QOC looks at hamiltonians through this lens

$$H(t) = H_d + \sum(u_1(t)H_{c1} + u_2(t)H_{c2} + ...) \tag{12}$$

H_d is easy, but it is a little more tricky to format the cos and sin elements correctly.

The equivalence is

$$u_1(t)H_{c1} + u_2(t)H_{c2} = \cos\phi\sigma^x_{01} - \sin\phi\sigma^y_{01} = c * D_0(t) \tag{13}$$

Note: c is just some constant

We know already that we can decompose $D(t)$ into a real and complex amplitude, $D(t) = e^{i\omega t} = \omega[cos(\phi) + i\sin(\phi)]$ Therefore, we can set $u_1(t) = \cos(\phi)$ and $u_2(t) = sin(\phi)$. Thus

$$H_{c1} = \frac{\Omega_{d_0}}{2}\sigma^x_{01} \tag{14}$$

$$H_{c2} = \frac{\Omega_{d_0}}{2}\sigma^y_{01} \tag{15}$$

$$D_0(t) = u_1(t)H_{c1} + iu_2(t)H_{c2} \tag{16}$$

$$Pulse\_amp(t) = u_1(t) + iu_2(t) \tag{17}$$

2 Notes that may be useful

$$H_c = [cos(\phi) * \sigma_x + sin(\phi) * \sigma_y()] \tag{18}$$
$$d(t) * H_c = H(t) \tag{19}$$
$$d(t)[cos(\phi(t)) * \sigma_x() + sin(\phi) * \sigma_y()] = H(t) \tag{20}$$
$$f(t) = d(t) * cos(\phi(t)) \tag{21}$$
$$[d(t)cos(\phi(t)) * \omega_d 0 * \sigma_x(), d(t)sin(\phi(t)) * \omega_d 0 * \sigma_y()] \tag{22}$$
$$[f(t) * \omega_d 0 * \sigma_x(), f1(t) * \omega_d 0 * \sigma_y()] \tag{23}$$
$$f(t) = d(t) * cos(\phi(t)) \tag{24}$$
$$f1(t) = d(t) * sin(\phi(t)) \tag{25}$$
$$D(t) = f(t) + i * f1(t) \tag{26}$$

Ignore U below

$$+\Omega_{d,0}(U_0^{(0,1)}(t))\sigma_0^X \tag{27}$$

2.1 Determining transformation matrix R for single qubit Athens

The matrix R is derived from equation (5). The two relationships that Fischer uses to make this derivation are the fact that R and $\sigma^z_{nn'}$ commute.

To find R, we look at equation 3.40 from Fischer:

$$[R, \sigma^x_{nn'} = \omega t(i\omega^y_{nn'})] \tag{28}$$

then we get the system of equations:

$$c_{01}(2i\sigma^y_{01}) + c_{12}(-i\sigma^y_{01}) = \omega t(i\sigma^y_{01}) \tag{29}$$
$$c_{01}(-i\sigma^y_{12}) + c_{12}(2\sigma^y_{01}) = \omega t(i\sigma^y_{12}) \tag{30}$$
$$\tag{31}$$

when we solve this system of equations we get: $c_{01} = c_{12}\omega t$

Thus $R = \omega t\,[\sigma^z_{01} + \sigma^z_{12}]$

3 Below is unfinished

3.1 Multiple drives and rotations

4 Rotating Frame Transformation for athens+

Note that for any more qubits the system looks the same, this is the minimal example. What does the lab Hamiltonian look like?

$$\mathcal{H}/\hbar = \sum_{i=0}^{4} \left(\frac{\omega_{q,i}}{2}(\mathbb{I} - \sigma_i^z) + \frac{\Delta_i}{2}(O_i^2 - O_i) + \Omega_{d,i} D_i(t) \sigma_i^X \right) \tag{32}$$

$$+ J_{1,2}(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) + J_{3,4}(\sigma_3^+\sigma_4^- + \sigma_3^-\sigma_4^+) \tag{33}$$

$$+ J_{0,1}(\sigma_0^+\sigma_1^- + \sigma_0^-\sigma_1^+) + J_{2,3}(\sigma_2^+\sigma_3^- + \sigma_2^-\sigma_3^+) \tag{34}$$

$$+ \Omega_{d,0}(U_0^{(0,1)}(t))\sigma_0^X + \Omega_{d,1}(U_1^{(1,0)}(t) + U_2^{(1,2)}(t))\sigma_1^X \tag{35}$$

$$+ \Omega_{d,2}(U_3^{(2,1)}(t) + U_4^{(2,3)}(t))\sigma_2^X + \Omega_{d,3}(U_6^{(3,4)}(t) + U_5^{(3,2)}(t))\sigma_3^X \tag{36}$$

$$+ \Omega_{d,4}(U_7^{(4,3)}(t))\sigma_4^X \tag{37}$$

$$\tag{38}$$

For this example, we are only concerned with the single qubit case (we will extend to 2q later). We are only going to model at most 2q at a time.

In this case, the lab hamiltonian looks like this:

$$\mathcal{H}/\hbar = \frac{\omega_{q,0}}{2}(\mathbb{I} - \sigma_0^z) + \frac{\Delta_0}{2}(O_0^2 - O_0) + \Omega_{d,0} D_0(t) \sigma_0^X \tag{39}$$

$$+ J_{0,1}(\sigma_0^+\sigma_1^- + \sigma_0^-\sigma_1^+) \tag{40}$$

$$+ \Omega_{d,0}(U_0^{(0,1)}(t))\sigma_0^X \tag{41}$$

Now the last term is the control channel, which we can ignore for the 1q situation. in addition, we can combine all the time independent parts to form H_0, giving us

$$\mathcal{H}/\hbar = H_0 + \Omega_{d_0} D_0(t) \sigma_0^x \tag{42}$$

Now we overlap this equation with (6), which we rewrite below.

$$H_{rot} = H_0 + \frac{\mathrm{d}R}{\mathrm{d}t} + \frac{1}{2}\Omega \sum_S g_{nn'} \left\{ \cos\phi \sigma_{nn'}^x - sin\phi \sigma_{nn'}^y \right\} \tag{43}$$

In other words, this is a more generalized form, and $D_0(t) = \cos\phi - i\sin\phi$ **I am a bit unsure but I think this is true** Also, note the following:

$$\Omega_{d_0} \approx \Omega \tag{44}$$
$$S = \{01\} \tag{45}$$
$$g_{01} = 1 \tag{46}$$

So the H_rot for the athens system is described by

$$H_{rot} = H_0 + \frac{\mathrm{d}R}{\mathrm{d}t} + \Omega d_0[] \tag{47}$$

4.1 Rotating Wave Approximation

4.2 What about the drive term?

The drive term that we see in the lab frame (after droppingcounter rotating terms and replacing with RWA) is

$$H_{lab} \approx H_0 + \frac{\Omega}{2}\sum_S g_{nn'}\cos(\omega t + \phi)\sigma^x_{nn'} - \sin(\omega t + \phi)\sigma^y_{nn'}) \tag{48}$$

The drive term that we see from athens is:

$$\Omega_{d_0} D_0(t)\sigma^x_0 \tag{49}$$

I'm pretty sure that this works out fine with the RWA (part 1? Fischer says use it to get to here but it still has the t terms so not sure) and we get

$$H_{rot} = H_0 + \frac{\Omega_{d_0}}{2}\left[\cos(\omega t + \phi)\sigma^x_{01} - \sin(\omega t + \phi)\sigma^y_{01}\right] \tag{50}$$

4.3 Finding R for single q athens

To find R, we look at equation 3.40 from Fischer:

$$[R, \sigma^x_{nn'} = \omega t(i\omega^y_{nn'})] \tag{51}$$

then we get the system of equations:

$$c_{01}(2i\sigma^y_{01}) + c_{12}(-i\sigma^y_{01}) = \omega t(i\sigma^y_{01}) \tag{52}$$
$$c_{01}(-i\sigma^y_{12}) + c_{12}(2\sigma^y_{01}) = \omega t(i\sigma^y_{12}) \tag{53}$$
$$\tag{54}$$

when we solve this system of equations we get: $c_{01} = c_{12}\omega t$
Thus $R = \omega t \left[\sigma^z_{01} + \sigma^z_{12}\right]$
Therefore, our hamiltonian is almost in the rotating frame:
The final step is to

4.4 Converting to format for qutip

In order to run qutip we need to factor out the t

we start with (56) and focus on the drive term, setting $\cos(\omega t + \phi)$ to $f(t)$ and $\sin(\omega t + \phi)$ to $f'(t)$

$$\begin{aligned} H_{rot} &= H_0 + \omega\left[\sigma^z_{01} + \sigma^z_{12}\right] \\ &+ \frac{\Omega_{d_0}}{2}\left[f(t) + f'(t)\right] \end{aligned} \tag{55}$$

4.5 The rotating frame transformation

Starting with equation (3.26) from Fischer, we see

$$\begin{aligned} H_{lab} &\approx H_0 + \omega\left[\sigma^z_{01} + \sigma^z_{12}\right] \\ &+ \frac{\Omega_{d_0}}{2}\left[\cos(\omega t + \phi)\sigma^x_{01} - \sin(\omega t + \phi)\sigma^y_{01}\right] \end{aligned} \tag{56}$$

we start by making the assumption that their exists some R such that

$$|\psi\rangle_{rot} = e^{-iR}\,|\psi\rangle_{lab} \tag{57}$$

We take the derivative and find the hamiltonian

$$H_{rot} = e^{-iR} H_{lab} e^{ir} + \frac{\mathrm{d}R}{\mathrm{d}t} \tag{58}$$

The terms inside of the sum in the hamiltonian are orthogonal to each other. Therefore the ω oscillation must be removed separately.

Ultimately we seek some R that allows the transformation which removes the time dependence

$$\begin{aligned} &e^{-iR}\left[\cos(\omega t + \phi)\sigma^x_{01} - \sin(\omega t + \phi)\sigma^y_{01}\right] e^{iR} \\ &= \cos(\phi)\sigma^x_{01} - \sin(\phi)\sigma^y_{01} \end{aligned} \tag{59}$$

... find the R section

Now that we have R, we get

$$\begin{aligned} H_{rot} &= H_0 + \omega\left[\sigma^z_{01} + \sigma^z_{12}\right] \\ &+ \frac{\Omega_{d_0}}{2}\left[\cos\phi\sigma^x_{01} - \sin\phi\sigma^y_{01}\right] \end{aligned} \tag{60}$$

In order to convert this to a format more easily digestible for qutip we need to replace the cos and sin, which we do

$H(t) = H_d + \sum(u_1(t)H_{c1} + u_2(t)H_{c2} + ...)$

Goal:

$$\cos\phi(t)\sigma^x_{01} - \sin\phi(t)\sigma^y_{01} = a * f(t) + b * g(t) \tag{61}$$
$$\cos\phi(t)^2 + \sin\phi^2 = 1 \tag{62}$$
$$f(t) \neq f(s(t)) \qquad \& \qquad g(t) \neq g(u(t)) \tag{63}$$
$$s(t), u(t) \notin \text{first order} \tag{64}$$

$$f(t) = a\cos x - b\sin x \tag{65}$$
$$f(t) = \frac{a}{2}\left[e^{ix} + e^{-ix}\right] - \frac{b}{2i}\left[e^{ix} - e^{-ix}\right] \qquad\qquad D(t) = f(t) \tag{66}$$

5 Conclusions

We worked hard, and achieved very little.

6 Rotating Frame Transformation

In this writeup we follow Fischer[CITE HERE] pretty carefully, the significance of this work is to apply his method to a specific case, the IBM Q chip hamiltonian. To do this, we set up the equivalencies between the base hamiltonian Fischer solves, then follow his walkthrough, and then when necessary plug in the IBM Q chip hamiltonian elements.

Fischer starts with the hamiltonian (3.24)

$$H_{lab} = H_0 + \Omega \cos(\omega t + \phi) \sum_S g_{nn'} \sigma^x_{nn'} \tag{67}$$

In this initial stage, we are only considered with a single transformation, and only one qubit.

Thus the IBM hamiltonian we observe is

$$H_{lab} = H_0 + \Omega_{d_0} D_0(t) \sigma^x_0 \tag{68}$$

The mapping between these hamiltonians gives these equivalencies

$$S = \{01\} \tag{69}$$

$$\Omega = 1 \tag{70}$$

$$g_{01} = \Omega_{d_0} \tag{71}$$

$$cos(\omega t + \phi) = D_0(t) \tag{72}$$

Now we follow Fischer's transformation process, plugging in where necessary.

6.1 RWA

The first step is to transform the cos element of the hamiltonian according to the RWA. Fischer in eq 3.26 notes that we can add a factor of $\frac{1}{2}$ and add in a sin term as shown to get the σ_y term as well. This is basically applying the 2 level RWA. (Talk this through with Thomas and Zach cause kind of weird) – "We drop the counter-rotating compenents of each driving term, replacing them with terms in the rwa" pretty sure this is just fast-forwarding a step because he already went through it in the 2 level case.

$$H_{lab} \approx H_0 + \frac{\Omega}{2} \sum_s g_{nn'} \{\cos((\omega t + \phi)) \sigma^x_{nn'} - \sin(\omega t + \phi) \sigma^y_{nn'}\} \tag{73}$$

The goal at this point is to find some basis transformation that will remove the oscillation at ω. We make the ansatz

$$|\psi\rangle_{rot} = e^{-iR} |\psi\rangle_{lab} \tag{74}$$

R is a matrix that we find on the side, we take the derivative of this ansatz equation, and find a transformed hamiltonian

$$H_{rot} = e^{-iR} H_{lab} e^{iR} + \frac{\mathrm{d}R}{\mathrm{d}t} \tag{75}$$

The desired transformation is (also note from fischer the terms in the sum are orthogonal and must be removed separately)

$$\begin{aligned} &e^{-iR} \left[\cos(\omega t + \phi)\sigma_{01}^{x} - \sin(\omega t + \phi)\sigma_{01}^{y}\right] e^{iR} \\ &= \cos(\phi)\sigma_{01}^{x} - \sin(\phi)\sigma_{01}^{y} \end{aligned} \tag{76}$$

Now, assuming that R has been found, the resulting hamiltonian is

$$H_{rot} = H_0 + \frac{\Omega_{d_0}}{2} \left[\cos(\omega t + \phi)\sigma_{01}^{x} - \sin(\omega t + \phi)\sigma_{01}^{y}\right] \tag{77}$$

In our single transition version

$$H_{rot} = H_0 + \frac{\Omega_{d_0}}{2} \left[\cos(\omega t + \phi)\sigma_{01}^{x} - \sin(\omega t + \phi)\sigma_{01}^{y}\right] \tag{78}$$

6.2 Finding R