MA302 Electromagnetism

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Preface

These are the lecture notes for the third-year Maths module MA302 Electromagnetism taught at the University of Warwick. These notes are available both as a PDF and a static website (which should be suitable for screen-reading devices), you can access both at https://brosaplanella.github.io/MA302-Electromagnetism/. I might update the notes as we go, fixing typos and improving explanations. You can keep track of those changes in the CHANGELOG. Further material is available on Moodle for registered students.

These lectures notes, which aim to be self–contained, are inspired by three main sources:

1. D. Tong, Electromagnetism, Cambridge University Press, Cambridge, 2025.
2. R.P. Feynman, R.B. Leighton, M.L. Sands, The Feynman lectures on physics, Definitive ed, Pearson Addison Wesley, San Francisco, 2006.
3. Lecture notes of the B7.2 Electromagnetism course at the University of Oxford, written by James Sparks and Erik Panzer.

The first two are good references if you want to read more about the topic, probably Tong's book is closer in structure to these lecture notes. In both cases, they cover a lot more material than this module does.

Aims and structure

The main aims of this module are:

- Provide the student with the background necessary to understand basic electromagnetism concepts and Maxwell's equations.
- Apply this knowledge to write and solve models for simple electromagnetism setups.
- Highlight the connections of electromagnetism to practical applications in our day-to-day lives.

We will look at electromagnetism from a mathematical perspective, and use it to better understand the world around us, which is what applied mathematics is about.

In Chapter 1 we will provide some motivation to the topic and recap some basic results from vector calculus that we will use in this module.

We will start with electrostatics in Chapter 2, which is the study of static electric charges. We will introduce some fundamental concepts like electric charge, electric field and electrostatic potential; and derive some results like Gauss' law. In Chapter 3 we will put into practice what we learned about electrostatics, by developing some concepts further and using them to understand some real applications, like conductors or capacitors.

Next, we will turn our attention to magnetostatics (Chapter 4), which is the study of steady magnetic fields. As magnetic fields are produced by charges in motion, we will introduce the concepts of current and current density. We will also derive some key results, like the Biot-Savart law, Gauss'law for magnetism or Ampère's law. Similarly to electrostatics, in Chapter 5 we will focus on applications of magnetostatics.

In Chapter 6 we will bring time into the equation, and generalise our previous results to allow for time-dependence, leading to the Maxwell's equations. We will talk about induction and displacement currents, but the spotlight will be on light[1]. We will derive the governing equation for electromagnetic waves, and introduce some results, though if you want to learn a lot more about wave you should probably sign up for MA301 Waves and Metamaterials.

We will conclude with **?@sec-matter**, in which we will extend the Maxwell's equations to account for real macroscale materials, rather than just charges in motion, and explain some of the phenomena that arise.

This is the first time I teach this module, and there will doubtless be errors and typos in these notes. There are also probably sections that could be better explained. If you spot anything or you have any suggestions, please do let me know via email.

Dr Ferran Brosa Planella, Autumn 2025

[1] Pun intended.

1 Introduction to electromagnetism

What is electromagnetism & motivation

1.1 Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.2}$$

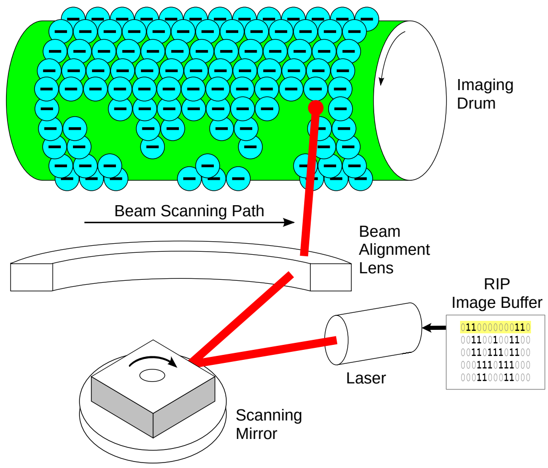
$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}, \tag{1.3}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial}{\partial t}\boldsymbol{E}\right). \tag{1.4}$$

1.2 Recap on vector calculus

2 Electrostatics

We start our journey in the field of electromagnetism, which is the branch of the subject concerned with stationary electric charges. This is the simplest case but present in our day-to-day lives in many ways: from laser printing to explaining static cling (see Figure 2.1).



(a) Laser printers use electrostatic forces to attach the toner particles to the imaging drum. Dale Mahalko, CC BY 3.0, via Wikimedia Commons.

(b) Styrofoam peanuts attaching to a cat's fur due to electrostatic forces. Sean McGrath from Saint John, NB, Canada, CC BY 2.0, via Wikimedia Commons.

Figure 2.1: Examples of electrostatics.

2.1 Point charges and Coulomb's law

As you may already know, subatomic particles like the proton and electrons, have a physical property called *electric charge*. Electric charge can be positive or negative and it is *quantized*, that means that it comes in integer multiples of the elementary charge e which, in SI units, is defined as $e \approx 1.602 \times 10^{-19}$ C (this unit is called Coulomb[1]). Protons have a charge of $+e$ while electrons have a charge of $-e$, and all charges in

[1]The unit is named after the french physicist Charles-Augustin de Coulomb, who formulated what today is known as Coulomb's law (coming next).

matter made of atoms arise from them. Charges[2] with the same sign attract each other, while charges of opposite sign repel each other.

We want to model the force between point charges. By point charges we mean that we will represent these electrically charges as points in $\mathbb{R}^3$, with their position in space denoted by the vector $\boldsymbol{r} \in \mathbb{R}^3$ and (the magnitude of) their charge denoted by q_i.

Definition 2.1 (Coulomb's law)**.** Given to point charges q_1 and q_2 positioned at $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathbb{R}^3$, respectively, each charge experiences a force

$$\boldsymbol{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|^2} \boldsymbol{e}_{12} = -\boldsymbol{F}_2, \tag{2.1}$$

where the constant $\epsilon_0 \approx 8.854 \times 10^{-12}\ \mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}$ is called the *permittivity of free space*, and $\boldsymbol{e}_{12}$ is the unit vector pointing from q_2 to q_1, which can be written as

$$\boldsymbol{e}_{12} = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}.$$

Remark 2.1.

1. The law only holds if $\boldsymbol{r}_1 \neq \boldsymbol{r}_2$ (otherwise the force would be infinitely large), therefore charges cannot sit on top of each other.
2. The force that the first charge exerts on the second has the same magnitude and opposite direction to the force that the second charge exerts on the first one, thus satisfying Newton's third law.
3. You may have noticed that Coulomb's law is very similar to Newton's law of universal gravitation (it is inversely proportional to the square of the distance between charges). However, charges (as opposed to masses) can take both positive and negative values. Therefore, the electrostatic force between charges can be attractive or repulsive.
4. Neutral particles (that is particles without an electric charge) do not experience electromagnetic force.

We defined Coulomb's law for a set of two charges. For three or more charges, we can use the Principle of Superposition.

Definition 2.2 (Principle of superposition)**.** Given N point charges q_i at positions $\boldsymbol{r}_i$, with $i \in 1, \ldots, N$, an additional charge q at position $\boldsymbol{r}$ experiences a force

$$\boldsymbol{F} = \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0} \frac{q q_i}{|\boldsymbol{r} - \boldsymbol{r}_i|^2} \frac{\boldsymbol{r} - \boldsymbol{r}_i}{|\boldsymbol{r} - \boldsymbol{r}_i|}. \tag{2.2}$$

[2] As a shorthand, we will refer to particles with an electrical charge as "charges".

Therefore, the total force on charge q is the sum of the forces exerted by all other charges (i.e. we can superpose the forces from each charge).

2.2 Electric fields and electrostatic potentials

Let's now introduce the concept of *electric field*. Even though it looks like a simple rewriting of the equation, it provides a very valuable new point of view.

Definition 2.3 (Electric field)**.** Given a set of charges q_i at positions $\boldsymbol{r}_i$, with $i \in 1, \ldots, N$, we define the *electric field* at a given point $\boldsymbol{r} \in \mathbb{R}^3$ as

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0} \frac{q_i}{|\boldsymbol{r} - \boldsymbol{r}_i|^2} \frac{\boldsymbol{r} - \boldsymbol{r}_i}{|\boldsymbol{r} - \boldsymbol{r}_i|}. \tag{2.3}$$

The electric field can be interpreted as the force that a *unit test charge* (that is, a virtual charge with $q = 1$) would experience at the point in space $\boldsymbol{r}$. It is called a test charge as its charge is not part of the set of charges we are considering (we use it only to "test" how the set of charges behaves). The electric field is a vector field defined in $\boldsymbol{r} \in \mathbb{R}^3 \setminus \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\}$.

Remark 2.2. Note that $\boldsymbol{F} = q\boldsymbol{E}$ yields Equation 2.2. This expression also allows us to deduce that the the electric field is measured (in SI units) in $\mathrm{N\,C^{-1}}$.

Now is the turn to introduce another useful concept: the electrostatic potential. You are probably familiar with the concept of potential, for example the gravitational potential. In a similar manner, we can define the electrostatic potential.

Definition 2.4 (Electrostatic potential)**.** The *electrostatic potential* for a given electric field $\boldsymbol{E}$ is defined as the function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\boldsymbol{E} = -\nabla\phi$.

Theorem 2.1. *For a single point charge q at $\boldsymbol{r}_0$ the electrostatic potential is given by*

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\boldsymbol{r} - \boldsymbol{r}_0|}. \tag{2.4}$$

The proof immediately follows from applying Definition 2.4 to Equation 2.4. This result can be easily extended to multiple point charge using the principle of superposition. We will further discuss the electrostatic potential in Section 2.5.

2.3 Gauss' law for point charges

As we said earlier, the definition of the electric field is a lot more than a simple relabelling, and in this section we will see why. Before that, we need to recall a definition from vector calculus.

Definition 2.5 (Flux)**.** Given a surface $\Sigma \subset \mathbb{R}^3$ with *outward*[3] unit normal vector $\boldsymbol{n}$, we define the *flux* of the electric field $\boldsymbol{E}$ through Σ as the integral $\int_\Sigma \boldsymbol{E} \cdot \boldsymbol{n} \mathrm{d}S$.

With this definition, we can now state one of the central results of electrostatics

Theorem 2.2 (Gauss' law)**.** *For any closed surface $\Sigma = \partial\Omega$ (that is the surface bounding a region $\Omega \in \mathbb{R}^3$), we have*

$$\int_\Sigma \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}A = \frac{1}{\epsilon_0} \sum_{i=1}^{N} q_i, \tag{2.5}$$

where q_i is the set of charges contained in Ω. For convenience, we define the total charge in Ω as $Q = \sum_{i=1}^N q_i$.

Proof. Let's start by considering a single point charge q at a position $\boldsymbol{r}_0$. Without loss of generality, we can assume $\boldsymbol{r}_0 = \boldsymbol{0}$ as we can always perform a change of coordinates. From Equation 2.3 we find that the charge produces an electric field

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}.$$

Let's also consider a ball of arbitrary radius $R > 0$ centred at the origin B_R, the flux of the electric field through its surface is[4]

$$\int_{\partial B_R} \boldsymbol{E} \cdot \boldsymbol{r} \, \mathrm{d}A = \frac{q}{4\pi\epsilon_0} \int_{\partial B_R} \frac{1}{|\boldsymbol{r}|^2} \mathrm{d}A = \frac{q}{4\pi R^2 \epsilon_0} \int_{\partial B_R} \mathrm{d}A = \frac{q}{\epsilon_0}. \tag{2.6}$$

This resembles the Gauss' law, but for a single charge and a very specific surface, so we now need to generalise the result. From the divergence theorem `REF`, for an arbitrary domain Ω bounded by $\Sigma = \partial\Omega$ we have

$$\int_\Omega \nabla \cdot \boldsymbol{E} \, \mathrm{d}V = \int_\Sigma \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}A.$$

[3] For a closed surface it is quite intuitive what outward means: pointing into the unbounded domain. For an open surface, the definition is arbitrary and we get to choose it. Note from the definition that this will simply change the sign of the flux.

[4] Recall that the outwards normal unit vector to the sphere is $\frac{\boldsymbol{r}}{|\boldsymbol{r}|}$.

Let's now compute the divergence of the electric field

$$\nabla \cdot \boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\left(\frac{3}{|\boldsymbol{r}|^3} - \frac{3}{|\boldsymbol{r}|^3}\right) = 0, \tag{2.7}$$

and note we have used `add relevant references from Chapter 1`. However, the electric field is singular at $\boldsymbol{r} = \boldsymbol{0}$, so we can only state that the divergence of the electric field is zero for $\boldsymbol{r} \in \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$.

Now let's consider assume our arbitrary domain $\Omega \in \mathbb{R}^3$ contains the ball B_R. The integral of Equation 2.7 over Ω can be split into

$$\int_\Omega \nabla \cdot \boldsymbol{E}\,\mathrm{d}V = \int_{\Omega\setminus B_R} \nabla \cdot \boldsymbol{E}\,\mathrm{d}V + \int_{B_R} \nabla \cdot \boldsymbol{E}\,\mathrm{d}V.$$

The first term is zero, as it the integration domain does not contain the origin, while we can apply the divergence theorem to the second term, obtaining

$$\int_\Omega \nabla \cdot \boldsymbol{E}\,\mathrm{d}V = \int_{B_R} \nabla \cdot \boldsymbol{E}\,\mathrm{d}V = \int_{\partial B_R} \boldsymbol{E} \cdot \boldsymbol{r}\,\mathrm{d}A = \frac{q}{\epsilon_0}, \tag{2.8}$$

where in the final step we have used Equation 2.6. Therefore

$$\int_\Sigma \boldsymbol{E} \cdot \boldsymbol{n}\mathrm{d}A = \frac{q}{\epsilon_0}. \tag{2.9}$$

Now consider a distribution of N charges q_i located at $\boldsymbol{r}_i$, respectively. By the principle of superposition Equation 2.2 we can write the electric field as the sum of the contributions for each charge. Then, similar to Equation 2.7, we can show that the divergence of the electric field will be zero in $\mathbb{R}^3 \setminus \bigcup_i \boldsymbol{r}_i$. Defining as B_i the ball of arbitrary radius R centred at $\boldsymbol{r}_i$ (i.e. around charge q_i), we can write

$$\int_\Omega \nabla \cdot \boldsymbol{E}\,\mathrm{d}V = \int_{\Omega\setminus\bigcup_{i=1}^N B_i} \nabla \cdot \boldsymbol{E}\,\mathrm{d}V + \sum_{i=1}^N \int_{B_i} \nabla \cdot \boldsymbol{E}\,\mathrm{d}V.$$

The first term is still zero as it doesn't include any of the charges. For each ball B_i we only need to consider the contribution of the charge q_i as it is the only singularity of the electric field in that ball, so Equation 2.8 generalises into

$$\int_{B_i} \nabla \cdot \boldsymbol{E}\,\mathrm{d}V = \int_{\partial B_i} \boldsymbol{E} \cdot \boldsymbol{n}\,\mathrm{d}A = \frac{q_i}{\epsilon_0}.$$

Therefore, we conclude

$$\int_{\Sigma} \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}A = \sum_{i=1}^{N} \frac{q_i}{\epsilon_0}.$$

□

2.4 Charge density

So far we have worked with a set of point charges. However, this is quite often impractical. For example, a macroscopic object has an enormous number of electrons and protons so it is not reasonable to consider them one by one. A more subtle point is that, from a quantum mechanics point of view, the position of electron be determined and instead they should be treated as a "blur" of charge. In either case, the concept of *charge density* comes handy.

Definition 2.6 (Charge density)**.** We define the *charge density* as a function $\rho(\boldsymbol{r}) : \Omega \in \mathbf{R}^3 \to \mathbf{R}$ that gives the charge per unit volume at a certain point in space.

We can then define the total charge in any arbitrary region $\Omega \subset \mathbb{R}^3$ as

$$Q = \int_{\Omega} \rho \, \mathrm{d}V.$$

Unless stated otherwise, we will assume that any charge density function ρ we encounter in this module is, at least, continuous. We will also assume that ρ has support Ω and that it is bounded. We can now redefine the electric field Equation 2.3 in terms of the charge density.

Definition 2.7 (Electric field)**.** Given a charge density $\rho : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$, we define the *electric field* at a given point $\boldsymbol{r} \in \mathbb{R}^3$ as

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_{\Omega} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^2} \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V', \tag{2.10}$$

where $\mathrm{d}V'$ is the infinitesimal volume element related to $\boldsymbol{r}'$.

Similarly, we can redefine the electrostatic potential for a charge density.

Theorem 2.3 (Electrostatic potential)**.** *The* electrostatic potential *for a charge density* $\rho(\boldsymbol{r})$ *is given by*

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_{\Omega} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V'. \tag{2.11}$$

Remark 2.3. It's worth pointing out that, as ρ has compact support, $\phi \sim r^{-1}$ as $|\boldsymbol{r}| \to \infty$ and thus it vanishes.

The proof also follows directly from Definition 2.4. Note that, when taking the gradient of ϕ, it acts on $\boldsymbol{r}$ only ($\boldsymbol{r}'$ is the integration variable and thus it's "invisible" from outside the integral).

It should come as no surprise that we can also rewrite Gauss' law for a charge density.

Theorem 2.4 (Gauss' law)**.** *For any closed surface $\Sigma = \partial\Omega$ (that is the surface bounding a region $\Omega \in \mathbb{R}^3$), we have*

$$\int_\Sigma \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}A = \frac{1}{\epsilon_0} \int_\Omega \rho \, \mathrm{d}V, \tag{2.12}$$

where $\rho : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ is the charge density in Ω. For convenience, we define the total charge in Ω as $Q = \int_\Omega \rho \, \mathrm{d}V$.

Using the divergence theorem on Equation 2.12 and rearranging we can write

$$\int_\Omega \left(\nabla \cdot \boldsymbol{E} - \frac{\rho}{\epsilon_0} \right) \mathrm{d}V = 0,$$

and given that this holds for any arbitrary domain $\Omega \subset \mathbb{R}^3$ we obtain the Gauss' law in differential form.

Theorem 2.5 (Gauss' law - differential form)**.** *The electric field generated by a given charge density $\rho : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ satisfies*

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}. \tag{2.13}$$

This is the first of the Maxwell's equations (Equation 1.1).

We have now two sets of definitions for electric field and Gauss' law: one for point charges and one for charge densities. But how do they relate to each other? The key is the *Dirac delta function*.

Definition 2.8 (Dirac delta function - 1D)**.** The *Dirac delta function* $\delta(x)$ in 1D is defined to satisfy the following properties:

$$\delta(x) = 0 \quad \text{if } x \neq 0,$$

and

$$\int_I \delta(x)\mathrm{d}x = \begin{cases} 1 & \text{if } 0 \in I, \\ 0 & \text{otherwise.} \end{cases}$$

In words, it is a function that is equal to zero everywhere except at the origin (where it is infinitely large), and its integral over $\mathbb{R}$ (or any interval I containing the origin) is equal to one.

Remark 2.4. One way of thinking about the Dirac delta is as the limit of a Gaussian probability distribution centred at the origin when the variance tends to zero (i.e. when you "squeeze" it).

Proposition 2.1. *The Dirac delta function satisfies:*

1.
$$\int_{-\infty}^{\infty} = f(x')\delta(x'-x)\mathrm{d}x' = f(x),$$

2.
$$\delta(ax) = \frac{1}{|a|}\delta(x),$$

3.
$$\int_{-\infty}^{x} \delta(x')\mathrm{d}x' = H(x),$$

where $f(x)$ is any continuous function, $a \neq 0$ is a constant and $H(x)$ is the Heaviside step function.

The proofs for these properties follow immediately from Definition 2.8. We can generalise the definition of the Dirac delta function to higher dimensions

Definition 2.9 (Dirac delta function)**.** The n-dimensional version Dirac delta function $\delta : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\delta(\boldsymbol{r}) = \Pi_{i=1}^{n}\delta(x_i) = \delta(x_1)\delta(x_2)\ldots\delta(x_n),$$

where $\boldsymbol{r} = (x_1, x_2, \ldots, x_n)$ in Cartesian coordinates.

Proposition 2.2. *The Dirac delta function in n-dimensions satisfies the properties*

1.
$$\delta(\boldsymbol{r}) = 0 \quad \text{if } \boldsymbol{r} \neq \boldsymbol{0},$$

2.

$$\int_\Omega \delta(\boldsymbol{r})\mathrm{d}V = \begin{cases} 1 & \textit{if } \mathbf{0} \in \Omega, \\ 0 & \textit{otherwise}, \end{cases}$$

3.

$$\int_\Omega f(\boldsymbol{r}')\delta(\boldsymbol{r}' - \boldsymbol{r})\mathrm{d}V = f(\boldsymbol{r}),$$

Let's now get back to the connection between point charges and charge densities. We can think of a point charge q at $\boldsymbol{r}_0$ as a charge density

$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r} - \boldsymbol{r}_0). \tag{2.14}$$

Substituting this definition in Equation 2.10 we obtain

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_\Omega \frac{q\delta(\boldsymbol{r}' - \boldsymbol{r}_0)}{|\boldsymbol{r} - \boldsymbol{r}'|^2}\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}\mathrm{d}V' = \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r} - \boldsymbol{r}_0|^2}\frac{\boldsymbol{r} - \boldsymbol{r}_0}{|\boldsymbol{r} - \ bfr_0|},$$

which corresponds to Equation 2.3 for a single point charge. We can proceed similarly for the electrostatic potential.

For the Gauss' law, we can substitute Equation 2.14 into Equation 2.12 and obtain

$$\int_\Sigma \boldsymbol{E} \cdot \boldsymbol{n}\mathrm{d}A = \frac{1}{\epsilon_0}\int_\Omega q\delta(\boldsymbol{r} - \boldsymbol{r}_0)\mathrm{d}V = \frac{q}{\epsilon_0},$$

as, by assumption, $\boldsymbol{r}_0 \in \Omega$. Therefore, we have recovered Equation 2.5 for a single point charge.

2.5 More on the electrostatic potential

Let's turn our attention back to the electrostatic potential. In Definition 2.4 we defined it in terms of the electric field, but it's usually more convenient to compute it directly from the charge density.

Proposition 2.3. *The electrostatic potential satisfies*

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}.$$

This equation is known as Poisson's equation.

Proof. Substituting Definition 2.4 into Equation 2.13 we get

$$\nabla \cdot (-\nabla\phi) = \frac{\rho}{\epsilon_0},$$

and thus

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}.$$

□

There is another interesting implication stemming from the definition of electrostatic potential. If $\boldsymbol{E} = -\nabla\phi$ we have

$$\nabla \times \boldsymbol{E} = \nabla \times (\nabla\phi) = 0, \tag{2.15}$$

as the curl of a gradient is always zero `CROSSREF`. Note that this is the steady-state version of the second of the Maxwell's equations Equation 1.3 (i.e. when $\frac{\partial \boldsymbol{B}}{\partial t} = 0$).

You may recall, for example when studying gravity, that when a force can be defined as the gradient of a scalar function (i.e. a potential), then the force is called *conservative*. Given that the electrostatic force is given by $\boldsymbol{F} = q\boldsymbol{E}$, and $\boldsymbol{E} = -\nabla\phi$, it must be conservative. Then, we can compute the work[5] done against the electrostatic force for a charge q moving along a path C starting at $\boldsymbol{r}_\mathrm{i}$ and ending at $\boldsymbol{r}_\mathrm{f}$ is

$$W = -\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -q\int_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = q\int_C \nabla\phi \cdot \mathrm{d}\boldsymbol{s} = q(\phi(\boldsymbol{r}_\mathrm{f} - \phi(\boldsymbol{r}_\mathrm{i}))). \tag{2.16}$$

Therefore, we deduce that the work done does not depend on the path C, only on its start and end points, and we can conclude that the electrostatic force is conservative.

Remark 2.5. Note that the electrostatic potential is defined up to a constant. This means that, if ϕ is a potential for a given electric field $\boldsymbol{E}$, then $\hat{\phi} = \phi + c$ (where c is a constant) is also a potential for $\boldsymbol{E}$. Therefore, the quantity that is well-defined (and we can measure) physically is the potential difference between two points, which we call *voltage*.

When working with electrostatic potentials, it is up to us to define a reference for the potential, that is the value of the potential at a given point, which allows us to fix the constant. For example, in Definition 2.4 we assumed that the potential is zero as $|\boldsymbol{r}| \to \infty$.

[5] Recall that the work is defined as the energy transferred to or from an object via the application of force along a displacement. Note the importance of the displacement, as if there is no displacement, there is no work.

Definition 2.10 (Field lines)**.** Given a vector field, a field line is a line that at each point is tangent to the vector field.[6]

Field lines are a very useful way to visualise electric (and magnetic) fields.

Definition 2.11 (Equipotential surfaces)**.** Surfaces of constant ϕ are called *equipotentials*.[7]

Proposition 2.4. *The electric field (i.e. the field lines) are always normal to an equipotential surface.*

Proof. Define $\boldsymbol{t}$ to be a tangent vector to the equipotential at a given point $\boldsymbol{r}$. From the definition of the equipotential, we know that the derivative of ϕ in the tangent direction $\boldsymbol{t}$ must be zero (as the potential is constant along an equipotential surface). Thus $(\nabla\phi)\cdot\boldsymbol{t} = 0$ and from the definition of the electrostatic potential we conclude that $\boldsymbol{E}\cdot\nabla\phi = 0$ so the electric field is perpendicular to the equipotential surface (and so are the field lines). □

2.6 Electrostatic energy

To finish this chapter on electrostatics, let's talk about the electrostatic energy. Note that the electrostatic potential at a given point can be interpreted as the potential energy required to bring a unit charge to that point from some reference point (usually infinity).

However, we want to extend this concept from a single charge to any electrostatic configuration. As usual, let's start considering point charges to build intuition about the general form. We start with a charge q_1 at $\boldsymbol{r}_1$. Its potential is

$$\phi_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\boldsymbol{r}-\boldsymbol{r}_1|}.$$

Now let's consider a charge q_2, which we move from infinity to a point $\boldsymbol{r}_2$. From Equation 2.16, the work done against the electric field is

$$W_2 = q_2\phi_1(\boldsymbol{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{|\boldsymbol{r}_2-\boldsymbol{r}_1|}.$$

[6] If you are taking MA3D1, you will notice that the streamlines are another example of field lines, in this case for a velocity field.

[7] If considering a 2D problem, equipotential surfaces reduce to equipotential curves.

Let's now do the same for a charge q_3 that needs to be placed at $\boldsymbol{r}_3$, in the presence of q_1 and q_2:

$$W_3 = q_3(\phi_1(\boldsymbol{r}_3) + \phi_2(\boldsymbol{r}_3)) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_3 q_1}{|\boldsymbol{r}_3 - \boldsymbol{r}_1} + \frac{q_3 q_2}{|\boldsymbol{r}_3 - \boldsymbol{r}_2|}\right),$$

which we deduce fro the principle of superposition (ϕ_2 is the potential of the charge q_2). The total work done so far is $W = W_2 + W_3$. By induction, we can deduce that for N charges $q_1, \ldots, q_N$ at $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$ the total work to assemble them is

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{j<i}\frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}.$$

By symmetry on i and j, we can write

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{j\neq i}\frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}.$$

Rewriting it in terms of electrostatic potentials we get

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i \phi^{(i)}, \tag{2.17}$$

where

$$\phi^{(i)} = \frac{1}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \tag{2.18}$$

is defined as the potential generated by all charges except for q_i evaluated at $\boldsymbol{r}_i$.

Now let's take the continuum limit, to consider the potential created by a charge density instead (Equation 2.11). Then Equation 2.17 becomes

$$W = \frac{1}{2}\int_{\Omega}\rho\phi \mathrm{d}V$$

.

Using Gauss' law (#eq-Gauss-law-charge-density) and some basic vector calculus identities, we obtain

$$\phi\frac{\rho}{\epsilon_0} = \phi\nabla\cdot\boldsymbol{E} = \nabla\cdot(\phi\boldsymbol{E}) - \nabla\phi\cdot\boldsymbol{E} = \nabla\cdot(\phi\boldsymbol{E}) + \boldsymbol{E}\cdot\boldsymbol{E} = \nabla\cdot(\phi\boldsymbol{E}) + |\boldsymbol{E}|^2.$$

Substituting into Equation 2.18 and using the divergence theorem we get

$$W = \frac{\epsilon_0}{2}\left(\int_\Sigma \phi \boldsymbol{E}\cdot\boldsymbol{n}\mathrm{d}A + \int_\Omega |\boldsymbol{E}|^2\mathrm{d}V\right). \tag{2.19}$$

As we want to know the energy of the whole charge configuration, we take Ω to be a ball centred at the origin with radius r. Taking the limit $r\to\infty$ and recalling that in that limit $\phi \sim \frac{1}{r}$, the surface integral term vanishes, and Equation 2.19 reduces to

$$W = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} |\boldsymbol{E}|^2\mathrm{d}V.$$

If the integral exists, we say that the charge configuration has finite energy W.

Definition 2.12 (Energy density)**.** The energy density of a charge configuration is defined by

$$\mathcal{E} = \frac{\epsilon_0}{2}|\boldsymbol{E}|^2,$$

and the total energy of the configuration can be written as

$$W = \int_{\mathbb{R}^3} \mathcal{E}\mathrm{d}V.$$

3 Applications of electrostatics

In the previous chapter we introduced some machinery to understand electrostatics. The aim of this chapter is to put this machinery to work, so we can use it to understand some of the phenomena we see and experience in our day-to-day lives.

3.1 Equilibrium in an electrostatic field

Let's first consider whether we can reach equilibrium in an electrostatic field, i.e. can we "trap" an electric charge at a given point just by placing other charges in the right places? To be precise, by "trap" we mean that the charge will lie in stable equilibrium at that point, meaning that if we slightly perturb it, it will go back to place (instead of flying away). We also do not consider the case where two charges lie on top of each other, which would actually give us equilibrium.

Example 3.1. Consider two point charges of charge q located at $\boldsymbol{r}_+ = (1, 0)$ and $\boldsymbol{r}_- = (-1, 0)$, respectively.[1] We assume these charges remain fixed in place somehow. By the principle of superposition, the electric field generated by these two charges is

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{\boldsymbol{r}-\boldsymbol{r}_+}{|\boldsymbol{r}-\boldsymbol{r}_+|^3} + \frac{\boldsymbol{r}-\boldsymbol{r}_-}{|\boldsymbol{r}-\boldsymbol{r}_-|^3}\right).$$

Now let's introduce another charge q at the origin. The force this charge will experience is

$$\boldsymbol{F} = q\boldsymbol{E}(\boldsymbol{0}) = \boldsymbol{0},$$

so the new charge will remain at the origin (no force acting on it). However, if we perturb the charge position, for example placing it at $(0, \delta)$, where $\delta > 0$ is a small quantity, the force becomes

$$\boldsymbol{F} = \left(0, \frac{q^2}{2\pi\epsilon_0}\frac{\delta}{(1+\delta^2)^{\frac{3}{2}}}\right),$$

[1] For simplicity, we constrain ourselves to the 2D case.

and thus the charge will be pushed upwards, away from the origin. Therefore, we found an equilibrium point, but not a stable one, not meeting our definition of "trapping".

You can try as hard as you want to find a charge arrangement that would give us electrostatic equilibrium, but it is actually impossible. Let's prove it by assuming otherwise and reaching a contradiction.

Assume there is a charge configuration in which there is a point in empty space $\boldsymbol{r}_*$ that is stable for a charge q (wlog[2] we assume $q > 0$). Empty space means that the charge density is zero in a neighbourhood of $\boldsymbol{r}_*$. As the point is stable, if we place the particle slightly off $\boldsymbol{r}_*$ it should be pushed back into position by the electric field. Therefore the particle should see an inwards force towards $\boldsymbol{r}_*$. Therefore, for an arbitrary surface Σ surrounding $\boldsymbol{r}_*$ and contained in the free space, we must have

$$\int_\Sigma \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}A < 0.$$

However, by Gauss' law, the right-hand side must be proportional to the charge contained within Σ, which we assumed was zero. This is a contradiction, and therefore electrostatic equilibrium is not possible.

Note that, by using forces other than the electrostatic ones, one can construct equilibrium configurations. For example, in Example 3.1, if we constrain the new charge to lie in the horizontal axis (by imposing some sort of force), then we have a stable configuration as for a small displacement δ along the horizontal axis (in either direction) will result into a force in the horizontal direction with magnitude

$$F = -\frac{q}{\pi\epsilon_0} \frac{\delta}{(\delta-1)^2(\delta+1)^2},$$

pushing the charge back to the origin.

3.2 Conductors

Let's now talk about conductors.

Definition 3.1 (Electrical Conductor)**.** An *electrical conductor* (or conductor, in short) is a material in which some charges are free to move (e.g. electrons in a metal). We will describe it mathematically as a given region in space that contains charges which can move freely.[3]

[2] wlog = without loss of generality.
[3] The conductor might also contain some immobile charges.

The are several consequences stemming from this definition, let's unpack them one by one.

Proposition 3.1. *In a static configuration, the electric field inside a conductor is equal to zero, i.e.* $\boldsymbol{E} \equiv \boldsymbol{0}$.

Proof. If the electric field inside the conductor was non-zero, any charges inside would move, but we do not allow it as we are considering a static configuration. □

From Proposition 3.1, we can deduce a series of corollaries.

Corollary 3.1. *The electrostatic potential* ϕ *inside a conductor must be constant (proof follows directly from the definition of* ϕ*).*

Corollary 3.2. *The charge density inside the conductor must be identically zero, i.e.* $\rho \equiv 0$ *(proof follows directly from Gauss' law). Therefore, any net charge of a conductor must sit at its surface.*

Corollary 3.3. *The surface of the conductor is an equipotential (proof follows directly from* $\phi =$ *const). Therefore, the electric field is perpendicular to the surface, which agrees with the fact that surface charges must be static.*

The physical interpretation of these results is that, when a conductor is in the presence of an electric field, its free charges will rearrange in such a way (across the conductor surface) so they cancel out the electric field inside of the conductor.

All this naturally leads to the introduction of *surface charges*. Similar to the charge density ρ, which is defined as charge per unit of volume, we can define the surface charge density σ as the charge per unit of area. Then, we can naturally extend Definition 2.7 for a charge density σ on a surface Σ as

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_{\Sigma} \frac{\sigma(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^2} \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}A'. \tag{3.1}$$

However, now the electric field has some interesting properties.

Proposition 3.2. *Consider the electric field Equation 3.1 generated by a surface charge density* σ *on a surface* Σ*. Then the electric field* $\boldsymbol{E}$ *must be continuous across* Σ *in the tangential direction, but not in the normal direction.*

In particular, the discontinuity can be written as

$$\boldsymbol{E}^+ \cdot \boldsymbol{N} - \boldsymbol{E}^- \cdot \boldsymbol{n} = \frac{\sigma}{\epsilon_0}. \tag{3.2}$$

Here the subscripts + and − represent different sides of the surface (arbitrarily labelled), and $\boldsymbol{n}$ is the normal unit vector to the surface pointing into the + side.

Proof.

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> ADD SKETCH OF CURVES

Let's start with the discontinuity in the normal direction. Consider a small cylinder Ω of height δ and cross-sectional area δA. For simplicity we will assume that, as outlined in Figure `REF FIG`, in the region of interest Σ is flat, the cylinder axis is perpendicular to the surface, and the surface intersects the cylinder at its midpoint; but the same argument holds for more generic situations.[4] By Gauss' law

$$\int_{\partial\Omega} \boldsymbol{E} \cdot \boldsymbol{n}\, \mathrm{d}A = \frac{Q}{\epsilon_0}, \tag{3.3}$$

where, remember, Q is the total charge inside the cylinder Ω. Let's now decompose the surface of the cylinder into three surfaces: the "top lid" $\partial\Omega^+$, the "bottom lid" $\partial\Omega^-$, and the side $\partial\Omega_{\text{side}}$. We can split the integral into

$$\int_{\partial\Omega} \boldsymbol{E} \cdot \boldsymbol{n}\, \mathrm{d}A = \int_{\partial\Omega^+} \boldsymbol{E} \cdot \boldsymbol{n}^+\, \mathrm{d}A + \int_{\partial\Omega^-} \boldsymbol{E} \cdot \boldsymbol{n}^-\, \mathrm{d}A + \int_{\partial\Omega_{\text{side}}} \boldsymbol{E} \cdot \boldsymbol{n}_{\text{side}}\, \mathrm{d}A.$$

Now we take the limit $\delta \to 0$ (i.e. we collapse the cylinder onto the surface). Then, the side component vanishes, and the top and bottom lid collapse into each other (and onto the surface). We can rewrite Equation 3.3 as

$$\int_{\partial\Omega^+} \boldsymbol{E} \cdot \boldsymbol{n}^+\, \mathrm{d}A + \int_{\partial\Omega^-} \boldsymbol{E} \cdot \boldsymbol{n}^-\, \mathrm{d}A = \int_{\partial\Omega^+} \left(\boldsymbol{E}^+ \cdot \boldsymbol{n}^+\right) - \left(\boldsymbol{E}^- \cdot \boldsymbol{n}^+\right)\, \mathrm{d}A,$$

where we have used that $\partial\Omega^+ \equiv \partial\Omega^-$ but $\boldsymbol{n}^+ = -\boldsymbol{n}^-$. The total charge Q is given by

$$Q = \int_{\partial\Omega^+} \sigma \mathrm{d}A,$$

so combining everything we obtain

$$\int_{\partial\Omega^+} \left(\boldsymbol{E}^+ \cdot \boldsymbol{n}^+\right) - \left(\boldsymbol{E}^- \cdot \boldsymbol{n}^+\right)\, \mathrm{d}A = \int_{\partial\Omega^+} \frac{\sigma}{\epsilon_0} \mathrm{d}A.$$

As this equality must hold for any cylinder Ω, we conclude that

[4] This is usually known as a Gaussian cylinder or Gaussian pillbox.

$$\boldsymbol{E}^{+} \cdot \boldsymbol{N} - \boldsymbol{E}^{-} \cdot \boldsymbol{n} = \frac{\sigma}{\epsilon_0}.$$

Now let's show that $\boldsymbol{E}$ must be continuous in the tangential direction. Consider, instead, the rectangular loop C in Figure `REF FIG`, with height δ and width δL. The surface enclosed by the loop is defined as S. We compute the integral of E around the loop and apply Stokes' theorem:

$$\int_C \boldsymbol{E} \cdot \boldsymbol{t} \, \mathrm{d}s = \int_S (\nabla \times \boldsymbol{E}) \cdot \boldsymbol{n} \, \mathrm{d}A.$$

But from Equation 2.15 we have that $\nabla \times \boldsymbol{E} = \boldsymbol{0}$ so the integral needs to be equal to zero. By taking $\delta \to 0$ and applying a similar argument as we did with the cylinder, we obtain

$$\boldsymbol{E}^{+} \cdot \boldsymbol{t} - \boldsymbol{E}^{-} \cdot \boldsymbol{t} = 0,$$

and thus $\boldsymbol{E}$ is continuous in the tangential direction. □

Remark 3.1. Similarly to a surface charge density, we can define a line charge density λ, where represents the charge per unit length along a line.

3.2.1 Examples

Example 3.2 (Line charge)**.**

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> Add diagram

Consider an infinitely long line charge distribution, with charge density λ. We want to compute the electric field it produces. We will work in cylindrical polar coordinates and, wlog, we will assume the charge is distribution along the z axis (i.e. at $r = 0$).

By symmetry, we can conclude that the electric field should only depend on the radial coordinate and, moreover, the only non-zero component of the electric field is also along the radial coordinate. Therefore $\boldsymbol{E} = E(r)\boldsymbol{e}_r$.

We will again use a Gaussian surface for the argument, but this time we take an arbitrary cylinder of radius r and length L with its axis along the z axis. By Gauss' law, we have

$$\int_{\partial\Omega} \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}A = E(r) 2\pi r L = \frac{\lambda L}{\epsilon_0},$$

and thus

$$E(r) = \frac{\lambda}{2\pi\epsilon_0 r}.$$

So we conclude that the electric field is given by

$$\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_0 r}\boldsymbol{e}_r.$$

Example 3.3 (Parallel plate capacitor)**.**

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Let's now consider two flat parallel surfaces. We will take the plates to be perpendicular to the x axis, and a distance d apart. The plates have an area A, and we assume that $d \ll \sqrt{A}$, so we can ignore any effects arising from the edge of the plates[5].

Each plate has a total charge Q, but with opposite sign. Therefore, the surface charge distribution is defined as $\pm\sigma = \pm Q/A$. We know[6] that the electric field produced by an infinite plate with surface charge density σ is uniform and perpendicular to the plate (pointing away from it) with magnitude

$$\|\boldsymbol{E}\| = \pm\frac{\sigma}{2\epsilon_0}.$$

Then, by superposition, the electric field produced by the two plates is

$$\boldsymbol{E} = \begin{cases} \frac{\sigma}{\epsilon_0}\boldsymbol{e}_x, & \text{between the plates,} \\ \boldsymbol{0}, & \text{otherwise.} \end{cases}$$

This configuration is known as a capacitor, and it is typically used to store small amounts of energy (and many other things). We can then consider what's the electrostatic potential in the capacitor. As the electric field is only in the x direction, we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = -E \implies \phi = -\frac{\sigma}{\epsilon_0}x + c,$$

where c is an integration constant. We define the *voltage* or *potential difference* as the difference in potential between two points. In this case, the voltage between the two plates is

[5] To formalise this argument, you would need to do an asymptotic analysis similar to what you learned in MA269 Asymptotics and Integral Transforms (if you took it).

[6] This is an exercise in the problem sheets.

$$V = \phi(0) - \phi(d) = \frac{\sigma d}{\epsilon_0}.$$

We can then define the *capacitance* C of the capacitor as the charge in the capacitor divided by the potential difference:

$$C = \frac{Q}{V} = \frac{A\epsilon_0}{d}.$$

The capacitance is dictates how much energy the capacitor can store, from Definition 2.12:

$$W = \frac{\epsilon_0 A}{2} \int_0^d \left(\frac{\sigma}{\epsilon_0}\right)^2 \mathrm{d}x = \frac{Q^2}{2C}.$$

3.2.2 Application: the Faraday cage

Let's use what we have learned about conductors to understand how a Faraday cage works. Let's consider a box made of a conducting material. Inside the box there are no charges. As shown in Figure 3.1, when we apply an electric field on the outside of the box, the charges on the box will redistribute to produce an electric field that will cancel out the field inside of the box. Therefore, anything or anyone inside the box is shielded from any external electric fields.

As the name suggests, though, Faraday cages are typically cages, not boxes, but the effect still holds. The mathematical analysis in that case is a bit more complicated, if you want to learn more, you can read this article by Jon Chapman, Dave Hewett and Nick Trefethen.

Faraday cages play an important role in everyday life by shielding sensitive electronic devices from electromagnetic interference. A familiar example is the way cars act as Faraday cages, protecting occupants from lightning strikes during storms. Lightning is a sudden discharge between very strongly electrically charged areas (in this case, the clouds and the ground). Therefore, the lightning is just the consequence of a very strong electric field. This video by Top Gear is an entertaining (though a bit dated now) demonstration about a car behaving like a Faraday cage.

3.3 Boundary value problems

So far, we have considered problems where we are given a charge distribution ρ everywhere in space, and we just computed the potential ϕ from Theorem 2.3. Instead, we now want to consider situations where the charge distribution is given in a region of space Ω only, along with some conditions on the potential at the boundary $\partial\Omega$. This type of problems

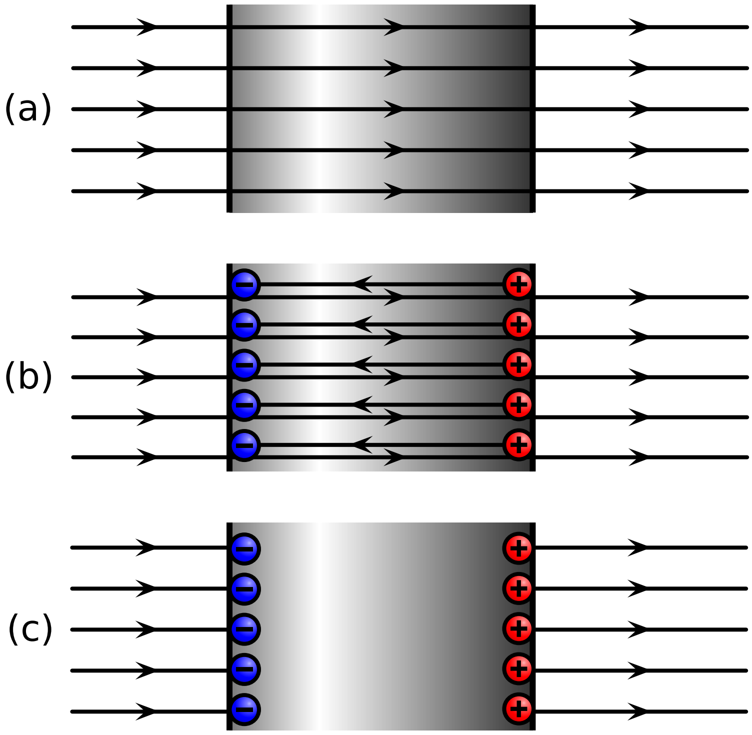

Figure 3.1: Diagram of a Faraday cage in operation. MikeRun, CC BY-SA 4.0, via Wikimedia Commons

are called, quite naturally, boundary value problems. More intuitively, it means that we will assume we have an infinite source/sink of charges outside Ω, and that they will do *whatever they need to do* to satisfy the imposed boundary condition.

In practice, this means solving Proposition 2.3, i.e.

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}, \tag{3.4}$$

along with some boundary conditions. There are two types of boundary conditions we will consider.

Definition 3.2 (Dirichlet boundary conditions)**.** The value of ϕ is fixed on $\partial\Omega$. Physically, this represents that the boundary is made of a conductive material.

Definition 3.3. The value of $\frac{\partial \phi}{\partial n} = \nabla\phi \cdot \boldsymbol{n}$ is fixed on $\partial\Omega$. This is equivalent to fixing $\boldsymbol{E} \cdot \boldsymbol{n}$. Physically, this represents that the boundary is made of an insulating material.

One could also mix the boundary conditions, having Dirichlet boundary conditions on part of the boundary and Neumann conditions in the rest.

It's natural to wonder whether Equation 3.4 with Dirichlet or Neumann (or mixed) boundary conditions has a unique solution.

Theorem 3.1. *The Poisson equation (Equation 3.4) with either Dirichlet or Neumann boundary conditions has a unique solution for ϕ. For Neumann boundary conditions, this solution is defined up to an arbitrary (or unphysical) additive constant.*

Proof. Suppose that there are two solutions ϕ_1 and ϕ_2 to the problem. Now define the function $f = \phi_1 - \phi_2$. It should satisfy

$$\nabla^2 f = 0 \tag{3.5}$$

with either zero Dirichlet or Neumann boundary conditions.

Consider the identity

$$\nabla \cdot (f \nabla f) = |\nabla f|^2 + f \nabla^2 f,$$

which is basically a distributive property of the divergence. Multiplying Equation 3.5 by f and integrating over Ω we get

$$0 = \int_\Omega f \nabla^2 f \mathrm{d}V = \int_\Omega \left(\nabla \cdot (f \nabla f) - |\nabla f|^2 \right) \mathrm{d}V = \int_{\partial\Omega} f \nabla f \cdot \boldsymbol{n} \, \mathrm{d}A - \int_\Omega |\nabla f|^2 \mathrm{d}V,$$

where in the last step we have used the divergence theorem on the first term in the integral. The surface integral that stems from it vanishes, as at the boundary either $f = 0$ or $\nabla f \cdot \boldsymbol{n} = 0$ (by the boundary conditions). Then, we conclude

$$\int_\Omega |\nabla f|^2 \mathrm{d}V = 0,$$

and therefore we must have $\nabla f = \mathbf{0}$ everywhere in Ω, which means f is constant. If we have Dirichlet boundary conditions (even if only at a subset of the boundary) we can conclude $f = 0$, if we have Neumann conditions everywhere, then we can't determine the value of the constant. However, this constant has no physical meaning (we can arbitrarily define the origin of the potential, as the only thing that matters are the potential differences). $\square$

3.4 Green's functions

Let's now discuss some methods to solve boundary value problems. We will start with Green's functions.

Definition 3.4 (Green's function)**.** A *Green's function* for Poisson's equation is a function $G(\boldsymbol{r}, \boldsymbol{r}')$ satisfying

$$\nabla'^2 G(\boldsymbol{r}, \boldsymbol{r}') = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{3.6}$$

where $\boldsymbol{r}, \boldsymbol{r}' \in \Omega$. Note that the Laplacian ∇'^2 is defined in terms of the prime variable. This means it does not affect $\boldsymbol{r}$, only $\boldsymbol{r}'$.

From `CROSSREF` we know that a particular solution to Equation 3.6 is

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

Note, however, that this solution is not unique. The general solution to Equation 3.6 is any function

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} + F(\boldsymbol{r}, \boldsymbol{r}'), \tag{3.7}$$

where $F(\boldsymbol{r}, \boldsymbol{r}')$ solves

$$\nabla'^2 F(\boldsymbol{r}, \boldsymbol{r}') = 0.$$

Proposition 3.3. *The solution of Equation 3.4 in a region Ω can be written as*

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_\Omega G(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V' + \frac{1}{4\pi}\int_{\partial\Omega}\left(G(\boldsymbol{r},\boldsymbol{r}')\frac{\partial\phi(\boldsymbol{r}')}{\partial n'} - \phi(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n'}\right)\mathrm{d}A'. \tag{3.8}$$

Proof. By the properties of the divergence (similar to the argument in the proof of Theorem 3.1), we have that

$$\phi(\boldsymbol{r}')\nabla'^2 G(\boldsymbol{r},\boldsymbol{r}') - G(\boldsymbol{r},\boldsymbol{r}')\nabla'^2\phi(\boldsymbol{r}') = \nabla'\cdot\left(\phi(\boldsymbol{r}')\nabla' G(\boldsymbol{r},\boldsymbol{r}') - G(\boldsymbol{r},\boldsymbol{r}')\nabla'\phi(\boldsymbol{r}')\right).$$

Integrating over Ω and applying the divergence theorem on the right hand side we obtain

$$\int_\Omega\left(\phi(\boldsymbol{r}')\nabla'^2 G(\boldsymbol{r},\boldsymbol{r}') - G(\boldsymbol{r},\boldsymbol{r}')\nabla'^2\phi(\boldsymbol{r}')\right)\mathrm{d}V' = \int_{\partial\Omega}\left(\phi(\boldsymbol{r}')\nabla' G(\boldsymbol{r},\boldsymbol{r}') - G(\boldsymbol{r},\boldsymbol{r}')\nabla'\phi(\boldsymbol{r}')\right)\cdot\boldsymbol{n}'\,\mathrm{d}A'.$$

By Equation 3.6 and Equation 3.4 we can rewrite the left hand side as

$$\int_\Omega\left(\phi(\boldsymbol{r}')\nabla'^2 G(\boldsymbol{r},\boldsymbol{r}') - G(\boldsymbol{r},\boldsymbol{r}')\nabla'^2\phi(\boldsymbol{r}')\right)\mathrm{d}V' = -4\pi\phi(\boldsymbol{r}) + \frac{1}{\epsilon_0}\int_\Omega G(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V'.$$

After some manipulation, we conclude

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_\Omega G(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V' + \frac{1}{4\pi}\int_{\partial\Omega}\left(G(\boldsymbol{r},\boldsymbol{r}')\frac{\partial\phi(\boldsymbol{r}')}{\partial n'} - \phi(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n'}\right)\mathrm{d}A',$$

where, for convenience, we have rewritten $\nabla' G(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{n}$ as $\frac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n'}$ and similarly for ϕ. □

This result holds for any choice of the Green's function which, remember, is not uniquely defined. Therefore we will try to make a suitable choice that simplifies Equation 3.8. For example, note that the second integral requires both ϕ and $\frac{\partial\phi}{\partial n'}$. If we have Dirichlet boundary conditions we will have the former but not the latter, and viceversa for Neumann boundary conditions. Therefore our choice of G will be aimed at eliminating the term for which we do not have information.

Definition 3.5. For Dirichlet boundary conditions, it is convenient to choose the Green's function $G_D(\boldsymbol{r}, \boldsymbol{r}')$ to satisfy homogeneous Dirichlet boundary conditions, i.e.

$$G_D(\boldsymbol{r}, \boldsymbol{r}') = 0, \quad \text{for } \boldsymbol{r}' \in \partial\Omega, \ \boldsymbol{r} \in \Omega.$$

Then, Equation 3.8 reduces to

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_\Omega G_D(\boldsymbol{r}, \boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V' - \frac{1}{4\pi} \int_{\partial\Omega} \phi(\boldsymbol{r}') \frac{\partial G_D(\boldsymbol{r}, \boldsymbol{r}')}{\partial n'} \mathrm{d}A'.$$

Definition 3.6. For Neumann boundary conditions, it is convenient to choose the Green's function $G_N(\boldsymbol{r}, \boldsymbol{r}')$ to satisfy

$$\frac{\partial G_D}{\partial n'} = -\frac{4\pi}{A}, \quad \text{for } \boldsymbol{r}' \in \partial\Omega, \ \boldsymbol{r} \in \Omega,$$

where $A = \int_{\partial\Omega} \mathrm{d}A'$ is the area of $\partial\Omega$. The reason for this choice is that the surface integral of $\frac{\partial G_D}{\partial n'}$ needs to be 4π, otherwise Equation 3.6 would not be satisfied.

Then, Equation 3.8 reduces to

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_\Omega G_N(\boldsymbol{r}, \boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V' + \frac{1}{4\pi} \int_{\partial\Omega} G(\boldsymbol{r}, \boldsymbol{r}') \frac{\partial \phi(\boldsymbol{r}')}{\partial n'} \mathrm{d}A' + \langle\phi\rangle,$$

where $\langle\phi\rangle = \frac{1}{A}\int_{\partial\Omega} \phi(\boldsymbol{r}')\mathrm{d}A'$ is the average of ϕ over the surface. But remember from Theorem 3.1 that for Neumann boundary conditions the solution is defined up to a constant, so we can arbitrarily set the value $\langle\phi\rangle$. Setting $\langle\phi\rangle = 0$ is usually a very reasonable choice.

You may be wondering whether we can always find the Green's functions for the Dirichlet and Neumann case. This requires solving the Laplace's equation with the suitable boundary conditions to determine $F(\boldsymbol{r}, \boldsymbol{r}')$ in Equation 3.7. This is possible as long as $\partial\Omega$ is "nice enough".[7] We may not be able to write such solutions explicitly, unfortunately, but even in these cases, writing the solution as in Proposition 3.3 is useful to understand its behaviour.

[7] In this module it will be "nice enough" unless otherwise stated. For example, $\partial\Omega$ being smooth is "nice enough".

3.5 Method of images

If Green's functions allowed us to write down the solution for pretty much any boundary value problem, albeit usually not explicitly, the next method we will introduce is the opposite: it is a clever trick that will allow us to find explicit solutions to some simple (yet common and useful) charge configurations. This is called the *method of images*.

Let's go back to the discrete case: imagine we have a set of charges q_i at positions $\boldsymbol{r}_i \in \Omega$, for $i = 1 \dots N$. We can use superposition to compute the potential of the set from Equation 2.4, but this potential will (most likely) not satisfy the boundary conditions we would like to impose. The method of images consists on adding a set of *virtual charges* outside the domain, such that the combined effect of the actual and virtual charges satisfies the required boundary condition.

This is much more easily explained through an example.

Example 3.4 (Charged particle near a conducting plane)**.** Consider a grounded conductor that fills the left half of the space, i.e. we have $\phi = 0$ for $x \leq 0$. We place a charge q at a point $(d, 0, 0)$, and we want to know what is the induced electrostatic potential for $x > 0$.

If the conductor was not there, we already know that the potential would be

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{(x-d)^2 + y^2 + z^2}},$$

but this clearly does not satisfy the boundary condition at $x = 0$ as

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{d^2 + y^2 + z^2}} \neq 0.$$

The clever trick is to assume the conductor is not there and, instead, place a charge $-q$ at $(-d, 0, 0)$. Then, the potential of the pair of charges is

$$\phi = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{\sqrt{(x-d)^2 + y^2 + z^2}} - \frac{q}{\sqrt{(x+d)^2 + y^2 + z^2}} \right),$$

and we can easily check that it satisfies $\phi = 0$ at $x = 0$. Now, of course this potential does not describe the actual potential of the conductor, but if we constrain ourselves to $x > 0$ this potential works just fine. That has a quick solution though, just define a piecewise potential that is zero if $x \leq 0$ and is the potential above otherwise.

This solution is extremely handy, and we can now use it to better understand how a charge behaves in the vicinity of a conductor. The first thing we can do is compute the electric field:

$$\boldsymbol{E} = -\nabla\phi = \frac{q}{4\pi\epsilon_0}\left(\frac{(x-d,y,z)}{((x-d)^2+y^2+z^2)^{\frac{3}{2}}} - \frac{(x+d,y,z)}{((x+d)^2+y^2+z^2)^{\frac{3}{2}}}\right), \quad \text{for } x>0.$$

Now let's calculate the surface charge distribution. We need to use Proposition 3.2. As our surface is the $y-z$ plane, the normal vector is $\boldsymbol{n} = \boldsymbol{e}_x$, so we only need the x-component of the electric field E_x. On the left of the boundary we have $E_x^-|_{x=0} = 0$. On the right of the boundary we get from the expression above

$$E_x^+\Big|_{x=0} = -\frac{q}{2\pi\epsilon_0}\frac{d}{(d^2+y^2+z^2)^{\frac{3}{2}}}.$$

Therefore, using Equation 3.2 we get

$$\sigma = \epsilon_0\, E_x^+\Big|_{x=0} = -\frac{q}{2\pi}\frac{d}{(d^2+y^2+z^2)^{\frac{3}{2}}}.$$

Finally, let's compute the force that the charge will experience from the conductor. We can again use the potential induced by our virtual charge to write

$$\boldsymbol{F} = \frac{-q^2}{16\pi\epsilon_0 d^2}\boldsymbol{e}_x.$$

Therefore, the (actual) charge experiences an attractive force towards the conductor.

> Warning
>
> Add diagram to example above.

4 Magnetostatics

So far we have considered situations where charges are static, so the next step is to consider what happens when charges move around. However, when charges move around, they create magnetic fields! This can create some quite interesting (and messy) interactions between electric and magnetic fields. We will unpick this little by little starting with magnetostatics, that is the study of magnetic fields that do not depend on time.

4.1 Electric current & conservation of charge

Consider a point charge. It's pretty straightforward to picture this charge moving around the space with a velocity $\boldsymbol{v}(t)$. Let's now think what would happen when we have infinitely many charges, so we consider a charge density $\rho(\boldsymbol{r})$. Similarly, we can define a velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$, which gives us the average[1] drift velocity of a charge at point $\boldsymbol{r}$ and time t. In this chapter we focus on time independent problems, so we drop time, i.e. $\boldsymbol{v}(\boldsymbol{r})$. This means that the charges can still move around, but the velocity does not change in time (though it can change in space). In short, the charge at a position $\boldsymbol{r}$ will always have the same velocity $\boldsymbol{v}(\boldsymbol{r})$.

Definition 4.1 (Current density)**.** Given a distribution of charge with density ρ and a velocity (vector) field $\boldsymbol{v}$, the electric current density $\boldsymbol{J}$ is defined as

$$\boldsymbol{J} = \rho \boldsymbol{v}.$$

If you stop to consider what are the dimensions of the current density, you will find that they are charge per unit time and unit area[2]. It is therefore a flux, that is (in very rigorous terms) *amount of stuff through a surface per unit time*, where in this case *stuff* is charge.

It is convenient to define electric current (or current, for short).

[1]Real charges (e.g. electrons) will have some random motion, but they can still have an overall velocity, for example, induced by an electric field. We ignore (i.e. average out) the random motion and focus only on the overall *drift* velocity.

[2]In SI units we have $\mathrm{A\ m^{-2}}$, where $\mathrm{A} = \mathrm{C\ s^{-1}}$ are amperes (name after Mr Ampère, who we will meet in a few pages).

Definition 4.2 (Electric current)**.** The electric current I through a surface Σ is given by

$$I = \int_{\Sigma} \boldsymbol{J} \cdot \boldsymbol{n} \, \mathrm{d}A.$$

In SI units, current is measured in amperes (A).

The physical interpretation of the current is the amount of charge per unit time flowing through the surface Σ. Therefore, current is not a vector quantity but a scalar. Depending on which way we define the normal vector $\boldsymbol{n}$, we may have positive or negative current. This arbitrarity is fine: we basically define a direction for current and, if it ends up being negative, it means charge is flowing in the opposite direction.

You are probably familiar with the idea of current when you think of electric cables. Charges inside might be doing all sorts of things, but what matters in the end is how much charge gets through the cross-section of the cable, i.e. the current.

A very important property of electric charge is that it is conserved. Like with mass, this means that charge cannot be created nor destroyed. The idea of conservation of *stuff* underpins many areas of physics[3] and, luckily for us, we have a very convenient way of representing it mathematically.

The equation that encapsulates conservation of *stuff* is called the continuity equation. Even though it is a very general equation, here we will state it and derive it in the context of electric fields.[4]

Theorem 4.1 (Continuity equation)**.** *Given a charge density ρ with a current density $\boldsymbol{J}$, it must satisfy*

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \boldsymbol{J} = 0. \tag{4.1}$$

Proof. Consider a bounded region $\Omega \in \mathbb{R}^3$, bounded by the surface $\partial\Omega$. The total charge in Ω is given by

$$Q = \int_{\Omega} \rho \, \mathrm{d}V.$$

Because charge is conserved, the only way by which Q can change is either by charge coming in or charge coming out through the boundary $\partial\Omega$. This is defined by the current density $\boldsymbol{J}$, so current through $\partial\Omega$ is equal to minus the rate of change of Q:

[3] You will encounter it in quite a few modules, e.g. MA3D1.
[4] In MA3D1 you will derive the generic form.

$$\int_{\partial\Omega} \boldsymbol{J} \cdot \boldsymbol{n} \, \mathrm{d}A = -\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_{\Omega} \frac{\partial \rho}{\partial t} \, \mathrm{d}V. \tag{4.2}$$

The minus sign is because we define the normal vector $\boldsymbol{n}$ to point outwards of Ω, therefore, if $\boldsymbol{J} \cdot \boldsymbol{n}$ is positive, it means charge is leaving Ω and therefore Q should decrease.

Applying the divergence theorem to the left-hand side of Equation 4.2 we obtain

$$\int_{\Omega} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} \right) \mathrm{d}V = 0,$$

and given that Ω is arbitrarily defined we must have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0.$$

□

In magnetostatics we have $\frac{\partial \rho}{\partial t} = 0$, therefore the continuity equation reduces to

$$\nabla \cdot \boldsymbol{J} = 0. \tag{4.3}$$

Currents satisfying this property are called *steady currents*.

4.2 Lorentz force & magnetic field

When charges move around, they generate a magnetic field, which we will denote with $\boldsymbol{B}(\boldsymbol{r}, t)$ (though in this chapter we concert only with static problems, i.e. $\boldsymbol{B}(\boldsymbol{r})$). The magnetic field, in turn, exerts a force over electric charges. In SI units the magnetic field is measure in *teslas* (T)[5], which correspond to $\mathrm{N\,s\,m^{-1}\,C^{-1}}$.

Definition 4.3 (Lorentz force law)**.** Given a point charge q under the effects of an electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. The charge moves with a velocity $\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$, where $\boldsymbol{r}(t)$ is the position of the charge at a time t. Then, the charge experiences a force

$$\boldsymbol{F}(\boldsymbol{r}, t) = q\boldsymbol{E}(\boldsymbol{r}, t) + q\boldsymbol{u}(t) \times \boldsymbol{B}(\boldsymbol{r}, t).$$

[5] Named after Nikola Tesla, not after a certain famous car brand...

You may have noticed we haven't given a "proper" definition of the magnetic field. One may regard the Lorentz force law as the implicit definition of a magnetic field.

Note that when the charge is static (i.e. $\boldsymbol{u} \equiv \mathbf{0}$), the Lorentz force law gives

$$\boldsymbol{F} = q\boldsymbol{E},$$

as expected from electrostatics. Note as well that the component related to the magnetic field actually does no work on the particle. We can rewrite the definition of work (Equation 2.16) as

$$W = -\int_0^t \boldsymbol{F} \cdot \boldsymbol{u} \, \mathrm{d}\tilde{t},$$

which represents the work on the charge moving with velocity $\boldsymbol{u}$ during the time interval $[0, t]$. Then,

$$W = -\int_0^t q\boldsymbol{E} \cdot \boldsymbol{u} \, \mathrm{d}\tilde{t},$$

and the component related to the magnetic field vanishes because $\boldsymbol{u}(t) \times \boldsymbol{B}(\boldsymbol{r}, t)$ is, by the properties of the cross product, perpendicular to $\boldsymbol{u}(t)$. Therefore, its dot product with $\boldsymbol{u}(t)$ is equal to zero.

4.3 Biot-Savart law

In the definition of the Lorentz force we have introduced the concept of magnetic field, but we have not yet given a mathematical definition like we did for the electric field Definition 2.3. This is the goal of this section.

Remember that the electric field can be interpreted as the force that a set of *static* charges exerts on a unit test charge. Similarly, the magnetic field is the force that a set of *moving* charges exerts on a unit test charge. Its mathematical expression is given by the Biot-Savart law

Definition 4.4 (Biot-Savart law)**.** A charge q at position $\boldsymbol{r}_0$ and moving with a velocity $\boldsymbol{v}$ produces a magnetic field[6]

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 q}{4\pi} \frac{\boldsymbol{v} \times (\boldsymbol{r} - \boldsymbol{r}_0)}{|\boldsymbol{r} - \boldsymbol{r}_0|^3},$$

[6] I am deliberately vague about whether the velocity can depend on time or not. It will be easier to discuss this when we reformulate the Biot-Savart law in terms of currents.

where μ_0 is a parameter called *permeability of free space*. Its value is $\mu_0 \approx 1.257 \times 10^{-6}\,\mathrm{N\,A^{-2}}$.

Note the similarities between Definition 2.3 and Definition 4.4.

The Principle of Superposition also applies to magnetic fields. Therefore, the magnetic field generated by a steady current density $\boldsymbol{J}$ defined in a domain Ω is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V'. \tag{4.4}$$

In the particular case of a steady current I through a wire C we can write

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int_C \frac{\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3},$$

where $\mathrm{d}\boldsymbol{r}'$ is the differential of length along the curve C.

Example 4.1 (Straight wire)**.** Let's consider a steady current I through an infinitely long straight wire. Wlog we will assume the wire is along the z axis (i.e. $x = y = 0$). Then, we can work in cylindrical coordinates (r, θ, z). In that case, the differential of length in the integral can be written as $\mathrm{d}\boldsymbol{r}' = \boldsymbol{e}_z\mathrm{d}z$. Then, we have

$$\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}') = r\mathrm{d}z\boldsymbol{e}_\theta,$$

i.e. the magnetic field will point in the angular direction. Then, from Definition 4.4 we have

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\boldsymbol{e}_\theta \int_{-\infty}^{\infty} \frac{r}{(r^2+z^2)^{\frac{3}{2}}}\mathrm{d}z = \frac{\mu_0 I}{2\pi r}\boldsymbol{e}_\theta.$$

Warning

Clarify example above and add sketch

4.4 Gauss' law for magnetism

We have now derived three of Maxwell's equation (at least their static versions); there is only Equation 1.2 left. This equation is often referred to as the Gauss' law for magnetism and, as the name suggests, it is very similar to Gauss' law for electric fields (Theorem 2.5).

Theorem 4.2 (Gauss' law for magnetism)**.** *Any magnetic field $\boldsymbol{B}$ satisfies*

$$\nabla \cdot \boldsymbol{B} = 0.$$

Proof. We will prove Gauss' law for magnetism for a point charge, but the proof for a current density would follow similarly, just with more complicated algebra. Wlog we set $\boldsymbol{r}_0 = \boldsymbol{0}$, so

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 q}{4\pi} \frac{\boldsymbol{v} \times \boldsymbol{r}}{|\boldsymbol{r}|^3}$$

First, note that

$$\nabla \cdot \left(\frac{\boldsymbol{v} \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \right) = \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \cdot (\nabla \times \boldsymbol{v}) - \boldsymbol{v} \cdot \left(\nabla \times \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \right)$$

using `CROSSREF`, but as $\boldsymbol{v}$ does not depend on $\boldsymbol{r}$ the first term vanished. Recall from `CROSSREF` that

$$\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} = -\nabla \frac{1}{|\boldsymbol{r}|},$$

therefore

$$\nabla \cdot \left(\frac{\boldsymbol{v} \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \right) = 0,$$

as the curl of a gradient is zero (`CROSSREF`). This shows that $\nabla \cdot \boldsymbol{B} = 0$ everywhere except at $\boldsymbol{r} = \boldsymbol{0}$ due to the singularity in the definition of $\boldsymbol{B}$. To resolve it, we consider the ball of radius R centred at the origin, B_R. Then

$$\int_{\partial B_R} \boldsymbol{B} \cdot \boldsymbol{n} \, \mathrm{d}A = \frac{\mu_0 q}{4\pi} \int_{\partial B_R} \left(\frac{\boldsymbol{v} \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \right) \cdot \boldsymbol{n} \, \mathrm{d}A = 0,$$

where in the final step we have used that $\boldsymbol{v} \times \boldsymbol{r}$ is perpendicular to the normal to the ball surface (which is in the radial direction). Then, by the divergence theorem, we find

$$\int_{B_R} \nabla \cdot \boldsymbol{B} \, \mathrm{d}V = 0,$$

and thus we conclude that $\nabla \cdot \boldsymbol{B} = 0$ everywhere. □

4.5 Ampère's law

So far we have defined the magnetic field for a given charge and for a current in a wire. However, we would like a more generic equation that, given an electric current, tells us the magnetic field it induces. This is precisely what Ampère's law[7] does.

Theorem 4.3 (Ampère's law)**.** *The magnetic field $\boldsymbol{B}$ induced by a steady electric current density $\boldsymbol{J}$ satisfies*

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

Note that this is equivalent to Equation 1.4 if the electric field $\boldsymbol{E}$ is steady.

Historically, this was derived by Ampère (and others) using experimental and theoretical tools. However, we can derive it from the Biot-Savart law (Equation 4.4). But before we need to introduce a couple of lemmas.

Lemma 4.1. *The vector field*

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_\Omega \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' \tag{4.5}$$

satisfies $\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r})$.

Proof. Let's compute the derivative of the i-th component of $\boldsymbol{A}$ with respect to the j-th coordinate:

$$\frac{\partial A_i}{\partial x_j} = -\frac{\mu_0}{4\pi} \int_\Omega \frac{\boldsymbol{J}_i(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} (x_j - x_j') \mathrm{d}V',$$

where we are using $\boldsymbol{r} = (x_1, x_2, x_3)$ and $\boldsymbol{r}' = (x_1', x_2', x_3')$.

From `CROSSREF` we have

[7] Named after André-Marie Ampère (1775-1836), a French physicist and mathematician.

$$\nabla \times \boldsymbol{A} = \sum_{j=1}^{3} \boldsymbol{e}_j \times \frac{\partial \boldsymbol{A}}{\partial x_j} = -\sum_{i,j=1}^{3} (\boldsymbol{e}_i \times \boldsymbol{e}_j) \frac{\partial A_i}{\partial x_j}.$$

And we observe that, after some close manipulation, the last term is equal to $\boldsymbol{B}(\boldsymbol{r})$ from Equation 4.4. □

Lemma 4.2. *For steady currents $\boldsymbol{J}$ supported inside a bounded region Ω we have $\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0$.*

Proof. First, note that if $\boldsymbol{r} \neq \boldsymbol{r}'$ we can write

$$\nabla \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) = -\nabla' \left(|\boldsymbol{r} - \boldsymbol{r}'| \right),$$

where ∇' is the gradient with respect to the $\boldsymbol{r}'$ variable.

From Equation 4.5

$$\begin{aligned}
\nabla \cdot \boldsymbol{A} &= \frac{\mu_0}{4\pi} \int_\Omega \boldsymbol{J}(\boldsymbol{r}') \cdot \nabla \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi} \int_\Omega \boldsymbol{J}(\boldsymbol{r}') \cdot \nabla' \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi} \int_\Omega \nabla' \left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \mathrm{d}V' + \frac{\mu_0}{4\pi} \int_\Omega \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \left(\nabla' \cdot \boldsymbol{J}(\boldsymbol{r}') \right) \mathrm{d}V'.
\end{aligned}$$

The last term vanishes as it satisfies Equation 4.3. For the remaining term, we can use divergence theorem:

$$-\frac{\mu_0}{4\pi} \int_\Omega \nabla' \left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \mathrm{d}V' = -\frac{\mu_0}{4\pi} \int_{\partial\Omega} \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \cdot \boldsymbol{n} \, \mathrm{d}A'.$$

But, by definition, $\boldsymbol{J}$ is supported inside Ω, therefore the boundary integral vanishes and we conclude

$$\nabla \cdot \boldsymbol{A} = 0.$$

□

of Ampère's law. From Lemma 4.1 we have

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}).$$

Using the identity `CROSSREF`, which applies to any vector field $\boldsymbol{A}$, we obtain

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}.$$

Applying Lemma 4.2 we deduce

$$\nabla \times \boldsymbol{B} = -\nabla^2 \boldsymbol{A}.$$

Now we only need to compute $\nabla^2 \boldsymbol{A}$ from Equation 4.5:

$$\nabla^2 \boldsymbol{A} = \frac{\mu_0}{4\pi} \int_\Omega \boldsymbol{J}(\boldsymbol{r}')\nabla^2 \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) \mathrm{d}V' = \frac{\mu_0}{4\pi} \int_\Omega \boldsymbol{J}(\boldsymbol{r}') \left(-4\pi\delta(\boldsymbol{r} - \boldsymbol{r}')\right) \mathrm{d}V' = -\mu_0 \boldsymbol{J}(\boldsymbol{r}).$$

We have used `CROSSREF` in the second step.

Combining everything we conclude

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{4.6}$$

which is Ampère's law (Theorem 4.3). □

Corollary 4.1 (Ampère's law - alternative formulation)**.** *For any simple*[8] *closed curve* $C = \partial\Sigma$ *bounding a surface* Σ

$$\int_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \mu_0 \int_\Sigma \boldsymbol{J} \cdot \boldsymbol{n} \mathrm{d}A = \mu_0 I,$$

where I *is the current through* Σ*.*

[8] i.e. that it doesn't intersect itself.

Example 4.2 (Straight wire – 2)**.** Let's consider again the current through a straight wire (Example 4.1), but now let's approach it using Corollary 4.1. If we look at the symmetries of the problem, we notice it is invariant under vertical translations and rotations around the z-axis, so we can conclude that the magnetic field only depends on the radial direction. Let's further assume that the only non-zero component is on the angular direction:

$$\boldsymbol{B} = B(r)\boldsymbol{e}_\theta.$$

For our curve C we choose a circle of arbitrary radius r in the $x - y$ plane with the centre on the z axis. Corollary 4.1 gives

$$\int_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = B(r) \int_0^{2\pi} r \mathrm{d}\theta = 2\pi R B(r) = \mu_0 I,$$

therefore we conclude that

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi r}\boldsymbol{e}_\theta.$$

4.6 Magnetostatic vector potential

In the derivation of Ampère's law we introduced the vector field $\boldsymbol{A}$, purely for convenience. It turns out though, that there is a lot more to it.

Definition 4.5 (Magnetostatic vector potential)**.** The *vector potential* of a static magnetic field $\boldsymbol{B}$ is defined as vector field $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

Note the parallelisms (and differences) with the electrostatic potential in Section 2.5. There, we had a scalar potential ϕ from which we computed the electric field as $\boldsymbol{E} = -\nabla\phi$. Now, we have a vector potential[9] $\boldsymbol{A}$ from which we compute the magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

Note as well that by defining the magnetic field from the vector potential it automatically satisfies $\nabla \cdot \boldsymbol{B} = 0$, as the divergence of a curl is always zero. The implication works both

[9] In some books you may encounter a *magnetic scalar potential*, which may confuse you. This scalar potential is used as a calculation technique in some very specific problems, and we will not use it in this module. If you want to learn more, there is a very good discussion in section 3.2.1 of David Tong's book.

ways and any field $\boldsymbol{B}$ that can be written as the curl of a potential satisfies $\nabla \cdot \boldsymbol{B} = 0$. CROSSREF

The choice for $\boldsymbol{A}$ is far from unique, similarly to the definition of the electric potential. In this case, the argument is a bit more subtle, as we can not only add constants, but also vector fields of the form $\nabla\chi$ (for a given function $\chi(\boldsymbol{x})$):

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi \implies \nabla \times \boldsymbol{A}' = \nabla \times \boldsymbol{A} + \nabla \times (\nabla\chi) = \nabla \times \boldsymbol{A}.$$

Definition 4.6 (Gauge transformation)**.** The transformation

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$$

for any given function χ is called a *gauge transformation*.

Proposition 4.1. *We can always find a gauge transformation χ such that $\boldsymbol{A}'$ satisfies*

$$\nabla \cdot \boldsymbol{A}' = 0.$$

Such transformation is called Coulomb gauge*.*

Proof. Suppose that we have a vector field $\boldsymbol{A}$ already satisfying $\nabla \times \boldsymbol{A} = \boldsymbol{B}$, but with $\nabla \cdot \boldsymbol{A} = \psi(\boldsymbol{x})$.

Consider the transformed vector field $\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$, taking the divergence we obtain

$$\nabla \cdot \boldsymbol{A}' = \nabla \cdot \boldsymbol{A} + \nabla^2\chi = \psi + \nabla^2\chi.$$

Then, to make $\nabla \cdot \boldsymbol{A}' = 0$ we simply need to choose χ such that

$$\nabla^2\chi = -\psi.$$

This is the Poisson equation, and we know that it has a unique solution (up to an additive constant). $\square$

There is a lot to unpack from gauges when one goes into quantum physics (in fact, there is a class of quantum field theories called gauge theories). However, in this module we will use gauges purely as a mathematical tool: choosing the *right* gauge allows us to write equations in a simpler form. For example, Ampère's law for the vector field yields

$$\mu_0\boldsymbol{J} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A},$$

and taking the Coulomb gauge allows us to simplify the equation to

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J},$$

which is the Poisson's equation and we know how to solve it.

Example 4.3 (Straight wire – 3)**.** Let's consider yet again the current through a straight wire (Example 4.1). Now we want to compute the vector potential $\boldsymbol{A}$. We found out in the previous examples that

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi r} \boldsymbol{e}_\theta.$$

As $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, we can write the following system of equations for the components of $\boldsymbol{A}$:

$$\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} = 0,$$

$$\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r},$$

$$\frac{1}{r}\left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right) = 0.$$

It is not straightforward how to solve this equation, but we can construct a suitable solution (remember, it will not be unique) by taking

$$\boldsymbol{A} = -\frac{\mu_0 I}{2\pi} \log\left(\frac{r}{r_0}\right) \boldsymbol{e}_z,$$

where r_0 is an integrating constant.

5 Applications of magnetostatics

Having introduced the basic concepts for magnetostatics, let's now put them into practice to better understand magnetism.

5.1 Magnetic monopoles

Let's start by looking more carefully at the Gauss' law for magnetic fields:

$$\nabla \cdot \boldsymbol{B} = 0.$$

What this equation tells us, physically, is that there is no such thing as a *magnetic point charge* (more commonly known as a *monopole*). To see this, let's compare it with Gauss' law for electric fields:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}.$$

In this case, the point charges (or *electric monopoles*, to continue the parallelism) are the sources and sinks of electric field. Therefore, going back to the magnetic field equation, we conclude magnetic monopoles can't exist.

Mathematically it might still make sense to allow for magnetic monopoles, and from a physical point of view they also are a useful tool to understand some effects (especially going down to quantum mechanics). However, no one has ever been able to observe a magnetic monopole in real life, so one should not worry very much about them.

5.2 Magnetic dipoles

Now let's talk about magnetic dipoles. Consider the magnetic vector potential generated by a current I flowing through a small circle C. From Equation 4.5, we have

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi} \int_C \frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

We now want to see what happens when we make this circle infinitely small. Wlog we take the C to be centred at the origin and lying on the (x, y)-plane. We can then parameterise it as $\boldsymbol{r}' = (R\cos\theta', R\sin\theta', 0)$ for $\theta' \in [0, 2\pi)$, where $R > 0$ is the radius of the circle. If we take a Taylor expansion in the limit of $R \ll 1$, we can write the integrand as

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{\sqrt{(x-R\cos\theta')^2+(y-R\sin\theta')^2+z^2}} = \frac{1}{\sqrt{x^2+y^2+z^2}}+R\frac{x\cos\theta'+y\sin\theta'}{(x^2+y^2+z^2)^{\frac{3}{2}}}+O(R^2).$$

Using that $\mathrm{d}\boldsymbol{r}' = (-R\sin\theta', -R\cos\theta', 0)\mathrm{d}\theta'$, we can write the vector potential as

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu_0 I}{4\pi}\int_C \left(\frac{1}{\sqrt{(x-R\cos\theta')^2+(y-R\sin\theta')^2+z^2}}\right)\mathrm{d}\boldsymbol{r}' \\
&= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}+R\frac{x\cos\theta'+y\sin\theta'}{(x^2+y^2+z^2)^{\frac{3}{2}}}+O(R^2)\right](-R\sin\theta', -R\cos\theta', 0)\mathrm{d}\theta' \\
&= \frac{\mu_0 I}{4\pi r^3}\left[\pi R^2(-y, x, 0)+O(R^2)\right] \\
&= \frac{\mu_0 I}{4\pi r^3}\left[\pi R^2\boldsymbol{e}_z\times\boldsymbol{r}+O(R^2)\right]
\end{aligned} \tag{5.1}$$

The first term in the integral vanishes, as the integral is with respect to the tilde variables. Therefore, we are effectively taking the integral of a constant over a closed curve. In the last line we have just rewritten the result as a cross product for convenience.

Continuing with rewriting things in a convenient way, let's introduce the *magnetic dipole moment*:

$$\boldsymbol{m} = I\pi R^2\boldsymbol{n},$$

where $\boldsymbol{n}$ is the vector normal to the surface enclosed by C (in our calculations above we have $\boldsymbol{n} = \boldsymbol{e}_z$ due to our choice of coordinates). Now let's take the limit $\boldsymbol{R} \to 0$ while taking $I \to \infty$ in such a way that $I\pi R^2$ remains constant. THen, as the higher order terms vanish, the potential Equation 5.1 can be written as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{|\boldsymbol{r}|^3}.$$

Calculating the magnetic field of the dipole we obtain, after some algebra,

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla\times\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(-\frac{\boldsymbol{m}}{|\boldsymbol{r}|^3}+\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r}}{|\boldsymbol{r}|^5}\right). \tag{5.2}$$

5.3 Magnetic forces

Now let's turn our attention to magnetic forces. We will consider two cases: forces between currents and forces between dipoles.

5.3.1 Force between currents

We first consider the force between two current distributions $\boldsymbol{J}_1(\boldsymbol{r})$ and $\boldsymbol{J}_2(\boldsymbol{r})$ defined in Ω_1 and Ω_2, respectively. Take the magnetic field produced by $\boldsymbol{J}_1$, from the Biot-Savart law (Definition 4.4) we have

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\Omega_1} \frac{\boldsymbol{J}_1(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V'.$$

I we restrict ourselves to the case where we have a current I_1 through a curve (wire) C_1, we can write it as

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I_1}{4\pi}\int_{C_1} \frac{\mathrm{d}\boldsymbol{r}_1 \times (\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3}.$$

Now let's bring in the current distribution $\boldsymbol{J}_2$, which will experience a force due to $\boldsymbol{J}_1$ of

$$\boldsymbol{F} = \int_{\Omega_2} \boldsymbol{J}_2(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r})\mathrm{d}V,$$

or, if we again restrict to the case where we have a current I_2 through a curve C_2,

$$\boldsymbol{F} = I_2\int_{C_2} \mathrm{d}\boldsymbol{r} \times \boldsymbol{B}(\boldsymbol{r}).$$

Combining the two results we obtain

$$\boldsymbol{F} = \frac{\mu_0}{4\pi} I_1 I_2 \int_{C_1}\int_{C_2} \mathrm{d}\boldsymbol{r}_2 \times \left(\mathrm{d}\boldsymbol{r}_1 \times \frac{\boldsymbol{r}_2-\boldsymbol{r}_1}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^3}\right). \tag{5.3}$$

Notice the symmetry of the expression: if we were to compute the force on $\boldsymbol{J}_1$ produced by $\boldsymbol{J}_2$ we would reach the same result.

Example 5.1 (Two straight wires)**.** Consider the case were we have to infinite, straight, parallel wires a distance d from each other, carrying currents I_1 and I_2 respectively. We want the force applied to the second wire.

Wlog we can write $\boldsymbol{r}_1 = (0, 0, z_1)$ and $\boldsymbol{r}_2 = (d, 0, z_2)$, and derive that $\mathrm{d}\boldsymbol{r}_1 = \mathrm{d}z_1\boldsymbol{e}_z$ and $\mathrm{d}\boldsymbol{r}_2 = \mathrm{d}z_2\boldsymbol{e}_z$. Then, Equation 5.3 becomes

$$\boldsymbol{F} = \frac{\mu_0}{4\pi} I_1 I_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \boldsymbol{e}_z \times \left(\boldsymbol{e}_z \times \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3} \right) \mathrm{d}z_2 \mathrm{d}z_1.$$

We have $\boldsymbol{r}_2 - \boldsymbol{r}_1 = d\boldsymbol{e}_x + (z_2 - z_1)\mathrm{d}\boldsymbol{e}_z$, and using the properties of the cross product we obtain

$$\begin{aligned}
\boldsymbol{F} &= \frac{\mu_0}{4\pi} I_1 I_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-d\boldsymbol{e}_x}{(d^2 + (z_2 - z_1)^2)^{\frac{3}{2}}} \mathrm{d}z_2 \mathrm{d}z_1 \\
&= -\frac{\mu_0}{2\pi d} I_1 I_2 \int_{-\infty}^{\infty} \boldsymbol{e}_x \int_{-\infty}^{\infty} \mathrm{d}z_2.
\end{aligned}$$

The second integral is infinite, which makes sense as it represents the length of the second wire. But if we consider only the force per unit length on the wire, we get

$$\boldsymbol{f} = -\frac{\mu_0}{2\pi d} I_1 I_2 \int_{-\infty}^{\infty} \boldsymbol{e}_x.$$

If the currents have the same sign (i.e. $I_1 I_2 > 0$), then the force is attractive, while if the currents have opposite sign it is repulsive.

5.3.2 Force between dipoles

Let's now calculate the force that a given magnetic field $\boldsymbol{B}$ exerts on a dipole. We will use the same argument that we used in the definition of the dipole (Section 5.2), in which we took the current around an infinitely small circular wire. From the Lorentz force law (Definition 4.3), the force on the dipole is

$$\boldsymbol{F} = \int_C I \mathrm{d}\boldsymbol{r}' \times \boldsymbol{B}(\boldsymbol{r}').$$

Using the parameterisation $\boldsymbol{r}' = (R\cos\theta', R\sin\theta', 0)$ for $\theta' \in [0, 2\pi)$, and taking the Taylor expansion in the limit of small R we get

$$
\begin{aligned}
\boldsymbol{F} &= I \int_0^{2\pi} (-R\sin\theta', R\cos\theta', 0) \times \left[\boldsymbol{B}(\boldsymbol{0}) + \frac{\partial}{\partial x}\boldsymbol{B}(\boldsymbol{0}) R\cos\theta' + \frac{\partial}{\partial y}\boldsymbol{B}(\boldsymbol{0}) R\sin\theta' + O(R^2)\right] \mathrm{d}\theta' \\
&= I\pi R^2 \left[\boldsymbol{e}_y \times \frac{\partial}{\partial x}\boldsymbol{B}(\boldsymbol{0}) - \boldsymbol{e}_x \times \frac{\partial}{\partial y}\boldsymbol{B}(\boldsymbol{0}) + O(R^3)\right] \\
&= I\pi R^2 \left[\frac{\partial}{\partial x}B_z(\boldsymbol{0})\boldsymbol{e}_x + \frac{\partial}{\partial y}B_z(\boldsymbol{0})\boldsymbol{e}_y - \left(\frac{\partial}{\partial x}B_x(\boldsymbol{0}) + \frac{\partial}{\partial y}B_y(\boldsymbol{0}) + O(R^3)\right)\right] \\
&= I\pi R^2 \left[\nabla B_z(\boldsymbol{0}) - (\nabla\cdot\boldsymbol{B}(\boldsymbol{0}))\,\boldsymbol{e}_z + O(R^3)\right].
\end{aligned}
$$

Taking the limit $R \to 0$ while keeping the dipole moment $\boldsymbol{m} = I\pi R^2 \boldsymbol{e}_z$ constant we obtain

$$
\boldsymbol{F} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B}) - (\nabla\cdot\boldsymbol{B})\boldsymbol{m} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B}) = \nabla(\boldsymbol{m}\cdot\boldsymbol{B}), \tag{5.4}
$$

where in the final step we have used that $\nabla\cdot\boldsymbol{B} = 0$.

This force is conservative, and we can define the potential

$$
V_{\text{dipole}} = -\boldsymbol{m}\cdot\boldsymbol{B}, \qquad \Longrightarrow \qquad \boldsymbol{F} = -\nabla V_{\text{dipole}}.
$$

The torque around the origin (using `CROSSREF`) is

$$
\boldsymbol{\tau} = \int_C \boldsymbol{r}' \times (I\mathrm{d}\boldsymbol{r}' \times \boldsymbol{B}(\boldsymbol{r}')) = I\int_C \mathrm{d}\boldsymbol{r}'\,(\boldsymbol{r}'\cdot\boldsymbol{B}(\boldsymbol{r}')).
$$

By a similar argument as before, we can write

$$
\begin{aligned}
\boldsymbol{\tau} &= I \int_0^{2\pi} (-R\sin\theta', R\cos\theta', 0) \left[B_x(\boldsymbol{0}) R\cos\theta' + B_y(\boldsymbol{0}) R\sin\theta' + O(R^2)\right] \mathrm{d}theta' \\
&= I\pi R^2 \left[(-B_y(\boldsymbol{0}), B_x(\boldsymbol{0}), 0) + O(R)\right],
\end{aligned}
$$

and taking the limit

$$
\boldsymbol{\tau} = \boldsymbol{m}\times\boldsymbol{B}.
$$

Note that both for the force and the forque, the expressions are analogous to those of the electric dipole, but replacing $\boldsymbol{p}$ with $\boldsymbol{m}$ and $\boldsymbol{E}$ with $\boldsymbol{B}$. This also implies that magnetic dipoles will try to align with the magnetic field.

Example 5.2 (Force between dipoles)**.** Now let's consider the force between to dipoles with moment $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ separated by a distance $\boldsymbol{d}$ (with $|\boldsymbol{d}| = d$). The magnetic field produced by the first dipole is given by Equation 5.2. Then, using Equation 5.4 we obtain that the force on the second dipole is

$$\boldsymbol{F} = \frac{\mu_0}{4\pi} \nabla \left(-\frac{\boldsymbol{m}_1 \cdot \boldsymbol{m}_2}{d^3} + \frac{3(\boldsymbol{m}_1 \cdot \boldsymbol{d})(\boldsymbol{m}_2 \cdot \boldsymbol{d})}{d^5} \right) .$$

We can compute the explicit expression (note that, here, the gradient applies on the vector $\boldsymbol{d}$). After some careful[1] manipulation, we obtain

$$\boldsymbol{F} = \frac{3\mu_0}{4\pi d^4} \left((\boldsymbol{m}_1 \cdot \hat{\boldsymbol{d}})\boldsymbol{m}_2 + (\boldsymbol{m}_2 \cdot \hat{\boldsymbol{d}})\boldsymbol{m}_1 + (\boldsymbol{m}_1 \cdot \boldsymbol{m}_2)\hat{\boldsymbol{d}} - 5(\boldsymbol{m}_1 \cdot \hat{\boldsymbol{d}})(\boldsymbol{m}_2 \cdot \hat{\boldsymbol{d}})\hat{\boldsymbol{d}} \right) ,$$

where $\hat{\boldsymbol{d}} = \boldsymbol{d}/d$ is the unit vector pointing from $\boldsymbol{m}_1$ to $\boldsymbol{m}_2$.

5.4 Application: permanent magnets

All the discussion we have had up to now was about magnetic fields arising from electric currents. But the magnets we are familiar with are the kind of thing we buy on our holidays to stick on our fridges. So where are the currents that produce the magnetic field in those magnets?

The answer to this question involves quantum mechanics. Even though quantum mechanics is out of scope for this model, let's try to give some intuition about it.

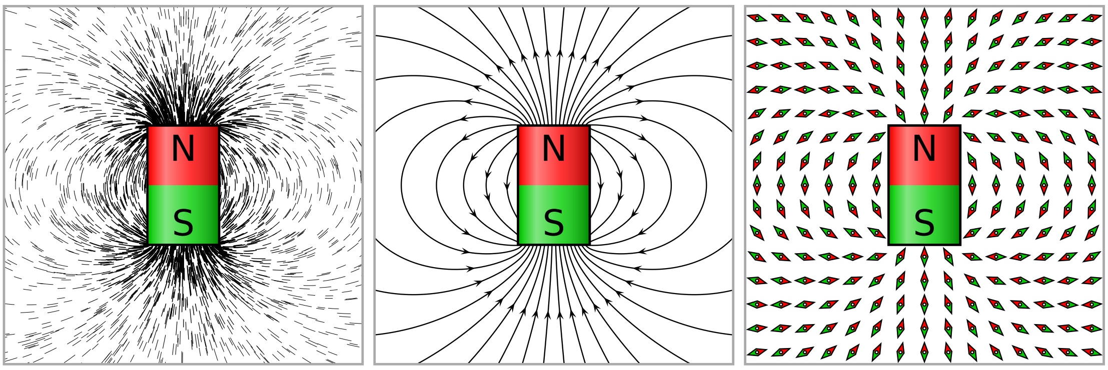

Figure 5.1: Diagram of the magnetic field produced by a bar (or straight) magnet. Left: iron fillings. Centre: field lines. Right: representation as compass needles. Geek3, CC BY-SA 4.0, via Wikimedia Commons.

[1] And painful!

The magnetism of such materials arises from the *spin* of the electrons. Electrons have an inherent angular momentum called spin. If you picture the electrons like tiny spheres you can imagine the spin in the same way as the Earth's spin[2]. It is this spin that produces a magnetic field. Typically, these magnetic fields point in all directions so their contributions basically cancel out at the macroscopic scale. However, when the spins of these many (many!) electrons align, the material produces a noticeable magnetic field.

[2]As you probably know electrons are a lot more complicated in real life, so this analogy only will take us thus far.

6 Electrodynamics

Up to now, we only considered static situations. We could already infer from the stationary Maxwell's equations that there are some connections between electric and magnetic fields: they are both produced by electric charges, the former by their existence, the latter by their motion.

However, the interaction between electric and magnetic fields is much deeper, and this is only apparent when we let them change in time. This is the focus of this chapter.

6.1 Faraday's Law of Induction

Let's first consider how Equation 2.15 is modified when we allow both $\boldsymbol{E}$ and $\boldsymbol{B}$ to change in time. This is given by the Faraday's law[1].

Theorem 6.1 (Faraday's law)**.** *The relationship between the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ is given by*

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Proof. This is not an actual proof, but we will show it for a particular case to get the idea. Generalising the proof is much trickier and out of the scope of this module.

Consider the electric and magnetic fields generated by a set of charges moving with constant velocity $\boldsymbol{v}$ (i.e. moving along a straight line with constant speed). By the Lorentz force law (Definition 4.3), an observer that is not moving will measure a following force over a charge q:

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}.$$

Now consider an observer moving with the same constant velocity $\boldsymbol{v}$ as the charges. To that observer, the charges will look static and thus will only produce an electric field $\boldsymbol{E}'$. They will measure a force over a charge q

[1] Named after Michael Faraday (1791-1867) the English chemist and physicist. He actually started the Royal Institution Christmas Lectures, which still carry on today. You can watch past lectures at the Royal Institution website

$$\boldsymbol{F}' = q\boldsymbol{E}'.$$

Now since we assume the two observer should be measuring the same force[2] we must have

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}.$$

For the second observer, the electric field is static, so Equation 2.15 applies. Then

$$\begin{aligned} \mathbf{0} = \nabla \times \boldsymbol{E}' &= \nabla \times \boldsymbol{E} + \nabla \times (\nabla \times \boldsymbol{B}) \\ &= \nabla \times \boldsymbol{E} + \boldsymbol{v}(\nabla \cdot \boldsymbol{B}) - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} \\ &= \nabla \times \boldsymbol{E} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B}, \end{aligned} \tag{6.1}$$

where in the second line we have used `CROSSREF` and in the third line we have applied Theorem 4.2.

For the first observer (the static one) all charges move with constant velocity $\boldsymbol{v}$, so the magnetic field at position $\boldsymbol{r} + \boldsymbol{v}\tau$ and time τ is the same as the magnetic field at position $\boldsymbol{r}$ and time t:

$$\boldsymbol{B}\left(\boldsymbol{r} + \boldsymbol{v}\tau, t + \tau\right) = \boldsymbol{B}(\boldsymbol{r}, t).$$

This is true for all τ, so we can divide by τ and take the limit $\tau \to 0$, obtaining[3]

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\,\boldsymbol{B} = \mathbf{0}.$$

Substituting this back into Equation 6.1 we obtain

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

□

Corollary 6.1. *For any simple closed curve $C = \partial\Sigma$ bounding a fixed surface Σ, the Faraday law can be rewritten as*

$$\int_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \boldsymbol{B} \cdot \boldsymbol{n}\,\mathrm{d}A.$$

[2] Things get trickier in the relativistic world.

[3] We are omitting the details here, but this is the definition of the total derivation, widely used in physics, which account for the intrinsic change of $\boldsymbol{B}$ in time (first term) and the change in time due to the motion in space (second term).

Let's now introduce a couple of useful definitions.

Definition 6.1 (Magnetic flux)**.** The integral $\Phi = \int_\Sigma \boldsymbol{B} \cdot \boldsymbol{n} \mathrm{d}A$ is called the magnetic flux through Σ.

Definition 6.2 (Electromotive force (emf))**.** The integral $\mathcal{E}_{\mathrm{emf}} = \int_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r}$ is called the electromotive force.[4]

Then, Faraday's law can be rewritten as

$$\mathcal{E}_{\mathrm{emf}} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}.$$

This is simply a change in notation, but you may encounter this form in books.

Faraday's law tells us that if we change the magnetic flux through Σ then we will induce a current. There are multiple ways of doing so physically. For example, one could change the magnetic field (e.g. by moving a magnet around) or maybe change the surface Σ (e.g. by moving the wires enclosing it). There is another effect though. When a current flows in a wire, it will create its own magnetic field that will oppose the change that has induced the current in the first place. This is called *Lenz's law*.

We can illustrate it as follows:

$$\text{change in } \boldsymbol{B} \xrightarrow{\text{Faraday}} \boldsymbol{E} \xrightarrow{\text{Lorentz}} \text{current} \xrightarrow{\text{Ampère}} \boldsymbol{B}.$$

⚠ Warning

Consider adding

- Magnetostatic energy
- Inductance
- Resistance

[4]Not confusingly at all, the electromotive force is not actually a force. It is the tangential component of the force per unit charge integrated along C.

6.2 Displacement current

The last step before being able to write the Maxwell's equations is to revise Ampère's law (Theorem 4.3) to account for time-dependent electric fields. To do this, let's start with its integral form (Corollary 4.1):

$$\int_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \mu_0 \int_\Sigma \boldsymbol{J} \cdot \boldsymbol{n} \mathrm{d}A.$$

Remember that C is a simple closed curve bounding Σ. But there are infinitely many surfaces bounded by C, so let's consider another surface Σ' such that $\partial\Sigma' = C$. Then

$$0 = \int_\Sigma \boldsymbol{J} \cdot \boldsymbol{n} \mathrm{d}A - \int_{\Sigma'} \boldsymbol{J} \cdot \boldsymbol{n} \mathrm{d}A = \int_S \boldsymbol{J} \cdot \boldsymbol{n} \mathrm{d}A, \tag{6.2}$$

where $S = \Sigma \cup \Sigma'$ is a closed surface. This tells us that the flux of $\boldsymbol{J}$ through any closed surface must be zero. Another way of looking at this is to consider the volume Ω enclosed by S (i.e. $\partial\Omega = S$). Then, we can write

$$\int_S \boldsymbol{J} \cdot \boldsymbol{n} \, \mathrm{d}A = \int_\Omega \nabla \cdot \boldsymbol{J} \mathrm{d}V = 0,$$

where we have applied the divergence theorem in the first equality and the steady current condition (Equation 4.3). But let's now drop the static assumption. In that case, we need to use Equation 4.1 instead, so

$$\begin{aligned}
\int_S \boldsymbol{J} \cdot \boldsymbol{n} \, \mathrm{d}A = \int_\Omega \nabla \cdot \boldsymbol{J} \mathrm{d}V &= -\int_\Omega \frac{\partial \rho}{\partial t} \mathrm{d}V \\
&= -\epsilon_0 \int_\Omega \frac{\partial}{\partial t} \left(\nabla \cdot \boldsymbol{E}\right) \mathrm{d}V \\
&= -\epsilon_0 \int_S \frac{\partial \boldsymbol{E}}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}A \\
&= -\epsilon_0 \int_\Sigma \frac{\partial \boldsymbol{E}}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}A + \epsilon_0 \int_{\Sigma'} \frac{\partial \boldsymbol{E}}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}A,
\end{aligned}$$

where we have used the continuity equation in the second line (Equation 4.1) and Gauss' law (Equation 2.13) in the third line, before applying the divergence theorem again and splitting the surface integral over S back into Σ and Σ'.

We conclude that

$$\int_\Sigma \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) \cdot \boldsymbol{n} \, \mathrm{d}A = \int_{\Sigma'} \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) \cdot \boldsymbol{n} \, \mathrm{d}A,$$

which suggests we can write Ampère's law Equation 4.6 as

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right).$$

This is one of the Maxwell's equation (Equation 1.4). This extra term that we picked up, $\partial \boldsymbol{E} \partial t$, is called the *displacement current*.

There is a quite interesting historical note about this equation. While the other three Maxwell's equations (and the static version of Ampère's law) were discovered mostly through experiments (even though mathematicians were still needed to write them as equations), the displacement current term Equation 1.4 was discovered purely by reasoning by Maxwell, and that's probably why all four equations are now named after him. Moreover, the displacement current *must* be included, otherwise the equations are not consistent, but we will discuss this in the next section.

6.3 Maxwell's equations

So we have finally derived Maxwell's equations in their general form. We have Gauss' law for electric fields

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{6.3}$$

and its equivalent for magnetic fields

$$\nabla \cdot \boldsymbol{B} = 0. \tag{6.4}$$

We have not said anything about these equations in the time-dependent case, but it turns out they still hold (we just let $\boldsymbol{E}$, $\boldsymbol{B}$ and ρ depend on time).

We have now introduced Faraday's law

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{6.5}$$

and modified Ampère's law to account for time-dependent fields

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right). \tag{6.6}$$

These four equations, jointly with the Lorentz force law

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right),$$

describe the whole of electromagnetism[5].

Let's took a closer look at the equations. The first two are scalar equations, while the latter two are vector equations. This means we have effectively 8 equations. Let's look at the unknowns now: each field has three components that we need to determine, so we have six unknowns in total. The system is overdetermined, so we should have two consistency conditions if we hope to get a solution.

First let's compute the time derivative of Equation 6.4,

$$0 = \frac{\partial}{\partial t}\left(\nabla \cdot \boldsymbol{B}\right) = \nabla \cdot \left(\frac{\partial \boldsymbol{B}}{\partial t}\right) = -\nabla \cdot \left(\nabla \times \boldsymbol{E}\right),$$

which makes it consistent with Equation 6.5.

The second condition is probably a lot more interesting. Let's now take the divergence of Equation 6.6, which yields

$$0 = \nabla \cdot \left(\frac{1}{\mu_0}\nabla \times \boldsymbol{B}\right) = \nabla \cdot \boldsymbol{J} + \epsilon_0 \frac{\partial}{\partial t}\left(\nabla \cdot \boldsymbol{E}\right) = \nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t},$$

where in the last step we have used Equation 6.3. Therefore, the continuity equation is encapsulated in the Maxwell's equations. This means that we can only get a solution to the Maxwell's equation if ρ and $\boldsymbol{J}$ satisfy the continuity equation (they would not be physical otherwise). This is why the displacement current is necessary in Equation 6.6.

6.4 Light

We are now in a position to study what is probably one of the most astonishing outcomes of the Maxwell's equations: the governing equation for light. The Maxwell's equations tell us that the electric and magnetic fields are closely coupled together, and that oscillations on one affect the other.

Our starting point will be the Maxwell's equations in vacuum, that is assuming that there are no charges (i.e. $\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$). To derive the equations governing the behaviour of light, let's start by taking the time derivative of Equation 6.6:

$$\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{\partial}{\partial t}\left(\nabla \times \boldsymbol{B}\right) = \nabla \times \frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left(\nabla \times \boldsymbol{E}\right),$$

[5] At least in the non-relativistic world.

where, remember, we are assuming $\boldsymbol{J} = \mathbf{0}$. Now let's use the identity

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla (\nabla \times \boldsymbol{E}) - \nabla^2 \boldsymbol{E}.$$

But by Equation 6.3 we conclude that the first term vanishes (remember, we assume $\rho = 0$), so we can write

$$\frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \nabla^2 \boldsymbol{E} = \mathbf{0},$$

where we defined

$$c \sqrt{\frac{1}{\mu_0 \epsilon_0}}.$$

Similarly, we can take the time derivative of Equation 6.5 to obtain

$$\frac{1}{c^2} \frac{\partial^2 \boldsymbol{B}}{\partial t^2} - \nabla^2 \boldsymbol{B} = \mathbf{0}.$$

Therefore, both the electric and magnetic fields satisfy the wave equation[6]. Let's look at the parameter c we defined. Recall the values for the constants

$$\epsilon_0 \approx 8.854 \times 10^{-12}\ \mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2} \quad \text{and} \quad \mu_0 \approx 1.257 \times 10^{-6}\ \mathrm{N}\,\mathrm{A}^{-2},$$

so the value of c is

$$c \approx 2.99 \times 10^8\ \mathrm{m}\,\mathrm{s}^{-1},$$

which is the speed of light!

Warning

Consider adding polarisation here

[6] If you are interested in learning more about waves, you should take MA301 Waves and Metamaterials

6.5 Electromagnetic potentials and gauge transformations

We also need to think on how the introduction of the time dependency affects the electric and magnetic potentials we introduced in Definition 2.4 and Definition 4.5.

Since Equation 6.4 still holds, we can define the magnetic potential $\boldsymbol{A}(\boldsymbol{r}, t)$ as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{6.7}$$

Using Equation 6.5 we obtain

$$0 = \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right).$$

Therefore, we can introduce a scalar potential $\phi(\boldsymbol{r}, t)$ such that

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \phi.$$

Then, the electric field can be in terms of both potentials as

$$\boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t}. \tag{6.8}$$

Definition 6.3 (Gauge transformation)**.** The transformations

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla \chi, \quad \text{and} \quad \phi \to \phi' = \phi - \frac{\partial \chi}{\partial t}$$

for any given function χ are called a *gauge transformations*. These can be seen as a generalisation of Definition 4.6, and they leave Equation 6.7 and Equation 6.8 invariant.

Definition 6.4 (Lorenz gauge)**.** The Lorenz[7] gauge for $\boldsymbol{A}$ and ϕ is the condition

$$\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0.$$

This is analogous to the Coulomb gauge (Proposition 4.1), and it will allow us to choose a form for χ so we can write Equation 6.7 and Equation 6.8 in a more convenient form.

[7] Do not confuse Mr Ludvig Lorenz (1829-1891) with Mr Hendrik Lorentz (1853-1928).

Theorem 6.2. *In Lorenz gauge, Maxwell's equations can be written as*

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{\rho}{\epsilon_0},$$

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}.$$

Proof. Taking Equation 6.3 and plugging in Equation 6.8,

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \boldsymbol{E} = \nabla \cdot \left(-\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}\right) = -\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t},$$

where in the last step we have used the Lorenz gauge.

Similarly, taking Equation 6.6 and substituting Equation 6.7, we obtain

$$\nabla \times \boldsymbol{B} = \nabla\left(\nabla \cdot \boldsymbol{A}\right) - \nabla^2 \boldsymbol{A} = \mu_0 \left[\boldsymbol{J} - \epsilon_0 \left(\frac{\partial}{\partial t}\nabla\phi + \frac{\partial^2 \boldsymbol{A}}{\partial t^2}\right)\right].$$

But from Lorenz gauge we have

$$\nabla\left(\nabla \cdot \boldsymbol{A}\right) = -\epsilon_0\mu_0 \frac{\partial}{\partial t}\nabla\phi,$$

therefore we can write

$$-\nabla^2 \boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \mu_0 \boldsymbol{J}.$$

□

6.6 Electromagnetic energy and the Poynting vector

Electromagnetic waves carry energy. In fact, most of the energy we get in the Earth comes, in origin, from the light of the Sun. The aim of this section is to calculate this energy.

Definition 6.5 (Electromagnetic energy density)**.** The electromagnetic energy density for a given field is

$$\mathcal{E} = \frac{\epsilon}{2}\boldsymbol{E} \cdot \boldsymbol{E} + \frac{1}{2\mu_0}\boldsymbol{B} \cdot \boldsymbol{B}.$$

Do not confuse the electromagnetic energy density $\mathcal{E}$ with the electromotive force $\mathcal{E}_{\text{emf}}$. In fact, the latter will not appear again in these notes.

Theorem 6.3 (Poynting's theorem)**.** *The electromagnetic energy density $\mathcal{E}$ satisfies*

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{E} \cdot \boldsymbol{J}, \tag{6.9}$$

where

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B},$$

is called the Poynting vector*.*

Proof. From Definition 6.5, taking the time derivative of the electromagnetic energy density we obtain

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial t} &= \epsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + \frac{1}{\mu_0} \boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \\
&= \frac{1}{\mu_0} \boldsymbol{E} \cdot (\nabla \times \boldsymbol{B} - \mu_0 \boldsymbol{J}) - \frac{1}{\mu_0} \boldsymbol{B} \cdot (\nabla \times \boldsymbol{E}) \\
&= -\nabla \cdot \left(\frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} \right) - \boldsymbol{E} \cdot \boldsymbol{J},
\end{aligned}$$

where in the second line we have used Equation 6.5 and Equation 6.6, and the third line uses `CROSSREF`. □

Note that that Equation 6.9 is analogous to Equation 4.1. Therefore, one can continue the analogy, and say that the Poynting vector $\boldsymbol{S}$ is to the electromagnetic energy density $\mathcal{E}$, what the current density $\boldsymbol{J}$ is to the charge density ρ. In other words, $\boldsymbol{S}$ is the flow of energy carried by the electromagnetic field. In the particular case where there are no currents, Equation 6.9 states that the electromagnetic energy is conserved. There is another interpretation of the Poynting vector though: it is the momentum stored in the electromagnetic field.