Green's Function and Magnitization for Fermions

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Free Particle

The finite tepurature Greens function of imaginary time in momentum space for uniform space and time is (eq 3.2.1 Mahan):

$$G_{\alpha\beta}(\vec{p},\tau) = -< T\psi_\alpha(\vec{p},\tau)\psi^\dagger_\beta(\vec{p},0)>, \quad \tau \in (-\beta,\beta) \tag{1}$$

Where $< ... >= Tr(e^{-H\beta}...)$. For a free particle the hamiltonian operator is:

$$H = \sum_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\psi^\dagger_\alpha(\vec{p})(H_0 - \mu N)\psi_\beta(\vec{p}) \tag{2}$$

Where H_0 is the free particle hamiltonian μ is the chemical potential and N is the number operator. Using the time evolution equation for $\psi_\alpha(\vec{p},\tau)$ and equation 3.2.8 of Mahan:

$$\begin{aligned}
G_{\alpha\beta}(\vec{p},\tau) &= -< Te^{\tau H}\psi_\alpha(\vec{p})e^{-\tau H}\psi^\dagger_\beta(\vec{p})> \\
&= -\Theta(\tau) < e^{-\tau\xi(\vec{p})}\psi_\alpha(\vec{p})\psi^\dagger_\beta(\vec{p})> + \Theta(-\tau) < e^{-\tau\xi(\vec{p})}\psi^\dagger_\beta(\vec{p})\psi_\alpha(\vec{p})> \\
&= -\Theta(\tau)e^{-\tau\xi(\vec{p})} < \psi_\alpha(\vec{p})\psi^\dagger_\beta(\vec{p})> + \Theta(-\tau)e^{-\tau\xi(\vec{p})} < \psi^\dagger_\beta(\vec{p})\psi_\alpha(\vec{p})>
\end{aligned}$$

We can now use the relations $< \psi_\alpha(\vec{p})\psi^\dagger_\beta(\vec{p}) >= (1 - n(\vec{p}))\delta_{\alpha\beta}$ and $< \psi^\dagger_\beta(\vec{p})\psi_\alpha(\vec{p}) >= n(\vec{p})\delta_{\alpha\beta}$. Where $n(\vec{p}) = (e^{\beta\xi(\vec{p})} + 1)^{-1}$. Dropping the spin indicies.

$$G(\vec{p},\tau) = -\Theta(\tau)e^{-\tau\xi(\vec{p})}(1 - n(\vec{p})) + \Theta(-\tau)e^{-\tau\xi(\vec{p})}n(\vec{p}) \tag{3}$$

The transformation to Matsubara energies is defined as:

$$G(\vec{p},\omega_m) = \frac{1}{2}\int_{-\beta}^{\beta} d\tau e^{i\tau\epsilon_m}G(\vec{p},\tau) = \frac{1}{i\omega_m - \xi(\vec{p})} \tag{4}$$

Spin Particle in Magnetic Field

We start by taking the τ derivitive of the Green's function.

$$-\frac{\partial}{\partial\tau}G_{\alpha\beta}(\vec{x},\vec{x}',\tau) = \delta(\vec{x}-\vec{x}')\delta(\tau)\delta_{\alpha\beta} - < T[\psi_\alpha(\vec{x},\tau), H]\psi^\dagger_\beta(\vec{x}',0)> \tag{5}$$

In this case the equation for H is

$$H = \sum_{\gamma\delta}\int d^3x\psi^\dagger_\gamma(\vec{x})(H_0 - \mu N - \mu_B\vec{\sigma}\cdot\vec{B})\psi_\delta(\vec{x}) = \sum_{\gamma\delta}\int d^3x((\epsilon(\vec{p}) - \mu)\delta_{\gamma\delta} - \mu_B\vec{\sigma}_{\gamma\delta}\cdot\vec{B})\psi^\dagger_\gamma(\vec{x})\psi_\delta(\vec{x}) \tag{6}$$

Inserting this into the second term on the right hand side of equation 5 and using the time evolved operator equation yields:

$$< Te^{\tau H}\sum_{\gamma\delta}\int d^3x''((\epsilon(\vec{p''})-\mu)\delta_{\gamma\delta}-\mu_B\vec{\sigma}_{\gamma\delta}\cdot\vec{B})[\psi_\alpha(\vec{x}),\psi^\dagger_\gamma(\vec{x''})\psi_\delta(\vec{x''})]e^{-\tau H}\psi^\dagger_\beta(\vec{x}') > \tag{7}$$

The commutator can be resolved by using the identity $[A,BC]=\{A,B\}C-B\{A,C\}$ and the relations $\{\psi_\alpha(\vec{x}),\psi^\dagger_\gamma(\vec{x'})\}=\delta(\vec{x}-\vec{x'})\delta_{\alpha\gamma}$ and $\{\psi_\alpha(\vec{x}),\psi_\delta(\vec{x'})\}=0$. Thus, equation 7 becomes:

$$< T[\psi_\alpha(\vec{x},\tau),H]\psi^\dagger_\beta(\vec{x}',0) >=< Te^{\tau H}\sum_\delta((\epsilon(\vec{p})-\mu)\delta_{\alpha\delta}-\mu_B\vec{\sigma}_{\alpha\delta}\cdot\vec{B})\psi_\delta(\vec{x})e^{-\tau H}\psi^\dagger_\beta(\vec{x}') > \tag{8}$$

Equation 5 now becomes:

$$-\frac{\partial}{\partial\tau}G_{\alpha\beta}(\vec{x},\vec{x}',\tau)=\delta(\vec{x}-\vec{x}')\delta(\tau)\delta_{\alpha\beta}-\sum_\delta((\epsilon(\vec{p})-\mu)\delta_{\alpha\delta}-\mu_B\vec{\sigma}_{\alpha\delta}\cdot\vec{B}) < T\psi_\delta(\vec{x},\tau)\psi^\dagger_\beta(\vec{x}') > \tag{9}$$

The Transformation to momentum space is defined as $G_{\delta\beta}(\vec{p},\vec{p}',\tau)=\int d^3x\int d^3x' G_{\delta\beta}(\vec{x},\vec{x}',\tau)e^{-i\vec{p}\cdot\vec{x}}e^{i\vec{p}'\cdot\vec{x}'}$.

$$-\frac{\partial}{\partial\tau}G_{\alpha\beta}(\vec{p},\vec{p}',\tau)=(2\pi)^3\delta(\vec{p}-\vec{p}')\delta(\tau)\delta_{\alpha\beta}+\sum_\delta((\epsilon(\vec{p})-\mu)\delta_{\alpha\delta}-\mu_B\vec{\sigma}_{\alpha\delta}\cdot\vec{B})G_{\delta\beta}(\vec{p},\vec{p}',\tau) \tag{10}$$

defining $\xi(\vec{p})=\epsilon(\vec{p})-\mu$ and rearranging:

$$\sum_\delta(-\frac{\partial}{\partial\tau}-\xi(\vec{p}))\delta_{\alpha\delta}+\mu_B\vec{\sigma}_{\alpha\delta}\cdot\vec{B})G_{\delta\beta}(\vec{p},\vec{p}',\tau)=(2\pi)^3\delta(\vec{p}-\vec{p}')\delta(\tau)\delta_{\alpha\beta} \tag{11}$$

Transforming this to Matsubara energies:

$$\sum_\delta((i\omega_m-\xi(\vec{p}))\delta_{\alpha\delta}+\mu_B\vec{\sigma}_{\alpha\delta}\cdot\vec{B})G_{\delta\beta}(\vec{p},\vec{p}',\omega_m)=\frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')\delta_{\alpha\beta} \tag{12}$$

Equation 11 is a 2x2 matrix equation of the form, $A*G=\frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')I$. We can invert this to get $G=\frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')A^{-1}$. Setting $\vec{B}=B_0\hat{z}$.

$$A=\begin{pmatrix} i\omega_m-\xi+\mu_B B_0 & 0 \\ 0 & i\omega_m-\xi(\vec{p})-\mu_B B_0\end{pmatrix}=(i\omega_m-\xi(\vec{p}))I+\mu_B\sigma_z\cdot\vec{B} \tag{13}$$

$$G(\vec{p},\vec{p}',\omega_m)=\frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')A^{-1}=\frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')\frac{(i\omega_m-\xi(\vec{p}))I-\mu_B\sigma_z B_0}{(i\omega_m-\xi(\vec{p}))^2-\mu_B^2B_0^2} \tag{14}$$

This leads to defining

$$G^0(\vec{p},\tau)=\frac{1}{2}\frac{(i\omega_m-\xi(\vec{p}))I-\mu_B\sigma_z B_0}{(i\omega_m-\xi(\vec{p}))^2-\mu_B^2B_0^2} \tag{15}$$

This is a diagonal matrix with entries

$$G^0_{11}(\vec{p},\tau)=\frac{1}{2}\frac{1}{i\omega_m-\xi(\vec{p})+\mu_B B_0}$$
$$G^0_{22}(\vec{p},\tau)=\frac{1}{2}\frac{1}{i\omega_m-\xi(\vec{p})-\mu_B B_0}$$

Non-Uniform Magnetic Field

Again, we start with the τ derivitive of the Greens function for uniform time to arrive at

$$\sum_{\delta} \big((-\frac{\partial}{\partial\tau} - \xi(\hat{\vec{p}}))\delta_{\alpha\delta} + \mu_B \vec{\sigma_{\alpha\delta}} \cdot \vec{B}(\vec{x})\big) G_{\delta\beta}(x,x') = \delta(\vec{x}-\vec{x}')\delta(\tau)\delta_{\alpha\beta} \tag{16}$$

We now wish to transform to momentum space. The first two terms on the left hand side transform simply, but the $\vec{B}(\vec{x})G_{\delta\beta}(x,x')$ term needs to be evaluated. Using $\vec{B}(\vec{x}) = B_0\hat{z} + \delta\vec{B}(\vec{x})$:

$$\begin{aligned}
&\int d^3x \int d^3x' \vec{B}(\vec{x}) G_{\delta\beta}(\vec{x},\vec{x}',\tau) e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{p}'\cdot\vec{x}'} \\
&= \vec{B_0} G_{\delta\beta}(\vec{p},\vec{p}',\tau) + \int d^3x \delta\vec{B}(\vec{x}) G_{\delta\beta}(\vec{x},\vec{p}',\tau) e^{-i\vec{p}\cdot\vec{x}} \\
&= \vec{B_0} G_{\delta\beta}(\vec{p},\vec{p}',\tau) + \int d^3x \int \frac{d^3q}{(2\pi)^3} \delta\vec{B}(\vec{q}) G_{\delta\beta}(\vec{x},\vec{p}',\tau) e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{x}} \\
&= \vec{B_0} G_{\delta\beta}(\vec{p},\vec{p}',\tau) + \int \frac{d^3q}{(2\pi)^3} \delta\vec{B}(\vec{q}) G_{\delta\beta}(\vec{p}-\vec{q},\vec{p}',\tau)
\end{aligned}$$

If we also transform to Matsubara energies, equation 16 becomes:

$$\begin{aligned}
\sum_{\delta} \big((i\omega_m - \xi(\vec{p}))\delta_{\alpha\delta} + \mu_B \sigma_{z\alpha\delta} B_0\big) G_{\delta\beta}(\vec{p},\vec{p}',\omega_m) + \mu_B \vec{\sigma_{\alpha\delta}} \cdot \int \frac{d^3q}{(2\pi)^3} \delta\vec{B}(\vec{q}) G_{\delta\beta}(\vec{p}-\vec{q},\vec{p}',\omega_m) \\
= \frac{(2\pi)^3}{2} \delta(\vec{p}-\vec{p}')\delta_{\alpha\beta}
\end{aligned}$$

We can then write the Greens function as a perterbation $G_{\delta\beta}(\vec{p},\vec{p}',\omega_m) = (2\pi)^3 G^0_{\delta\beta}(\vec{p},\omega_m)\delta(\vec{p}-\vec{p}') + \delta G_{\delta\beta}(\vec{p},\vec{p}',\omega_m)$. Where G^0 is from the uniform field case (equation 15).

$$\begin{aligned}
\sum_{\delta} \big((i\omega_m - \xi(\vec{p}))\delta_{\alpha\delta} + \mu_B \sigma_{z\alpha\delta} \cdot \vec{B_0}\big)\big((2\pi)^3 G^0_{\delta\beta}(\vec{p},\vec{p}',\omega_m) + \delta G_{\delta\beta}(\vec{p},\vec{p}',\omega_m)\big) \\
+\mu_B \vec{\sigma_{\alpha\delta}} \cdot \delta\vec{B}(\vec{p}-\vec{p}') G^0_{\delta\beta}(\vec{p}',\vec{p}',\omega_m) + \mu_B \vec{\sigma_{\alpha\delta}} \cdot \int \frac{d^3q}{(2\pi)^3} \delta\vec{B}(\vec{q}) \delta G_{\delta\beta}(\vec{p}-\vec{q},\vec{p}',\omega_m)\big) = \frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')\delta_{\alpha\beta}
\end{aligned}$$

Keeping only terms up to first order in $\delta\vec{B}(\vec{x})$ and using the result from the uniform field we have:

$$\sum_{\delta} \big((i\omega_m - \xi(\vec{p}))\delta_{\alpha\delta} + \mu_B \vec{\sigma_{\alpha\delta}} \cdot \vec{B_0}\big)\delta G_{\delta\beta}(\vec{p},\vec{p}',\omega_m) + \mu_B \vec{\sigma_{\alpha\delta}} \cdot \delta\vec{B}(\vec{p}-\vec{p}') G^0_{\delta\beta}(\vec{p}',\omega_m) = 0 \tag{17}$$

This is a matrix equation of the form $A(\vec{p}) * \delta G = -\mu_B \sigma \cdot \delta\vec{B}(\vec{p}-\vec{p}')G^0$ where A is the same matrix as for the uniform field ($G^0 = \frac{1}{2}A^{-1}$). The result is:

$$\delta G(\vec{p},\vec{p}',\omega_m) = -\frac{\mu_B}{2} G^0(\vec{p},\omega_m) \vec{\sigma} \cdot \delta\vec{B}(\vec{p}-\vec{p}') G^0(\vec{p}',\omega_m) \tag{18}$$

Magnetization

The magnetization is defined as $\vec{M}(\vec{x}) = \mu_B \sum_{\alpha\beta} < \vec{\sigma_{\beta\alpha}} \psi^\dagger_\beta(\vec{x})\psi_\alpha(\vec{x}) >= \mu_B \sum_{\alpha\beta} \vec{\sigma_{\beta\alpha}} G_{\alpha\beta}(\vec{x},\vec{x},\tau = -0i)$. We wish to transform this to momentum space:

$$\vec{M}(\vec{x}) = \frac{\mu_B}{(2\pi)^6} \sum_{\alpha\beta} \vec{\sigma_{\beta\alpha}} \int d^3p \int d^3p' G_{\alpha\beta}(\vec{p},\vec{p}',\tau = -0i) e^{i(\vec{p}-\vec{p}')\cdot\vec{x}} \tag{19}$$

$$\vec{M}(\vec{q}) = \frac{\mu_B}{(2\pi)^6}\sum_{\alpha\beta}\sigma\vec{_{\beta\alpha}}\int d^3p\int d^3p' G_{\alpha\beta}(\vec{p},\vec{p}',\tau=-0i)\int d^3x e^{i(\vec{p}-\vec{p}'-\vec{q})\cdot\vec{x}} \tag{20}$$

$$= \frac{\mu_B}{(2\pi)^6}\sum_{\alpha\beta}\sigma\vec{_{\beta\alpha}}\int d^3p\int d^3p' G_{\alpha\beta}(\vec{p},\vec{p}',\tau=-0i)(2\pi)^3\delta(\vec{p}-\vec{p}'-\vec{q}) \tag{21}$$

$$= \frac{\mu_B}{(2\pi)^3}\sum_{\alpha\beta}\sigma\vec{_{\beta\alpha}}\int d^3p G_{\alpha\beta}(\vec{p},\vec{p}-\vec{q},\tau=-0i) \tag{22}$$

$$= \mu_B\sum_{\alpha\beta}\sigma\vec{_{\beta\alpha}}\int d^3p (G^0_{\alpha\beta}(\vec{p},\vec{p}-\vec{q},\tau=-0i)+\delta G_{\alpha\beta}(\vec{p},\vec{p}-\vec{q},\tau=-0i)/(2\pi)^3) \tag{23}$$

We can now change to a sum over Matsubara energies ($\omega_m = 2\pi(m+1/2)T$) so $\delta G(\vec{p},\tau=-0i) = T\sum_{\omega_m}\delta G(\vec{p},\omega_m)$. If we Consider the ith component of the magentization:

$$\vec{M}_i(\vec{q}) = \mu_B T\sum_{\alpha\beta}\sum_{\omega_m}\int d^3p\Big[\vec{\sigma}_{i\beta\alpha}G^0_{\alpha\beta}(\vec{p},\omega_m) - \frac{\mu_B}{2(2\pi)^3}\vec{\sigma}_{i\beta\alpha}\sum_{\delta\gamma}\sum_j G^0_{\alpha\delta}(\vec{p},\omega_m)\vec{\sigma}_{j\delta\gamma}G^0_{\gamma\beta}(\vec{p}-\vec{q},\omega_m)\delta\vec{B}(\vec{q})_j\Big]$$
$$= \vec{M}_0(\vec{p})_i + \sum_j \mathcal{X}(\vec{p})_{ij}\delta\vec{B}(\vec{q})_j$$

The susceptibility is:

$$\mathcal{X}(\vec{q})_{ij} = -\frac{\mu_B^2 T}{2(2\pi)^3}\sum_{\alpha\beta\delta\gamma}\sum_{\omega_m}\int d^3p\vec{\sigma}_{i\beta\alpha}G^0_{\alpha\delta}(\vec{p},\omega_m)\vec{\sigma}_{j\delta\gamma}G^0_{\gamma\beta}(\vec{p}-\vec{q},\omega_m) \tag{24}$$

Since G^0 is a diagnol matrix equation 24 can be simplified

$$\mathcal{X}(\vec{q})_{ij} = -\frac{\mu_B^2 T}{2(2\pi)^3}\sum_{\alpha\beta}\sum_{\omega_m}\int d^3p\vec{\sigma}_{i\beta\alpha}G^0_{\alpha\alpha}(\vec{p},\omega_m)\vec{\sigma}_{j\alpha\beta}G^0_{\beta\beta}(\vec{p}-\vec{q},\omega_m) \tag{25}$$

We can also do the sum over matsubara energies using complex integration with $z = i\omega_m$, $T\sum_{\omega_m}B(z) = \frac{1}{2i\pi}\int dz B(z)f(z)$ where $f(z)$ is the fermi function $f(\xi) = (e^{\xi/T}+1)^{-1}$. Upon investigation, one finds that the susceptibility tensor is diagonal and that $\mathcal{X}_{xx} = \mathcal{X}_{yy}$. We also can symmetrize the momentum integral. If we define $\xi_\pm = \xi(\vec{p}\pm\vec{q}/2)$ the susceptibility is.

$$\mathcal{X}(\vec{q})_{zz} = -\frac{\mu_B^2}{2(2\pi)^3}\int d^3p\Big[\frac{f(\xi_+-\mu_B B_0) - f(\xi_- - \mu_B B_0) + f(\xi_+ + \mu_B B_0) - f(\xi_- + \mu_B B_0)}{\xi_+ - \xi_-}\Big]$$
$$\mathcal{X}(\vec{q})_{xx} = -\frac{\mu_B^2}{2(2\pi)^3}\int d^3p\Big[\frac{f(\xi_+-\mu_B B_0) - f(\xi_- + \mu_B B_0)}{\xi_+ - \xi_- - 2\mu_B B_0} + \frac{f(\xi_+ + \mu_B B_0) - f(\xi_- - \mu_B B_0)}{\xi_+ - \xi_- + 2\mu_B B_0}\Big]$$

Superconducting Phase

We now assume that our sample is in the uniform superconducting phase (S wave superconductor) with Hamiltonian:

$$H = \sum_{\gamma\delta}\int d^3x\psi^\dagger_\gamma(\vec{x})(H_0 - \mu N - \mu_B\sigma_z B_0)\psi_\delta(\vec{x}) + \frac{1}{2}\int d^3x' V_{\gamma\delta}(\vec{x}-\vec{x}')\psi^\dagger_\alpha(\vec{x}')\psi^\dagger_\gamma(\vec{x})\psi_\delta(\vec{x})\psi_\beta(\vec{x}')$$

Where $V_{\gamma\delta}(\vec{x}-\vec{x}')$ is the spin dependent attractive superconducting potential. We also must define a new set of Greens functions for the superconducting state:

$$\begin{aligned} G_{\alpha\beta}(\vec{x},\vec{x}',\tau) &= -< T\psi_\alpha(\vec{x},\tau)\psi^\dagger_\beta(\vec{x}') > \\ \bar{G}_{\alpha\beta}(\vec{x},\vec{x}',\tau) &= -< T\psi^\dagger_\alpha(\vec{x},\tau)\psi_\beta(\vec{x}') > \\ F_{\alpha\beta}(\vec{x},\vec{x}',\tau) &= -< T\psi_\alpha(\vec{x},\tau)\psi_\beta(\vec{x}') > \\ \bar{F}_{\alpha\beta}(\vec{x},\vec{x}',\tau) &= -< T\psi^\dagger_\alpha(\vec{x},\tau)\psi^\dagger_\beta(\vec{x}') > \end{aligned}$$

In order to proceed we must take the mean field approximation and define the superconducting order parameter $\Delta_{\alpha\beta}(\vec{x},\vec{x}') = V_{\alpha\beta}(\vec{x}-\vec{x}') < \psi_\beta(\vec{x}')\psi_\alpha(\vec{x}) >$. The mean field Hamiltonian is:

$$H_{mf} = \sum_{\gamma\delta}\int d^3x\psi^\dagger_\gamma(\vec{x})(H_0-\mu N-\mu_B\sigma_z B_0)\psi_\delta(\vec{x})+\frac{1}{2}\int d^3x'\left[\psi^\dagger_\gamma(\vec{x})\psi^\dagger_\delta(\vec{x}')\Delta_{\gamma\delta}(\vec{x},\vec{x}')+\psi_\delta(\vec{x}')\psi_\gamma(\vec{x})\Delta^*_{\delta\gamma}(\vec{x},\vec{x}')\right] \tag{26}$$

To find the Green's function we proceed as before and try to find the commutator $[\psi_\alpha(\vec{x},\tau), H_{mf}]$. We have found the first part of this previously, but need to find $[\psi_\alpha(\vec{x},\tau), V_{sc}]$ and $[\psi^\dagger_\alpha(\vec{x},\tau), V_{sc}]$. Pulling out the time dependence and keeping in mind that Δ is a fermionic operator:

$$\begin{aligned} [\psi_\alpha(\vec{x}), V_{sc}] &= \frac{1}{2}\sum_{\gamma\delta}\int d^3x'\int d^3x''\Delta_{\gamma\delta}(\vec{x}',\vec{x}'')[\psi_\alpha(\vec{x}),\psi^\dagger_\gamma(\vec{x}')\psi^\dagger_\delta(\vec{x}'')] + \Delta^*_{\delta\gamma}(\vec{x}',\vec{x}'')[\psi_\alpha(\vec{x}),\psi_\delta(\vec{x}'')\psi_\gamma(\vec{x}')] \\ &= \frac{1}{2}\sum_{\gamma\delta}\int d^3x'\delta_{\alpha\gamma}\Delta_{\gamma\delta}(\vec{x},\vec{x}')\psi^\dagger_\delta(\vec{x}') - \delta_{\alpha\delta}\Delta_{\gamma\delta}(\vec{x}',\vec{x})\psi^\dagger_\gamma(\vec{x}') \\ &= \sum_\delta\int d^3x'\Delta_{\alpha\delta}(\vec{x},\vec{x}')\psi^\dagger_\delta(\vec{x}') \\ [\psi^\dagger_\alpha(\vec{x}), V_{sc}] &= \sum_\delta\int d^3x'\Delta^*_{\alpha\delta}(\vec{x},\vec{x}')\psi_\delta(\vec{x}') \end{aligned}$$

Plugging this into the equation of motion for the Greens function:

$$\begin{aligned} -\frac{\partial}{\partial\tau}G_{\alpha\beta}(\vec{x},\vec{x}',\tau) = \delta(\vec{x}-\vec{x}')\delta(\tau)\delta_{\alpha\beta}\mathcal{I}_{ph} &+ \sum_\delta((\epsilon(\vec{p})-\mu)\delta_{\alpha\delta} - \mu_B\sigma_{z\alpha\delta}\vec{B})G_{\delta\beta}(\vec{x},\vec{x}',\tau) \\ &+ \sum_\delta\int d^3y\Delta_{\alpha\delta}(\vec{x},\vec{y})\bar{F}_{\delta\beta}(\vec{y},\vec{x}',\tau) \end{aligned}$$

Where $\mathcal{I}_{ph}$ is the identity matrix in particle/hole space. Transforming to momentum and energy space:

$$\begin{aligned} \sum_\delta\int\frac{d^3k}{(2\pi)^3}\left[(i\omega_m-\xi(\vec{p}))\delta_{\alpha\delta}+\mu_B\sigma_{z\alpha\delta}B_0)(2\pi)^3\delta(\vec{k}-\vec{p})G_{\delta\beta}(\vec{k},\vec{p}',\omega_m) - \Delta_{\alpha\delta}(\vec{p},\vec{k})\bar{F}_{\delta\beta}(\vec{k},\vec{p}',\omega_m)\right] \\ = \frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')\delta_{\alpha\beta}\mathcal{I}_{ph} \end{aligned}$$

If we assume that Δ is uniform in space (ie. $\Delta(\vec{x},\vec{x}') = \Delta(|\vec{x}-\vec{x}'|)$), and recall that since Δ is a fermionic operator $\Delta(-\vec{p}) = \Delta(\vec{p})$, then the momentum/energy equation is:

$$\sum_\delta\left[(i\omega_m-\xi(\vec{p}))\delta_{\alpha\delta}+\mu_B\sigma_{z\alpha\delta}B_0)G_{\delta\beta}(\vec{p},\vec{p}',\omega_m) - \Delta_{\alpha\delta}(\vec{p})\bar{F}_{\delta\beta}(\vec{p},\vec{p}',\omega_m)\right] = \frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')\delta_{\alpha\beta}$$

Working through the rest of the Greens function equations of motion yields a matrix equation (sum over δ is implied).

$$\begin{pmatrix} (i\omega_m - \xi(\vec{p}))\delta_{\alpha\delta} + \mu_B \vec{\sigma}_{z\alpha\delta} B_0 & -\Delta_{\alpha\delta}(\vec{p}) \\ -\Delta^*_{\alpha\delta}(\vec{p}) & (i\omega_m + \xi(\vec{p}))\delta_{\alpha\delta} - \mu_B \sigma_{z\alpha\delta} B_0 \end{pmatrix} \begin{pmatrix} G_{\delta\beta}(\vec{p},\vec{p}',\omega_m) & F_{\delta\beta}(\vec{p},\vec{p}',\omega_m) \\ \bar{F}_{\delta\beta}(\vec{p},\vec{p}',\omega_m) & \bar{G}_{\delta\beta}(\vec{p},\vec{p}',\omega_m) \end{pmatrix}$$
$$= \frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')\delta_{\alpha\beta}\mathcal{I}_{ph}$$

Now we use the same definition of the matrix A as (13), and define $A' = \begin{pmatrix} i\omega_m + \xi - \mu_B B_0 & 0 \\ 0 & i\omega_m + \xi(\vec{p}) + \mu_B B_0 \end{pmatrix}$. We also choose the order parameter to be a singlet state which means it has spin structure $\Delta(\vec{p}) = \Delta(\vec{p}) i\sigma_y$. Using these definitions we can invert the matrix equation to get the superconducting Greens functions.

$$G(\vec{p},\vec{p}',\omega_m) = \frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')\frac{\left(i\omega_m - \xi - \frac{|\Delta|^2(i\omega_m+\xi)}{(i\omega_m+\xi)^2-\mu_B^2B_0^2}\right)\mathcal{I} - b\left(1+\frac{|\Delta|^2}{(i\omega_m+\xi)^2-\mu_B^2B_0^2}\right)\sigma_z}{\left(i\omega_m - \xi - \frac{|\Delta|^2(i\omega_m+\xi)}{(i\omega_m+\xi)^2-\mu_B^2B_0^2}\right)^2 - b^2\left(1+\frac{|\Delta|^2}{(i\omega_m+\xi)^2-\mu_B^2B_0^2}\right)^2}$$

$$\bar{G}(\vec{p},\vec{p}',\omega_m) = \frac{(2\pi)^3}{2}\delta(\vec{p}-\vec{p}')\frac{\left(i\omega_m + \xi - \frac{|\Delta|^2(i\omega_m-\xi)}{(i\omega_m-\xi)^2-\mu_B^2B_0^2}\right)\mathcal{I} + b\left(1+\frac{|\Delta|^2}{(i\omega_m-\xi)^2-\mu_B^2B_0^2}\right)\sigma_z}{\left(i\omega_m + \xi - \frac{|\Delta|^2(i\omega_m-\xi)}{(i\omega_m-\xi)^2-\mu_B^2B_0^2}\right)^2 - b^2\left(1+\frac{|\Delta|^2}{(i\omega_m-\xi)^2-\mu_B^2B_0^2}\right)^2}$$

$$\begin{aligned} F(\vec{p},\vec{p}',\omega_m) = {} & \frac{(2\pi)^3\delta(\vec{p}-\vec{p}')\Delta(\vec{p})}{2((i\omega_m-\xi)^2-b^2)(')}\Bigg[\left(i\omega_m + \xi - \frac{|\Delta|^2(i\omega_m-\xi)}{(i\omega_m-\xi)^2-\mu_B^2B_0^2}\right)((i\omega_m-\xi)i\sigma_y - b\sigma_x) \\ & - b\left(1+\frac{|\Delta|^2}{(i\omega_m-\xi)^2-\mu_B^2B_0^2}\right)((i\omega_m-\xi)\sigma_x - bi\sigma_y)\Bigg] \\ \bar{F}(\vec{p},\vec{p}',\omega_m) = {} & -\frac{(2\pi)^3\delta(\vec{p}-\vec{p}')\Delta^*(\vec{p})}{2((i\omega_m+\xi)^2-b^2)('')}\Bigg[\left(i\omega_m - \xi - \frac{|\Delta|^2(i\omega_m+\xi)}{(i\omega_m+\xi)^2-\mu_B^2B_0^2}\right)((i\omega_m+\xi)i\sigma_y + b\sigma_x) \\ & + b\left(1+\frac{|\Delta|^2}{(i\omega_m+\xi)^2-\mu_B^2B_0^2}\right)((i\omega_m+\xi)\sigma_x + bi\sigma_y)\Bigg] \end{aligned}$$

Where $(')$ and $('')$ is the denominator of $\bar{G}$ and G respectfully.