HW 6: Physics 545

Ben Rosemeyer

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1a

To show this relation we need use the fact that the distribution function $n(\epsilon_p) = n(\epsilon_p - \mu) = n(\xi_p)$ is a function of ϵ_p and μ so that $\frac{\delta n_p}{\delta \epsilon_p} = \frac{\delta \xi_p}{\delta \epsilon_p}\frac{\delta \mu}{\delta \xi_p}\frac{\delta n_p}{\delta \mu} = -\frac{\delta n_p}{\delta \mu}$.

As $q \to 0$, the differences in the numerator and denominator of the density-density correlation become differentials so:

$$\chi(q \to 0) = -\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\frac{\delta n_p}{\delta \epsilon_p} = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\frac{\delta n_p}{\delta \mu} = \frac{1}{\delta \mu}\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\delta n_p \quad (1)$$

$$\boxed{\chi(q \to 0) = \frac{\delta n}{\delta \mu}} \quad (2)$$

Where we used the relations above and the definition of $\delta n = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\delta n_p$. Thus we have the desired result $\kappa_T = \frac{1}{n^2}\frac{\delta n}{\delta \mu} = \frac{1}{n^2}\chi(q \to 0)$

1b

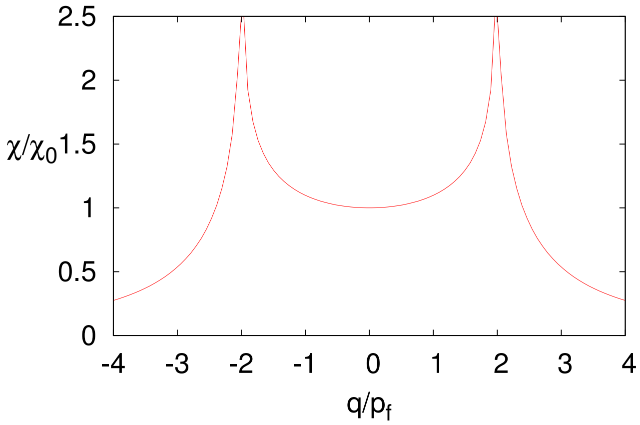
We can do the integral for χ noting that the distribution functions limit the integration to two intervals $[-p_f - q/2, -p_f + q/2]$ (+ from $n_{p+q/2}$), $[p_f - q/2, p_f + q/2]$ (- from $n_{p-q/2}$).

$$\chi(q) = -\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\frac{n_{p+q/2} - n_{p-q/2}}{\epsilon_{p+q/2} - \epsilon_{p-q/2}} \quad (3)$$

$$= -\int_{-p_f-q/2}^{-p_f+q/2} \frac{dp}{2\pi\hbar}\frac{1}{2qp} + \int_{p_f-q/2}^{p_f+q/2} \frac{dp}{2\pi\hbar}\frac{1}{2qp} \quad (4)$$

$$\boxed{\chi(q) = \frac{1}{2\pi\hbar q}\, ln\left|\frac{p_f + q/2}{p_f - q/2}\right|} \quad (5)$$

The discontinuity is at $q^* = 2p_f$ and we define $\chi_0 = \frac{1}{2\pi\hbar p_f}$



2

The Hamiltonian can be written as $\mathcal{H} = \sum_{q>0,\sigma} \epsilon_{q\sigma} \hat{a}^\dagger_{q\sigma} \hat{a}_{q\sigma}$, where $\sigma = \{s, c\}$ denotes the spin or charge channel, and the energies are $\epsilon_{q\sigma} = qv_\sigma$. Here it is important to note that the operators $\hat{a}_{q\sigma}$ are BOSONIC and the distribution function $n_{q\sigma} =< \hat{a}^\dagger_{q\sigma} \hat{a}_{q\sigma} >= \frac{1}{e^{(\epsilon_{q\sigma}-\mu)/T}-1}$, which requires $\epsilon_{q\sigma} > \mu$ ($q > \mu/v_\sigma$)

To calculate the specific heat ($C_v = T\frac{\delta S}{\delta T}$) we start by writting the entropy S:

$$S = \sum_{q\sigma} (n_{q\sigma} + 1)\, ln(n_{q\sigma} + 1) - n_{q\sigma}\, ln(n_{q\sigma}) \tag{6}$$

$$\delta S = -\sum_{q\sigma} \delta n_{q\sigma}\, ln\left[\frac{n_{q\sigma}}{n_{q\sigma}+1}\right] \tag{7}$$

$$\delta S = \sum_{q\sigma} \delta n_{q\sigma}\, (\epsilon_{q\sigma} - \mu)/T \tag{8}$$

Plugging the equation for δS into C_v we see that there will be a term like $\frac{\delta n_{q\sigma}}{\delta T}$. If we neglect higher order corrections to the temperature deviations (ie $\frac{\delta}{\delta T}(\epsilon_{q\sigma} - \mu) \approx 0$) we have:

$$\frac{\delta n_{q\sigma}}{\delta T} = \frac{e^{(\epsilon_{q\sigma}-\mu)/T}(\epsilon_{q\sigma} - \mu)/T^2}{(e^{(\epsilon_{q\sigma}-\mu)/T} - 1)^2} = \frac{(\epsilon_{q\sigma} - \mu)/T^2}{sinh^2(\epsilon_{q\sigma} - \mu)/2T)} \tag{9}$$

The specific heat is

$$C_v = \sum_{q\sigma} \frac{[(\epsilon_{q\sigma} - \mu)/T]^2}{sinh^2(\epsilon_{q\sigma} - \mu)/2T)} \tag{10}$$

$$= \sum_{\sigma} \int_{\mu/v_\sigma}^{\infty} \frac{dq}{2\pi\hbar} \frac{[(\epsilon_{q\sigma} - \mu)/T]^2}{sinh^2(\epsilon_{q\sigma} - \mu)/2T)} \tag{11}$$

$$= 8T \sum_{\sigma} \int_{0}^{\infty} \frac{dx}{2\pi\hbar v_\sigma} \frac{x^2}{sinh^2(x)} \tag{12}$$

$$\boxed{C_v = \frac{2T\pi}{3\hbar} \sum_{\sigma} \frac{1}{v_\sigma}} \tag{13}$$

We can compare this result to the 1D fermi liquid case by following the notes from class. Doing this we arrive at the following equation for C_v

$$C_v = 4 \int_{0}^{\infty} \frac{dp}{2\pi\hbar} [(\epsilon_p - \mu)/T]^2 cosh^{-2}[(\epsilon_p - \mu)/2T] \tag{14}$$

$$= \frac{\sqrt{m}}{\sqrt{2}\pi\hbar} \int_{0}^{\infty} \frac{d\epsilon}{\sqrt{\epsilon}} [(\epsilon - \mu)/T]^2 cosh^{-2}[(\epsilon - \mu)/2T] \tag{15}$$

$$= \frac{4T\sqrt{2m}}{\pi\hbar} \int_{-\mu}^{\infty} \frac{dx}{\sqrt{Tx + \mu}} x^2 cosh^{-2}[x] \tag{16}$$

At this point we note that $cosh^{-2}[x]$ is strongly peaked near x=0 allowing us to extend the lower integration limit to $-\infty$. We can also taylor expand for small Tx so that $\frac{1}{\sqrt{Tx+\mu}} \approx \frac{1}{\sqrt{\mu}}$ and set $\mu = \epsilon_f$. Thus, the leading order temperature behavior of C_v for 1D Fermi liquid is:

$$\boxed{\frac{4T\pi}{3v_f\hbar}} \tag{17}$$

The temperature dependence is the same! In fact the results are identical if the spin/charge velocities satisfy $v_c = v_s = v_f$