HW 10: Physics 545

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The total entropy is the sum of the phonons + rotons. The distribution of the rotons with spin degeneracy g_r is $n_{\mathbf{p}} \approx g_r \xi e^{-\beta \frac{(p-p_0)^2}{2m}}$, with $\xi = e^{-\beta\Delta}$. We are assuming the temperature is small compared to the energy gap, $\beta\Delta >> 1$, $\xi << 1$, and $\beta = 1/T$. The entropy of the rotons bosons is:

$$
\begin{aligned}
S_r &= \frac{g_r}{(2\pi\hbar)^3} \int d^3\mathbf{p} \quad (1+n_{\mathbf{k}})ln(1+n_{\mathbf{k}}) - n_{\mathbf{k}} ln(n_{\mathbf{k}}) && (1)\\
&= \frac{4g_r\pi}{(2\pi\hbar)^3} \int dp \quad p^2 \Big[(1+\xi e^{-\beta\frac{(p-p_0)^2}{2m}}) ln(1+\xi e^{-\beta\frac{(p-p_0)^2}{2m}}) && (2)\\
&\qquad -\xi e^{-\beta\frac{(p-p_0)^2}{2m}} \Big(ln(\xi) - \beta \frac{(p-p_0)^2}{2m} \Big) \Big] && (3)\\
&\approx \frac{4g_r\pi\xi}{(2\pi\hbar)^3} \int dp \quad p^2 \ e^{-\beta\frac{(p-p_0)^2}{2m}} \Big(\beta\Delta + \beta\frac{(p-p_0)^2}{2m} \Big) && (4)
\end{aligned}
$$

And the first term in the integral can be discarded in the low temperature expansion $1 + \xi e^{-\beta\frac{(p-p_0)^2}{2m}} \approx 1$ because the gaussian is always less than 1.

We make the substitution $x = (p - p_0)/c$, $c = \sqrt{2mT}$

$$
\begin{aligned}
S_r &\approx \frac{4g_r\pi\xi c}{(2\pi\hbar)^3 T} \int_{-p_0/c}^{\infty} dx \quad (cx+p_0)^2 \ e^{-x^2} \Big(\Delta + x^2 T \Big) && (5)\\
&\approx \frac{4g_r\pi\xi c}{(2\pi\hbar)^3 T} \int_{-p_0/c}^{\infty} dx \quad \Big(c^2 T x^4 + 2TCp_0 x^3 (\Delta c^2 + T p_0^2) x^2 + 2cp_0\Delta x + p_0^2 \Delta \Big) \ e^{-x^2} && (6)
\end{aligned}
$$

Now, we keep only the lowest order in T (noting that $c \propto \sqrt{T}$) and extending the lower bound $-\frac{p_0}{\sqrt{2mT}} \to -\infty$.

$$
S_r \approx \frac{g_r p_0^2 \Delta}{\hbar^3} \sqrt{\frac{m}{2\pi^3 T}} e^{-\Delta/T} \qquad (7)
$$

The phonon contribution can be calculated similarly using the phonon distribution to be $n_{\mathbf{p}} = (e^{\beta up} - 1)^{-1}$:

$$\begin{aligned}
S_p &= \frac{g_p}{(2\pi\hbar)^3}\int d^3\mathbf{p} \quad (1+n_{\mathbf{k}})ln(1+n_{\mathbf{k}}) - n_{\mathbf{k}}ln(n_{\mathbf{k}}) && (8)\\
&= \frac{4g_r\pi}{(2\pi\hbar)^3}\int dp \quad p^2(e^{\beta up}-1)^{-1}\Big[e^{\beta up}\Big(\beta up - ln(e^{\beta up}+1)\Big) && (9)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + ln(e^{\beta up}-1)\Big] && (10)\\
&= \frac{4g_r\pi}{(2\pi\hbar)^3}\int dp \quad \beta up^3 e^{\beta up}(e^{\beta up}-1)^{-1} - p^2 ln(e^{\beta up}-1) && (11)
\end{aligned}$$

The change of variables is to $x = ap$, $a = \beta u = u/T$

$$\begin{aligned}
S_p &= \frac{4g_r\pi}{a(2\pi\hbar)^3}\int dx \quad x^3 e^x(e^x-1)^{-1} - x^2 ln(e^x-1) && (12)\\
S_p &= \frac{4g_r\pi}{a^3(2\pi\hbar)^3}\Big[-\frac{x^3}{3}ln(e^x-1)\Big|_0^\infty + \frac{4}{3}\int dx \quad x^3e^x(e^x-1)^{-1}\Big] && (13)\\
S_p &= \frac{2g_r\pi^2T^3}{45u^3\hbar^3} && (14)
\end{aligned}$$

Mathematica gives result of the integral in brackets to be $4\pi^4/45$.

Becuase the entropy is an additive property, the total phonon-roton entropy is $S_{tot} = S_p + S_r$:

$$S_{tot} = \frac{2g_r\pi^2T^3}{45u^3\hbar^3} + \frac{g_rp_0^2\Delta}{\hbar^3}\sqrt{\frac{m}{2\pi^3T}}e^{-\Delta/T} \tag{15}$$

The heat capacity is $C_v = T\frac{\partial S}{\partial T}$

$$C_v = \frac{6g_r\pi^2T^3}{45u^3\hbar^3} + \frac{g_rp_0^2\Delta}{\hbar^3}\sqrt{\frac{m}{2\pi^3}}\Big[-\frac{1}{2}T^{-3/2} + \Delta T^{-5/2}\Big]e^{-\Delta/T} \tag{16}$$

For the density of the normal component, we use the expression from class:

For the phonon distribution:

$$\rho_n^{phon} = \frac{4\pi}{3(2\pi\hbar)^3}\int dp \quad p^4\Big(-\frac{\partial n}{\partial \epsilon}\Big) \tag{17}$$

$$\rho_n = -\frac{4\pi}{3u(2\pi\hbar)^3}\int dp \quad p^4\Big(\frac{\partial n_p}{\partial p}\Big) \tag{18}$$

$$\rho_n = \frac{16\pi}{3u(2\pi\hbar)^3}\int dp \quad p^3 n_p \tag{19}$$

$$\rho_n = \frac{16\pi}{3ua^4(2\pi\hbar)^3}\int dp \quad \frac{x^3}{e^x-1} \tag{20}$$

$$\rho_n = \frac{2\pi^2T^4}{45u^5\hbar^3} \tag{21}$$

$$\tag{22}$$

And we have used integration by parts in the last step. The integral is the DEBYE INTEGRAL$= \pi^4/15$.

And for the roton distribution, using the same substitutions as for the roton entropy:

$$\rho_n = \frac{4\pi\beta\xi}{3(2\pi\hbar)^3}\int dp \quad p^4 e^{-\beta(p-p_0)^2/2m} \tag{23}$$

$$\rho_n = \frac{4\pi\beta\xi c}{3(2\pi\hbar)^3}\int_{-p_0/c}^{\infty} dx \quad (cx+p_0)^4 e^{-x^2} \tag{24}$$

$$\rho_n \approx \frac{4\pi\beta\xi c}{3(2\pi\hbar)^3}\int_{-\infty}^{\infty} dx \quad \Big(c^4x^4 + 4p_0c^3x^3 + 6p_0^2c^2x^2 + 4p_0^3cx + p_0^4\Big)e^{-x^2} \tag{25}$$

Again, we use the low temperature limit to extend the lower limit $-p_0/c \to \infty$ and we keep only the lowest order in T which is the last term in the brackets:

$$\rho_n^{rot} \approx \frac{p_0^4}{6\hbar^3}\sqrt{\frac{2m}{\pi^3T}}e^{-\beta\Delta} \tag{26}$$

2a

The energy of a single vortex is given by $E_v = \frac{1}{2}\int d^3\mathbf{r}\rho_s\mathbf{v}_s^2$. We can write this as the energy per length along the cylinder ϵ_v

$$\epsilon_v = \frac{\rho_s}{2}\int drd\phi \quad r\Theta(r-r_c)^2\Gamma_n^2/r^2 \tag{27}$$

$$\epsilon_v = \rho_s\Gamma_n^2\pi\int dr \quad \Theta(r-r_c)^2/r \tag{28}$$

$$\epsilon_v = \rho_s\Gamma_n^2\pi\Big[ln(r)\Big]_{r_c}^{R} \tag{29}$$

$$\epsilon_v = \pi\rho_s\left(\frac{\hbar n}{m}\right)^2 ln(R/r_c) \tag{30}$$

and we use $\Gamma_n = \frac{\hbar}{m}n$ and $\mathbf{v}_s = \frac{\Gamma}{r}\hat{r}$.

The superfluid momentum is $\mathbf{p}_s = m\mathbf{v}_s = \frac{\hbar n}{r}\hat{\phi}$, and angular momentum is $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. The total angular momentum of the vortex is:

$$\mathbf{L}_s = \int d^3\mathbf{r} \quad \mathbf{r}\times\mathbf{p} \tag{31}$$

$$\mathbf{L}_s = \int drd\theta dz \quad r((r\hbar n/r)\hat{z} - (z\hbar n/r)\hat{r}) \tag{32}$$

$$\tag{33}$$

the $\hat{r}$ term integrates to 0, and the $\hat{z}$ part we write as angular momentum per length $\mathbf{l}_s$:

$$\mathbf{l}_s = \pi\hbar n(R^2 - r_c^2)\hat{z} \tag{34}$$

The critical frequency ω_c is when $\epsilon_s = |\mathbf{l}_s|\omega_c$:

$$\omega_c = \rho_s\left(\frac{\hbar n}{m^2}\right)\frac{ln(R/r_c)}{R^2 - r_c^2} \tag{35}$$

2b

The circulation along a circle of radius r inside the star is:

$$\begin{aligned} 2\pi\Gamma_r &= \int_C \mathbf{v}_s\cdot d\mathbf{l} = \int\int\vec{\nabla}\times\mathbf{v}_s\cdot d\mathbf{s} && (36)\\ &= 2\pi\Gamma_s N(r) && (37)\end{aligned}$$

$\Gamma_s = \hbar/m$ is the single quantized circulation of one vortex and m is the mass of the condensate particles (probably neutrons i guess...)

$N(r) = \pi r^2 n_v$ is the number of vortices contained in the circle of radius r with n_v the area density of vortices which is constant.

The average velocity of this circulation is $\mathbf{v}_s = \frac{\Gamma_r}{r}\hat{\phi}$, and the angular momentum associated with this circulation of all the particles with total mass M is:

$$\mathbf{L} = M\Gamma_r \hat{z} = 2\pi^2 \hbar n_v r^2 \frac{M}{m} \hat{z} \tag{38}$$

Extending the radius of the circle to the edge of the sphere $r = R$, and assuming the mass $M = M_{star}$. Then we can equate this angular momentum with that of the spinning sphere of mass M and frequency Ω, $\mathbf{L} = \frac{2}{5}MR^2\Omega\hat{z}$

$$n_v = \frac{m\Omega}{5\pi^2\hbar} \tag{39}$$