Spin Susceptibility Calculation for the Inhomogeneous Superconducting state

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Spin Susceptibility

Calculation of the spin susceptibility yields (see other inhomogeneous write-up):

$$
\begin{aligned}
\chi(x,x') = -\mu_e \sum_{s,s',p,k} \sigma^{\beta}_{ss'}\sigma^{\alpha}_{s's}\Bigg[&\frac{(f_{ks}-f_{ps'})u^*_p(x')u_p(x)u_k(x')u^*_k(x)}{\omega_{ks}-\omega_{ps'}} \\
&-\frac{(1-f_{ps'}-f_{k-s})u^*_p(x')u_p(x)v^*_k(x')v_k(x)}{\omega_{ps'}+\omega_{k-s}} \\
&+\frac{(-1+f_{p-s'}+f_{ks})v_p(x')v^*_p(x)u_k(x')u^*_k(x)}{\omega_{p-s'}+\omega_{ks}} \\
&+\frac{(f_{p-s'}-f_{k-s})v_p(x')v^*_p(x)v^*_k(x')v_k(x)}{\omega_{p-s'}-\omega_{k-s}}\Bigg] \\
+ss'\sigma^{\beta}_{ss'}\sigma^{\alpha}_{-s-s'}\Bigg[&-\frac{(1-f_{k-s}-f_{ps'})u^*_k(x')v_k(x)v^*_p(x')u_p(x)}{\omega_{ps'}+\omega_{k-s}} \\
&+\frac{(f_{k-s}-f_{p-s'})u^*_k(x')v_k(x)v^*_p(x)u_p(x')}{\omega_{p-s'}-\omega_{k-s}} \\
&+\frac{(f_{ps'}-f_{ks})u^*_k(x)v_k(x')v^*_p(x')u_p(x)}{\omega_{ks}-\omega_{ps'}} \\
&+\frac{(-1+f_{ks}+f_{p-s'})u^*_k(x)v_k(x')v^*_p(x)u_p(x')}{\omega_{ks}+\omega_{p-s'}}\Bigg]
\end{aligned}
$$

In the absence of applied field and for zero temperature this reduces to:

$$\chi(x,x') = 2\mu_e \sum_{p,k} \frac{u_p^*(x')u_p(x)v_k^*(x')v_k(x)}{\omega_p+\omega_k} + \frac{u_k^*(x)u_k(x')v_p^*(x)v_p(x')}{\omega_p+\omega_k} - \frac{u_k^*(x')v_k(x)v_p^*(x')u_p(x)}{\omega_p+\omega_k} - \frac{u_k^*(x)v_k(x')v_p^*(x)u_p(x')}{\omega_k+\omega_p}$$

To continue toward the simplified equation for χ we must write the sum over p and k as an integral over energies E_1 and E_2 with density of states $N(E) = E/\sqrt{E^2-\Delta^2}$

$$\chi(x,x') = 2\mu_e \int_{E_1,E_2,\phi_1,\phi_2} N(E_1)N(E_2)$$

$$\frac{u_1^*(x')u_1(x)v_2^*(x')v_2(x) + u_2^*(x)u_2(x')v_1^*(x)v_1(x') - u_2^*(x')v_2(x)v_1^*(x')u_1(x) - u_2^*(x)v_2(x')v_1^*(x)u_1(x')}{E_1+E_2}$$

We proceed by writting the Bogolioubov amplitudes u and v as plane wave solutions to the Bogolioubov-DeGenne equations
$u_E(x) = \sqrt{(1+(p_x^2-1)/E)/2}e^{ip_xx}$
$v_E(x) = \sqrt{(1-(p_x^2-1)/E)/2}e^{ip_xx}$
Here we use the notation p_x to mean the momentum p obtained from quasi-particle with energy E using the dispersion at position x. That is to say for x in the superconductor, $E = \sqrt{(p^2-1)**2+\Delta^2}$ and in the normal state, $E = abs(p^2-1)$.

$$\chi(x,x') = 2\mu_e \int_{E_1,E_2,\phi_1,\phi_2} \frac{N(E_1)N(E_2)}{E_1+E_2}$$

$$\Big((u_{1,x'}u_{1,x}v_{2,x'}v_{2,x} - u_{2,x'}v_{2,x}v_{1,x'}u_{1,x}) \exp[i(-k_{1,x'}x' + k_{1,x}x - k_{2,x'}x' + k_{2,x}x)]$$

$$+ (u_{2,x}u_{2,x'}v_{1,x}v_{1,x'} - u_{2,x}v_{2,x'}v_{1,x}u_{1,x'}) \exp[i(-k_{1,x}x + k_{1,x'}x' - k_{2,x}x + k_{2,x'}x')]\Big)$$

Now we exchange coordinates x and x' for the center of mass $R = (x+x')/2$ and relative $r = x - x'$ coordinates and set $R = 0$ without loss of generality.

$$\chi(R=0,r) = 2\mu_e \int\limits_{E_1,E_2,\phi_1,\phi_2} \frac{N(E_1)N(E_2)}{E_1+E_2}$$
$$\Big((u_{1,-r/2}u_{1,r/2}v_{2,-r/2}v_{2,r/2} - u_{2,-r/2}v_{2,r/2}v_{1,-r/2}u_{1,r/2}) \exp[i(+k_{1,-r/2}r + k_{1,r/2}r + k_{2,-r/2}r + k_{2,r/2}r)/2]$$
$$+(u_{2,r/2}u_{2,-r/2}v_{1,r/2}v_{1,-r/2} - u_{2,r/2}v_{2,-r/2}v_{1,r/2}u_{1,-r/2}) \exp[i(-k_{1,r/2}r - k_{1,-r/2}r - k_{2,r/2}r - k_{2,-r/2}r)/2]\Big)$$

For the first pass we will calculate in the homogeneous limit which allows us to neglect the positional subscript (r/2, -r/2).

$$\chi(R=0,r) = 2\mu_e \int\limits_{E_1,E_2,\phi_1,\phi_2} \frac{N(E_1)N(E_2)}{E_1+E_2}$$
$$\Big((u_1^2v_2^2 - u_2v_2v_1u_1) \exp[i(+k_1r + k_1r + k_2r + k_2r)/2]$$
$$+(u_2^2v_1^2 - u_2v_2v_1u_1) \exp[i(-k_1r - k_1r - k_2r - k_2r)/2]\Big)$$

Writting with the real and imaginary parts:

$$\chi(R=0,r) = 4\mu_e \int\limits_{E_1,E_2,\phi_1,\phi_2} \frac{N(E_1)N(E_2)}{E_1+E_2}$$
$$\Big((u_1v_2 - u_2v_1)^2 \cos((k_1+k_2)r)$$
$$+i(u_1^2v_2^2 - u_2^2v_1^2) \sin((k_1+k_2)r)\Big)$$