HW 4: Physics 545

Ben Rosemeyer

February 26, 2015

1a

The linearized transport equation for a given mode $(\omega, \mathbf{q})$, is given with a collision integral

$$I_{\mathbf{p}} = \frac{1}{\tau}(\delta n_{\mathbf{p}} - \frac{\delta n^0}{\delta \epsilon_p}(\nu_0 - \nu_1 P_1(\hat{q}\cdot\hat{p}))) \quad (1)$$

Where the ν's are defined as $\delta n_{\mathbf{p}} = \frac{\delta n^0}{\delta \epsilon_p}\nu_{\hat{p}} = \frac{\delta n^0}{\delta \epsilon_p}\sum_l \nu_l P_l(\hat{q}\cdot\hat{p})$

$$(s - \hat{q}\cdot\hat{p})\nu_l P_l(\hat{q}\cdot\hat{p}) - \hat{q}\cdot\hat{p}\int \frac{d\Omega_{\hat{p}'}}{4\pi} F_l^s \nu_{l'} P_l(\hat{p}'\cdot\hat{p}) P_{l'}(\hat{q}\cdot\hat{p}') - \hat{q}\cdot\hat{p}U = -iI_{\mathbf{p}}/qv_f \quad (2)$$

$$(s - \hat{q}\cdot\hat{p})\nu_l P_l(\hat{q}\cdot\hat{p}) - \hat{q}\cdot\hat{p}\frac{F_l^s \nu_l}{2l+1} P_l(\hat{q}\cdot\hat{p}) - \hat{q}\cdot\hat{p}U = -iI_{\mathbf{p}}/qv_f \quad (3)$$

Now we can use the identity for Legendre polynomials $xP_l(x) = \frac{l+1}{2l+1}P_{l+1}(x) + \frac{l}{2l+1}P_{l-1}(x)$, $x = \hat{q}\cdot\hat{p}$

$$s\nu_l P_l(x) - \nu_l\left(1 + \frac{F_l^s}{2l+1}\right)\left[\frac{l+1}{2l+1}P_{l+1}(x) + \frac{l}{2l+1}P_{l-1}(x)\right] - \hat{q}\cdot\hat{p}U = -iI_{\mathbf{p}}/qv_f \quad (4)$$

If we exploit the orthogonality of Legendre polynomials we can get for each mode l:

$$\nu_l - \frac{1}{s}\left[\frac{l}{2l-1}\nu_{l-1}\left(1 + \frac{F_{l-1}^s}{2l-1}\right) + \frac{l+1}{2l+3}\nu_{l+1}\left(1 + \frac{F_{l+1}^s}{2l+3}\right) + U\delta_{l1}\right] = -iI_l \quad (5)$$

Where $I_l = \frac{1}{\omega\tau}(\nu_l - (\nu_0\delta_{l0} - \nu_1\delta_{l1})$ and we note that $I_0 = I_1 = 0$.

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writing this explicitly for $l = 0...3$, assuming that $F^s_{l>2} = 0$

$$\nu_0 - \frac{1}{3s}\nu_1\left(1 + \frac{F_1^s}{3}\right) = 0 \tag{6}$$

$$\nu_1 - \frac{1}{s}\left[\nu_0\left(1 + F_0^s\right) + \frac{2}{5}\nu_2\left(1 + \frac{F_2^s}{5}\right) + U\right] = 0 \tag{7}$$

$$\nu_2 - \frac{1}{s}\left[\frac{2\nu_1}{3}\left(1 + \frac{F_1^s}{3}\right) + \frac{3}{7}\nu_3\right] = -\frac{i\nu_2}{\omega\tau} \tag{8}$$

$$\nu_3 - \frac{1}{s}\left[\frac{3\nu_2}{5}\left(1 + \frac{F_2^s}{5}\right) + \frac{4}{9}\nu_4\right] = -\frac{i\nu_3}{\omega\tau} \tag{9}$$

We can show the particle conservation equation is the same as equation 6.

$$n(\dot{r}, t) + \nabla\mathbf{j}(r,t) = 0 \tag{10}$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\left[-i\omega\delta n_{\mathbf{p}} + i\mathbf{q}\cdot\mathbf{v}_p(1 + F_1^s/3)\delta n_{\mathbf{p}}\right] = 0 \tag{11}$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\delta(\epsilon_{\mathbf{p}} - \epsilon_f)\left[s\nu_l P_l(\hat{p}\cdot\hat{q}) - P_1(\hat{p}\cdot\hat{q})(1 + F_1^s/3)\nu_l P_l(\hat{p}\cdot\hat{q})\right] = 0 \tag{12}$$

$$s\nu_0 - \frac{1}{3}\nu_1(1 + F_1^s/3) = 0 \tag{13}$$

We can also show that the momentum conservation equation is the same as equation 7:

$$\dot{g}_i + \nabla_j\pi_{ij} + n(r,t)\nabla_i U = 0 \tag{14}$$

Where $\dot{g}_i = m\dot{j}_i = -i\omega\frac{N_0}{3}mv_f(1 + F_1^s/3)q_i\nu_1 = -iN_0\frac{q_i}{3}p_f\omega\nu_1$. In the last step we use the relation for m^*/m. And we use the notation $q_i = \hat{q}\cdot\hat{\imath}$ and note $q_iq_i = 1$. Now we must find $\nabla_j\pi_{ij}$:

$$\nabla_j\pi_{ij} = iN_0 q\nu_l q_j v_f p_f \int\frac{d\Omega_p}{4\pi}p_ip_j\left[P_l(\hat{p}\cdot\hat{q}) + F_{l'}^s\int\frac{d\Omega_{p'}}{4\pi}P_{l'}(\hat{p}'\cdot\hat{p})P_l(\hat{p}\cdot\hat{q})\right] \tag{15}$$

$$= iN_0 q v_f p_f \nu_l[1 + F_l^s/(2l+1)]\left\{q_j\int\frac{d\Omega_p}{4\pi}p_ip_jP_l(\hat{p}\cdot\hat{q})\right\} \tag{16}$$

From HW 3 we know how to get the term in curly brackets, and that only the $l = 0$ and $l = 2$ terms survive:

$$l = 0: \quad q_j\int\frac{d\Omega_p}{4\pi}p_ip_j = q_i/3 \tag{17}$$

for l=2 (summation over j, k, l implied in first term):

$$q_j \int \frac{d\Omega_p}{4\pi} p_i p_j P_l(\hat{p}\cdot\hat{q}) = \frac{1}{2}\int \frac{d\Omega_p}{4\pi} p_i p_j (3 q_k p_k q_l p_l - 1) \tag{18}$$

$$= \frac{q_j}{2}\Big(\frac{3}{15} q_k q_l (\delta_{ij}\delta_{kl} + \delta_{ik}\delta jl + \delta_{il}\delta jk) - \delta_{ij}/3\Big) \tag{19}$$

$$= \frac{q_j}{2}\Big(\frac{3}{15}(\delta_{ij} + 2q_i q_j) - \delta_{ij}/3\Big) \tag{20}$$

$$= \frac{q_j}{2}\Big(-\frac{2}{15}\delta_{ij} + \frac{6}{15} q_i q_j\Big) \tag{21}$$

$$= \frac{2}{15} q_i \tag{22}$$

The last term is the external potential, and we only keep $n^0(r,t)$ for linear response to perturbation U:

$$n(r,t)\nabla_i U = n^0(r,t) iqq_i U \tag{23}$$

$$= iqq_i U\Big\{\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{e^{(\epsilon_p-\epsilon_f)/T}+1}\Big\} \tag{24}$$

The bracketed term is evaluated at $T = 0$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{e^{(\epsilon_p-\epsilon_f)/T}+1} = \frac{N_0}{\sqrt{\epsilon_f}}\int_0^{\epsilon_f} d\epsilon\sqrt{\epsilon} = \frac{2}{3}N_0\epsilon_f = \frac{1}{3}N_0 v_f p_f \tag{25}$$

Plugging all this into the momentum equation and canceling N_0 and p_f everywhere:

$$-i\frac{q_i}{3}\omega\nu_1 + iqv_f\nu_0[1+F_0^s]q_i/3 + iqv_f\nu_2[1+F_2^s/5]\frac{2}{15}q_i + iqv_f q_i U/3 = 0 \tag{26}$$

$$s\nu_1 - \nu_0[1+F_0^s] - \nu_2[1+F_2^s/5]\frac{2}{5} - U = 0 \tag{27}$$

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It is convenient to define $G_n = (1+F_n^s/(2n+1))/(2n+1)$ so the equations can be written:

$$\nu_0 - \frac{G_1}{s}\nu_1 = 0 \tag{28}$$

$$\nu_1 - \frac{1}{s}[G_0\nu_0 + 2G_2\nu_2 + U] = 0 \tag{29}$$

$$\nu_2 - \frac{1}{s}[2G_1\nu_1 + \frac{3}{7}\nu_3] = -\frac{i\nu_2}{\omega\tau} \tag{30}$$

$$\nu_3 - \frac{1}{s}[3G_2\nu_2 + \frac{4}{9}\nu_4] = -\frac{i\nu_3}{\omega\tau} \tag{31}$$

The $l = 0$ equation has an easy solution $\nu_1 = s\nu_0/G_1$. To show that we can terminate the series at $l = 2$ we solve the $l = 2$ equation for ν_2 and plug that result into the $l = 3$ equation:

$$\nu_2 = \frac{2G_1\nu_1 + 3\nu_3/7}{s(1+i/\omega\tau)} = \frac{2s\nu_0 + 3\nu_3/7}{s(1+i/\omega\tau)} \tag{32}$$

$$\nu_3 = \frac{3G_2\nu_2 + 4\nu_4/9}{s(1+i/\omega\tau)} = \frac{6G_1G_2\nu_1 + 9G_2\nu_3/7}{s^2(1+i/\omega\tau)^2} + \frac{4\nu_4/9}{s(1+i/\omega\tau)} \tag{33}$$

Now we can resolve for ν_3 in the equation above and write out the first few equations for ν_l in terms of ν_0 and ν_{l+1} in the limit of $s >> 1$

$$\begin{aligned}
\nu_3 &= \frac{s[6G_2\nu_0 + 4(1+i/\omega\tau)\nu_4/9]}{s^2(1+i/\omega\tau)^2 - 9G_2/7} \approx \frac{6G_2\nu_0 + 4(1+i/\omega\tau)\nu_4/9}{s(1+i/\omega\tau)^2} &(34)\\
\nu_2 &= \frac{2\nu_0}{(1+i/\omega\tau)} + \frac{3\nu_3/7}{s(1+i/\omega\tau)} &(35)\\
\nu_1 &= s\nu_0/G_1 &(36)
\end{aligned}$$

So, in terms of ν_0, we have $\nu_1 \propto s\nu_0$, $\nu_2 \propto \nu_0$, $\nu_3 \propto \nu_0/s$ and for large s, $\nu_3 \to 0$. Now we can write the equations for $\nu_{l=0..2}$ in matrix form

$$\begin{pmatrix} 1 & -G_1/s & 0 \\ -G_0/s & 1 & -2G_2/s \\ 0 & -2G_1/s & 1+i/(\omega\tau) \end{pmatrix} \begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ U/s \\ 0 \end{pmatrix} \tag{37}$$

We wish to find the normal modes of this system which exist when $U \to 0$. Thus, we need to find when the determinant of the matrix on the LHS is zero. The result is and equation

$$(1+i/(\omega\tau)) - 4G_1G_2/s^2 - (1+i/(\omega\tau))G_1G_0/s^2 = 0 \tag{38}$$

$$\frac{1}{s} = \frac{qv_f}{\omega} = \sqrt{\frac{(1+\frac{i}{\omega\tau})}{A + \frac{iB}{\omega\tau}}} \tag{39}$$

Where we defined $A = 4G_1G_2 + G_0G_1$ and $B = G_0G_1$.

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Now we can Taylor expand the two limits of $\omega\tau$ and drop all terms which are quadratic and above

CASE 1, $1/\omega\tau << 1$, zero sound:

$$\begin{aligned} \frac{qv_f}{\omega} &= \sqrt{\frac{1}{A}}\left(1+\frac{i}{\omega\tau}\right)^{1/2}\left(1+\frac{iB}{A\omega\tau}\right)^{-1/2} && (40)\\ &\approx \sqrt{\frac{1}{A}}(1+\frac{i}{2\omega\tau})(1-\frac{iB}{2A\omega\tau}) && (41)\\ &= \sqrt{\frac{1}{A}}(1+\frac{i}{2\omega\tau}-\frac{iB}{2A\omega\tau}) && (42) \end{aligned}$$

Now we can write out the real and imaginary parts of q for these modes $q = q' + iq''$, and the sound speed ($c_0 = \omega/q'$)

$$q' = \frac{\omega}{v_f\sqrt{A}} \Rightarrow tempurature\ independent \quad (43)$$

$$c_0 = v_f\sqrt{A} \quad (44)$$

$$q'' = \frac{1}{2\tau v_f\sqrt{A}}\left(1-\frac{B}{A}\right) \propto T^2 \Rightarrow frequency\ independent \quad (45)$$

CASE 2, $\omega\tau << 1$, first sound:

$$\begin{aligned} \frac{qv_f}{\omega} &= \sqrt{\frac{1}{B}}\left(-i\omega\tau+1\right)^{1/2}\left(-\frac{iA\omega\tau}{B}+1\right)^{-1/2} && (46)\\ &\approx \sqrt{\frac{1}{B}}(1-\frac{i\omega\tau}{2})(1+\frac{iA\omega\tau}{2B}) && (47)\\ &= \sqrt{\frac{1}{B}}(1-\frac{i\omega\tau}{2}+\frac{iA\omega\tau}{2B}) && (48) \end{aligned}$$

Now we can write out the real and imaginary parts of q and sound speed:

$$q' = \frac{\omega}{v_f\sqrt{B}} \Rightarrow tempurature\ independent \quad (49)$$

$$c_1 = v_f\sqrt{B} \quad (50)$$

$$q'' = \frac{\omega^2\tau}{2v_f\sqrt{B}}\left(\frac{A}{B}-1\right) \propto \frac{\omega^2}{T^2} \quad (51)$$

The sound speeds obey the relation from class:

$$\frac{c_0^2-c_1^2}{c_1^2} = (A-B)/B = 4G_1G_2/(G_0G_1) = \frac{4(1+F_2^2/5)}{5(1+F_0^s)} \quad (52)$$