Approximation of Integrals near

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Superconducting state

$$\delta\chi_{\|}(\mathbf{q}) = -\frac{1}{\chi_0}\sum_{\mathbf{k},s}\frac{(f(\epsilon_{k_-s})-f(\epsilon_{k_+s}))(u_{k_+}u_{k-}+v_{k+}v_{k-})^2}{\epsilon_{k_-s}-\epsilon_{k_+s}} - \frac{(1-f(\epsilon_{k_-s})-f(\epsilon_{k_+\bar{s}}))(u_{k_+}v_{k-}-v_{k+}u_{k-})^2}{\epsilon_{k_-s}+\epsilon_{k_+\bar{s}}} - \frac{f(\xi_{k_-s})-f(\xi_{k_+s})}{\xi_{k_-s}-\xi_{k_+s}} \tag{1}$$

$$\delta\chi_{\perp}(\mathbf{q}) = -\frac{1}{\chi_0}\sum_{\mathbf{k},s}\frac{(f(\epsilon_{k_-s})-f(\epsilon_{k_+\bar{s}}))(u_{k_+}u_{k-}+v_{k+}v_{k-})^2}{\epsilon_{k_-s}-\epsilon_{k_+\bar{s}}} - \frac{(1-f(\epsilon_{k_-s})-f(\epsilon_{k_+s}))(u_{k_+}v_{k-}-v_{k+}u_{k-})^2}{\epsilon_{k_-s}+\epsilon_{k_+s}} - \frac{f(\xi_{k_-s})-f(\xi_{k_+\bar{s}})}{\xi_{k_-s}-\xi_{k_+\bar{s}}} \tag{2}$$

We make the following assumptions: $\boxed{\xi_{k\pm} = \pm v_f k_x, \quad \Delta_{k\pm} = \pm v_d k_y}$.

The excitation energies are then equal $\epsilon_{k_+} = \epsilon_{k_-} = -\sqrt{v_f^2k_x^2+v_d^2k_y^2}$Now we can plug these into the formula for the $u's$ and $v's$.:

$$u_{k_\pm} = sgn(\Delta_{k_\pm})\sqrt{\frac{1}{2}\big(1+\xi_{k_\pm/\epsilon_{k_\pm}}\big)} = sgn(\Delta_{k_\pm})\sqrt{\frac{1}{2}\big(1\pm v_fk_x/\sqrt{v_f^2k_x^2+v_d^2k_y^2}\big)}$$
$$v_{k_\pm} = \sqrt{\frac{1}{2}\big(1-\xi_{k_\pm/\epsilon_{k_\pm}}\big)} = \sqrt{\frac{1}{2}\big(1\mp v_fk_x/\sqrt{v_f^2k_x^2+v_d^2k_y^2}\big)}$$

Now we can see that $u_{k_+} = sgn(\Delta_{k_+})v_{k_-}$ and $u_{k_-} = sgn(\Delta_{k_-})v_{k_+}$. Plugging these into the formula for $\delta\chi$ causes the first term to be zero:$u_{k_+}u_{k-}+v_{k+}v_{k-} = sgn(\Delta_{k_+})sgn(\Delta_{k_-})v_{k+}v_{k-}+v_{k+}v_{k-} = 0$.
The second term, however survives:
$(u_{k_+}v_{k-}-v_{k+}u_{k-})^2 = (sgn(\Delta_{k_+})v_{k_-}^2 - sgn(\Delta_{k_-})v_{k+}^2)^2 = (v_{k_-}^2+v_{k+}^2)^2 = 1$
So

$$\delta\chi_{\|}(\mathbf{q}) = -\frac{1}{\chi_0}\sum_{\mathbf{k},s} -\frac{1-f(\epsilon_{k_-s})-f(\epsilon_{k_+\bar{s}})}{2\epsilon_{k_+}} - \frac{f(\xi_{k_-s})-f(\xi_{k_+s})}{kq\cos(\theta)} \tag{3}$$

$$\delta\chi_{\perp}(\mathbf{q}) = -\frac{1}{\chi_0}\sum_{\mathbf{k},s} -\frac{1-f(\epsilon_{k_-s})-f(\epsilon_{k_+s})}{2\epsilon_{k_-}+2s\mu_eH} - \frac{f(\xi_{k_-s})-f(\xi_{k_+\bar{s}})}{kq\cos(\theta)+2s\mu_eH} \tag{4}$$

Now we need to find the regions of integration for both the normal state component and the superconducting component. The function $1-f(\epsilon_{k_-})-f(\epsilon_{k_+})$ which appears in the superconducting term is approximately equal to 1 in the region near the intersections of the fermi surfaces Γ_{k_+} and Γ_{k_-}. We take advantage of this by switching to and elliptic coordinate system and cut off the integration at some energy Λ. Now we can write the sums as integrals, using a normalized area of integration $(A/(2\pi\hbar)^2 = 1)$. The superconducting terms are:

$$\sum_{\mathbf{k},s}\frac{1}{2\epsilon_{k_+}}=\int dk_xdk_y\frac{1}{\sqrt{v_f^2k_x^2+v_d^2k_y^2}}=\frac{1}{v_dv_f}\int dxdy\frac{1}{\sqrt{x^2+y^2}}=\frac{2\pi}{v_dv_f}\int_0^{\sqrt{\Lambda}}dr\frac{r}{\sqrt{r^2}} \tag{5}$$

$$=\frac{2\pi\sqrt{\Lambda}}{v_dv_f} \tag{6}$$

$$\sum_{\mathbf{k},s}\frac{1}{2\epsilon_{k_-}+2s\mu_eH}=\frac{1}{2}\sum_s\int dk_xdk_y\frac{1}{\sqrt{v_f^2k_x^2+v_d^2k_y^2}+s\mu_eH}=\frac{1}{2v_dv_f}\sum_s\int dxdy\frac{1}{\sqrt{x^2+y^2}+s\mu_eH} \tag{7}$$

$$=\frac{\pi}{v_dv_f}\sum_s\int_0^{\sqrt{\Lambda}}dr\frac{r}{r+s\mu_eH}=\frac{\pi}{v_dv_f}\sum_s\Big[r-s\mu_eH\ln|r+s\mu_eH|\Big]_{r=0}^{\sqrt{\Lambda}} \tag{8}$$

$$=\frac{\pi}{v_dv_f}\sum_s\sqrt{\Lambda}-s\mu_eH\ln|\sqrt{\Lambda}/(s\mu_eH)+1| \tag{9}$$

$$=\frac{2\pi\sqrt{\Lambda}}{v_dv_f}-\frac{\pi}{v_dv_f}\mu_eH\ln\left|\frac{\sqrt{\Lambda}+\mu_eH}{-\sqrt{\Lambda}+\mu_eH}\right| \tag{10}$$

Now we calculate the normal state components:

$$\sum_{\mathbf{k},s}\frac{f(\xi_{k_-s})-f(\xi_{k_+s})}{kq\cos(\theta)}=\sum_s\int dk_xdk_y\frac{f(\xi_{k_-s})-f(\xi_{k_+s})}{k_xq}=4\sum_s\int_0^{\sqrt{\Lambda}}dk_x\int_0^{\alpha_{\parallel s}k_x}dk_y\frac{1}{k_xq} \tag{11}$$

$$=4\sum_s\frac{\alpha_{\parallel s}\sqrt{\Lambda}}{q} \tag{12}$$

$$\sum_{\mathbf{k},s}\frac{f(\xi_{k_-s})-f(\xi_{k_+\bar{s}})}{kq\cos(\theta)+2s\mu_eH}=\sum_s\int dk_xdk_y\frac{f(\xi_{k_-s})-f(\xi_{k_+\bar{s}})}{k_xq+2s\mu_eH}=4\sum_s\int_0^{\sqrt{\Lambda}}dk_x\int_0^{\alpha_\perp k_x}dk_y\frac{1}{k_xq+2s\mu_eH} \tag{13}$$

$$=(4\alpha_\perp/q)\sum_s\sqrt{\Lambda}-(2s\mu_eH/q)\ln|\sqrt{\Lambda}/(2s\mu_eH/q)+1| \tag{14}$$

$$=(8\alpha_\perp\sqrt{\Lambda}/q)-(8\mu_eH\alpha_\perp/q^2)\ln\left|\frac{\sqrt{\Lambda}+(2\mu_eH/q)}{-\sqrt{\Lambda}+(2\mu_eH/q)}\right| \tag{15}$$