Electronic Spin Susceptibility Near Superconductoring Domain Wall

Benjamin M. Rosemeyer and Anton B. Vorontsov
Department of Physics, Montana State University, Montana 59717, USA
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We calculate the wave-vector dependent electronic spin susceptibility $\chi_{\alpha\beta}(\mathbf{q},\mathbf{R})$ around a Superconductor-Normal metal interface at zero temperature. We consider 1D free electrons subject to a BCS type Hamiltonian with a step function profile for $\Delta(\mathbf{R}) = sgn(x)\Delta_0$.

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INTRODUCTION

EQUATIONS

Our model is described by a Hamiltonian with two parts. A homogeneous normal part $\mathcal{H}_0$, and an inhomogeneous superconducting part $\mathcal{H}_1$ which we write in mean field.

$$\begin{aligned}\mathcal{H}_0 &= \sum_\alpha \int dx \psi^\dagger_\alpha(x)\left[\frac{-\hbar^2}{2m}\nabla^2 - \mu\right]\psi_\alpha(x) \\ \mathcal{H}_1 &= \int dxdx'\left[\Delta(x,x')\psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x') + h.c.\right]\end{aligned} \tag{1}$$

and susceptibility is a two particle correlation function [1]:

$$\chi_{\alpha\beta}(\mathbf{x},\mathbf{x}',t) = \frac{i\mu_{\mathrm{B}}^2}{\hbar}\langle[S_\alpha(\mathbf{x},t), S_\beta(\mathbf{x}',0)]\theta(t)\rangle_0 \tag{2}$$

where $\mathbf{S}(\mathbf{x},t) = \sum_{\mu\nu}\psi^\dagger_\mu(\mathbf{x},t)\,\boldsymbol{\sigma}_{\mu\nu}\,\psi_\nu(\mathbf{x},t)$

DERIVATION

We wish to compute the steady state, position ($\mathbf{R}$) and vector ($\mathbf{q}$) dependent susceptibility $\chi_{\alpha\beta}(\mathbf{R},\mathbf{q})$ from $\chi_{\alpha\beta}(\mathbf{x},\mathbf{x}',t)$. The vector bit will be the fourier transform of the relative position $\mathbf{q} \leftarrow FT \rightarrow \mathbf{r} = \mathbf{x} - \mathbf{x}'$, and the position is the center of mass coordinate $\mathbf{R} = (\mathbf{x}+\mathbf{x}')/2$.

We proceed by using the Bogoliubov transformation which diagnolizes the hamiltonian, producing energies ϵ_λ:

$$\psi_\mu(\mathbf{x},\mathbf{t}) = \sum_\lambda u_\lambda(\mathbf{x})\gamma_{\lambda\mu}(t) + (i\sigma_2)_{\mu\nu}v_\lambda(\mathbf{x})^*\gamma_{\lambda\nu}(t)^\dagger \tag{3}$$

$$\tag{4}$$

(here $\{\epsilon_\lambda\}$ are the eigenvalues of the Hamiltonian upon using this transformation) which results in new quasiparticle spectrum $\mathcal{H}_0 = \sum_{\lambda\mu}\epsilon_{\lambda\mu}\gamma^\dagger_{\lambda\mu}\gamma_{\lambda\mu}$, with $\epsilon_{\lambda\mu} = \epsilon_\lambda$ Here it is important to note that $u(\mathbf{x})_\lambda$, $v(\mathbf{x})_\lambda$ are no longer plane-wave solutions as in the case for homogeneous superconductivity, but can be themselves expanded as plane-waves $u(\mathbf{x})_\lambda = \sum_{\mathbf{k}} u_{\mathbf{k},\lambda}e^{i\mathbf{kx}}$. For energies $\epsilon_\lambda >> \Delta_0$ the $u_{\mathbf{k},\lambda}$ behave like dirac delta functions ($u_{\mathbf{k},\lambda} \rightarrow \delta(\mathbf{k}-\mathbf{k}_\lambda)$, $|\mathbf{k}_\lambda| = \sqrt{1 \pm \sqrt{\epsilon_\lambda^2 - \Delta^2}}$) to recover the normal dispersion relation and density of states.

It is only energies near Δ_0 which are modified. These modifications are generally known and the Andreev Approximation method works well to find these wave functions. $u_\lambda \propto e^{i\mathbf{p}_f + i\kappa\mathbf{x}}$, $\kappa = \frac{1}{v_f}\sqrt{\epsilon_\lambda - \Delta_0}$

It is also convenient to define the products of u/v's and gammas in the following way:

$$\Gamma_{u\lambda} = u(\mathbf{x})_\lambda\gamma_{\lambda\mu} \tag{5}$$

$$\Gamma_{v\lambda} = (i\sigma_2)_{\mu\nu}v(\mathbf{x})^*_\lambda\gamma^\dagger_{\lambda\nu} \tag{6}$$

OK! Now we can plug these things into 2:

$$\chi_{\alpha\beta}(\mathbf{x},\mathbf{x}',t) = \frac{i\mu_{\mathrm{B}}^2}{\hbar}\sum_{\mu\mu'\nu\nu'}\boldsymbol{\sigma}^\alpha_{\mu\nu}\boldsymbol{\sigma}^\beta_{\mu'\nu'} \tag{7}$$

$$\langle[\psi^\dagger_\mu(\mathbf{x},t)\,\psi_\nu(\mathbf{x},t), \psi^\dagger_{\mu'}(\mathbf{x}',0)\,\psi_{\nu'}(\mathbf{x}',0)]\theta(t)\rangle_0 \tag{8}$$

The correlations evaluate according to Wicks Theorem:

$$\begin{aligned}<[\psi^\dagger_\mu\psi_\nu, \psi^\dagger_{\mu'}\psi_{\nu'}]> &= <\psi^\dagger_\mu\psi_\nu\psi^\dagger_{\mu'}\psi_{\nu'}> - <\psi^\dagger_{\mu'}\psi_{\nu'}\psi^\dagger_\mu\psi_\nu> \\ &= <\psi^\dagger_\mu\psi_\nu><\psi^\dagger_{\mu'}\psi_{\nu'}> - <\psi^\dagger_\mu\psi^\dagger_{\mu'}><\psi_\nu\psi_{\nu'}> \\ &+ <\psi^\dagger_\mu\psi_{\nu'}><\psi_\nu\psi^\dagger_{\mu'}> - <\psi^\dagger_{\mu'}\psi_{\nu'}><\psi^\dagger_\mu\psi_\nu> \\ &+ <\psi^\dagger_{\mu'}\psi^\dagger_\mu><\psi_{\nu'}\psi_\nu> - <\psi^\dagger_{\mu'}\psi_\nu><\psi_{\nu'}\psi^\dagger_\mu> \\ &= - <\psi^\dagger_\mu\psi^\dagger_{\mu'}><\psi_\nu\psi_{\nu'}> + <\psi^\dagger_\mu\psi_{\nu'}><\psi_\nu\psi^\dagger_{\mu'}> \\ &+ <\psi^\dagger_{\mu'}\psi^\dagger_\mu><\psi_{\nu'}\psi_\nu> - <\psi^\dagger_{\mu'}\psi_\nu><\psi_{\nu'}\psi^\dagger_\mu>\end{aligned}$$

At this point we can see that what's left is only inner products of primed ($\mathbf{x}'$, or coordinate 2) with unprimed ($\mathbf{x}$, or coordinate 1) operators.

now we insert the sum over momentums which defines the ψ operators.

$$\frac{i\mu_B^2}{\hbar}\sum_{\mu\nu\mu'\nu'}\theta(t)\sigma^\alpha_{\mu\nu}\sigma^\beta_{\mu\nu}$$
$$\begin{aligned}
\big[&- < (\Gamma_{u\mu}+\Gamma_{v\mu})^\dagger(\Gamma_{u\mu'}+\Gamma_{v\mu'})^\dagger >\\
& < (\Gamma_{u\nu}+\Gamma_{v\nu})(\Gamma_{u\nu'}+\Gamma_{v\nu'}) >\\
&+ < (\Gamma_{u\mu}+\Gamma_{v\mu})^\dagger(\Gamma_{u\nu'}+\Gamma_{v\nu'}) >\\
& < (\Gamma_{u\nu}+\Gamma_{v\nu})(\Gamma_{u\mu'}+\Gamma_{v\mu'})^\dagger >\\
&+ < (\Gamma_{u\mu'}+\Gamma_{v\mu'})^\dagger(\Gamma_{u\mu}+\Gamma_{v\mu})^\dagger >\\
& < (\Gamma_{u\nu'}+\Gamma_{v\nu'})(\Gamma_{u\nu}+\Gamma_{v\nu}) >\\
&- < (\Gamma_{u\mu'}+\Gamma_{v\mu'})^\dagger(\Gamma_{u\nu}+\Gamma_{v\nu}) >\\
& < (\Gamma_{u\nu'}+\Gamma_{v\nu'})(\Gamma_{u\mu}+\Gamma_{v\mu})^\dagger > \big]
\end{aligned}$$

To compute these inner products we use the definition of Γ's (and γ's $<\gamma^\dagger_\mu\gamma_\nu>=\delta_{\mu\nu}f(\epsilon_\mu)e^{-i\omega_\mu t}$). From these, we find that the only non-vanishing inner products are:

$$\begin{aligned}
<\Gamma^\dagger_{u\mu}\Gamma_{u\nu'}> &= \delta_{\mu\nu'}u^*_\mu\, u_\mu\, f(\epsilon_\mu)e^{-i\omega_\mu t}\\
<\Gamma^\dagger_{u\mu}\Gamma^\dagger_{v\nu'}> &= (i\sigma_2)_{\nu'\mu}\, u^*_\mu\, v_{-\mu}\, f(\epsilon_\mu)e^{-i\omega_\mu t}\\
<\Gamma_{v\mu}\Gamma_{u\nu'}> &= (i\sigma_2)_{\mu\nu'}\, v^*_\mu\, u_{-\mu}\, f(\epsilon_{-\mu})e^{-i\omega_{-\mu} t}\\
<\Gamma_{v\mu}\Gamma^\dagger_{v\nu'}> &= \delta_{\mu\nu'}\, v^*_\mu\, v_\mu\, f(\epsilon_{-\mu})e^{-i\omega_{-\mu} t}
\end{aligned}$$

$$\frac{i\mu_B^2}{\hbar}\sum_{\mu\nu\mu'\nu'}\theta(t)\sigma^\alpha_{\mu\nu}\sigma^\beta_{\mu\nu}$$
$$\begin{aligned}
\big[&-(<\Gamma^\dagger_{u\mu}\Gamma^\dagger_{v\mu'}>+<\Gamma^\dagger_{v\mu}\Gamma^\dagger_{u\mu'}>)\\
&(<\Gamma_{v\nu}\Gamma_{u\nu'}>+<\Gamma_{u\nu}\Gamma_{v\nu'}>)\\
&+(<\Gamma^\dagger_{u\mu}\Gamma_{u\nu'}>+<\Gamma^\dagger_{v\mu}\Gamma_{v\nu'}>)\\
&(<\Gamma_{v\nu}\Gamma^\dagger_{v\mu'}>+<\Gamma_{u\nu}\Gamma^\dagger_{u\mu'}>)\\
&+(<\Gamma^\dagger_{u\mu'}\Gamma^\dagger_{v\mu}>+<\Gamma^\dagger_{v\mu'}\Gamma^\dagger_{u\mu}>)\\
&(<\Gamma_{v\nu'}\Gamma_{u\nu}>+<\Gamma_{u\nu'}\Gamma_{v\nu}>)\\
&-(<\Gamma^\dagger_{u\mu'}\Gamma_{u\nu}>+<\Gamma^\dagger_{v\mu'}\Gamma_{v\nu}>)\\
&(<\Gamma_{v\nu'}\Gamma^\dagger_{v\mu}>+<\Gamma_{u\nu'}\Gamma^\dagger_{u\mu}>)\big]
\end{aligned}$$

$$\begin{aligned}
\frac{i\mu_B^2}{\hbar}\sum_{\mu\nu}\theta(t)\sigma^\alpha_{\mu\nu}\sigma^\beta_{\mu\nu}\Bigg[&+\delta_{\mu\nu'}\delta_{\nu\mu'}\Bigg[(u^*_\mu(\mathbf{x})\,u_\mu(\mathbf{x}')\,f_\mu e^{i\omega_\mu t}+v_\mu(\mathbf{x})\,v^*_\mu(\mathbf{x}')\,(1-f_{-\mu})e^{-i\omega_{-\mu}t})\\
&\quad(v^*_\nu(\mathbf{x})\,v_\nu(\mathbf{x}')\,f_{-\nu}e^{i\omega_{-\nu}t}+u_\nu(\mathbf{x})\,u^*_\nu(\mathbf{x}')\,(1-f_\nu)e^{-i\omega_\nu t})\\
&\quad-(u^*_\nu(\mathbf{x}')\,u_\nu(\mathbf{x})\,f_\nu e^{-i\omega_\nu t}+v_\nu(\mathbf{x}')\,v^*_\nu(\mathbf{x})\,(1-f_{-\nu})e^{i\omega_{-\nu}t})\\
&\quad(v^*_\mu(\mathbf{x}')\,v_\mu(\mathbf{x})\,f_{-\mu}e^{-i\omega_{-\mu}t}+u_\mu(\mathbf{x}')\,u^*_\mu(\mathbf{x})\,(1-f_\mu)e^{i\omega_\mu t})\Bigg]\\
&+(i\sigma_2)_{\mu\mu'}(i\sigma_2)_{\nu'\nu}\Bigg[(u^*_{-\mu}(\mathbf{x}')\,v_\mu(\mathbf{x})\,f_{-\mu}e^{-i\omega_{-\mu}t}-v_{-\mu}(\mathbf{x}')\,u^*_\mu(\mathbf{x})\,(1-f_\mu)e^{i\omega_\mu t})\\
&\quad(v^*_{-\nu}(\mathbf{x}')\,u_\nu(\mathbf{x})\,f_\nu e^{-i\omega_\nu t}-u_{-\nu}(\mathbf{x}')\,v^*_\nu(\mathbf{x})\,(1-f_{-\nu})e^{i\omega_{-\nu}t})\\
&\quad-(u^*_\mu(\mathbf{x})\,v_{-\mu}(\mathbf{x}')\,f_\mu e^{i\omega_\mu t}-v_\mu(\mathbf{x})\,u^*_{-\mu}(\mathbf{x}')\,(1-f_{-\mu})e^{-i\omega_{-\mu}t})\\
&\quad(v^*_\nu(\mathbf{x})\,u_{-\nu}(\mathbf{x}')\,f_{-\nu}e^{i\omega_{-\nu}t}-u_\nu(\mathbf{x})\,v^*_{-\nu}(\mathbf{x}')\,(1-f_\nu)e^{-i\omega_\nu t})\Bigg]
\end{aligned}$$

Additionally, we consider the two tensor components which are perpendicular to ($\alpha\beta=xx$ or yy) and parallel to ($\alpha\beta=zz$) the applied field $\mathbf{H}=H_0\hat{z}$. In either case all the spin coefficients evaluate to $+1$, the only difference is in the spin pairing. xx pairs opposite spins ($\nu=-\mu$) and zz pairs like spins ($\nu=\mu$).

$$
\begin{aligned}
\frac{i\mu_B^2}{\hbar}\theta(t)\sum_{\mu\nu} & (u_\mu^*(\mathbf{x})\, u_\mu(\mathbf{x}')\, f_\mu e^{i\omega_\mu t} + v_\mu(\mathbf{x})\, v_\mu^*(\mathbf{x}')\,(1-f_{-\mu})e^{-i\omega_{-\mu} t}) \\
& (v_\nu^*(\mathbf{x})\, v_\nu(\mathbf{x}')\, f_{-\nu} e^{i\omega_{-\nu} t} + u_\nu(\mathbf{x})\, u_\nu^*(\mathbf{x}')\,(1-f_\nu)e^{-i\omega_\nu t}) \\
& -(u_\nu^*(\mathbf{x}')\, u_\nu(\mathbf{x})\, f_\nu e^{-i\omega_\nu t} + v_\nu(\mathbf{x}')\, v_\nu^*(\mathbf{x})\,(1-f_{-\nu})e^{i\omega_{-\nu} t}) \\
& (v_\mu^*(\mathbf{x}')\, v_\mu(\mathbf{x})\, f_{-\mu} e^{-i\omega_{-\mu} t} + u_\mu(\mathbf{x}')\, u_\mu^*(\mathbf{x})\,(1-f_\mu)e^{i\omega_\mu t}) \\
& +(u_{-\mu}^*(\mathbf{x}')\, v_\mu(\mathbf{x})\, f_{-\mu} e^{-i\omega_{-\mu} t} - v_{-\mu}(\mathbf{x}')\, u_\mu^*(\mathbf{x})\,(1-f_\mu)e^{i\omega_\mu t}) \\
& (v_{-\nu}^*(\mathbf{x}')\, u_\nu(\mathbf{x})\, f_\nu e^{-i\omega_\nu t} - u_{-\nu}(\mathbf{x}')\, v_\nu^*(\mathbf{x})\,(1-f_{-\nu})e^{i\omega_{-\nu} t}) \\
& -(u_\mu^*(\mathbf{x})\, v_{-\mu}(\mathbf{x}')\, f_\mu e^{i\omega_\mu t} - v_\mu(\mathbf{x})\, u_{-\mu}^*(\mathbf{x}')\,(1-f_{-\mu})e^{-i\omega_{-\mu} t}) \\
& (v_\nu^*(\mathbf{x})\, u_{-\nu}(\mathbf{x}')\, f_{-\nu} e^{i\omega_{-\nu} t} - u_\nu(\mathbf{x})\, v_{-\nu}^*(\mathbf{x}')\,(1-f_\nu)e^{-i\omega_\nu t})
\end{aligned}
$$

$$
\begin{aligned}
\frac{i\mu_B^2}{\hbar}\sum_{\mu\nu}\theta(t)\Big[& -u_{\mu 1}^*\, u_{\mu 2}\, v_{\nu 1}^*\, v_{\nu 2}\,(1-f_{-\nu 1}-f_{\mu 1})e^{i(\omega_{\mu 1}+\omega_{-\nu 1})t} \\
& + u_{\mu 1}^*\, u_{\mu 2}\, u_{\nu 1}\, u_{\nu 2}^*\,(f_{\mu 1}-f_{\nu 1})e^{i(\omega_{\mu 1}-\omega_{\nu 1})t} \\
& + v_{\mu 1}\, v_{\mu 2}^*\, v_{\nu 1}^*\, v_{\nu 2}\,(f_{-\nu 1}-f_{-\mu 1})e^{i(\omega_{-\nu 1}-\omega_{-\mu 1})t} \\
& + v_{\mu 1}\, v_{\mu 2}^*\, u_{\nu 1}\, u_{\nu 2}^*\,(1-f_{\nu 1}-f_{-\mu 1})e^{-i(\omega_{\nu 1}+\omega_{-\mu 1})t} \\
& - u_{-\mu 2}^*\, v_{\mu 1}\, v_{-\nu 2}^*\, u_{\nu 1}\,(1-f_{\nu 1}-f_{-\mu 1})e^{-i(\omega_{-\mu 1}+\omega_{\nu 1})t} \\
& + u_{-\mu 2}^*\, v_{\mu 1}\, v_{\nu 1}^*\, u_{-\nu 2}\,(f_{-\nu 1}-f_{-\mu 1})e^{i(\omega_{-\nu 1}-\omega_{-\mu 1})t} \\
& + u_{\mu 1}^*\, v_{-\mu 2}\, v_{-\nu 2}^*\, u_{\nu 1}\,(f_{\mu 1}-f_{\nu 1})e^{i(\omega_{\mu 1}-\omega_{\nu 1})t} \\
& + u_{\mu 1}^*\, v_{-\mu 2}\, v_{\nu 1}^*\, u_{-\nu 2}\,(1-f_{-\nu 1}-f_{\mu 1})e^{i(\omega_{\mu 1}+\omega_{-\nu 1})t}\Big]
\end{aligned}
$$

At this point we are ready to go to a steady state solution by integrating out the time

$$
\chi = \lim_{\eta\to 0^+}\int_0^\infty \chi(t)e^{-\eta t}dt \tag{9}
$$

$$
\begin{aligned}
\frac{\mu_B^2}{\hbar}\sum_{\mu\nu}\theta(t)\Big[&- u_\mu^*(\mathbf{x})\,u_\mu(\mathbf{x}')\,v_\nu^*(\mathbf{x})\,v_\nu(\mathbf{x}')\,\frac{1-f_{-\nu}-f_\mu}{\omega_\mu+\omega_{-\nu}+i\eta} \\
&+ u_\mu^*(\mathbf{x})\,u_\mu(\mathbf{x}')\,u_\nu(\mathbf{x})\,u_\nu(\mathbf{x}')^*\,\frac{f_\mu-f_\nu}{\omega_\mu-\omega_\nu+i\eta} \\
&+ v_\mu(\mathbf{x})\,v_\mu^*(\mathbf{x}')\,v_\nu^*(\mathbf{x})\,v_\nu(\mathbf{x}')\,\frac{f_{-\nu}-f_{-\mu}}{\omega_{-\nu}-\omega_{-\mu}+i\eta} \\
&+ v_\mu(\mathbf{x})\,v_\mu^*(\mathbf{x}')\,u_\nu(\mathbf{x})\,u_\nu(\mathbf{x}')^*\,\frac{1-f_\nu-f_{-\mu}}{\omega_\nu+\omega_{-\mu}-i\eta} \\
&- u_{-\mu}^*(\mathbf{x}')\,v_\mu(\mathbf{x})\,v_{-\nu}^*(\mathbf{x}')\,u_\nu(\mathbf{x})\,\frac{1-f_\nu-f_{-\mu}}{\omega_{-\mu}+\omega_\nu-i\eta} \\
&+ u_{-\mu}^*(\mathbf{x}')\,v_\mu(\mathbf{x})\,v_\nu^*(\mathbf{x})\,u_{-\nu}(\mathbf{x}')\,\frac{f_{-\nu}-f_{-\mu}}{\omega_{-\nu}-\omega_{-\mu}+i\eta} \\
&+ u_\mu^*(\mathbf{x})\,v_{-\mu}(\mathbf{x}')\,v_{-\nu}^*(\mathbf{x}')\,u_\nu(\mathbf{x})\,\frac{f_\mu-f_\nu}{\omega_\mu-\omega_\nu+i\eta} \\
&+ u_\mu^*(\mathbf{x})\,v_{-\mu}(\mathbf{x}')\,v_\nu^*(\mathbf{x})\,u_{-\nu}(\mathbf{x}')\,\frac{1-f_{-\nu}-f_\mu}{\omega_\mu+\omega_{-\nu}+i\eta}\Big]
\end{aligned}
$$

Next we can see that for S and D-wave cases, there is $\pm$ symmetry for momentums and we can write all momentums as positive (NOTE: this would not be ok for P-wave superconductors). We will also switch some of the spin indicies around ($\mu \leftrightarrow -\mu$, $\nu \leftrightarrow -\nu$), which we can do becuase we are summing all μ and ν anyway. In addition, we restrict ourselves to the real part of the susceptibility by taking $\eta = 0^+$.

$$
\begin{aligned}
&\frac{\mu_B^2}{\hbar}\sum_{\mu\nu\mu\nu} \\
&\Big[-u_\mu^*(\mathbf{x})\,u_\mu(\mathbf{x}')\,v_\nu^*(\mathbf{x})\,v_\nu(\mathbf{x}') - u_\mu^*(\mathbf{x}')\,v_{-\mu}(\mathbf{x})\,v_\nu^*(\mathbf{x}')\,u_{-\nu}(\mathbf{x}) \\
&\quad + v_{-\mu}(\mathbf{x})\,v_{-\mu}^*(\mathbf{x}')\,u_{-\nu}(\mathbf{x})\,u_{-\nu}(\mathbf{x}')^* + u_\mu^*(\mathbf{x})\,v_{-\mu}(\mathbf{x}')\,v_\nu^*(\mathbf{x})\,u_{-\nu}(\mathbf{x}')\Big]\frac{1-f_{-\nu}-f_\mu}{\omega_\mu+\omega_{-\nu}} \\
&+\Big[u_\mu^*(\mathbf{x})\,u_\mu(\mathbf{x}')\,u_\nu(\mathbf{x})\,u_\nu(\mathbf{x}')^* + v_{-\mu}(\mathbf{x})\,v_{-\mu}^*(\mathbf{x}')\,v_{-\nu}^*(\mathbf{x})\,v_{-\nu}(\mathbf{x}') \\
&\quad + u_\mu^*(\mathbf{x}')\,v_{-\mu}(\mathbf{x})\,v_{-\nu}^*(\mathbf{x})\,u_\nu(\mathbf{x}') + u_\mu^*(\mathbf{x})\,v_{-\mu}(\mathbf{x}')\,v_{-\nu}^*(\mathbf{x}')\,u_\nu(\mathbf{x})\Big]\frac{f_\mu-f_\nu}{\omega_\mu-\omega_\nu}
\end{aligned}
$$

Now we can deal with the spacial dependence of the u/v's. To simplify further we can pick the center of mass frame in which to calculate (ie $\mathbf{R} = (\mathbf{x}+\mathbf{x}')/2 = 0$) so that the coordinates $\mathbf{x} = \mathbf{r}/2$ and $\mathbf{x}' = -\mathbf{r}/2$ where $\mathbf{r} = \mathbf{x}-\mathbf{x}'$ is the relative coordinate. In doing this, we will vary the position of the interface relative to the center of mass without loss of generality. The result is:

Where now $\pm\nu 1$ is a coordinate for momentum ν and spin $\pm\mu$. It's also important to notice that now the u's and v's now depend on $\pm\mathbf{r}/2$ as well. (coordinates $\mu 1$ and $\nu 1$ are momentums at $\mathbf{r}/2$ for energies $\epsilon_{\mu 1}$ and $\epsilon_{\nu 1}$ respectively, coordinates $\mu 2$ and $\nu 2$ are momentums at $-\mathbf{r}/2$ for energies $\epsilon_{\mu 1}$ and $\epsilon_{\nu 1}$ respectively.

$$\frac{\mu_B^2}{\hbar}\sum_{\mu\nu\mu\nu}$$
$$\Big[(u_{\mu1}\,v_{\mu2}\,v_{\nu1}\,u_{\nu2}-u_{\mu1}\,u_{\mu2}\,v_{\nu1}\,v_{\nu2})e^{i(-\mu1\mathbf{r}/2-\mu2\mathbf{r}/2-\nu1\mathbf{r}/2-\nu2\mathbf{r}/2)}$$
$$+(u_{\mu2}\,v_{\mu1}\,v_{\nu2}\,u_{\nu1}-v_{\mu1}\,v_{\mu2}\,u_{\nu1}\,u_{\nu2})e^{i(\mu1\mathbf{r}/2+\mu2\mathbf{r}/2+\nu1\mathbf{r}/2+\nu2\mathbf{r}/2)}\Big]\frac{1-f_{-\nu1}-f_{\mu1}}{\omega_{\mu1}+\omega_{-\nu1}}$$
$$+\Big[(u_{\mu1}\,u_{\mu2}\,u_{\nu1}\,u_{\nu2}+u_{\mu1}\,v_{\mu2}\,v_{\nu2}\,u_{\nu1})e^{i(-\mu1\mathbf{r}/2-\mu2\mathbf{r}/2+\nu1\mathbf{r}/2+\nu2\mathbf{r}/2)}$$
$$+(v_{\mu1}\,v_{\mu2}\,v_{\nu1}\,v_{\nu2}+u_{\mu2}\,v_{\mu1}\,v_{\nu1}\,u_{\nu2})e^{i(\mu1\mathbf{r}/2+\mu2\mathbf{r}/2-\nu1\mathbf{r}/2-\nu2\mathbf{r}/2)}\Big]\frac{f_{\mu1}-f_{\nu1}}{\omega_{\mu1}-\omega_{\nu1}}$$

The final simplification is made by assuming that the variation of the order parameter is only in the $\hat{x}$ direction and that the $\hat{y}$ direction is homogeneous. We can thereby easily fourier transform the y component to arrive at assuming that $\mu1=\mu2$ and $\nu1=\nu2$ when integrating only $\mathbf{r}_y$:

$$\chi(\mathbf{r}_x,q_y,\mathbf{R}=0)=\frac{\mu_B^2}{\hbar}\sum_{\mu\nu_x\mu}$$
$$\Big[(u_{\mu1}\,v_{\mu2}\,v_{\nu1}\,u_{\nu2}-u_{\mu1}\,u_{\mu2}\,v_{\nu1}\,v_{\nu2})e^{i(-\mu1_x\mathbf{r}_x/2-\mu2_x\mathbf{r}_x/2-\nu1_x\mathbf{r}_x/2-\nu2_x\mathbf{r}_x/2)}\Big|_{-\mu1_y-\nu1_y-q_y=0}$$
$$+(u_{\mu2}\,v_{\mu1}\,v_{\nu2}\,u_{\nu1}-v_{\mu1}\,v_{\mu2}\,u_{\nu1}\,u_{\nu2})e^{i(\mu1_x\mathbf{r}_x/2+\mu2_x\mathbf{r}_x/2+\nu1_x\mathbf{r}_x/2+\nu2_x\mathbf{r}_x/2)}\Big|_{\mu1_y+\nu1_y-q_y=0}\Big]\frac{1-f_{-\nu1}-f_{\mu1}}{\omega_{\mu1}+\omega_{-\nu1}}$$
$$+\Big[(u_{\mu1}\,u_{\mu2}\,u_{\nu1}\,u_{\nu2}+u_{\mu1}\,v_{\mu2}\,v_{\nu2}\,u_{\nu1})e^{i(-\mu1_x\mathbf{r}_x/2-\mu2_x\mathbf{r}_x/2+\nu1_x\mathbf{r}_x/2+\nu2_x\mathbf{r}_x/2)}\Big|_{-\mu1_y+\nu1_y-q_y=0}$$
$$+(v_{\mu1}\,v_{\mu2}\,v_{\nu1}\,v_{\nu2}+u_{\mu2}\,v_{\mu1}\,v_{\nu1}\,u_{\nu2})e^{i(\mu1_x\mathbf{r}_x/2+\mu2_x\mathbf{r}_x/2-\nu1_x\mathbf{r}_x/2-\nu2_x\mathbf{r}_x/2)}\Big|_{\mu1_y-\nu1_y-q_y=0}\Big]\frac{f_{\mu1}-f_{\nu1}}{\omega_{\mu1}-\omega_{\nu1}}$$

To find the Fourier Transform wrt the x coordinate we can use a fast fourier transform from the spacial domain $x\in[-L:L]$ to momentum space $q_x\in[-n\pi:n\pi]$ using $2N+1$ points ($N=nL$):

$$\chi(\mathbf{q})=\sum_{i=1}^{2*N+1}e^{-iq_x\mathbf{r}_x(i)}\chi(\mathbf{r}_x(i),q_y,\mathbf{R}=0)\qquad(10)$$

NOTE: currently I am only looking at the real part of $\chi(\mathbf{r},\mathbf{R}=0)$, so the exponentials turn into cosines and I have $\pm$ symmetry for x and my FT looks like

$$\chi(\mathbf{q})=\sum_{i=1}^{N+1}\cos(q_x\mathbf{r}_x(i))\chi(\mathbf{r}_x(i),q_y,\mathbf{R}=0)\qquad(11)$$

[1] G. D. Mahan, *Many-Particle Physics*, 3rd ed. (Plenum Publishers, 2000).