Electronic Spin Susceptibility Enhancement in Pauli Limited Unconventional Superconductors

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We calculate the wave-vector dependent electronic spin susceptibility $\chi_{\alpha\beta}(\mathbf{q}, \mathbf{H}_0)$ of a superconducting state in uniform magnetic field $\mathbf{H}_0$. We consider Pauli limited materials with d-wave symmetry, and a 2D cylindrical electronic Fermi surface. We find that both longitudinal and transverse components of the susceptibility tensor are enhanced over their normal state values in the high-field low-temperature region of the $H-T$ phase diagram. We identify several wave vectors, connecting field-produced hot spots on the Fermi surface, that correspond to the enhancement of either $\chi_\perp$ or $\chi_\parallel$ components.

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I. INTRODUCTION

The interplay of different orders, simultaneously present in the same system, is of a great interest to physics community and beyond. Such interactions present another level of complexity in emergent systems. They offer insight into the properties of the ordered phases, but also often pose a great challenge both mathematically and conceptionally.[?] The two most widely studied orders, that often appear together in many systems, are superconductivity and magnetism, both having their origin tightly connected to the behavior of the quantum mechanical spin. The behavior of these orders have been investigated under many different conditions. Ferromagnetic order and singlet superconductivity avoid each other since they have opposite spin structures[?] ; on the contrary triplet superconductivity is much less sensitive to the ferromagnetism[?] ; antiferromagnetic order with spatial period much smaller than the superconducting coherence length does not interfere with superconductivity, and the two orders may easily appear together[?] , and in some cases unconventional superconducting and spin-density wave orders even attract each other[?] .

Despite this great amount of knowledge, the interplay of antiferromagnetism (AFM) and superconductivity (SC) still often challenges our understanding of this phenomenon, for various reasons. For example, materials such as cuprate oxides have unusual normal state properties[?] , whereas iron-based superconductors have complex orbital multiband structure[?] .

One other, interesting and relatively recent example, is heavy-fermion material $CeCoIn_5$, which manifests a co-existent AFM and superconducting state (Q-phase,[?]) in the high-field low-temperature limit. Experimental results for $CeCoIn_5$ indicate that this co-existence is the result of strong attraction between superconducting unconventional d-wave order, and the AFM state[1–3]. The Q-phase and magnetism disappear when superconductivity is suppressed at the upper critical field H_{c2} through a first order transition to the normal state. The magnetic order appear inside the SC state through a second order transition. Moreover, although the normal state of $CeCoIn_5$ is non-magnetic, the proximity to magnetic instability can be demonstrated by applying doping to induce it[?] , or pressure, to destroy it[?] .

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Other results for $CeCoIn_5$ show that the magnetic ordering wave vector in the Q-phase displays little or no dependence on field (within experimental error), which indicates a relatively small or non-existent interaction with the flux lattice [3]. This phenomenon has not been directly addressed in many of the current theories. Our findings indicate that this behavior is more consistent with the longitudinal component of suscep tibility rather than the transverse. This is another area of study which has not been very thouroghly analyzed.

Many theories have been proposed to explain the experimental results seen in $CeCoIn_5$, including various manifestations of the FFLO state where the superconducting order parameter oscillates in real space. [4–9]. The FFLO state has long been believed to offer the possibility of magentic order, however, the predicted field dependence of the ordering vector is not seen in experiment, and recent experiments point to a homogeneous gap function [3,10].

Yet other theories offer insight but leave many questions unanswered. Such as: How does the ordering wave vector change with applied field? What are the necessary conditions on the order parameter? And what roles do the transverse and longitudinal components of the magnatism play? Here we provide a first principles approach to these problems by calculating the magnetic susceptibility of itinerant electrons in the D-wave superconducting state. In this paper we investigate the microscopic underlay of such interaction.

II. MODEL

To investigate the interplay of superconductivity and magnetic order in external field, we consider mean-field SC Hamiltonian of 2D electrons with cylindrical FS, in-

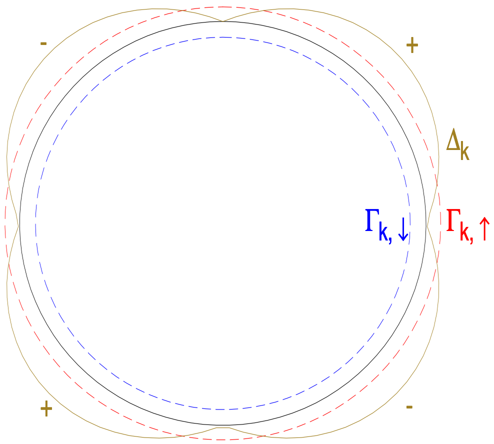

FIG. 1:

We consider 2D electrons with circular Fermi surface (FS). Zeeman interaction splits Fermi surfaces for up and down spins ($\Gamma_{\mathbf{k}\uparrow}, \Gamma_{\mathbf{k}\downarrow}$). The d-wave order parameter $\Delta_{\mathbf{k}} \propto \sin 2\theta_{\mathbf{k}}$ is also shown. Magnetic field produces pockets of low energy spin-down excitations near the nodes of the order parameter.

teracting with magnetic field through Zeeman term:

$$\mathcal{H} = \mathcal{H}_N + \mathcal{H}_{sc} + \mathcal{H}_Z \qquad (1)$$

$$\mathcal{H}_N = \sum_{\mathbf{k},s} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k},s} c_{\mathbf{k},s} \qquad (2)$$

$$\mathcal{H}_{sc} = \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}} c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow} + h.c. \right) \qquad (3)$$

$$\mathcal{H}_Z = \sum_{\mathbf{k},\mu,\nu} c^\dagger_{\mathbf{k}+\mathbf{q},\mu} \mu_B \boldsymbol{\sigma}_{\mu\nu} (\mathbf{H}_0 \delta_{\mathbf{q},0} + \delta \mathbf{H}_{\mathbf{q}}) c_{\mathbf{k},\nu}$$

where the electronic dispersion in the normal state is $\xi_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m^*} - \epsilon_F$, and μ_B is the magnetic moment of electron (Bohr magneton). We assumed a strong uniform magnetic field and perturbation at wavevector $\mathbf{q}$,

$$\mathbf{H}(\mathbf{R}) = \mathbf{H}_0 + \delta \mathbf{H}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}} . \qquad (4)$$

The resulting magnetization is due to the interaction term $V = -\mathbf{m} \cdot \mathbf{H}$, and also has uniform part and linear response to perturbation:

$$M_\alpha(\mathbf{R}) = M_{0\alpha}(\mathbf{H}_0) + \delta M_\alpha(\mathbf{R}) \qquad (5)$$

$$\delta M_\alpha(\mathbf{R}) = \chi_{\alpha\beta}(\mathbf{q}) \delta H_\beta e^{i\mathbf{q}\cdot\mathbf{R}} \qquad (6)$$

given by the standard expressions.[11]

$$\text{Add equation for } \mathbf{M}_0 \,, \text{Check expression for } \chi \,, \text{Check transformation} \qquad (7)$$

$$M_\alpha(t) = -i \int_{-\infty}^{t} < [m_\alpha(t), V(t')] > dt' \qquad (8)$$

$$\chi_{\alpha\beta}(\mathbf{x}, \mathbf{x}', \omega) = -i\mu_B^2 \sum_{\mu\mu'\nu\nu'} \int_{-\infty}^{0} dt' \; \sigma^\alpha_{\mu\mu'} \sigma^\beta_{\nu\nu'} e^{i\omega t'} \; \langle [\psi^\dagger_\mu(\mathbf{x}', t') \psi_{\mu'}(\mathbf{x}', t'), \psi^\dagger_\nu(\mathbf{x}, 0) \psi_{\nu'}(\mathbf{x}, 0)] \rangle \qquad (9)$$

We are mostly interested in the electronic susceptibility χ, since it determines the magnetic instability into an SDW (spin-density wave) state, or responsible for RKKY-type interaction between localized moments on Ce atoms. To calculate the susceptibility (9) in superconducting state, we diagonalize the Hamiltonian (3) by the Bogoliubov transformation[12], that defines the elctron operators $c_{\mathbf{k}\alpha}$ through new quasiparticle operators $\gamma_{\mathbf{k}\alpha}$:

$$c_{\mathbf{k}\mu} = u_{\mathbf{k}} \gamma_{\mathbf{k}\mu} + \sum_{\mu'} (i\sigma_2)_{\mu\mu'} v^*_{\mathbf{k}} \gamma^\dagger_{-\mathbf{k}\mu'} \qquad (10)$$

$$\epsilon_{\mathbf{k},\mu} = \sqrt{\Delta^2_{\mathbf{k}} + \xi^2_{\mathbf{k}}} + (\sigma_3)_{\mu\mu} \mu_B H_0 \qquad (11)$$

$$\xi_{\mathbf{k}} = e_{\mathbf{k}} - \epsilon_f \qquad (12)$$

$$u_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\Delta^2_{\mathbf{k}} + \xi^2_{\mathbf{k}}}}\right)} \qquad (13)$$

$$v_{\mathbf{k}} = \mathrm{sgn}(\Delta_{\mathbf{k}}) \sqrt{\frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\Delta^2_{\mathbf{k}} + \xi^2_{\mathbf{k}}}}\right)} \qquad (14)$$

The diagonalized Hamiltonian is

$$\mathcal{H} = \sum_{\mathbf{k}s} \epsilon_{\mathbf{k}s} \gamma^\dagger_{\mathbf{k}s} \gamma_{\mathbf{k}s} \qquad (15)$$

We then use this Hamiltonian and time-dependent operators to compute the magnetic susceptibility of electrons. In the presence of external magnetic field $\mathbf{H}_0$ the spin-rotational symmetry is broken even in magnetically isotropic system. Now one has to distinguish between two possibilities for the direction of the wave-vector dependent magnetization: (a) $\delta\mathbf{M}(\mathbf{q}) \parallel \mathbf{H}_0$ (longitudinal response), and (b) $\delta\mathbf{M}(\mathbf{q}) \perp \mathbf{H}_0$ (transverse response). In standard literature e.g.[?] it is discussed that the longitudinal response is usually smaller than the transverse, in case of antiferromagnetism. We will show, however, that the standard reasoning is not applicable here, where the magnetic field and presense of unconventional superconductivity create a new environment/terms in GL functional that stabilize the parallel component as well.

The general expressions for the real part of the longitudinal and transverse susceptibilities are:

$$\chi^{sc}_{\parallel}(\mathbf{q}) = -\mu_B^2 \sum_{\mathbf{k},s} \frac{(f(\epsilon_{\mathbf{k}_- s}) - f(\epsilon_{\mathbf{k}_+ s}))(u_{\mathbf{k}_+}u_{\mathbf{k}_-} + v_{\mathbf{k}_+}v_{\mathbf{k}_-})^2}{\epsilon_{\mathbf{k}_- s} - \epsilon_{\mathbf{k}_+ s}} - \frac{(1 - f(\epsilon_{\mathbf{k}_- s}) - f(\epsilon_{\mathbf{k}_+ (-s)}))(u_{\mathbf{k}_+}v_{\mathbf{k}_-} - v_{\mathbf{k}_+}u_{\mathbf{k}_-})^2}{\epsilon_{\mathbf{k}_- s} + \epsilon_{\mathbf{k}_+ (-s)}} \tag{16}$$

$$\chi^{sc}_{\perp}(\mathbf{q}) = -\mu_B^2 \sum_{\mathbf{k},s} \frac{(f(\epsilon_{\mathbf{k}_- s}) - f(\epsilon_{\mathbf{k}_+ (-s)}))(u_{\mathbf{k}_+}u_{\mathbf{k}_-} + v_{\mathbf{k}_+}v_{\mathbf{k}_-})^2}{\epsilon_{\mathbf{k}_- s} - \epsilon_{\mathbf{k}_+ (-s)}} - \frac{(1 - f(\epsilon_{\mathbf{k}_- s}) - f(\epsilon_{\mathbf{k}_+ s}))(u_{\mathbf{k}_+}v_{\mathbf{k}_-} - v_{\mathbf{k}_+}u_{\mathbf{k}_-})^2}{\epsilon_{\mathbf{k}_- s} + \epsilon_{\mathbf{k}_+ s}} \tag{17}$$

where $\chi_0 = 2N_F\mu_B^2$ is Pauli susceptibility in the normal state, $f(\epsilon) = (\exp(\epsilon/T) + 1)^{-1}$ is the Fermi distribution, and momenta are shifted by the magnetization wave vector $\mathbf{k}_\pm = \mathbf{k} \pm \mathbf{q}/2$. The derivation of this result is presented in Appendix A.

III. NORMAL STATE

We first review the results for the normal state susceptibility in magnetic field. By setting $\Delta_{\mathbf{k}} = 0$ in the general expression above, one obtains the familiar Lindhard function,

$$\chi_{\parallel} = -2\mu_B^2 \sum_{\mathbf{k},\mu} \frac{f(\xi_{\mathbf{k},\mu}) - f(\xi_{\mathbf{k+q},\mu})}{\xi_{\mathbf{k},\mu} - \xi_{\mathbf{k+q},\mu}} \tag{18}$$

$$\chi_{\perp} = -2\mu_B^2 \sum_{\mathbf{k},s} \frac{f(\xi_{\mathbf{k},\mu}) - f(\xi_{\mathbf{k+q},-\mu})}{\xi_{\mathbf{k},\mu} - \xi_{\mathbf{k+q},-\mu}} \tag{19}$$

where $\xi_{\mathbf{k}\mu} = \frac{k^2}{2m^*} - \epsilon_f + (\sigma_3)_{\mu\mu}\mu_B H_0$ are electron excitation energies in magnetic field (excluding orbital effects)

At zero temperature the Fermi functions are step-functions, and the summation over momenta can be done analytically; in two dimensions we get (details of the integration is presented in appendix B),

$$\frac{\chi^N_{\parallel}(q)}{\chi^N_0} = 1 - \frac{\Theta(q-2k_{f\uparrow})}{2}\sqrt{1-(\tfrac{2}{q})^2 k_{f\uparrow}^2} - \frac{\Theta(q-2k_{f\downarrow})}{2}\sqrt{1-(\tfrac{2}{q})^2 k_{f\downarrow}^2} \tag{20}$$

$$\frac{\chi^N_{\perp}(q)}{\chi^N_0} = 1 - \Theta(q - k_{f\uparrow} - k_{f\downarrow})\sqrt{1-(\tfrac{2}{q})^2(1-(k_{f\uparrow}^2 - k_{f\downarrow}^2)^2/(2q)^2)} \tag{21}$$

Where we have normalized everything so that $\epsilon_f = 1$ and $k^2_{f\uparrow\downarrow} = 1 \pm H$.

When a field is applied to the system the transverse and longitudinal components begin to differentiate themselves due to the way they connect spins (fig. 2). The transverse component connects opposite spins so there is only one critical q at which the paired surfaces touch each other at one point, $q = k_{f\uparrow} + k_{f\downarrow}$. The longitudinal component connects equal spins so there are two critical q's at which the surfaces touch each other $q_\uparrow = 2k_{f\uparrow}$ and $q_\downarrow = 2k_{f\downarrow}$[?] .

IV. SUPERCONDUCTING STATE

We proceed further into the superconducting state with D-wave symmetry. The inclusion of a Zeeman term in the superconducting Hamiltonian and a nodal order parameter ensures that for high enough fields the Zeeman splitting of the energy levels is enough to overcome the superconducting gap somewhere on the Fermi surface. Resulting in quasi-particle hole states with negative energy $\epsilon_{\mathbf{k},\mu} < 0$

Consequently, the $\epsilon_{\mathbf{k},\mu} = 0$ contour (fig. 3) defines the region inside which $\epsilon_{\mathbf{k},2} < 0$, and $f(\epsilon_{\mathbf{k},2}) = 1$. Only quasi particles with $\mu = 2$ provide this necessary condition

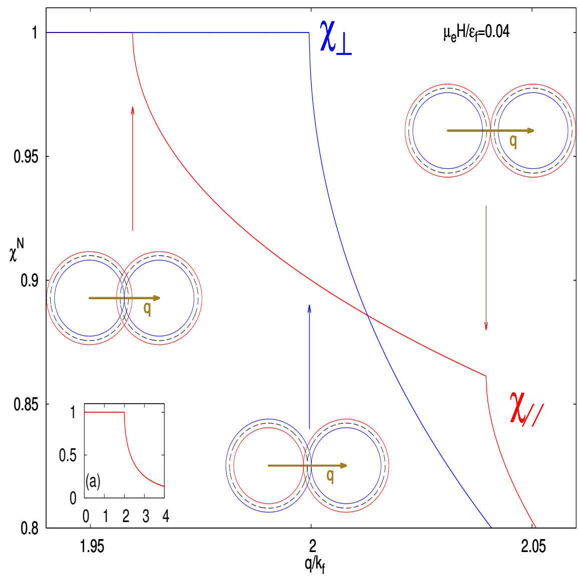

FIG. 2:

The 2D normal state susceptibility in zero field has a discontinuity in the first derivative for $q = 2k_f$ (a). Including Zeeman interaction, the location of these depend on the applied field and how the spins are paired for the different components (insets). The longitudinal component pairs equal spins so there are two discontinuities ($q_\pm = 2\sqrt{\epsilon_f \pm \mu_B H_0}$), while the transverse component pairs opposite spins leading to only one discontinuity ($q = \sqrt{\epsilon_f + \mu_B H_0} + \sqrt{\epsilon_f - \mu_B H_0}$)

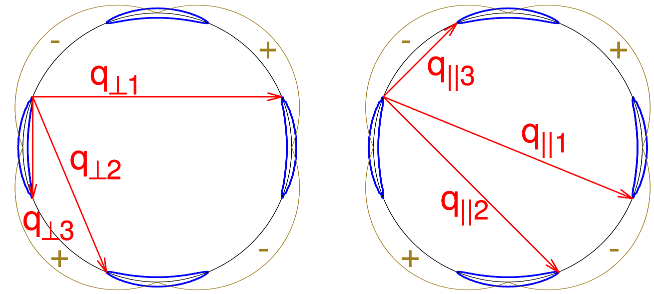

FIG. 3:

The critical wave vectors which connect points on the Fermi surface that have zero quasi-particle excitations for transverse and longitudinal afm. The blue outlines the region inside which $\epsilon_{\mathbf{k},2} < 0$, while the olive is a schematic of the order parameter.

which dictates the terms in equations 15 and 16 that provide the major contributions.

Analysis of the integrand for $\delta\chi_\alpha(\mathbf{q}) = \chi_\alpha(\mathbf{q})^{sc} - \chi_\alpha(\mathbf{q})^N$ indicates that value of is less than $(\Delta_{00}/\epsilon_f)^2$ in the majority of phase space and that the main contributions come from regions which are close to the intersections of the two fermi surfaces (fig. 4).[7–9]

V. RESULTS

We further search for the conditions on the wave vector for which χ^{sc} is maximized. This occurs for $\mathbf{q}$'s which connect points in phase space where both the quasi-particle excitation energy, $\epsilon_{\mathbf{k},\mu}$, and normal state excitation energy, $\xi_{\mathbf{k}\mu}$ are zero. Examining equations 15 and 16 at these points, one sees that it leads to large positive contributions at the points of intersection, the weight of which is determined by the phase factor (ie the u/v combination)(fig. 4).

As we discussed previously, the $\mu = 2$ excitations are negative, and provide the conditions for which we have defined the critical wave vectors. Thus we consider the terms in equations 15 and 16 which pair equal spins so that we pair two hole states. for the longitudinal component this is the first termm, with phase factor $(u_{\mathbf{k}_+}u_{\mathbf{k}_-} + -v_{\mathbf{k}_+}v_{\mathbf{k}_-})^2$, and for the transverse it is the second with phase factor $(u_{\mathbf{k}_+}v_{\mathbf{k}_-} - v_{\mathbf{k}_+}u_{\mathbf{k}_-})^2$.

Now we look to maximize the phase factor in each case. Under these conditions the only parameter left to choose is the sign of $\Delta_{\mathbf{k}}$ which is used in the definition v_k. To get the most positive contribution from the phase factors we need require that critical wave vectors in the longitudinal component connect points with the same sign $\Delta_{\mathbf{k}}$, while the transverse wave vectors connect points of opposite sign $\Delta_{\mathbf{k}}$.(fig. 3)

still need to figure out/make other figures to put into results and for what critical qs (tomorrow) We present our result in figure 5. At each temperature and field we calculate $\delta\chi$ by numerically integrating near the fermi surface intersections for a range of $\mathbf{q}$'s which are near $q_{\|2}$ and $q_{\perp 1}$ respectively. We then select the maximum $\chi^{sc} = \chi^N + \delta\chi$ in that range of $\mathbf{q}$'s.

We believe the behavior of the ideal wave vector $\mathbf{q}^*(H,T)$ is of particular interest because it provides the possibility of further experimental work which could shed more light on this phenomenon(fig. **??**).As temperature increases the critical wave vector begins to deviate away from the zero temperature case in such a way that it overlaps the fermi pockets more ($|\mathbf{q}|$ becomes smaller for both components).

The longitudinal wave vector $\mathbf{q}_{\|2}$deviates most significantly from the predicted value in direction, but not magnitude. In fact, it becomes purely nodal for $T > 0.25T_c$. This deviation seems more drastic than that of the transverse case. We propose two possible reasons for this result. Firstly, under an applied field, $\chi^N_{\|}$ begins to drop off for $|\mathbf{q}| < \mathbf{2k_f}$ (fig. 2). Thus, there is a competition between this drop off and the enhancement in $\delta\chi$ which may cause the system to prefer a $\mathbf{q}$ which is more nodal for

FIG. 4: $\delta\chi$ integrands

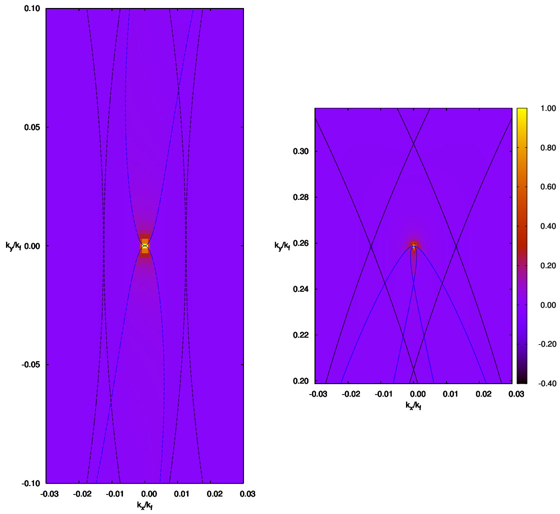

FIG. 5:

The integrand for the longitudinal (left) and transverse (right) component at zero temperature and for $\mathbf{q}_{\parallel 1}$ and $\mathbf{q}_{\perp 1}$. The black lines are the Zeeman split fermi surfaces for normal electrons and the blue lines are the pockets inside which there are negative quasi-particle excitations.

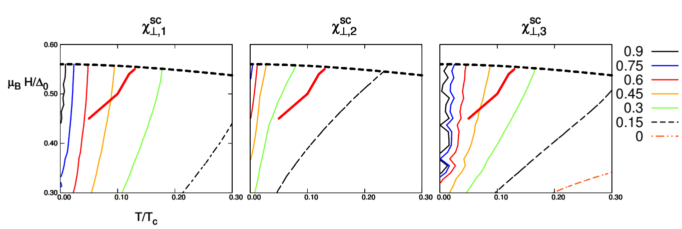

FIG. 6:

Percent increase of $\chi_\perp$ for the critical wave vectors in figure 3. The black bold dotted line is the first order phase transition for a Pauli limited d-wave superconductor, and the bold red line is a qualitative sketch of the Q-phase boundary of $CeCoIn_5$.
$\Delta_0/\epsilon_f = 0.005$

higher temperatures. Secondly, $\mathbf{q}_{\parallel 2}$ connects the points of zero excitation energy in such a way that the pockets are touching, not overlapping slightly as in the transverse case (fig. 4). For finite temperature the edges of these pockets get smeared out, thus affecting the longitudinal more dramatically and causing the system to find a more different maximum.

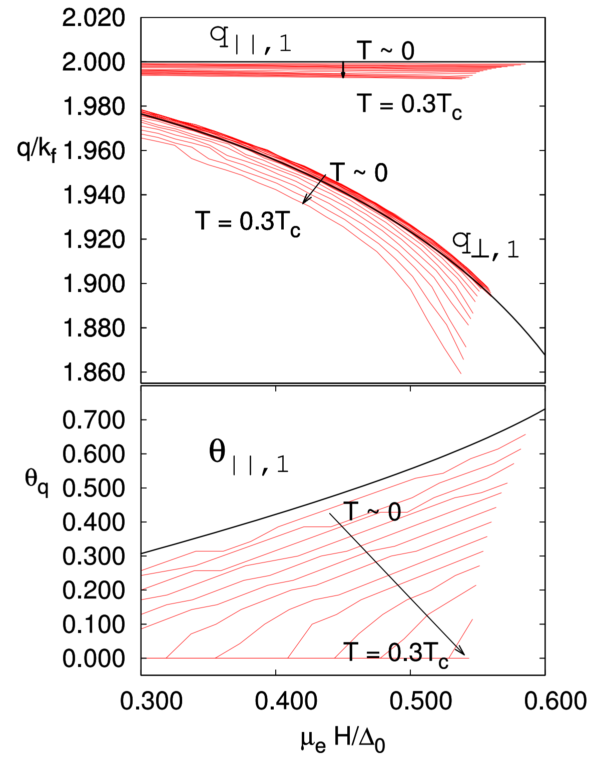

FIG. 7:

The ideal wave vector magnitude for both components depend on temperature and applied field. The black lines are the theoretical curves for $\mathbf{q}_{\parallel,1}$ and $\mathbf{q}_{\perp,1}$, and the red lines are found by adaptively finding the maximum in susceptibility.

VI. CONCLUSIONS

We find that for certain ordering wave vectors, the superconducting magnetic susceptibility becomes enhanced over the normal state. At zero temperature, these ideal wave vectors connect points on the zero field fermi surface corresponding to zero qusiparticle excitation energy $\epsilon_{\mathbf{k},s}$ which can only occur for order parameters which become sufficiently small.

Due to the terms in equation 3 which drive this enhancement we can also conclude that the enhancement in $\chi^{sc}_\perp$ can only occur for order parameters which are nodal (ie change sign) becuase the ideal $\mathbf{q}$'s connect points of zero excitation energy and opposite signs of the order parameter. This contrasts with the longitudinal component which prefers wave vectors which connect regions of the same sign of the order parameter. Thus, the longitudinal componenet may become enhanced for strongly anisotropic superconductors if the order parameter becomes sufficiently small.

We believe that the enhancement of χ may be enough to establish magnetic order inside the superconducting state. In the case of $CeCoIn_5$ it may also explain the collapse of AMF with superconductivity at H_{c2}. This conclusion follows from the equaiton for total susceptibility in terms of the first order expansion and the magnetic interaction(equation 6). If a normal state material were to have a susceptibility and magnetic interaction such that $1-J_{\alpha\beta}\chi_{\alpha\beta} << 1$, then a small enhancement of $\chi_{\alpha\beta}$ could be sufficent to induce magnetic order. In the case of itinerant electrons, this order would be in the form of a spin density wave (SDW). The origins of the magnetic interaction $J_{\alpha\beta}$ are not discussed here, but may be from coupling to the localized magnetic moments of the Ce atoms, (RKKY interaction CITE) or by some other mechanism. Our findings indicate that under certain conditions $\chi_{\alpha\beta}$ in the superconducting state can indeed increase over the normal state value for certain ideal ordering wave vectors, providing the necessary condition for a SDW.

$$\chi^{total}_{\alpha\beta} = \frac{\chi_{\alpha\beta}}{1 - J_{\alpha\beta}\chi_{\alpha\beta}} \tag{22}$$

Appendix A: Susceptibility in magnetic field

The presence of a magnetic field introduces a potential for particles with spin $V = -\vec{m}\cdot\vec{H}$. The magnetization due to this potential is given by $M_\alpha(t) = -i\int\limits_{-\infty}^{t} < [m_\alpha(t), V(t')] > dt'$, and the magnetic susceptibility is $\chi_{\alpha,\beta}(\mathbf{x},\mathbf{x}',t) = i\int\limits_{-\infty}^{t} < [m_\alpha(\mathbf{x},t), m_\beta(\mathbf{x}',t')] > dt'$. The magnetic moment is given by $m_\alpha(\mathbf{x},t) = \mu_e \sum\limits_{\mu,\mu'} \sigma^\alpha_{\mu,\mu'}\psi^\dagger_\mu(\mathbf{x},t)\psi_{\mu'}(\mathbf{x},t)$ [11]. Now we can proceed to calculate the susceptibitily. In the case of uniform space and time, we can assume that the product $m_\alpha(\mathbf{x},t), m_\beta(\mathbf{x}',t')$ is a function of $t-t'$ and $\mathbf{x}-\mathbf{x}'$. In this limit we get:

$$\chi_{\alpha\beta}(\mathbf{x},\omega) = -i\mu_B^2 \sum_{\mu\mu'\nu\nu'} \int_{-\infty}^{0} dt'\ \sigma^\alpha_{\mu\mu'}\sigma^\beta_{\nu\nu'} e^{i\omega t'}\ \langle[\psi^\dagger_\mu(\mathbf{x},t')\psi_{\mu'}(\mathbf{x}',t'), \psi^\dagger_\nu(0,0)\psi_{\nu'}(0,0)]\rangle$$

From now on we will use the μ and μ' subscripts to denote the spin for the $(\mathbf{x},t')$ coordinates while the ν and ν' subscripts denote the spin for the $(0,0)$ coordinates so that $\psi^\dagger_\mu = \psi^\dagger_\mu(\mathbf{x},t')$ and $\psi^\dagger_\nu = \psi^\dagger_\nu(0,0)$. The correlation function inside the integral evaluates as follows.

$$\begin{aligned}
< [\psi^\dagger_\mu\psi_{\mu'}, \psi^\dagger_\nu\psi_{\nu'}] > &= < \psi^\dagger_\mu\psi_{\mu'}\psi^\dagger_\nu\psi_{\nu'} > - < \psi^\dagger_\nu\psi_{\nu'}\psi^\dagger_\mu\psi_{\mu'} > \\
&= - < \psi^\dagger_\mu\psi^\dagger_\nu >< \psi_{\mu'}\psi_{\nu'} > + < \psi^\dagger_\mu\psi_{\nu'} >< \psi_{\mu'}\psi^\dagger_\nu > \\
&\quad + < \psi^\dagger_\nu\psi^\dagger_\mu >< \psi_{\nu'}\psi_{\mu'} > - < \psi^\dagger_\nu\psi_{\mu'} >< \psi_{\nu'}\psi^\dagger_\mu >
\end{aligned}$$

To evaluate these various correlation functions we use the Bogoliubov representation for the field operators $\psi_\mu = \sum\limits_{\mathbf{k}} u_{\mathbf{k}}\gamma_{\mathbf{k}\mu} + \sum\limits_{\mu'}(i\sigma_2)_{\mu\mu'}v^*_{\mathbf{k}}\gamma^\dagger_{-\mathbf{k}\mu'} = \sum\limits_{\mathbf{k}}\Gamma_{\mathbf{k},\mu}(x,t) - \Gamma^\dagger_{\mathbf{k},-\mu}(x,t)$, where $u_{\mathbf{k}}(\mathbf{x})$ and $v_{\mathbf{k}}(\mathbf{x})$ are, in general, complex functions and the γ's are new fermionic quasi-particle operators ($\Gamma_{\mathbf{k},\mu}(\mathbf{x},t) = \gamma_{\mathbf{k},\mu}(t)u_{\mathbf{k}}(\mathbf{x})$, $\Gamma^\dagger_{\mathbf{k},-\mu}(\mathbf{x},t) = \mu\gamma^\dagger_{-\mathbf{k},-\mu}(t)v^*_{\mathbf{k}}(\mathbf{x})$). We find the time dependence of the γ operators via the Heisenberg representation, $\frac{d}{dt}\gamma_{\mathbf{k}\mu} = \frac{i}{\hbar}[H,\gamma_{\mu\mathbf{k}}] = \frac{-i\epsilon_{\mathbf{k}\mu}}{\hbar}\gamma_{\mathbf{k}\mu}$ and $\frac{d}{dt}\gamma^\dagger_{\mathbf{k}\mu} = \frac{i\epsilon_{\mathbf{k}\mu}}{\hbar}\gamma^\dagger_{\mathbf{k}\mu}$.

$$\gamma_{\mathbf{k}\mu}(t) = \gamma_{\mathbf{k}\mu}e^{-i\omega_{\mathbf{k}\mu}t} \qquad \gamma^\dagger_{\mathbf{k}\mu}(t) = \gamma^\dagger_{\mathbf{k}\mu}e^{i\omega_{\mathbf{k}\mu}t}$$

And the correlation relations for the γ's are:

$$< \gamma^\dagger_{\mathbf{k}\mu}\gamma_{\mathbf{p}\nu} >= \delta_{\mathbf{pk}}\delta_{\mu\nu}f(\epsilon_{\mathbf{k}\mu}) \qquad < \gamma_{\mathbf{k}\mu}\gamma_{\mathbf{p}\nu} >=< \gamma^\dagger_{\mathbf{k}\mu}\gamma^\dagger_{\mathbf{p}\nu} >= 0$$

Where $f(\epsilon_{\mathbf{k}\mu}) = f_{\mathbf{k}\mu} = (e^{\beta\epsilon_{\mathbf{k}\mu}} + 1)^{-1}$ is the fermi distribution. Using the definition of Γ and omitting the time bit, we can deduce the following rules:

$$< \Gamma^\dagger_{\mathbf{k}\mu}\Gamma_{\mathbf{p}\nu} >= \delta_{\mathbf{pk}}\delta_{\mu\nu}f(\epsilon_{\mathbf{k}\mu})u^*_{\mathbf{k}}u_{\mathbf{p}} \quad < \Gamma^\dagger_{\mathbf{k}\mu}\Gamma_{\mathbf{p}-\nu} >= \beta\delta_{\mathbf{pk}}\delta_{\mu-\nu}f(\epsilon_{\mathbf{k}\mu})u^*_{\mathbf{k}}v_{\mathbf{p}} \quad < \Gamma^\dagger_{\mathbf{k}-\mu}\Gamma_{\mathbf{p}-\nu} >= \delta_{\mathbf{pk}}\delta_{\mu\nu}f(\epsilon_{\mathbf{k}\mu})v_{\mathbf{k}}v^*_{\mathbf{p}}$$

Going back to the sum of wick contractions, and dropping the quantum number subscript on the Γ's:

$$\begin{aligned}
< [\psi^\dagger_\mu\psi_{\mu'},\psi^\dagger_\nu\psi_{\nu'}] >= &-(< \Gamma^\dagger_\mu\Gamma_{-\nu} > + < \Gamma_{-\mu}\Gamma^\dagger_\nu >)(< \Gamma_{\mu'}\Gamma^\dagger_{-\nu'} > + < \Gamma^\dagger_{-\mu'}\Gamma_{\nu'} >)\\
&+(< \Gamma^\dagger_\mu\Gamma_{\nu'} > + < \Gamma_{-\mu}\Gamma^\dagger_{-\nu'} >)(< \Gamma_{\mu'}\Gamma^\dagger_{\nu} > + < \Gamma^\dagger_{-\mu'}\Gamma_{-\nu} >)\\
&+(< \Gamma^\dagger_\nu\Gamma_{-\mu} > + < \Gamma_{-\nu}\Gamma^\dagger_\mu >)(< \Gamma_{\nu'}\Gamma^\dagger_{-\mu'} > + < \Gamma^\dagger_{-\nu'}\Gamma_{\mu'} >)\\
&-(< \Gamma^\dagger_\nu\Gamma_{\mu'} > + < \Gamma_{-\nu}\Gamma^\dagger_{-\mu'} >)(< \Gamma_{\nu'}\Gamma^\dagger_{\mu} > + < \Gamma^\dagger_{-\nu'}\Gamma_{-\mu} >)
\end{aligned}$$

$$\begin{aligned}
=\sum_{\mathbf{kp}}\delta_{\mu\nu'}\delta_{\mu'\nu}\Big[&[T^*_{\mathbf{k}\mu}f_{\mathbf{k}\mu}u^*_{\mathbf{k}}(\mathbf{x})u_{\mathbf{k}}+T_{-\mathbf{k}-\mu}(1-f_{-\mathbf{k}-\mu})v_{-\mathbf{k}}(\mathbf{x})v^*_{-\mathbf{k}}][T_{\mathbf{p}\mu'}(1-f_{\mathbf{p}\mu'})u_{\mathbf{p}}(\mathbf{x})u^*_{\mathbf{p}}+T^*_{-\mathbf{p}-\mu'}f_{-\mathbf{p}-\mu'}v^*_{-\mathbf{p}}(\mathbf{x})v_{-\mathbf{p}}]\\
&-[T_{\mathbf{p}\mu'}f_{\mathbf{p}\mu'}u^*_{\mathbf{p}}u_{\mathbf{p}}(\mathbf{x})+T^*_{-\mathbf{p}-\mu'}(1-f_{-\mathbf{p}-\mu'})v_{-\mathbf{p}}v^*_{-\mathbf{p}}(\mathbf{x})][T^*_{k\mu}(1-f_{k\mu})u_{\mathbf{k}}u^*_{\mathbf{k}}(\mathbf{x})+T_{-\mathbf{k}-\mu}f_{-\mathbf{k}-\mu}v^*_{-\mathbf{k}}v_{-\mathbf{k}}(\mathbf{x})]\Big]\\
+\mu\mu'\delta_{\mu-\nu}\delta_{\mu'-\nu'}\Big[&-[T_{-\mathbf{k}-\mu}(1-f_{-\mathbf{k}-\mu})u^*_{-\mathbf{k}}v_{-\mathbf{k}}(\mathbf{x})-T^*_{\mathbf{k}\mu}f_{\mathbf{k}\mu}v_{\mathbf{k}}u^*_{\mathbf{k}}(\mathbf{x})][-T_{\mathbf{p}\mu'}(1-f_{\mathbf{p}\mu'})v^*_{\mathbf{p}}u_{\mathbf{p}}(\mathbf{x})+T^*_{-\mathbf{p}-\mu'}f_{-\mathbf{p}-\mu'}u_{-\mathbf{p}}v^*_{-\mathbf{p}}(\mathbf{x})]\\
&+[T_{-\mathbf{k}-\mu}f_{-\mathbf{k}-\mu}v_{-\mathbf{k}}(\mathbf{x})u^*_{-\mathbf{k}}-T^*_{\mathbf{k}\mu}(1-f_{\mathbf{k}\mu})u^*_{\mathbf{k}}(\mathbf{x})v_{\mathbf{k}}][T^*_{-\mathbf{p}-\mu'}(1-f_{-\mathbf{p}-\mu'})v^*_{-\mathbf{p}}(\mathbf{x})u_{-\mathbf{p}}-T_{\mathbf{p}\mu'}f_{\mathbf{p}\mu'}u_p(\mathbf{x})v^*_p]\Big]
\end{aligned}$$

Where $T_{k\mu}=e^{-i\omega_{\mathbf{k}\mu}\tau}$ which carries the time dependence from the γ operators.
Now we transform to frequency space by multiplying the entire integral by $e^{-\delta t'}$, and preform the integration in t'.

$$\begin{aligned}
\chi_{\alpha\beta}(\omega,q)=-\mu_B^2\sum_{\mathbf{kp}\mu\mu'}\sigma^\beta_{\mu\mu'}\sigma^\alpha_{\mu'\mu}\Big[&\frac{(f_{\mathbf{k}\mu}-f_{\mathbf{p}\mu'})u^*_{\mathbf{p}}u_{\mathbf{p}}(\mathbf{x})u_{\mathbf{k}}u^*_{\mathbf{k}}(\mathbf{x})}{\omega+\omega_{\mathbf{k}\mu}-\omega_{\mathbf{p}\mu'}+i\delta}+\frac{(1-f_{\mathbf{p}\mu'}-f_{-\mathbf{k}-\mu})u^*_{\mathbf{p}}u_{\mathbf{p}}(\mathbf{x})v^*_{-\mathbf{k}}v_{-\mathbf{k}}(\mathbf{x})}{\omega-\omega_{-\mathbf{k}-\mu}-\omega_{\mathbf{p}\mu'}+i\delta}\\
&+\frac{(-1+f_{-\mathbf{p}-\mu'}+f_{\mathbf{k}\mu})v_{-\mathbf{p}}v^*_{-\mathbf{p}}(\mathbf{x})u_{\mathbf{k}}u^*_{\mathbf{k}}(\mathbf{x})}{\omega+\omega_{-\mathbf{p}-\mu'}+\omega_{\mathbf{k}\mu}+i\delta}+\frac{(f_{-\mathbf{p}-\mu'}-f_{-\mathbf{k}-\mu})v_{-\mathbf{p}}v^*_{-\mathbf{p}}(\mathbf{x})v^*_{-\mathbf{k}}v_{-\mathbf{k}}(\mathbf{x})}{\omega+\omega_{-\mathbf{p}-\mu'}-\omega_{-\mathbf{k}-\mu}+i\delta}\Big]\\
+\mu\mu'\sigma^\beta_{\mu\mu'}\sigma^\alpha_{-\mu-\mu'}\Big[&\frac{(1-f_{-\mathbf{k}-\mu}-f_{\mathbf{p}\mu'})u^*_{-\mathbf{k}}v_{-\mathbf{k}}(\mathbf{x})v^*_{\mathbf{p}}u_{\mathbf{p}}(\mathbf{x})}{\omega-\omega_{\mathbf{p}\mu'}-\omega_{-\mathbf{k}-\mu}+i\delta}+\frac{(f_{-\mathbf{k}-\mu}-f_{-\mathbf{p}-\mu'})u^*_{-\mathbf{k}}v_{-\mathbf{k}}(\mathbf{x})v^*_{-\mathbf{p}}(\mathbf{x})u_{-\mathbf{p}}}{\omega+\omega_{-\mathbf{p}-\mu'}-\omega_{-\mathbf{k}-\mu}+i\delta}\\
&-\frac{(f_{\mathbf{p}\mu'}-f_{\mathbf{k}\mu})u^*_{\mathbf{k}}(\mathbf{x})v_{\mathbf{k}}v^*_{\mathbf{p}}u_{\mathbf{p}}(\mathbf{x})}{\omega+\omega_{\mathbf{k}\mu}-\omega_{\mathbf{p}\mu'}+i\delta}-\frac{(1-f_{\mathbf{k}\mu}-f_{-\mathbf{p}-\mu'})u^*_{\mathbf{k}}(\mathbf{x})v_{\mathbf{k}}v^*_{-\mathbf{p}}(\mathbf{x})u_{-\mathbf{p}}}{\omega+\omega_{\mathbf{k}\mu}+\omega_{-\mathbf{p}-\mu'}+i\delta}\Big]
\end{aligned}$$

Now we Fourier transform to momentum space by using the equations for u's and v's from the Bogolioubov equations $u_{\mathbf{k}}(\mathbf{x})=u_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}$ and $v_{\mathbf{k}}(\mathbf{x})=v_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}$, $\left(u_{\mathbf{k}}=\sqrt{\tfrac{1}{2}(1+\tfrac{\xi_{\mathbf{k}}}{\epsilon_{\mathbf{k}}})},\quad v_{\mathbf{k}}=sgn(\Delta_{\mathbf{k}})\sqrt{\tfrac{1}{2}(1-\tfrac{\xi_{\mathbf{k}}}{\epsilon_{\mathbf{k}}})}\right)$

$$\begin{aligned}
\chi_{\alpha\beta}(\omega,\mathbf{q})=-\mu_B^2\sum_{\mathbf{k},\mu,\mu'}\sigma^\beta_{\mu\mu'}\sigma^\alpha_{\mu'\mu}\Big[&\frac{(f_{\mathbf{k}\mu}-f_{\mathbf{k+q}\mu'})u^2_{\mathbf{k+q}}u^2_{\mathbf{k}}}{\omega+\omega_{\mathbf{k}\mu}-\omega_{\mathbf{k+q}\mu'}+i\delta}+\frac{(1-f_{\mathbf{k+q}\mu'}-f_{\mathbf{k}-\mu})u^2_{\mathbf{k+q}}v^2_{\mathbf{k}}}{\omega-\omega_{\mathbf{k}-\mu}-\omega_{\mathbf{k+q}\mu'}+i\delta}\\
&+\frac{(-1+f_{\mathbf{k+q}-\mu'}+f_{\mathbf{k}\mu})v^2_{\mathbf{k+q}}u^2_{\mathbf{k}}}{\omega+\omega_{\mathbf{k+q}-\mu'}+\omega_{\mathbf{k}\mu}+i\delta}+\frac{(f_{\mathbf{k+q}-\mu'}-f_{\mathbf{k}-\mu})v^2_{\mathbf{k+q}}v^2_{\mathbf{k}}}{\omega+\omega_{\mathbf{k+q}-\mu'}-\omega_{\mathbf{k}-\mu}+i\delta}\Big]\\
+\mu\mu'\sigma^\beta_{\mu\mu'}\sigma^\alpha_{-\mu-\mu'}\Big[&\frac{(1-f_{\mathbf{k}-\mu}-f_{\mathbf{k+q}\mu'})u_{\mathbf{k}}v_{\mathbf{k}}v_{\mathbf{k+q}}u_{\mathbf{k+q}}}{\omega-\omega_{\mathbf{k+q}\mu'}-\omega_{\mathbf{k}-\mu}+i\delta}+\frac{(f_{\mathbf{k}-\mu}-f_{\mathbf{k+q}-\mu'})u_{\mathbf{k}}v_{\mathbf{k}}v_{\mathbf{k+q}}u_{\mathbf{k+q}}}{\omega+\omega_{\mathbf{k+q}-\mu'}-\omega_{\mathbf{k}-\mu}+i\delta}\\
&-\frac{(f_{\mathbf{k+q}\mu'}-f_{\mathbf{k}\mu})u_{\mathbf{k}}v_{\mathbf{k}}v_{\mathbf{k+q}}u_{\mathbf{k+q}}}{\omega+\omega_{\mathbf{k}\mu}-\omega_{\mathbf{k+q}\mu'}+i\delta}-\frac{(1-f_{\mathbf{k}\mu}-f_{\mathbf{k+q}-\mu'})u_{\mathbf{k}}v_{\mathbf{k}}v_{\mathbf{k+q}}u_{\mathbf{k+q}}}{\omega+\omega_{\mathbf{k}\mu}+\omega_{\mathbf{k+q}-\mu'}+i\delta}\Big]
\end{aligned}$$

The susceptibility is then:

$$\chi_{\alpha\beta}(\omega,q) = -\mu_B^2 \sum_{\mathbf{k},\mu,\mu'} \sigma^{\beta}_{\mu\mu'}\sigma^{\alpha}_{\mu'\mu}\left[\frac{(f^+_{\mathbf{k}\mu} - f^+_{\mathbf{k+q}\mu'})u^2_{\mathbf{k+q}}u^2_{\mathbf{k}}}{\omega+\omega^+_{\mathbf{k}\mu}-\omega^+_{\mathbf{k+q}\mu'}+i\delta} + \frac{(f^-_{\mathbf{k}\mu} - f^+_{\mathbf{k+q}\mu'})u^2_{\mathbf{k+q}}v^2_{\mathbf{k}}}{\omega+\omega^-_{\mathbf{k}\mu}-\omega^+_{\mathbf{k+q}\mu'}+i\delta}\right.$$
$$\left. + \frac{(f^+_{\mathbf{k}\mu} - f^-_{\mathbf{k+q}\mu'})v^2_{\mathbf{k+q}}u^2_{\mathbf{k}}}{\omega+\omega^+_{\mathbf{k}\mu}-\omega^-_{\mathbf{k+q}\mu'}+i\delta} + \frac{(f^-_{\mathbf{k}\mu} - f^-_{\mathbf{k+q}\mu'})v^2_{\mathbf{k+q}}v^2_{\mathbf{k}}}{\omega+\omega^-_{\mathbf{k}\mu}-\omega^-_{\mathbf{k+q}\mu'}+i\delta}\right]$$
$$+\mu\mu'\sigma^{\beta}_{\mu\mu'}\sigma^{\alpha}_{-\mu-\mu'}\left[\frac{(f^-_{\mathbf{k}\mu} - f^+_{\mathbf{k+q}\mu'})u_{\mathbf{k}}v_{\mathbf{k}}v_{\mathbf{k+q}}u_{\mathbf{k+q}}}{\omega+\omega^-_{\mathbf{k}\mu}-\omega^+_{\mathbf{k+q}\mu'}+i\delta} - \frac{(f^-_{\mathbf{k}\mu} - f^-_{\mathbf{k+q}\mu'})u_{\mathbf{k}}v_{\mathbf{k}}v_{\mathbf{k+q}}u_{\mathbf{k+q}}}{\omega+\omega^-_{\mathbf{k}\mu}-\omega^-_{\mathbf{k+q}\mu'}+i\delta}\right.$$
$$\left. - \frac{(f^+_{\mathbf{k}\mu} - f^+_{\mathbf{k+q}\mu'})u_{\mathbf{k}}v_{\mathbf{k}}v_{\mathbf{k+q}}u_{\mathbf{k+q}}}{\omega+\omega^+_{\mathbf{k}\mu}-\omega^+_{\mathbf{k+q}\mu'}+i\delta} + \frac{(f^+_{\mathbf{k}\mu} - f^-_{\mathbf{k+q}\mu'})u_{\mathbf{k}}v_{\mathbf{k}}v_{\mathbf{k+q}}u_{\mathbf{k+q}}}{\omega+\omega^+_{\mathbf{k}\mu}-\omega^-_{\mathbf{k+q}\mu'}+i\delta}\right]$$

Where we have defined the following:

$$\begin{aligned} \delta &\to 0^- \\ \omega^+_{\mathbf{k}\mu} &= \omega_{\mathbf{k}\mu} \\ \omega^-_{\mathbf{k}\mu} &= -\omega_{\mathbf{k}-\mu} \\ f^+_{\mathbf{k}\mu} &= f(\epsilon_{\mathbf{k}\mu}) \\ f^-_{\mathbf{k}\mu} &= f(-\epsilon_{\mathbf{k}-\mu}) \end{aligned}$$

Appendix B: Normal state susceptibility

To find the normal state susceptibility at zero temperature we first orient our coordinates such that $\mathbf{q} = q\hat{x}$, and change from $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$ to $\mathbf{k}-\mathbf{q}/2$ and $\mathbf{k}+\mathbf{q}/2$. Converting the sums to integrals, and using symmetry about the x and y axes, we have:

$$\chi_{\|} = -8\mu_B^2 \sum_{\mu} \int_0^{\pi/2} d\phi \int kdk \frac{f(\epsilon_{\mathbf{k-q}/2,\mu}) - f(\epsilon_{\mathbf{k+q}/2,\mu})}{-2kq\cos\phi}$$
$$\chi_{\perp} = -8\mu_B^2 \sum_{\mu} \int_0^{\pi/2} d\phi \int kdk \frac{f(\epsilon_{\mathbf{k-q}/2,\mu}) - f(\epsilon_{\mathbf{k+q}/2,-\mu})}{-2kq\cos\phi + 2\mu\mu_B H}$$

Where we have also normalized the 2D area of integration $A/(2\pi\hbar)^2 = 1$

LONGITUDINAL

To determine the limits of integration on k, we need to solve the dispersion relation for k when $\epsilon_{\mathbf{k}\pm\mathbf{q}/2,s} = \epsilon_f$, the fermi energy. If we normalize the equation by multiplying and dividing it by k_f^2, the resulting limits of integration are:

$$k' = \mp\frac{q'\cos\phi}{2} \pm \sqrt{1 - sH' - (q'/2)^2\sin^2\phi} = K_{\pm\pm s}$$
$$\sin\phi_s = \frac{2}{q'}\sqrt{1-sH'}$$

Where $k' = k/k_f$, $q' = q/k_f$ and $H' = \mu_B H/k_f^2$. For the longitudinal component we must consider three regions of q:

$\boxed{\textbf{A})\ q' \in [0, 2\sqrt{1-H'}]}$

$$\chi_{\parallel} = 8\mu_B^2 \sum_s \int_0^{\pi/2} d\phi \int_0^{K_{++s}} dk' \frac{1}{2q'\cos\phi} - \int_0^{K_{-+s}} dk' \frac{1}{2q'\cos\phi} = 4\mu_B^2\pi$$

$\boxed{\textbf{B})\ q' \in [2\sqrt{1-H'}, 2\sqrt{1+H'}]}$

$$\chi_{\parallel} = 2\mu_B^2\pi + 8\mu_B^2 \int_0^{\phi_1} d\phi \int_{KK_{+-1}}^{K_{++1}} dk' \frac{1}{2q'\cos\phi}$$

$$= 2\mu_B^2\pi + \frac{8\mu_B^2}{q'} \int_0^{\phi_1} d\phi \frac{\sqrt{1 - H' - (q'/2)^2 \sin^2\phi}}{\cos\phi}$$

$$= 2\mu_B^2\pi + \frac{8\mu_B^2}{q'} \int_0^{\sin\phi_1} dx \frac{\sqrt{1 - H' - (q'/2)^2 x^2}}{1 - x^2}$$

$$\chi_{\parallel} = 4\mu_B^2\pi \left[1 - \frac{1}{2}\sqrt{1 - (1 - H')(2/q')^2}\right]$$

$\boxed{\textbf{C})\ q' \in [2\sqrt{1+H'}, \infty]}$

$$\chi_{\parallel} = 8\mu_B^2 \int_0^{\phi_{-1}} d\phi \int_{K_{+-(-1)}}^{K_{++(-1)}} dk' \frac{1}{2q'\cos\phi} + 8\mu_B^2 \int_0^{\phi_1} d\phi \int_{KK_{+-1}}^{K_{++1}} dk' \frac{1}{2q'\cos\phi}$$

$$\chi_{\parallel} = 4\mu_B^2\pi \left[1 - \frac{1}{2}\sqrt{1 - (1 + H')(2/q'^2)} - \frac{1}{2}\sqrt{1 - (1 - H')(2/q')^2}\right]$$

TRANSVERSE

Now we continue with the perpendicular component. To do this we move the origin such that at k_x=0 the s=1 and s=-1 surfaces intersect. The equation for this transformation is $k'_x \to k'_x - q'/2 + sH'/q'$, and the limits of integration are:

$$k = \pm(q'/2 - sH'/q')\cos\phi \pm \sqrt{1 - sH' - (q'/2 - sH'/q')^2 \sin^2\phi} = K_{\pm\pm s}$$

$$\sin\phi_s = \frac{2q'\sqrt{1 - sH'}}{q'^2 - 2sH'}$$

For this integration there are two regions:

$\boxed{\textbf{A})\ q' \in [0, \sqrt{1+H'} + \sqrt{1-H'}]}$

$$\chi_{\perp} = 8\mu_B^2 \int_0^{\pi/2} d\phi \int_0^{K_{++1}} dk' \frac{1}{2q'\cos\phi} - \int_0^{K_{-+1}} dk' \frac{1}{2q'\cos\phi} + \int_0^{K_{++(-1)}} dk' \frac{1}{2q'\cos\phi} - \int_0^{K_{-+(-1)}} dk' \frac{1}{2q'\cos\phi}$$

$$\chi_{\perp} = (8\mu_B^2/q') \int_0^{\pi/2} d\phi (q'/2 - H'/q') + (q'/2 + H'/q')$$

$$\chi_{\perp} = 4\mu_B^2\pi$$

$\boxed{\mathbf{B})\ q' \in [\sqrt{1+H'}+\sqrt{1-H'}, \infty]}$

$$\chi_\perp = 8\mu_B^2 \int_0^{\phi_1} d\phi \int_{K_{+-1}}^{K_{++1}} dk' \frac{1}{2q'\cos\phi} + \int_0^{\phi_{-1}} d\phi \int_{K_{+-(-1)}}^{K_{++(-1)}} dk' \frac{1}{2q'\cos\phi}$$

$$\chi_\perp = (8\mu_B^2/q') \int_0^{\phi_1} d\phi \frac{\sqrt{1-H'-(q'/2-H'/q')^2\sin^2\phi}}{\cos\phi} + \int_0^{\phi_{-1}} d\phi \frac{\sqrt{1+H'-(q'/2+H'/q')^2\sin^2\phi}}{\cos\phi}$$

$$\chi_\perp = (8\mu_B^2/q') \int_0^{\sin\phi_1} dx \frac{\sqrt{1-H'-(q'/2-H'/q')^2x^2}}{1-x^2} + \int_0^{\sin\phi_{-1}} dx \frac{\sqrt{1+H'-(q'/2+H'/q')^2x^2}}{1-x^2}$$

$$\chi_\perp = 4\mu_B^2\pi\left[1-\sqrt{1+(2H'/q'^2)^2-(2/q')^2}\right]$$

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