HW 7: Physics 545

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1a

The real space Yukawa potential is given by:

$$\begin{aligned} \Phi(\mathbf{r}) = & \tfrac{1}{(2\pi)^3} \int d^3k \quad e^{i\mathbf{k}\cdot\mathbf{r}} \tfrac{4\pi Q}{k^2+k_0^2} && (1) \\ & \tfrac{8\pi^2 Q}{(2\pi)^3} \int_0^\infty dk \quad \tfrac{k^2}{k^2+k_0^2} \int_{-1}^{1} dx \quad e^{ikrx} && (2) \\ & \tfrac{Q}{i2r\pi} \int_{-\infty}^{\infty} dk \quad \tfrac{k}{(k+ik_0)(k-ik_0)} \quad (e^{ikr} - e^{-ikr}) && (3) \end{aligned}$$

Where we use the even symmetry of 2 to get 3. Now we can use the complex plane of k to close the integral above (first term) and below (second term) and sum the residues:

$$\Phi(r) = \frac{2\pi i Q}{i2r\pi}\left[\frac{ik_0}{2ik_0}e^{-kr} - (-1)\frac{-ik_0}{-2ik_0}e^{-kr}\right] \tag{4}$$

$$\Phi(r) = \frac{Q}{r}e^{-kr} \tag{5}$$

1b

To see the differential equation we can write the fourier representation of the Coulomb law

$$k^2\Phi(k) = 4\pi\rho(\mathbf{k}) \tag{6}$$

To get the Yukawa potential we need $\rho(\mathbf{k}) = Q - \frac{k_0^k}{4\pi}\Phi(k)$ and we have

$$k^2\Phi(k) + k_0\Phi(k) = 4\pi Q \tag{7}$$

We can easily move this back to a real space differential equation

$$-\nabla^2\Phi(\mathbf{r}) + k_0\Phi(\mathbf{r}) = 4\pi Q\delta(\mathbf{r}) \tag{8}$$

2a

Homogeneous local charge neutrality implies that the ion and electron number densities satisfy $n_e(\mathbf{r}) = Zn_i(\mathbf{r})$. We can then write the Hartree terms all in terms of $n_e(r)$:

$$\hat{\nu}_{ee} = \frac{1}{2}\int d^3r \int d^3r' n_e(r) n_e(r') \frac{e^2}{|\mathbf{r}-\mathbf{r}'|} \quad (9)$$

$$\hat{\nu}_{ii} = \frac{1}{2}\int d^3r \int d^3r' n_e(r) n_e(r') \frac{e^2}{|\mathbf{r}-\mathbf{r}'|} \quad (10)$$

$$\hat{\nu}_{ei} = -\int d^3r \int d^3r' n_e(r) n_e(r') \frac{e^2}{|\mathbf{r}-\mathbf{r}'|} \quad (11)$$

It's pretty obvious that the sum of these is zero.

2b

We write the sum for $\Sigma(\mathbf{k})$ as an integral over $\mathbf{p} = \mathbf{k} + \mathbf{q}$ and use azimuthal symmetry for ϕ integral

$$\Sigma(\mathbf{k}) = -\frac{8\pi e^2}{(2\pi\hbar)^3} \int_0^{p_f} dp \quad p^2 \int_{-1}^{1} dx \frac{1}{p^2 + k^2 - 2kpx} \quad (12)$$

$$= \frac{e^2}{k\pi^2\hbar^3} \int_0^{p_f} dp \quad p\Big[ln|p-k| - ln|p+k|\Big] \quad (13)$$

$$= \frac{e^2}{k\pi^2\hbar^3}\Bigg[\int_{-k}^{p_f - k} dp \quad (x+k) ln|x| - \int_{k}^{p_f + k} dp \quad (x-k) ln|x|\Bigg] \quad (14)$$

Now let us write only the term in brackets and combine the integrals noting their intervals and symmetries (if any) and integrate by parts

$$\Big[*\Big] = 2k \int\limits_0^{p_f-k} dx\, ln|x| - \int\limits_{p_f-k}^{p_f+k} dx\,(x-k)ln|x| \tag{15}$$

$$= 2k\Big[(p_f-k)ln|p_f-k| - (p_f-k)\Big] \tag{16}$$

$$-\Big[x(x/2-k)ln|x|\Big|_{p_f-k}^{p_f+k} - \int\limits_{p_f-k}^{p_f+k} dx\,(x/2-k)\Big] \tag{17}$$

$$= 2k\Big[(p_f-k)ln|p_f-k| - (p_f-k)\Big] \tag{18}$$

$$-\Big[\tfrac{1}{2}(p_f^2-k^2)ln|p_f+k| - \tfrac{1}{2}(p_f-k)(p_f-3k)ln|p_f-k| \tag{19}$$

$$-p_f k + 2k^2)\Big] \tag{20}$$

$$= -kp_f + \tfrac{1}{2}(p_f^2-k^2)ln\Big|\tfrac{p_f-k}{p_f+k}\Big| \tag{21}$$

Now we can write the self energy:

$$\Sigma(k) = \frac{e^2}{\pi^2\hbar^3}\Big[-p_f + \frac{p_f^2-k^2}{k}ln\Big|\frac{p_f-k}{p_f+k}\Big|\Big] \tag{22}$$

and it's contribution to the group velocity

$$\delta_k\Sigma(k) = \frac{e^2}{\pi^2\hbar^3}\Big[-\Big(\frac{p_f^2}{k^2}+1\Big)ln\Big|\frac{p_f-k}{p_f+k}\Big| + \frac{(p_f+k)^2}{k}\Big] \tag{23}$$

Which is indeed divergent for $k = p_f$ because of the $ln|p_f-k|$ term