HW 3: Physics 545

Ben Rosemeyer

February 10, 2015

1

First we note that for a single excitation at momentum $\mathbf{k}$ we have $\delta n_{\mathbf{p}} = V\delta_{\mathbf{p},\mathbf{k}}$.

To begin with we will find the three quantities as a single sum over the excitations, similar to the class notes which have that result for the current $\mathbf{j}$. To do this we need only consider the result from the second term in $\delta\bar{n}_{\mathbf{p}} = \delta n_{\mathbf{p}} - \frac{\partial n^0_{\mathbf{p}}}{\partial \epsilon_{\mathbf{p}}}\delta\epsilon_{\mathbf{p}}$. We also note the lack of all antisymmetric spin interactions so that only the symmetric ones contribute. $P_l(x)$ is the lth Legendre Polynomial and f^s_l is the Fermi Liquid parameter for the symmetric spin interaction in the lth channel.

Particle Current

$$
\begin{aligned}
& -\tfrac{1}{V}\sum_{\mathbf{p}}(\nabla_{\mathbf{p}}\epsilon^0_{\mathbf{p}})\tfrac{\partial n^0_{\mathbf{p}}}{\partial\epsilon_{\mathbf{p}}}\delta\epsilon_{\mathbf{p}} && (1)\\
= \quad & -\tfrac{1}{V}\sum_{\mathbf{p}}(\mathbf{p}/m^*)(-\delta(\xi_{\mathbf{p}}))\tfrac{2}{V}\sum_{\mathbf{p}',l} f^s_l P_l(\hat{p}\cdot\hat{p'})\delta n_{\mathbf{p}'} && (2)\\
= \quad & \tfrac{1}{V}\sum_{\mathbf{p}',l} 2N_f(p_f/m^*)\delta n_{\mathbf{p}'} f^s_l \int d\Omega_{\mathbf{p}}\hat{p}P_l(\hat{p}\cdot\hat{p'}) && (3)\\
= \quad & \tfrac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f/m^*)\delta n_{\mathbf{p}'}\hat{p'} f^s_1/3 && (4)\\
= \quad & \tfrac{1}{V}\sum_{\mathbf{p}'}(\mathbf{p}'/m^*)\delta n_{\mathbf{p}'} F^s_1/3 && (5)
\end{aligned}
$$

between lines 3 and 4 we used the result from HW 2 to do the solid angle integration, and we define $F^s_l = 2N_f f^s_l$ as a dimensionless Fermi Liquid parameter. Combining this will the first term of $\delta\bar{n}_{\mathbf{p}}$ and switching the sum over $\mathbf{p}'$ to $\mathbf{p}$ above, we have the result from lecture.

$$\mathbf{j} = \frac{1}{V}\sum_{\mathbf{p}}\delta n_{\mathbf{p}}\frac{\mathbf{p}}{m^*}(1 + F^s_l/3) \tag{6}$$

$\boxed{\mathbf{j} = \frac{\mathbf{k}}{m^*}(1 + F_l^s/3)}$

For Galilean invariance we go to the co-moving frame such that the excitation momentum $\mathbf{k} = 0$. This frame has velocity $\mathbf{U} = \mathbf{k}/m$, and the ratio $\frac{m^*}{m} = 1 + F_l^s/3$ from class. plugging in the ratio of $\frac{m^*}{m}$ into $\mathbf{j}$ yields $\mathbf{j} = \mathbf{U}$ and thus we have Galilean Invariance.

Momentum Current

$$-\tfrac{1}{V}\sum_{\mathbf{p}} p_i \tfrac{\partial \epsilon^0_{\mathbf{p}}}{\partial p_j}\tfrac{\partial n^0_{\mathbf{p}}}{\partial \epsilon_{\mathbf{p}}}\delta\epsilon_{\mathbf{p}} \tag{7}$$

$$= \quad -\tfrac{1}{V}\sum_{\mathbf{p}} p_i (p_j/m^*)(-\delta(\xi_{\mathbf{p}}))\tfrac{2}{V}\sum_{\mathbf{p}',l} f_l^s P_l(\hat{p}\cdot\hat{p'})\delta n_{\mathbf{p}'} \tag{8}$$

$$= \quad \tfrac{1}{V}\sum_{\mathbf{p}',l} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'} f_l^s \int d\Omega_{\mathbf{p}}(\hat{p}\cdot\hat{i})(\hat{p}\cdot\hat{j})P_l(\hat{p}\cdot\hat{p'}) \tag{9}$$

$$\tag{10}$$

To do the solid angle integral above, we write the Legendre Polynomial as an expansion of spherical harmonics $P_l(\hat{p}\cdot\hat{p'}) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\hat{p})Y_l^{m^*}(\hat{p'})$, and also decompose the products $\hat{p}_i\hat{p}_j$ into spherical harmonics:

$$\hat{p}_x\hat{p}_x = \sin^2(\theta)\cos^2(\phi) = -\frac{4}{30}Y_2^{0^*} + \frac{1}{3}Y_0^{0^*} + \frac{1}{30}(Y_2^{2^*} - Y_2^{-2^*}) \tag{11}$$

$$\hat{p}_y\hat{p}_y = \sin^2(\theta)\sin^2(\phi) = -\frac{4}{30}Y_2^{0^*} + \frac{1}{3}Y_0^{0^*} - \frac{1}{30}(Y_2^{2^*} - Y_2^{-2^*}) \tag{12}$$

$$\hat{p}_z\hat{p}_z = \cos^2(\theta) = \frac{4}{15}Y_2^{0^*} + \frac{2}{3}Y_0^{0^*} \tag{13}$$

$$\hat{p}_x\hat{p}_y = \sin^2(\theta)\cos(\phi)\sin(\phi) = \frac{1}{30}(-Y_2^{2^*} - Y_2^{-2^*}) \tag{14}$$

$$\hat{p}_x\hat{p}_z = \sin(\theta)\cos(\theta)\cos(\phi) = \frac{1}{30}(-Y_2^{1^*} + Y_2^{-1^*}) \tag{15}$$

$$\hat{p}_y\hat{p}_z = \sin(\theta)\cos(\theta)\sin(\phi) = \frac{1}{30}(Y_2^{1^*} + Y_2^{-1^*}) \tag{16}$$

From these it is easy to see that only the $l = 0$ and $l = 2$ Fermi Liquid parameters will survive. After integration of the solid angle using orthogonality of the spherical harmonics we have

$$\hat{p}_x\hat{p}_x : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\Big(- \frac{4}{5*30}Y_2^{0^*} + \frac{1}{3}Y_0^{0^*} + \frac{1}{5*30}(Y_2^{2^*} - Y_2^{-2^*})\Big) \quad (17)$$

$$\hat{p}_y\hat{p}_y : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\Big(- \frac{4}{5*30}Y_2^{0^*} + \frac{1}{3}Y_0^{0^*} - \frac{1}{5*30}(Y_2^{2^*} - Y_2^{-2^*})\Big) \quad (18)$$

$$\hat{p}_z\hat{p}_z : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\Big(\frac{4}{5*15}Y_2^{0^*} + \frac{2}{3}Y_0^{0^*}\Big) \quad (19)$$

$$\hat{p}_x\hat{p}_y : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\frac{1}{5*30}(-Y_2^{2^*} - Y_2^{-2^*}) \quad (20)$$

$$\hat{p}_x\hat{p}_z : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\frac{1}{5*30}(-Y_2^{1^*} + Y_2^{-1^*}) \quad (21)$$

$$\hat{p}_y\hat{p}_z : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\frac{1}{5*30}(Y_2^{1^*} + Y_2^{-1^*}) \quad (22)$$

Finally, we can write the spherical harmonic combinations as products of cartesian basis

$$\hat{p}_x\hat{p}_x : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\frac{1}{5}(f_2^s\hat{p'}_x\hat{p'}_x - \frac{4(f_2^s - f_0^s)}{15}) \quad (23)$$

$$\hat{p}_y\hat{p}_y : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\frac{1}{5}(f_2^s\hat{p'}_y\hat{p'}_y - \frac{4(f_2^s - f_0^s)}{15}) \quad (24)$$

$$\hat{p}_z\hat{p}_z : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\frac{1}{5}(f_2^s\hat{p'}_z\hat{p'}_z - \frac{4(f_2^s - f_0^s)}{15}) \quad (25)$$

$$\hat{p}_x\hat{p}_y : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\frac{1}{5}(f_2^s\hat{p'}_x\hat{p'}_y) \quad (26)$$

$$\hat{p}_x\hat{p}_z : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\frac{1}{5}(f_2^s\hat{p'}_x\hat{p'}_z) \quad (27)$$

$$\hat{p}_y\hat{p}_z : \frac{1}{V}\sum_{\mathbf{p}'} 2N_f(p_f^2/m^*)\delta n_{\mathbf{p}'}\frac{1}{5}(f_2^s\hat{p'}_y\hat{p'}_z) \quad (28)$$

The momentum current for a single excitation is then

$$\pi_{xx} = \frac{k_x^2}{m^*} + (p_f^2/m^*)\frac{1}{5}(f_2^s \hat{k}_x \hat{k}_x - \frac{4(f_2^s - f_0^s)}{15}) \tag{29}$$

$$\pi_{yy} = \frac{k_y^2}{m^*} + (p_f^2/m^*)\frac{1}{5}(f_2^s \hat{k}_y \hat{k}_y - \frac{4(f_2^s - f_0^s)}{15}) \tag{30}$$

$$\pi_{zz} = \frac{k_z^2}{m^*} + (p_f^2/m^*)\frac{1}{5}(f_2^s \hat{k}_z \hat{k}_z - \frac{4(f_2^s - f_0^s)}{15}) \tag{31}$$

$$\pi_{xy} = \pi_{yx} = \frac{k_x k_y}{m^*} + (p_f^2/m^*)\frac{1}{5}(f_2^s \hat{k}_x \hat{k}_y) \tag{32}$$

$$\pi_{xz} = \pi_{zx} = \frac{k_x k_z}{m^*} + (p_f^2/m^*)\frac{1}{5}(f_2^s \hat{k}_x \hat{k}_z) \tag{33}$$

$$\pi_{yz} = \pi_{zy} = \frac{k_y k_z}{m^*} + (p_f^2/m^*)\frac{1}{5}(f_2^s \hat{k}_y \hat{k}_z) \tag{34}$$

Energy Current

$$-\tfrac{1}{V}\sum_{\mathbf{p}} \epsilon_{\mathbf{p}}^0 (\nabla_{\mathbf{p}} \epsilon_{\mathbf{p}}^0) \tfrac{\partial n_{\mathbf{p}}^0}{\partial \epsilon_{\mathbf{p}}} \delta\epsilon_{\mathbf{p}} \tag{35}$$

$$= \quad -\tfrac{1}{V}\sum_{\mathbf{p}} \epsilon_{\mathbf{p}}^0 (\mathbf{p}/m^*)(-\delta(\xi_{\mathbf{p}}))\tfrac{2}{V}\sum_{\mathbf{p}',l} f_l^s P_l(\hat{p}\cdot\hat{p'})\delta n_{\mathbf{p}'} \tag{36}$$

$$= \quad \tfrac{1}{V}\sum_{\mathbf{p}',l} 2N_f \epsilon_f (p_f/m^*) \delta n_{\mathbf{p}'} f_l^s \int d\Omega_{\mathbf{p}} \hat{p} P_l(\hat{p}\cdot\hat{p'}) \tag{37}$$

$$= \quad \tfrac{1}{V}\sum_{\mathbf{p}'} 2N_f \epsilon_f (p_f/m^*) \delta n_{\mathbf{p}'} \hat{p'} f_1^s/3 \tag{38}$$

$$= \quad \tfrac{1}{V}\sum_{\mathbf{p}'} \epsilon_f (\mathbf{p}'/m^*) \delta n_{\mathbf{p}'} F_1^s/3 \tag{39}$$

$$\mathbf{q} = \frac{1}{V}\sum_{\mathbf{p}} \delta n_{\mathbf{p}} \frac{\mathbf{p}}{m^*}\left(\epsilon_{\mathbf{p}}^0 + \epsilon_f \frac{F_1^s}{3}\right) \tag{40}$$

$$\boxed{\mathbf{q} = \frac{\mathbf{k}}{m^*}\epsilon_f\left(1 + \frac{F_1^s}{3}\right)}$$