HW 8: Physics 545

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1

The imaginary part of the dielectric in RPA is $\epsilon'' = -\frac{4\pi e^2}{q^2}\chi''(\mathbf{q},\omega)$ which requres the imaginary part of the susceptibility($''$ denotes imaginary part and $'$ denotes real part). For the distribution function we use Maxwell Boltzman $n_{\mathbf{p}} = e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}$, $\beta = (k_B T)^{-1}$:

$$\chi''(\mathbf{q},\omega) = -\pi\sum_{\mathbf{p}}(n_{\mathbf{p}-\mathbf{q}/2} - n_{\mathbf{p}+\mathbf{q}/2})\,\delta(\hbar\omega + \epsilon_{\mathbf{p}-\mathbf{q}/2} - \epsilon_{\mathbf{p}+\mathbf{q}/2}) \tag{1}$$

$$= -\frac{4\pi^2 e^{-\beta(q^2/8m-\mu)}}{(2\pi\hbar)^3}\int_{-1}^{1} dx\int_{0}^{\infty} dp\, p^2 e^{-\beta p^2/2m} sinh\Big(\frac{\beta qpx}{2m}\Big)\delta(\hbar\omega - \frac{pqx}{m}) \tag{2}$$

$$= -\frac{4\pi^2 m e^{-\beta(q^2/8m-\mu)}}{(2\pi\hbar)^3 q} sinh\Big(\frac{\beta\hbar\omega}{2}\Big)\int_{0}^{\infty} dp\, p e^{-\beta p^2/2m} \tag{3}$$

$$= -\frac{4\pi^2 m^2 e^{-\beta(q^2/8m-\mu)}}{(2\pi\hbar)^3 \beta q} sinh\Big(\frac{\beta\hbar\omega}{2}\Big) \tag{4}$$

Now we wish to use convenient dimensionless variables to plot with. $\Omega = \beta\hbar\omega$, $Q = \sqrt{\frac{\beta}{8m}}q$

$$\chi''(Q,\Omega) = -\Big[\frac{4\pi^2 m^{3/2} e^{\beta\mu}}{(2\pi\hbar)^3\sqrt{8\beta}}\Big]\frac{e^{-Q^2}}{Q} sinh\Big(\frac{\Omega}{2}\Big) \tag{5}$$

$$= -C(T,\mu,m)\frac{e^{-Q^2}}{Q} sinh\Big(\frac{\Omega}{2}\Big) \tag{6}$$

The Q and Ω free term $C(T,\mu,m)$ is the amplitude and could also be plotted vs T and μ.

Setting $\hbar = 1$, the imaginary part of the dielectric is:

$$\epsilon''(Q,\Omega) = \frac{4\pi e^2}{8mQ^2\beta}\Big[\frac{4\pi^2 m^{3/2} e^{\beta\mu}}{(2\pi)^3\sqrt{8\beta}}\Big]\frac{e^{-Q^2}}{Q} sinh\Big(\frac{\Omega}{2}\Big) \tag{7}$$

$$= \Big[\frac{e^2 m^{1/2} e^{\beta\mu}}{8\sqrt{2}\beta^{3/2}}\Big]\frac{e^{-Q^2}}{Q^3} sinh\Big(\frac{\Omega}{2}\Big) \tag{8}$$

$$\tag{9}$$

The problem asks to plot:

$$\omega\epsilon''(Q,\Omega) = \left[\frac{e^2 m^{1/2} e^{\beta\mu}}{8\sqrt{2}\beta^{5/2}}\right]\frac{e^{-Q^2}}{Q^3}\Omega sinh\left(\frac{\Omega}{2}\right) \tag{10}$$

2a

REAL PART We wish to find $\epsilon(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{q^2}\chi(\mathbf{q},\omega)$ and therefore must compute the susceptibility by shifting the origin to the center of the two Fermi Surfaces:

$$\chi(\mathbf{q},\omega) = \sum_{\mathbf{p}} \frac{n_{\mathbf{p}-\mathbf{q}/2} - n_{\mathbf{p}+\mathbf{q}/2}}{\hbar\omega + \epsilon_{\mathbf{p}-\mathbf{q}/2} - \epsilon_{\mathbf{p}+\mathbf{q}/2}} \tag{11}$$

$$= \int_0^{k_f} dk \int_0^{2\pi} d\theta \frac{k}{\hbar\omega - \frac{q^2}{2m} - \frac{kq}{m}cos(\theta)} - \frac{k}{\hbar\omega + \frac{q^2}{2m} - \frac{kq}{m}cos(\theta)} \tag{12}$$

$$= \frac{m}{q}\int_0^{k_f} dk \int_0^{2\pi} d\theta \frac{k}{\lambda_- - kcos(\theta)} - \frac{k}{\lambda_+ - kcos(\theta)} \tag{13}$$

Where we defined $\lambda_\pm = \hbar\omega m/q \pm \frac{q}{2}$. To do the θ integral we can use complex definition $z = e^{i\theta}$ to get a contour integral around the unit circle.

$$= -\frac{im}{q}\int_0^{k_f} dk \quad k\int_C dz \frac{1}{\lambda_- z - \frac{k}{2}(z^2+1)} - \frac{1}{\lambda_+ z - \frac{k}{2}(z^2+1)} \tag{14}$$

$$= \frac{2im}{q}\int_0^{k_f} dk \int_C dz \frac{1}{-(2\lambda_-/k)z + z^2 + 1} - \frac{1}{-(2\lambda_+/k)z + z^2 + 1} \tag{15}$$

$$= \frac{2im}{q}\int_0^{k_f} dk \int_C dz \frac{1}{(z - z_{--})(z - z_{-+})} - \frac{1}{(z - z_{+-})(z - z_{++})} \tag{16}$$

Where $z_{s\pm} = (\lambda_s/k) \pm \sqrt{(\lambda_s/k)^2 - 1}$

And we require $|z_{s\pm}| < 1$ in order for the pole to be inside the unit circle contour.

$$= \quad -\frac{4\pi m}{q}\int_0^{k_f} dk\Big[\frac{1}{2\sqrt{(\lambda_-/k)^2-1}} \quad |z_{-+}|<1 \tag{17}$$

$$-\frac{1}{2\sqrt{(\lambda_-/k)^2-1}} \quad |z_{--}|<1 \tag{18}$$

$$-\frac{1}{2\sqrt{(\lambda_+/k)^2-1}} \quad |z_{++}|<1 \tag{19}$$

$$+\frac{1}{2\sqrt{(\lambda_+/k)^2-1}} \quad |z_{+-}|<1\Big] \tag{20}$$

Turning back to the integral, we do the k bit:

$$= \quad \frac{2\pi m}{q}\Big[-|\lambda_-|+\sqrt{\lambda_-^2-k_f^2} \quad |z_{-+}|<1 \tag{21}$$

$$+|\lambda_-|-\sqrt{\lambda_-^2-k_f^2} \quad |z_{--}|<1 \tag{22}$$

$$-|\lambda_+|-\sqrt{\lambda_+^2-k_f^2} \quad |z_{++}|<1 \tag{23}$$

$$+|\lambda_+|+\sqrt{\lambda_+^2-k_f^2} \quad |z_{+-}|<1\Big] \tag{24}$$

Now it is a matter of exactly when to keep the various residues above according to the magnitude of $|z_{s\pm}|$ and I leave that to the experimentalist to interpret HAHA!!!!

I'd plot with reduced units $Q = q/k_f$, $\hbar\omega/\epsilon_f$...

It is important to remember that we are considering retarded interactions so $\omega \to \omega + i\eta/\hbar$. After taylor expanding for small η, the condition $|z_{s\pm}| < 1$ for $Re(\lambda'_s/k)^2 < 1$ becomes

$|z_{s\pm}|^2 \approx 1 \pm C\eta$, $(\lambda'_s/k)^2 < 1$

Where C is some constant depending on *lambda* which doesn't matter, the main point here is to see that only the *minus* roots are inside of the contour in this case

The other case for $Re(\lambda'_s/k_f)^2 > 1$ results in the following inequality:

$$-k_f < \lambda \pm \sqrt{\lambda^2 - k_f^2} < k_f \tag{25}$$

and we take the "+" root if $\lambda_\pm < -k_f$ and "-" root if $\lambda_\pm < k_f$

2b

Imaginary part

$$
\begin{aligned}
\chi''(\mathbf{q},\omega) &= -\frac{\pi}{(2\pi\hbar)^2}\int d\theta dp \quad p(n_{\mathbf{p}-\mathbf{q}/2} - n_{\mathbf{p}+\mathbf{q}/2})\,\delta(\hbar\omega + \epsilon_{\mathbf{p}-\mathbf{q}/2} - \epsilon_{\mathbf{p}+\mathbf{q}/2}) && (26)\\
&= -\frac{\pi}{(2\pi\hbar)^2}\int d\theta dk \quad k(\delta(\hbar\omega - q^2/2m - kqcos(\theta)/m) && (27)\\
&\qquad -\delta(\hbar\omega + q^2/2m - kqcos(\theta)/m)) && (28)\\
&= -\frac{\pi}{(2\pi\hbar)^2}\int d\theta \quad \frac{m}{qcos(\theta)}\Big(k^*_- \qquad if\ k^*_- < k_f && (29)\\
&\qquad -k^*_+ \qquad if\ k^*_+ < k_f\Big) && (30)
\end{aligned}
$$

Where $k^*_\pm = \frac{m}{qcos(\theta)}(\hbar\omega \pm q^2/2m)$.

Again, I leave this to the experimentalists to interpret so on to the next calculation!