HW 12: Physics 545

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1a

The critical value of $|\psi|(T)$ is found by minimizing the free energy with respect to ψ^* for temperatures below T_c.

$$\frac{\partial F}{\partial \psi^*} = a(T - T_c)\psi + \beta\psi^2\psi^* = 0 \tag{1}$$

$$\Rightarrow |\psi|^2 = a(T_c - T)/\beta \tag{2}$$

The resulting condensation energy per volume is:

$$\begin{aligned} \Delta F(|\psi|)/V \quad / = &\quad -\beta(a(T_c - T)/\beta)^2 + \frac{\beta}{2}(a(T_c - T)/\beta)^2 & (3) \\ = &\quad -\frac{a^2(T_c - T)^2}{2\beta} & (4) \end{aligned}$$

and the critical field such that the magnetic energy = condensation energy is:

$$\Delta F(|\psi|) = -H_c^2/(8\pi) \tag{5}$$

$$H_c = a|T_c - T|\sqrt{\frac{4\pi}{\beta}} \tag{6}$$

1b

The coherence length $\xi(T)$ and London penetration depth $\lambda(T)$ where defined in class. They appeared as characteristic length coefficients in the differential equations for ψ and $\mathbf{A}$ respectively.

$$\xi(T) = \sqrt{\frac{K}{a|T - T_c|}} \tag{7}$$

$$\lambda(T) = \frac{\hbar c}{4e}\sqrt{\frac{\beta}{2\pi K a|T - T_c|}} \tag{8}$$

1c

The equation for $|\psi|$ in a magnetic field is found by the minimization of F w.r.t ψ^*

$$\frac{\partial F}{\partial \psi^*} = 0 \tag{9}$$

$$\Rightarrow K\left|\frac{\nabla}{i} - \frac{2e}{\hbar c}\mathbf{A}\right|^2 \psi + a(T - T_c)\psi + \beta|\psi|^2\psi = 0 \tag{10}$$

Up to linear order in ψ this reduces to the Schrodinger eigen-equation for a charged particle in magnetic field with $a(T-T_c)$ acting as the eigen-energy ϵ.

$$a(T - T_c)\psi = -K\left|\frac{\nabla}{i} - \frac{2e}{\hbar c}\mathbf{A}\right|^2 \psi \tag{11}$$

The solutions to such a problem are the quantized cyclotron orbits (aka Landau levels) $\epsilon_n = (\frac{1}{2} + n)\hbar\omega_c$ and $\omega_c = \frac{eH}{mc}$ is the cyclotron frequency of electrons.

When the lowest energy cyclotron orbit ϵ_0 is smaller than $a(T - T_c)$ this becomes the energetically favorable state and the critical value of field needed for this condition is H_{c2}.

$$a(T - T_c) = \frac{\hbar e H_{c2}}{2mc} \tag{12}$$

$$\Rightarrow H_{c2}(T) = \frac{2mca|T - T_c|}{\hbar e} \tag{13}$$

1d

We are interested in the critical condition $H_{c2} \geq H_c$. When viewed as an equality, this condition means that formation of cyclotron orbits and bulk magnetism is equally favorable and energetically preferred over the condensed superconducting state.

$$H_{c2} \geq H_c \tag{14}$$

$$\Rightarrow \frac{2mc}{\hbar e}\sqrt{\frac{\beta}{4\pi}} \geq 1 \tag{15}$$

From part b we can define the Ginzburg-Landau coefficient $\kappa = \lambda(T)/\xi(T)$ and write it in a way which lends itself easily to use the above condition

$$\kappa = \frac{1}{\sqrt{2}}\left[\frac{2mc}{\hbar e}\sqrt{\frac{\beta}{4\pi}}\right] \tag{16}$$

The term in brackets is exactly that which appears in the inequality above so we can write the condition is $\kappa \geq 1/\sqrt{2}$!

2a

The first step in this problem is to determine the coefficients of the two magnetizations M_x and M_d, where d denotes the diagonal magnetization $\mathbf{M} \propto (1,1,1)$. The variation of the free energy wrt $\mathbf{M}$ is:

$$\frac{\partial F}{\partial \mathbf{M}} = \sum_{i=xyz} 2a(T-T_c)M_i + 2\beta(\mathbf{M}\cdot\mathbf{M})M_i + 2b(T-T^*)M_i^3 \tag{17}$$

For the M_x direction the minimization equation is

$$0 = \left.\frac{\partial F}{\partial \mathbf{M}}\right|_{\mathbf{M}=M_x(1,0,0)} = 2a(T-T_c)M_x + 2\beta M_x^3 + 2b(T-T^*)M_x^3 \tag{18}$$

and the solution for magnetization magnitude is

$$M_x^2 = \frac{-a(T-T_c)}{\beta + b(T-T^*)} \tag{19}$$

with the restriction $T < T_c$ and $T > T^* - \beta/b$.

Similarly, for $\mathbf{M} = M_d(1,1,1)$, the minimization equation is

$$0 = \left.\frac{\partial F}{\partial \mathbf{M}}\right|_{\mathbf{M}=M_d(1,1,1)} = 3(2a(T-T_c)M_d + 6\beta M_d^3 + 2b(T-T^*)M_d^3) \tag{20}$$

and the solution for magnetization magnitude is

$$M_d^2 = \frac{-a(T-T_c)}{3\beta + b(T-T^*)} = M_x^2\frac{\beta + b(T-T^*)}{3\beta + b(T-T^*)} \tag{21}$$

with restriction $T < T_c$ and $T > T^* - 3\beta/b$

To find the energetically favorable orientation one must compare the free energy of both states:

$$F_{M_x} = a(T-T_c)M_x^2/2 = \frac{-a^2(T-T_c)^2}{2(\beta + b(T-T^*))} \tag{22}$$

$$F_{M_d} = 3a(T-T_c)M_d^2/2 = \frac{-3a^2(T-T_c)^2}{2(3\beta + b(T-T^*))} \tag{23}$$

The above shows that both magnetizations are favorable over the ground state for $T < T_c$. To find the most favorable we consider their difference

$$\Delta F \quad = \quad F_{M_x} - F_{M_d} = \frac{a^2 b(T-T_c)^2(T-T^*)}{(\beta + b(T-T^*))(3\beta + b(T-T^*))} \tag{24}$$

$\Delta F > 0$ for $T > T^*$ so the diagonal order is preferred. For $T < T^*$ the ordering along the axis is preferred.

2b

The specific heat jump at T_c for the magnetized state is:

$$\Delta C = -T \frac{\partial^2 F}{\partial T^2}\bigg|_{T_c} = \frac{3a^2 T_c}{3\beta + b(T_c - T^*)} \tag{25}$$

The entropy is $S = -\frac{\partial F}{\partial T}$

$$S_{M_x} = \frac{-a^2(T - T_c)}{2}\left[\frac{2(\beta + b(T - T^*)) - b(T - T_c)}{(\beta + b(T - T^*))^2}\right] \tag{26}$$

$$S_{M_d} = \frac{-3a^2(T - T_c)}{2}\left[\frac{2(3\beta + b(T - T^*)) - b(T - T_c)}{(3\beta + b(T - T^*))^2}\right] \tag{27}$$

and the jump at $T = T^*$ is:

$$\Delta S = S_{M_x} - S_{M_d}\bigg|_{T^*} = \frac{a^2 b(T^* - T_c)^2}{3\beta^2} \tag{28}$$

The latent heat for such a phase transition is $Q = T^* \Delta S$