HW 11: Physics 545

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We start by noting the spin structure of the wave function in a slightly different form than given: $\psi = i[\boldsymbol{\Delta}\cdot\boldsymbol{\sigma}][\hat{y}\cdot\boldsymbol{\sigma}]$. This definition lends itself better to solutions of earlier homework.

The average spin is

$$\begin{aligned}
\langle \mathbf{S}\rangle &= \frac{\hbar}{2}\int \frac{d^3k}{(2\pi\hbar)^3} Tr\Big[\psi(\mathbf{k})^\dagger \boldsymbol{\sigma}\psi(\mathbf{k})\Big] && (1)\\
&= \frac{\hbar}{2}\int \frac{d^3k}{(2\pi\hbar)^3} Tr\Big[[\hat{y}\cdot\boldsymbol{\sigma}][\boldsymbol{\Delta}^*\cdot\boldsymbol{\sigma}]\boldsymbol{\sigma}[\boldsymbol{\Delta}\cdot\boldsymbol{\sigma}][\hat{y}\cdot\boldsymbol{\sigma}]\Big] && (2)\\
&= \frac{\hbar}{2}\int \frac{d^3k}{(2\pi\hbar)^3} Tr\Big[[\boldsymbol{\Delta}^*\cdot\boldsymbol{\sigma}]\boldsymbol{\sigma}[\boldsymbol{\Delta}\cdot\boldsymbol{\sigma}]\Big] && (3)
\end{aligned}$$

Where we used the cyclic exchange symmetry of the trace.

Now, using the formula $\sigma_i\sigma_j = \delta_{ij} + 2i\epsilon_{ijk}\sigma_k$, one can write out the above in index notation

$$\begin{aligned}
\langle \mathbf{S}_j\rangle &= \frac{\hbar}{2}\int \frac{d^3k}{(2\pi\hbar)^3} Tr\Big[\boldsymbol{\Delta}_i^*\boldsymbol{\Delta}_k\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k\Big] && (4)\\
&= \frac{\hbar}{2}\int \frac{d^3k}{(2\pi\hbar)^3} Tr\Big[\boldsymbol{\Delta}_i^*\boldsymbol{\Delta}_k(\delta_{ij} + 2i\epsilon_{ijn}\sigma_n)\boldsymbol{\sigma}_k\Big] && (5)\\
&= \frac{\hbar}{2}\int \frac{d^3k}{(2\pi\hbar)^3} Tr\Big[\boldsymbol{\Delta}_i^*\boldsymbol{\Delta}_k(2i\epsilon_{ijn}\delta_{kn})\Big] && (6)
\end{aligned}$$

$$\langle \mathbf{S}\rangle = i\hbar\int \frac{d^3k}{(2\pi\hbar)^3} 2[-\boldsymbol{\Delta}^*\times\boldsymbol{\Delta}] \qquad (7)$$

2a

The states in Fock space are $|00\rangle$, $|11\rangle$, $|01\rangle$, $|10\rangle$. The result after acting with $h_{\mathbf{k}}$ operator is:

$$\begin{aligned} h_{\mathbf{k}}|00\rangle &= -\Delta_{\mathbf{k}}|11\rangle & (8)\\ h_{\mathbf{k}}|11\rangle &= -\Delta^*_{\mathbf{k}}|00\rangle + 2\xi_{\mathbf{k}}|11\rangle & (9)\\ h_{\mathbf{k}}|01\rangle &= \xi_{\mathbf{k}}|01\rangle & (10)\\ h_{\mathbf{k}}|10\rangle &= \xi_{\mathbf{k}}|10\rangle & (11) \end{aligned}$$

As you say, this proves that these are a complete set and we expect to find 4 eigenstates which are a superposition of these.

We can immediately write two of these from the last two equations becuase $h_{\mathbf{k}}$ does not alter these states, and their energies are $\epsilon_{\mathbf{k}} = \xi_{\mathbf{k}}$ (spin independent)

The other two states are linear combinations of the Cooper pair $|11\rangle$, and no pair $|00\rangle$ $\psi = A|11\rangle + B|00\rangle$. The resulting equations can be written in matrix form and solved.

$$\epsilon_{\mathbf{k}} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2\xi_{\mathbf{k}} & -\Delta^*_{\mathbf{k}} \\ -\Delta_{\mathbf{k}} & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{12}$$

The equation for $\epsilon_{\mathbf{k}}$ is $\epsilon^2_{\mathbf{k}} - 2\xi_{\mathbf{k}}\epsilon_{\mathbf{k}} - |\Delta_{\mathbf{k}}|^2 = 0$ which has two solutions

$$\epsilon^{\pm}_{\mathbf{k}} = \xi_{\mathbf{k}} \pm \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2} \tag{13}$$

We write the un-normalized eigenvectors from the second line of the matrix equation ($B = -\frac{\Delta_{\mathbf{k}}}{\epsilon^{\pm}_{\mathbf{k}}} A$)

$$\psi^{\pm} = A\Big(|11\rangle - \frac{\Delta_{\mathbf{k}}}{\epsilon^{\pm}_{\mathbf{k}}}|00\rangle\Big) \tag{14}$$

And after normalization:

$$A = \frac{\epsilon^{\pm}_{\mathbf{k}}}{\sqrt{\left(\epsilon^{\pm}_{\mathbf{k}}\right)^2 + \Delta^2_{\mathbf{k}}}} \tag{15}$$

Now we can write these eigenvectors using:

$$\begin{aligned} u^{\pm}_{\mathbf{k}} &= -\frac{\Delta_{\mathbf{k}}}{\sqrt{\left(\epsilon^{\pm}_{\mathbf{k}}\right)^2 + \Delta^2_{\mathbf{k}}}} & (16)\\ v^{\pm}_{\mathbf{k}} &= \frac{\epsilon^{\pm}_{\mathbf{k}}}{\sqrt{\left(\epsilon^{\pm}_{\mathbf{k}}\right)^2 + \Delta^2_{\mathbf{k}}}} & (17)\\ \psi^{\pm} &= u^{\pm}_{\mathbf{k}}|00\rangle + v^{\pm}_{\mathbf{k}}|11\rangle & (18) \end{aligned}$$

2b

The state with the lowest energy is $\epsilon_{\mathbf{k}}^{-} = \xi_{\mathbf{k}} - \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$ and acting with $b_{\mathbf{k}\uparrow} = u_{\mathbf{k}} a_{\mathbf{k}\uparrow} - v_{\mathbf{k}} a^{\dagger}_{-\mathbf{k}\downarrow}$ does indeed annihilate the BCS ground state $|BCS\rangle = u_{\mathbf{k}}|00\rangle + v_{\mathbf{k}}|11\rangle$

$$b_{\mathbf{k}\uparrow}|BCS\rangle \quad = \quad -v_{\mathbf{k}} u_{\mathbf{k}}|01\rangle + u_{\mathbf{k}} v_{\mathbf{k}}|01\rangle = 0 \tag{19}$$

This means that this state is the ground state for these b operators.

We can define $b_{\mathbf{k}\downarrow} = u_{\mathbf{k}} a_{-\mathbf{k}\downarrow} + v_{\mathbf{k}} a^{\dagger}_{\mathbf{k}\uparrow}$ and use the anti-commutation relations of Fermions to see that:

$a_{-\mathbf{k}\downarrow}|11\rangle = a_{-\mathbf{k}\downarrow} a^{\dagger}_{-\mathbf{k}\downarrow} a^{\dagger}_{\mathbf{k}\uparrow}|00\rangle = -|10\rangle$

And this operator also annihilates the BCS state.

2c

If we combine the above relations we can see that the other eigenstates of Hamiltonian can be written as:

$$|10\rangle = b^{\dagger}_{\mathbf{k}\uparrow}|BCS\rangle \tag{20}$$

$$|01\rangle = b^{\dagger}_{\mathbf{k}\downarrow}|BCS\rangle \tag{21}$$

$$u_{\mathbf{k}}^{+}|00\rangle + v_{\mathbf{k}}^{+}|11\rangle = b^{\dagger}_{\mathbf{k}\uparrow}|BCS\rangle \tag{22}$$

And their excitation energy above the BCS energy $\epsilon_{\mathbf{k}}^{BCS} = \xi_{\mathbf{k}} - \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$, are:

$$\sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} \tag{23}$$

$$\sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} \tag{24}$$

$$2\sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} \tag{25}$$

respectively