Normal State Calculation of Magnetic Susceptibility

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The normal state susceptibility is given by the Lindhard formula:

$$\chi_{\parallel} = -2\mu_e^2 \sum_{\mathbf{k},s} \frac{f(\epsilon_{\mathbf{k},s}) - f(\epsilon_{\mathbf{k+q},s})}{\epsilon_{\mathbf{k},s} - \epsilon_{\mathbf{k+q},s}}$$

$$\chi_{\perp} = -2\mu_e^2 \sum_{\mathbf{k},s} \frac{f(\epsilon_{\mathbf{k},s}) - f(\epsilon_{\mathbf{k+q},\bar{s}})}{\epsilon_{\mathbf{k},s} - \epsilon_{\mathbf{k+q},\bar{s}}}$$

Where $\epsilon_{\mathbf{k},s} = k^2/(2m) + s\mu_B H$, and $f(\epsilon)$ is the fermi distribution funciton.

We first proceed with the calculation at zero temperature. We orient our coordinates such that $\mathbf{q} = q\hat{x}$, and change from $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$ to $\mathbf{k}-\mathbf{q}/2$ and $\mathbf{k}+\mathbf{q}/2$. Converting the sums to integrals, and using symmetry about the x and y axes, we have:

$$\chi_{\parallel} = -8\mu_e^2 \sum_s \int_0^{\pi/2} d\phi \int kdk \frac{f(\epsilon_{\mathbf{k-q}/2,s}) - f(\epsilon_{\mathbf{k+q}/2,s})}{-2kq\cos\phi}$$

$$\chi_{\perp} = -8\mu_e^2 \sum_s \int_0^{\pi/2} d\phi \int kdk \frac{f(\epsilon_{\mathbf{k-q}/2,s}) - f(\epsilon_{\mathbf{k+q}/2,\bar{s}})}{-2kq\cos\phi + 2s\mu_B H}$$

Where we have also normalized the 2D area of integration $A/(2\pi\hbar)^2 = 1$

1 Parallel

To determine the limits of integration on k, we need to solve the dispersion relation for k when $\epsilon_{\mathbf{k}\pm\mathbf{q}/2,s} = \mu$, the chemical potential. If we normalize the equation by multiplying and dividing it by k_f^2, the result is:

$$1 = k'^2 \pm k'q'\cos\phi + (q'/2)^2 + sH' \qquad \Rightarrow k' = \mp\frac{q'\cos\phi}{2} \pm \sqrt{(q'\cos\phi/2)^2 - ((q'/2)^2 + sH' - 1)}$$

$$\Rightarrow k' = \mp\frac{q'\cos\phi}{2} \pm \sqrt{1 - sH' - (q'/2)^2\sin^2\phi}$$

Where $k' = k/k_f$, $q' = q/k_f$ and $H' = \mu_B H/k_f^2$. Now we consider only the parallel component in three different regions:

$\boxed{q < 2\sqrt{1-H'}}$

$$\chi_\| = 8\mu_e^2 \sum_s \int_0^{\pi/2} d\phi \int_0^{\frac{q'\cos\phi}{2}+\sqrt{1-sH'-(q'/2)^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi} - \int_0^{\frac{-q'\cos\phi}{2}+\sqrt{1-sH'-(q'/2)^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi}$$

$$\chi_\| = 8\mu_e^2 \sum_s \int_0^{\pi/2} d\phi \frac{q'\cos\phi}{2q'\cos\phi} = 4\mu_e^2\pi$$

$\boxed{2\sqrt{1-H'} < q < 2\sqrt{1+H'}}$

$$\chi_\| = 2\mu_e^2\pi + 8\mu_e^2 \int_0^{\phi*} d\phi \int_{\frac{q'\cos\phi}{2}-\sqrt{1-H'-(q'/2)^2\sin^2\phi}}^{\frac{q'\cos\phi}{2}+\sqrt{1-H'-(q'/2)^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi}$$

$$\chi_\| = 2\mu_e^2\pi + \frac{8\mu_e^2}{q'} \int_0^{\phi*} d\phi \frac{\sqrt{1-H'-(q'/2)^2\sin^2\phi}}{\cos\phi}$$

$$\chi_\| = 2\mu_e^2\pi + \frac{8\mu_e^2}{q'} \int_0^{\sin\phi*} dx \frac{\sqrt{1-H'-(q'/2)^2x^2}}{1-x^2}$$

$$\chi_\| = 2\mu_e^2\pi + \frac{8\mu_e^2}{q'}\frac{\pi}{4}(q' - \sqrt{q'^2+4H'-4})$$

$$\chi_\| = 4\mu_e^2\pi - 2\mu_e^2\pi\sqrt{1-(1-H')(2/q')^2}$$

$\boxed{q > 2\sqrt{1+H'}}$

$$\chi_\| = 8\mu_e^2 \int_0^{\phi*} d\phi \int_{\frac{q'\cos\phi}{2}-\sqrt{1+H'-(q'/2)^2\sin^2\phi}}^{\frac{q'\cos\phi}{2}+\sqrt{1+H'-(q'/2)^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi} + 8\mu_e^2 \int_0^{\phi*} d\phi \int_{\frac{q'\cos\phi}{2}-\sqrt{1-H'-(q'/2)^2\sin^2\phi}}^{\frac{q'\cos\phi}{2}+\sqrt{1-H'-(q'/2)^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi}$$

$$\chi_\| = \frac{8\mu_e^2}{q'}\frac{\pi}{4}(q' - \sqrt{q'^2-4H'-4}) + \frac{8\mu_e^2}{q'}\frac{\pi}{4}(q' - \sqrt{q'^2+4H'-4})$$

$$\chi_\| = 4\mu_e^2\pi - 2\mu_e^2\pi\sqrt{1-(1+H')(2/q'^2)} - 2\mu_e^2\pi\sqrt{q'^2-(1-H')(2/q')^2}$$

2 Perpendicular

Now we continue with the perpendicular component. To do this we move the origin such that at k_x=0 the s=1 and s=-1 surfaces intersect. The equations for this transformation are:

$$s=1: \qquad k_x' \to k_x' - q'/2 + H'/q', \qquad k^2 = 1-H' \to k = (q'/2 - H'/q')\cos\phi \pm \sqrt{1-H'-(q'/2-H'/q')^2\sin^2\phi}$$

$$s=-1: \qquad k_x' \to k_x' - q'/2 - H'/q', \qquad k^2 = 1+H' \to k = (q'/2 + H'/q')\cos\phi \pm \sqrt{1+H'-(q'/2+H'/q')^2\sin^2\phi}$$

For this integration there are two regions:

$$\boxed{q' < \sqrt{1+H'} + \sqrt{1-H'}}$$

$$\chi_\perp = 8\mu_e^2 \int\limits_0^{\pi/2} d\phi \int\limits_0^{(q'/2-H'/q')\cos\phi+\sqrt{1-H'-(q'/2-H'/q')^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi}$$
$$- \int\limits_0^{-(q'/2-H'/q')\cos\phi+\sqrt{1-H'-(q'/2-H'/q')^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi}$$
$$+ \int\limits_0^{(q'/2+H'/q')\cos\phi+\sqrt{1+H'-(q'/2+H'/q')^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi}$$
$$- \int\limits_0^{-(q'/2+H'/q')\cos\phi+\sqrt{1+H'-(q'/2+H'/q')^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi}$$

$$\chi_\perp = (4\mu_e^2/q') \int\limits_0^{\pi/2} d\phi 2(q'/2 - H'/q') + 2(q'/2 + H'/q')$$

$$\chi_\perp = 4\mu_e^2\pi$$

$$\boxed{q' > \sqrt{1+H'} + \sqrt{1-H'}}$$

$$\chi_\perp = 8\mu_e^2 \int\limits_0^{\phi^*_{1,2}} d\phi \int\limits_{(q'/2-H'/q')\cos\phi-\sqrt{1-H'-(q'/2-H'/q')^2\sin^2\phi}}^{(q'/2-H'/q')\cos\phi+\sqrt{1-H'-(q'/2-H'/q')^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi}$$
$$+ \int\limits_{(q'/2+H'/q')\cos\phi-\sqrt{1+H'-(q'/2+H'/q')^2\sin^2\phi}}^{(q'/2+H'/q')\cos\phi+\sqrt{1+H'-(q'/2+H'/q')^2\sin^2\phi}} dk' \frac{1}{2q'\cos\phi}$$

$$\chi_\perp = (8\mu_e^2/q') \int\limits_0^{\phi^*_1} d\phi \frac{\sqrt{1-H'-(q'/2-H'/q')^2\sin^2\phi}}{\cos\phi} + \int\limits_0^{\phi^*_2} d\phi \frac{\sqrt{1+H'-(q'/2+H'/q')^2\sin^2\phi}}{\cos\phi}$$

$$\chi_\perp = (8\mu_e^2/q') \int\limits_0^{\sqrt{1-H'}/(q'/2-H'/q')} dx \frac{\sqrt{1-H'-(q'/2-H'/q')^2x^2}}{1-x^2} + \int\limits_0^{\sqrt{1+H'}/(q'/2+H'/q')} dx \frac{\sqrt{1+H'-(q'/2+H'/q')^2x^2}}{1-x^2}$$

$$\chi_\perp = (8\mu_e^2/q')(\pi/4q')\left[(-2H' + q^2 - \sqrt{4H'^2 - 4q'^2 + q'^4}) + (2H' + q'^2 - \sqrt{4H'^2 - 4q'^2 + q'^4})\right]$$

$$\chi_\perp = 4\mu_e^2\pi\left[1 - \sqrt{1 + (2H'/q'^2)^2 - (2/q')^2}\right]$$

3 Low Temperature Expansion

We now compute the low temperature expansion of χ for zero field using the Sommerfeld Expansion. We again write the susceptibility:

$$\begin{aligned}
\chi &= -4\mu_e^2 \sum_{\mathbf{k}} \frac{f(\epsilon_k) - f(\epsilon_{k+q})}{\epsilon_k - \epsilon_{k+q}} \\
&= -4\mu_e^2 \sum_{\mathbf{k}} \frac{f(\epsilon_k)}{\epsilon_k - \epsilon_{k+q}} - \frac{f(\epsilon_{k+q})}{\epsilon_k - \epsilon_{k+q}} \\
&= -4\mu_e^2 \sum_{\mathbf{k}} \frac{f(\epsilon_k)}{\epsilon_k - \epsilon_{k+q}} - \frac{f(\epsilon_k)}{\epsilon_{k-q} - \epsilon_k}
\end{aligned}$$

Where we have changed variables in the second term $\mathbf{k} \to \mathbf{k} - \mathbf{q}$. To find the Sommerfeld expansion we need to convert the integration to energies and add an imaginary part to the denominator which we will take to zero at the end.

$$\begin{aligned}
\chi &= 2\mu_e^2 \int_0^{2\pi} d\phi \int_0^{\infty} d\epsilon \left[\frac{1}{2kq\cos\phi + q^2} + \frac{1}{-2kq\cos\phi + q^2}\right] f(\epsilon_k) \\
&= (\mu_e^2/q) \int_0^{\infty} d\epsilon f(\epsilon) \int_0^{2\pi} d\phi \left[\frac{1}{\sqrt{\epsilon}\cos\phi + q/2 + i\delta/2} + \frac{1}{-\sqrt{\epsilon}\cos\phi + q/2 - i\delta/2}\right] \\
&= (\mu_e^2/q) \int_0^{\infty} d\epsilon \frac{f(\epsilon)}{\sqrt{\epsilon}} \int_0^{2\pi} d\phi \left[\frac{1}{\cos\phi + (q+i\delta)/2\sqrt{\epsilon}} - \frac{1}{\cos\phi - (q-i\delta)/2\sqrt{\epsilon}}\right]
\end{aligned}$$

Now we turn our attention to the ϕ integral, using complex integration around a unit circle in the complex plane ($z = e^{i\phi}$), and defining $a_\pm = (q \pm i\delta)/\sqrt{\epsilon}$

$$\begin{aligned}
&\int_\Gamma \frac{dz}{iz} \left[\frac{1}{(z+1/z)/2 + a_+/2} - \frac{1}{(z+1/z)/2 - a_-/2}\right] \\
=&(2/i) \int_\Gamma dz \left[\frac{1}{z^2 + a_+ z + 1} - \frac{1}{z^2 - a_- z + 1}\right]
\end{aligned}$$

Using subscript 1(2) to refer to the first (second) term, the poles are:

$$\begin{aligned}
z_1 &= -a_+/2 \pm \sqrt{(a_+/2)^2 - 1} \\
z_2 &= a_-/2 \pm \sqrt{(a_-/2)^2 - 1}
\end{aligned}$$

For $q > 2$ the roots which are inside of the unit circle are z_{1+} and z_{2-}. However, for $q < 2$ we need to carfully consider the magnitude of the poles. With this consideration, one finds that $|z_{1+}| < 1$ and $|z_{2+}| < 1$ while $|z_{1-}| > 1$ and $|z_{2-}| > 1$.

$\boxed{q < 2}$

$$
\begin{aligned}
&(2/i)\int_\Gamma dz\left[\frac{1}{z^2+a_+z+1}-\frac{1}{z^2-a_-z+1}\right]\\
=&4\pi Res\left[\frac{1}{(z-z_{1+})(z-z_{1-})}-\frac{1}{(z-z_{2+})(z-z_{2-})}\right]\\
=&4\pi\left[\frac{1}{z_{1+}-z_{1-}}-\frac{1}{z_{2+}-z_{2-}}\right]\\
=&(2\pi)\left[\frac{1}{\sqrt{(a_+/2)^2-1}}-\frac{1}{\sqrt{(a_-/2)^2-1}}\right]\\
=&(2\pi)\frac{\sqrt{(a_-/2)^2-1}-\sqrt{(a_+/2)^2-1}}{\sqrt{(a_+/2)^2-1}\sqrt{(a_-/2)^2-1}}\\
=&\frac{8\pi}{\sqrt{(q/2)^2/\epsilon-1}}
\end{aligned}
$$

$\boxed{q > 2}$

$$
\begin{aligned}
&(2/i)\int_\Gamma dz\left[\frac{1}{z^2+a_+z+1}-\frac{1}{z^2-a_-z+1}\right]\\
=&4\pi Res\left[\frac{1}{(z-z_{1+})(z-z_{1-})}-\frac{1}{(z-z_{2+})(z-z_{2-})}\right]\\
=&4\pi\left[\frac{1}{z_{1+}-z_{1-}}-\frac{1}{z_{2-}-z_{2+}}\right]\\
=&\frac{4\pi}{\sqrt{(q/2)^2/\epsilon-1}}
\end{aligned}
$$

Now we can insert this into the ϵ integral and approximate with the Sommerfeld expansion.

$$
\begin{aligned}
\chi &= (4\pi\mu_e^2/q)\int_0^\infty d\epsilon\frac{f(\epsilon)}{\sqrt{\epsilon}}\frac{1}{\sqrt{(q/2)^2/\epsilon-1}}\\
&= (4\pi\mu_e^2/q)\int_0^\infty d\epsilon\frac{f(\epsilon)}{\sqrt{(q/2)^2-\epsilon}}\\
&\approx -(8\pi\mu_e^2/q)\sqrt{(q/2)^2-\epsilon}\Big|_{\epsilon=0}^{1}+(\pi^2T^2/6)(4\pi\mu_e^2/q)\frac{d}{d\epsilon}\left[\frac{1}{\sqrt{(q/2)^2-\epsilon}}\right]_{\epsilon=1}\\
&= (8\pi\mu_e^2/q)\big(q/2-\sqrt{(q/2)^2-1}\big)+(\pi^2T^2/6)(4\pi\mu_e^2/q)\frac{d}{d\epsilon}\left[\frac{1}{\sqrt{(q/2)^2-\epsilon}}\right]_{\epsilon=1}
\end{aligned}
$$