Global Linear Stability of the Two-Dimensional Shallow-Water Equations: An Application of the Distributive Theorem of Roots for Polynomials on the Unit Circle

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ABSTRACT

This paper deals with the numerical stability of the linearized shallow-water dynamic and thermodynamic system using centered spatial differencing and leapfrog time differencing. The nonlinear version of the equations is commonly used in both 2D and 3D (split technique) numerical models. To establish the criteria, we employ the theorem of the root distributive theory of a polynomial proposed by Cheng (1966). The Fourier analysis or von Neumann method is applied to the linearized system to obtain a characteristic equation that is a sixth-order polynomial with complex coefficients. Thus, a series of necessary and sufficient criteria (but only necessary conditions for the corresponding nonlinear equations) are obtained by applying Cheng's theorem within the unit circle. It is suggested that the global stability should be determined by this set of criteria rather than the Courant–Friedrichs–Lewy (CFL) criterion alone. Each of the conditions has physical meaning: for instance, $h + \zeta > 0$, $|f|\Delta t < 1$, and $0 < \Delta t\hat{\beta}' < 1$, etc., must be satisfied as well, which helps define the model domain and the relation between damping coefficients and integration time step, where h is the undisturbed water depth, ζ the free surface elevation, f the Coriolis parameter, $\hat{\beta}'$ the sum of bottom friction coefficient and horizontal viscosity, and Δt the integrating time step. The full solution and the physical implications are given in the paper. Since Cheng's theorem was published in Chinese only and is of considerably theoretical and practical value in numerical stability analysis, the translation of the theorem is in appendix A.

1. Introduction

Two types of stability analyses, the von Neumann or Fourier method and the energy method or matrix method, have been widely used in both atmospheric numerical models (Arakawa and Lamb 1977; Fisher 1965; Haltiner and Williams 1980; Kasahara 1965; Lilly 1965) and oceanographic numerical models (Wang et al. 1994a). The difference between these two methods is that the former is easily performed on a linearized system, but leads only to necessary conditions for stability, whereas the second method can be applied to a nonlinear system to produce sufficient conditions for stability (Arakawa and Lamb 1977; Haltiner and Williams 1980). Possibly due to its simplicity, the Fourier method has been more widely used to derive necessary conditions of stability.

Even in a linear system, the stability conditions obtained by the Fourier method are not free of assumptions, as mentioned by Kasahara (1965). In particular, several assumptions (such as no rotation, no dissipation, and no advection) were made in deriving the stability conditions for a two-dimensional, rotating, advective, and dissipative system (Chen and Qin 1985; Irvine and Houghton 1971; Kasahara 1965). This limitation is due to the difficulty of the mathematics involved, since the resulting characteristic equation is commonly a sixth-order polynomial with complex coefficients (Kasahara 1965). This difficulty was avoided by applying some assumptions to lower the order of the equation and make it have real coefficients (Chen and Qin 1985; Irvine and Houghton 1971; Kasahara 1965). Thus, some of the roots, as will be seen later, were lost, although these missing roots describe physical processes in the model that have practical as well as theoretical implications. An examination of the global (or systematic) stability of this linear system without the above assumptions, therefore, is necessary.

In this study, the global stability conditions of the system as a whole without any simplifications (i.e., in a complete set) are investigated with the help of Cheng's theorem (Cheng 1966). This theorem is powerful and simple in application. We solve the sixth-order characteristic equation with complex coefficients.

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A series of constraint conditions or stability criteria are obtained and must be satisfied simultaneously. Thus, the global stability criteria, which includes the Courant–Friedrichs–Lewy (CFL) condition as a special case, are interpreted physically.

In the next section, the solution and discussion of the global stability constraints on this linear system are examined in detail. In section 3, conclusions and a discussion are given.

2. The global linear stability analysis

The dynamic and thermodynamic shallow-water equations were successfully applied to simulate the wintertime circulation and the thermal effect of the Kuroshio in the East China Sea (Wang and Yuan 1988), while the pure dynamic shallow-water equations were also used to simulate the propagation of continental shelf waves (Wang et al. 1988). This system, which is a rotating, dissipative set of equations with advection, can be written as (in Cartesian coordinates)

$$\frac{D_h\mathbf{V}_h}{Dt} + f\mathbf{k}\times\mathbf{V}_h = -g'\nabla\zeta + \frac{\alpha g}{2}D\nabla T - \beta\mathbf{V}_h + A_h\Delta\mathbf{V}_h + g\nabla\zeta_a + \frac{\boldsymbol{\tau}_a}{\rho D}, \qquad (1)$$

$$\frac{\partial\zeta}{\partial t} + \frac{\partial UD}{\partial x} + \frac{\partial VD}{\partial y} = 0, \qquad (2)$$

$$\frac{D_hT}{Dt} = K_h\Delta T + \frac{Q}{\rho C_p D}, \qquad (3)$$

where x, y, and z are positive eastward, northward, and upward; $\mathbf{V}_h = (U, V)$ is the horizontal velocity vector; f is the Coriolis parameter; $\mathbf{k}$ is the unit vector in the z direction; g is the gravity; $g' = g(1 - \alpha T)$; α is the thermal expansion coefficient of water; T is the water temperature; $\zeta_a = -P_a/\rho_a g$ is the barometric relation between surface elevation and sea level pressure, where ρ_a is the air density and P_a is the sea level pressure; $D = h + \zeta$ is the total water depth, where ζ is the sea level elevation and h is the water depth with respect to the mean sea level; $\boldsymbol{\tau}_a = \rho_a C_d(U_w^2 + V_w^2)^{1/2}\mathbf{V}_w$ is the wind stress vector over the sea surface, where C_d is the drag coefficient, and U_w and V_w the surface wind speed in the x and y directions, respectively; ρ is the water density; $\beta = C_b(U^2 + V^2)^{1/2}D^{-1}$ is the bottom friction, where C_b is the bottom friction coefficient and U and V the current velocity in the x and y directions, respectively; A_h and K_h are the horizontal viscosity and diffusivity coefficients, respectively; $Q = Q_i - Q_b - Q_l - Q_s$ is the net heat flux on the sea surface, where subscripts i, b, l, and s refer to the absorption of solar radiation by the ocean, the net loss of longwave radiative energy from the ocean, and the net losses of latent and sensible heat, respectively (Wang and Yuan 1988); and C_p is the specific heat at constant pressure. Term D_h/Dt is the total (absolute) derivative, with the subscript h denoting the horizontal direction; ∇ and Δ are the horizontal gradient and Laplacian operator, respectively.

Although the quadratic-conserving schemes for (1)–(3) (Irvine and Houghton 1971; Wang and Yuan 1988; Ford et al. 1991; Wang et al. 1994a) are able to suppress nonlinear instabilities with the help of some spatial and temporal filtering (Asselin 1972), the computational stability still depends on the relation between the time step and grid spacing. Furthermore, the computational stability is also constrained by the physical processes in the model, as it will be discussed later. In other words, the model physics must be well designed to maintain the real physics in addition to satisfying the CFL criterion. The combination of these mathematical and physical constraints is called the "global stability" (or systematic stability) in this paper.

a. The Fourier analysis and characteristic equations

Irvine and Houghton (1971), and Chen and Qin (1985) both used Fourier analysis to study a pure linear dynamic system. However, because some assumptions were made to simplify the characteristic equation, some of the important roots, which describe real physical processes in the model, were lost. One purpose of this paper is to identify these physical processes and to formulate the criteria for the global stability of the system, which replace the single CFL criterion.

Following Irvine and Houghton (1971), we assume that this system is a weakly nonlinear system that can be linearized. We consider only the global linear stability caused by the initial errors within the system itself. Thus, the atmospheric disturbances, the wind stresses in (1), and the heat flux in (3), as well as boundary errors, are neglected. Then we obtain the linearized forms of (1)–(3) (not shown). Note that most stability analyses ignore the effects of boundary conditions whose inappropriate implementation can make a scheme that is stable for the interior of a grid domain unstable.

Without loss of generality, we assume that $\Delta x = \Delta y = \Delta s$, and we set $S = \Delta t/\Delta s$. To compare our results easily with the results of Irvine and Houghton (1971) and Chen and Qin (1985), we also use the A grid (Arakawa and Lamb 1977) in which all variables are evaluated at the same grid. The same conclusions derived in this paper are also applied without any difficulty to the B and C grids (see appendix B).

After some finite-difference operations and rearrangements, we obtain the explicit difference equations for (1)–(3) as follows:

$$
\begin{aligned}
U_{i,j}^{n+1} = U_{i,j}^{n-1} &- S\hat{U}(U_{i+1,j}^{n} - U_{i-1,j}^{n}) \\
&- S\hat{V}(U_{i,j+1}^{n} - U_{i,j-1}^{n}) + 2\Delta t f V_{i,j}^{n} \\
&- gS(1 - \alpha\hat{T})(\zeta_{i+1,j}^{n} - \zeta_{i-1,j}^{n}) \\
&+ \frac{\alpha g S}{2}\hat{D}(T_{i+1,j}^{n} - T_{i-1,j}^{n}) - 2\Delta t\beta U_{i,j}^{n-1} \\
&+ \frac{2SA_h}{\Delta s}(U_{i+1,j}^{n-1} + U_{i-1,j}^{n-1} - 4U_{i,j}^{n-1} \\
&+ U_{i,j+1}^{n-1} + U_{i,j-1}^{n-1}), \quad (4)
\end{aligned}
$$

$$
\begin{aligned}
V_{i,j}^{n+1} = V_{i,j}^{n-1} &- S\hat{V}(V_{i+1,j}^{n} - V_{i-1,j}^{n}) \\
&- S\hat{V}(V_{i,j+1}^{n} - V_{i,j-1}^{n}) - 2\Delta t f U_{i,j}^{n} \\
&- gS(1 - \alpha\hat{T})(\zeta_{i,j+1}^{n} - \zeta_{i,j-1}^{n}) \\
&+ \frac{\alpha g S}{2}\hat{D}(T_{i,j+1}^{n} - T_{i,j-1}^{n}) - 2\Delta t\beta V_{i,j}^{n-1} \\
&+ \frac{2SA_h}{\Delta s}(V_{i+1,j}^{n-1} + V_{i-1,j}^{n-1} - 4V_{i,j}^{n-1} \\
&+ V_{i,j+1}^{n-1} + V_{i,j-1}^{n-1}), \quad (5)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{i,j}^{n+1} = \zeta_{i,j}^{n-1} &- \hat{U}S(\zeta_{i+1,j}^{n} - \zeta_{i-1,j}^{n}) \\
&- \hat{V}S(\zeta_{i,j+1}^{n} - \zeta_{i,j-1}^{n}) - \hat{D}S(U_{i+1,j}^{n} - U_{i-1,j}^{n} \\
&+ V_{i,j+1}^{n} - V_{i,j-1}^{n}), \quad (6)
\end{aligned}
$$

$$
\begin{aligned}
T_{i,j}^{n+1} = T_{i,j}^{n-1} &- \hat{U}S(T_{i+1,j}^{n} - T_{i-1,j}^{n}) \\
&- \hat{V}S(T_{i,j+1}^{n} - T_{i,j-1}^{n}) + \frac{2K_h S}{\Delta s}(T_{i+1,j}^{n-1} + T_{i-1,j}^{n-1} \\
&- 4T_{i,j}^{n-1} + T_{i,j+1}^{n-1} + T_{i,j-1}^{n-1}), \quad (7)
\end{aligned}
$$

where circumflex denotes the corresponding linearized value. The centered (leapfrog) differences are used for all time and space derivatives. Thus, the finite-difference equations are of second-order accuracy in space and of first-order accuracy in time because the bottom friction and diffusion terms are lagged by one time step in (4), (5), and (7). Note that the damping terms in (4), (5), and (7) are lagged by one time step following the previous studies (Wang and Yuan 1988; Ford et al. 1991; Wang 1993). Elliot (1976) has shown that the friction and diffusion terms must be lagged one time step to ensure stability of this explicit scheme. Unlike the implicit and semi-implicit schemes (Casulli 1990), this explicit scheme does not introduce artificial numerical diffusion.

The Fourier components of the error (4)–(7) are assumed to be

$$
(U, V, \zeta, T)_{k,m}^{n} = (U_0, V_0, \zeta_0, T_0)\lambda^n \times \exp[i(k\alpha_x + m\alpha_y)\Delta s], \quad (8)
$$

where $\lambda = \exp(i\sigma t)$ is called the characteristic value (σ is the radian frequency); $\alpha_x = 2\pi/L_x$ and $\alpha_y = 2\pi/L_y$ are the wavenumbers for the x and y directions, respectively, with the corresponding wavelengths being L_x and L_y; and k and m denote the grid numbers for the x and y directions, respectively, as i and j did before (the reason is that i here denotes an imaginary number). Substituting (8) into (4)–(7) yields the matrix form of the characteristic equation as follows (after some rearrangements):

$$
\begin{vmatrix}
D(\lambda) & -2\Delta t f\lambda & g(1-\alpha\hat{T})Z_1\lambda & -\frac{\alpha g}{2}\hat{D}Z_1\lambda \\
2\Delta t f\lambda & D(\lambda) & g(1-\alpha\hat{T})Z_2\lambda & -\frac{\alpha g}{2}\hat{D}Z_2\lambda \\
\hat{D}Z_1\lambda & \hat{D}Z_2\lambda & \lambda^2 + Z_4\lambda - 1 & 0 \\
0 & 0 & 0 & T(\lambda)
\end{vmatrix}
\times \begin{vmatrix} U_0 \\ V_0 \\ \zeta_0 \\ T_0 \end{vmatrix} = 0, \quad (9)
$$

where

$$
D(\lambda) = \lambda^2 + Z_4\lambda - \left(1 - 2\Delta t\beta - \frac{8A_h\Delta t}{\Delta s^2}Y\right), \quad (10)
$$

$$
T(\lambda) = \lambda^2 + Z_4\lambda + \frac{8K_h\Delta t}{\Delta s^2}Y - 1. \quad (11)
$$

For nontrivial solutions of (9), the determinant of the coefficient matrix must be zero. Expanding (9) gives (11) equal to zero [i.e., $T(\lambda) = 0$] and

$$
\begin{aligned}
&[\lambda^2 + Z_4\lambda - (1 - 2\Delta t\beta')]^2(\lambda^2 + Z_4\lambda - 1) \\
&\quad - \hat{D}g'(Z_1^2 + Z_2^2)\lambda^2[\lambda^2 + Z_4\lambda - (1 - 2\Delta t\beta')] \\
&\quad + 4\Delta t^2 f^2\lambda^2(\lambda^2 + Z_4\lambda - 1) = 0, \quad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
&Z_1 = i2S\sin(\alpha_x\Delta s), \quad Z_2 = i2S\sin(\alpha_y\Delta s), \\
&\qquad Z_4 = \hat{U}Z_1 + \hat{V}Z_2, \\
&Y = \sin^2\left(\frac{\alpha_x\Delta S}{2}\right) + \sin^2\left(\frac{\alpha_y\Delta s}{2}\right), \\
&g' = g(1 - \alpha\hat{T}), \quad \hat{\beta}' = \hat{\beta} + \frac{4A_h}{\Delta s^2}Y. \quad (13)
\end{aligned}
$$

We can see that the characteristic equation (11), $T(\lambda) = 0$, for the thermodynamic equation and (12) for the dynamic equations are independent. In other words, the numerical scheme for thermodynamic equation influences only the stability of that equation rather than affecting also the dynamic system, and vice versa. Thus, any numerical scheme may be chosen for the thermodynamic equation, such as the Euler forward scheme in time (Wang 1993) and the upwind scheme of second-order accuracy (Roache 1976), etc. But it is possible that in a two-way feedback system, any instability of thermodynamic system (say, temperature calculated from the diffusion equation) will result in instability of the velocity field in the dynamic system through baroclinic density gradient terms. Similarly, an unstable dynamic system will lead to an unstable thermodynamic system through the advection terms.

First, we rewrite (11) as follows:

$$b_{0,0}\lambda^2 + b_{1,0}\lambda + b_{2,0} = 0, \qquad (14)$$

where $b_{0,0} = 1$, $b_{1,0} = Z_4$, and $b_{2,0} = (8K_h\Delta tY/\Delta s^2) - 1$. We can see that this is a second-order polynomial with complex coefficients (i.e., in a complex system). According to Cheng's theorem in appendix A [(A13)], that is, $|b_{2,0}| < |b_{0,0}|$, through some operations we obtain

$$\Delta t < \frac{\Delta s^2}{8K_h}. \qquad (15)$$

This is, in fact, the solution of the seventh characteristic root $|\lambda_7| < 1$, which is identical to the solution of Blumberg and Mellor (1987). According to (A7) in appendix A, we now can lower the second-order polynomial (11) to be the first-order one, namely,

$$F_1(\lambda) = b_{0,1}\lambda + b_{1,1}, \qquad (16)$$

where $b_{0,1} = (8K_h\Delta tY/\Delta s^2)(2 - 8K_h\Delta tY/\Delta s^2)$ and $b_{1,1} = (8K_h\Delta tY/\Delta s^2)Z_4$. Again from (A13), through some algebraic operations we obtain the following solution:

$$\Delta t < \left(\frac{|\hat{U}| + |\hat{V}|}{\Delta s} + \frac{8K_h}{\Delta s^2}\right)^{-1}. \qquad (17)$$

This is the stability condition for $|\lambda_8| < 1$. We can see that this condition includes (15). In other words, this solution is more general, which was lost in a system with real coefficients by Blumberg and Mellor (1987). The condition (17) derived using Cheng's theorem in the complex coefficient space includes the solution (15) of Blumberg and Mellor (1987) as a special case obtained using a conventional manner in the real coefficient space.

Next, let us look for the solutions of (12). To this end, it is convenient to introduce the following abbreviations:

$$B = \Delta t\hat{\beta}', \quad F = \Delta tf,$$

$$\phi = S\sin(\alpha_x\Delta s), \quad \psi = S\sin(\alpha_y\Delta s),$$

$$L = \hat{U}\phi + \hat{V}\psi, \quad M = \hat{D}g'(\phi^2 + \psi^2). \qquad (18)$$

Then, we can write (12) as a sixth-order polynomial in λ with complex coefficients:

$$F_0(\lambda) = b_{0,0}\lambda^6 + b_{1,0}\lambda^5 + b_{2,0}\lambda^4 + b_{3,0}\lambda^3 + b_{4,0}\lambda^2 + b_{5,0}\lambda + b_{6,0} = 0, \qquad (19)$$

where

$$b_{0,0} = 1,$$
$$b_{1,0} = i6L,$$
$$b_{2,0} = -[1 + 2(1 - 2B) + 12L^2 - 4M - 4F^2],$$
$$b_{3,0} = -i4L[1 + 2(1 - 2B) + 2L^2 - 2M - 2F^2],$$
$$b_{4,0} = 2(1 - 2B) + (1 - 2B)^2 + 8(1 - 2B)L^2 + 4L^2 - 4M(1 - 2B) - 4F^2,$$
$$b_{5,0} = i2L[2(1 - 2B) + (1 - 2B)^2],$$
$$b_{6,0} = -(1 - 2B)^2. \qquad (20)$$

b. Solutions and discussions

The purpose of solving (19) is to determine the stability conditions, that is, to find the criteria for the characteristic roots to lie inside the unit circle. If all the characteristic roots lie inside the unit circle [i.e., $|\lambda_i| < 1$ ($i = 1, \cdots, 6$)], then the numerical scheme is said to be globally stable. In other words, systematic (or intrinsic) errors do not increase with time. (Note that the global stability is defined such that all roots fall *inside* the unit circle to avoid a possible case where there are multiple roots with $|\lambda_i| = 1$, that is, on the unit circle. For example, if an explicit scheme could be written as $\mathbf{x}^{n+1} = \mathbf{A}\mathbf{x}^n$, where $\mathbf{A}$ is a 2×2 matrix with $a_{11} = 1$, $a_{12} = \epsilon$, $a_{21} = 0$, $a_{22} = 1$, and $\mathbf{x}$ is an 1×2 matrix with $(x_1, x_2)^T$, where T and ϵ denote the transpose and a small positive number. Then $|\lambda_i| = 1$, but $x_1^n = x_1^0 + n\epsilon x_2^0$. Hence, the solution is growing.)

Fortunately, a theorem on the distributive theory of roots for polynomials, first proposed by Cheng (1966), provides a powerful tool for solving such nth-order polynomial with complex coefficients. The core idea of this theorem is to "force" all roots, which may be located inside, on, and outside the unit circle for a broad range of parameters, to fall inside the unit circle (Fig. 1), so that the corresponding constraints can be derived. The solution procedure is to find each root by lowering the equation order by order. To this end, we solve (19) and discuss the physical meaning of each condition below.

Step 1: Since $b_{0,0} = 1$ and $b_{6,0} = -(1 - 2B)^2$, it is very easy to obtain the necessary and sufficient condition for $|\lambda_1| < 1$ from $|b_{6,0}| < |b_{0,0}|$ [see (A13) of appendix A], that is,

$$0 < B = \Delta t\hat{\beta}' < 1. \qquad (21)$$

The physical interpretation of this condition is as follows. 1) $\hat{\beta}' > 0$ means that the bottom friction and horizontal eddy viscosity always exist and dissipate en-

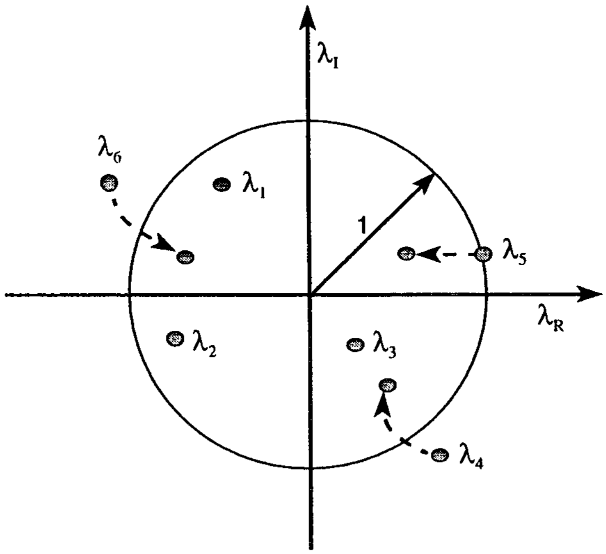

FIG. 1. Roots distribution in a unit circle of a complex plane for a broad range of parameters, such as U, V, Δt, Δs, H, etc. By applying the Cheng's theorem, all roots are "forced" to fall inside the unit circle (see the dashed arrows), so that the corresponding constraint for each root is derived.

ergy. Multiplying (21) by the kinetic energy E gives $0 < \Delta t\hat{\beta}'E < E$. This implies that kinetic energy dissipated per unit time Δt must be both greater than zero and less than the total kinetic energy. Thus, in the numerical model, either bottom or horizontal friction must exist but must not be too large. Either no friction or too much friction would result in computational instability. 2) Since both the bottom friction and the horizontal viscosity are positive [although the transfer of kinetic energy from small-scale motions to large-scale flow due to baroclinic instability in atmosphere and ocean may appear as negative viscosity (Holton 1979)], we have $\hat{\beta} = C_b(U^2 + V^2)^{1/2}(h + \zeta)^{-1} > 0$. Since the bottom friction coefficient $C_b > 0$, we have

$$D = h + \zeta > 0. \tag{22}$$

This constraint states that only within a computational domain full of fluid (i.e., "wet points") can the computation maintain stability. This condition is less relevant in atmospheric models and large-scale ocean models without the free surface since the atmospheric thickness is always much greater than the topography or mountain elevation. However, it becomes extremely important in coastal ocean models, when the free surface, ζ, is considered (Cheng et al. 1993). In simulations of tidal dynamics and storm surges in coastal seas and bays, the displacement of the free surface is usually of the same order as the water depth near the coastal boundaries.

Step 2: By taking $\xi_0' = 0$ (i.e., at the center of the unit circle), we can lower the polynomial (19) by one order according to Cheng's theorem [see (A7) of appendix A] and thus obtain a fifth-order polynomial in λ through some trivial rearrangements:

$$\begin{aligned} F_1(\lambda) &= \frac{1}{\lambda}[F_0^*(0)F_0(\lambda) - F(0)F_0^*(\lambda)] \\ &= b_{0,1}\lambda^5 + b_{1,1}\lambda^4 + b_{2,1}\lambda^3 + b_{3,1}\lambda^2 \\ &\qquad + b_{4,1}\lambda + b_{5,1} = 0, \end{aligned} \tag{23}$$

where

$$\begin{aligned} b_{0,1} &= 1 - I^4, \\ b_{1,1} &= i2L(3 - 2I^3 - I^4), \\ b_{2,1} &= I^2(2I + I^2 - 8L^2I + 4L^2 - 4MI - 4F^2) \\ &\quad - (1 - 2I + 12L^2 - 4M + 4F^2), \\ b_{3,1} &= -i4L(1 + 2I + 2L^2 - 2M - 2F^2)(1 - I^2), \\ b_{4,1} &= 2I + 4L^2(1 + 2I - 3I^2) \\ &\quad - 8MBI + 4F^2(I^2 - 1) - 2I^3, \\ b_{5,1} &= i8LBI, \end{aligned} \tag{24}$$

and $I = 1 - 2B$ has been used. From $|b_{5,1}| < |b_{0,1}|$, we have

$$|L| < \left|\frac{1 - 3B + 4B^2 - 2B^3}{1 - 2B}\right|, \quad \text{when} \quad 4B < 1. \tag{25}$$

This is the relation between L and B. Since $B < 1/4$, then $|L| < 1$, that is,

$$\frac{\Delta t}{\Delta s} \leqslant (|\hat{U}| + |\hat{V}|)^{-1}. \tag{26}$$

We see that $B < 1/4$ slightly modifies the right side of the inequality in (21). Equation (26) implies that the model Courant number (defined as $|\bar{V}|\Delta t/\Delta s$ in this paper) must be smaller than unity. In other words, $\Delta s/\Delta t$ in the model should be fast enough to override the short waves ($2\Delta s$ waves or high-frequency oscillations) generated by the advection terms (i.e., nonlinear terms).

Step 3: Similarly, by taking $\xi_1' = 0$, we have $F_1(0) = i8LBI \equiv iF_1(0)_i$, $F_1^*(0) = (1 - I^4)$. From (A7) of appendix A, we can construct a fourth-order polynomial again by lowering (23) by one order as follows:

$$F_2(\lambda) = b_{0,2}\lambda^4 + b_{1,2}\lambda^3 + b_{2,2}\lambda^2 + b_{3,2}\lambda + b_{4,2} = 0, \tag{27}$$

with the coefficients

$$b_{0,2} = F_1^{*2}(0) - F_1^2(0)_i,$$

$$b_{1,2} = i[2LHFF_1^*(0) - RF_1(0)_i],$$

$$b_{2,2} = PF_1^*(0) + 4LTF_1(0)_i,$$
$$b_{3,2} = -i[4LQTF_1^*(0) + PF_1(0)_i],$$
$$b_{4,2} = RF_1^*(0) - 2LHF_1(0)_i, \tag{28}$$

where

$$H = 3 - 2I^3 - I^4,$$
$$P = I^2(2I + I^2 + 8L^2I + 4L^2 - 4MI - 4F^2) - (1 + 2I + 12L^2 - 4M - 4F^2),$$
$$Q = 1 + 2I + 2L^2 - 2M - 2F^2,$$
$$R = 2I + 4L^2(1 + 2I - 3I^2) - 8MBI + 4F^2(I^2 - 1) - 2I^3,$$
$$T = 1 - I^2. \tag{29}$$

From the inequality $|b_{4,2}| < |b_{0,2}|$, when

$$|F| = |f|\Delta t < 1, \tag{30}$$

we have $0 \leqslant M < 1$ or

$$0 \leqslant \frac{\Delta t}{\Delta s} < (2g'\hat{D})^{-1/2}. \tag{31}$$

The physical interpretation of (30) is given below. Multiplying (30) by the velocity vector **V** indicates that the computation can be stable when the velocity due to the Coriolis acceleration (the earth rotation, i.e., inertial frequency) over a unit time is smaller than the original velocity. Since the system considered here is not in geostrophic balance, this is the constraint, $|F| = |f|\Delta t < 1$. If $|f|\Delta t > 1$, then the system becomes approximately in geostrophic balance (Merilees 1976). Thus, this condition is valid anywhere as long as the time-dependent evolution in the rotating system is included.

Equation (31) is the typical CFL criterion when the rotation is neglected (i.e., $f = 0$) (Chen and Qin 1985; Irvine and Houghton 1971). We obtain this condition without this assumption $f = 0$ by requiring $|F| = |f|\Delta t < 1$. This means that the CFL criterion intrinsically exists in the rotating and dissipative system and not just in the nonrotating and nondissipative system. As is well known, the CFL criterion states that $\Delta s/\Delta t$ of the model must exceed the speed of the fast gravity wave.

Step 4: By repeating the same procedure as above, we can construct a third-order polynomial

$$F_3(\lambda) = b_{0,3}\lambda^3 + b_{1,3}\lambda^2 + b_{2,3}\lambda + b_{3,3} = 0, \tag{32}$$

with complex coefficients (not shown due to their trivial expressions). Again, from $|b_{3,3}| < |b_{0,3}|$, through some trivial algebraic operations and rearrangements and according to the typical scaling $O(B, F, L, M) < 1$, we can obtain the following inequality:

$$M\left(1 + \frac{3}{2}B\right) < (1 - L)^2 + O(\epsilon), \tag{33}$$

where $\epsilon = B^2 + L^2 + M^2 + F^2$. For the sake of having a real root, $M < 1/2$ has been required, from which we see that this is a further refinement of (31). Neglecting the higher-order terms, we obtain the following inequality:

$$\frac{\Delta t}{\Delta s}\left[|\bar{V}| + (2\hat{D}g')^{1/2}\left(1 + \frac{3}{4}\Delta t\hat{\beta}'\right)\right] < 1, \tag{34}$$

where

$$|\bar{V}| = |\hat{U}| + |\hat{V}|, \tag{35}$$

and the inertial wave effect, $(f\Delta s)^2(2g'\hat{D})^{-1}$, in the parentheses on the left-hand side of (34) has been neglected because the surface gravity wave speed is much greater than the inertial wave speed. For instance, if $f = 10^{-4}$ s^{-1}, $\Delta s = 10$ km, $g = 9.8$ m s^{-2}, and $\hat{D} = 100$ m, then this term is 0.5×10^{-3}, much smaller than the other three terms (see also Irvine and Houghton 1971).

It is interesting to note that the condition (34) includes three effects together, "linearized advection" terms ($2\Delta s$ waves), gravity waves (the fast waves), and the dissipation terms (damping). It is clear that $\Delta s/\Delta t$ in a numerical model must exceed or override the "linear sum" of the advection speed, the gravity wave speed, and the gravity wave damping rate due to the eddy viscosity. This condition includes (26) and the CFL criterion, as shown in (31). Thus, it can be called the modified CFL criterion.

Step 5: By the same procedure as above and through very trivial algebraic work, we can construct a second-order polynomial with complex coefficients (not shown since the expressions for the coefficients are fairly trivial). If $M < 1/2$ (the same as step 4) and $B < 1$, then $|b_{2,4}| < |b_{0,4}|$ holds. As we can see, this condition is the same constraint as (21). In other words, the damping coefficient in the model is not allowed to be large.

Step 6: Repeating the same procedure finally results in a first-order polynomial (not shown since the coefficients are again very trivial). If $B < 1$ (step 5) and $L < 1$ (step 4), then $|b_{1,5}| < |b_{0,5}|$ is true.

3. Conclusions and discussion

In the preceding section, an attempt has been made to examine the global linear stability conditions for the 2D shallow-water dynamic and thermodynamic system, which includes rotation, dissipation, and advection (i.e., in a complete set), without any assumptions. The main conclusion, obtained with the help of Cheng's theorem, is that the explicit centered finite difference equations, as shown in (4)–(7), are globally stable only if all roots are located inside the unit circle $[|\lambda_i| < 1\ (i = 1, \cdots, 8)]$, that is,

(a) $0 < B = \Delta t \hat{\beta}'$

$$= \Delta t \left[\frac{C_b(\hat{U}^2 + \hat{V}^2)^{1/2}}{h + \zeta} + \frac{8A_h}{\Delta s^2} \right] < 1,$$

(b) $h + \zeta > 0,$

(c) $\dfrac{\Delta s}{\Delta t} > |\bar{V}|,$

(d) $|f|\Delta t < 1,$

(e) $\dfrac{\Delta s}{\Delta t} > (2\hat{D}g')^{1/2},$

(f) $\dfrac{\Delta s}{\Delta t} > \left[|\bar{V}| + (2\hat{D}g')^{1/2}\left(1 + \frac{3}{4}\Delta t \hat{\beta}'\right) \right],$

(g) $\dfrac{\Delta s}{\Delta t} > |\bar{V}| + \dfrac{8K_h}{\Delta s}.$ (36)

In other words, the explicit centered finite-difference schemes that simultaneously satisfy the criteria (a)–(g) allow the computation to remain stable (Wang 1993). This is why the global stability must be considered rather than the CFL criterion alone [i.e., criterion (e)]. We observe that criteria (c) and (e) are included in criterion (f). This indicates that criterion (f) is more general and stricter than (c) and (e). Also, (f) is generally stricter than (g). It should be pointed out that many researchers have only used the CFL criterion [i.e., criterion (e)] as the stability condition for the fully nonlinear shallow water equations. This is not reasonable either physically or mathematically and may lead to conceptual confusion. As we can see, the explicit quadratic-conserving finite-difference equations, in addition to satisfying (f), must obey the physical criteria (a), (b), and (d) as well. Otherwise, the computation will be unstable due to global instability. Thus, the criteria (a), (b), (d), (f) (for the dynamic equations), and (g) (for the thermodynamic equation) must be simultaneously satisfied in the numerical schemes to maintain the global linear stability. The following discussion may be helpful for us to understand the concept of global stability.

1) In the thermodynamic equation, (g) includes the result of Blumberg and Mellor (1987), that is, (15). It is clear that when the advection term is neglected, as a special case, (g) is identical to their result.

2) If the thermodynamic equation is neglected, that is, T = const and the thermal expansion coefficient of water $\alpha = 0$, so that $g' = g$, then the system degenerates into the pure dynamic shallow-water equations (Irvine and Houghton 1971; Chen and Qin 1985; Wang et al. 1988). The global linear stability criteria are (a), (b), (d), and (f).

3) In a pure dynamic system, and also if the bottom friction and horizontal viscosity are neglected, that is, $\hat{\beta}' = 0$ (i.e., in a nondissipative subset), then criterion (f) degenerates into the result of Irvine and Houghton (Irvine and Houghton 1971). We can see that their result is only one particular criterion of our global stability criteria.

4) In a pure dynamic system, and also if the advection terms, Coriolis parameter, and bottom and horizontal viscosity are neglected, that is, $Z_4 = 0$, $f = 0$, $\hat{\beta}' = 0$ (i.e., in a nonrotating, nondissipative, and nonadvective system, or subset), then criterion (f) degenerates into the result of Chen and Qin (1985), which is actually (e). Their result is therefore also only one particular case of our criteria. We can see that the assumptions made by Irvine and Houghton (1971) and Chen and Qin (1985) were not necessary and that the criteria (a), (b), and (d) were lost in their solutions due to these assumptions.

5) In a pure dynamic system, when $f = 0$, and the horizontal eddy viscosity is neglected (i.e., in a nonrotating subset), the system degenerates into the one of Casulli (1990), even though the semi-implicit numerical schemes were employed. The criterion (c) is his Eq. (26). However, the criterion (b) is an assumption of his study. Under this assumption, the finite-difference equations (or matrix) consisting of a linear five-diagonal system can remain symmetric and strictly diagonally dominant with positive elements on the main diagonal and negative ones elsewhere. We note that the criterion (b) is very important in coastal ocean numerical models.

Again, it is noted that $h + \zeta > 0$ is physically plausible. Since the elevations of tides and storms surges are typically a few meters, the water depth in a bay or along the coastline in the computation domain are usually of the same order (a few meters also). Since h and ζ are comparable, this criterion becomes extremely significant, particularly for a model domain that includes tidal marshland, mudflats, or small islands (e.g., San Francisco Bay, Cheng et al. 1993). This criterion is also widely used to determine the moving boundary in some coastal ocean models. One suggestion is that if a model includes such a domain in which there are tidal marshland or mudflats, etc., the moving boundary must be designed according to $h + \zeta > 0$. Otherwise, if the boundary is fixed, the water depths must be clipped or "artificially" modified to satisfy $h + \zeta > 0$ according to the observed tidal amplitude.

Once again, too small or too large eddy viscosity coefficients in the model with explicit quadratic-conserving schemes explicitly violates real physical principles and results in global instability, as discussed in the preceding section. This is criterion (a). For example, in a 2D thermodynamic and dynamic sea ice model (Wang et al. 1994b), maximum ice normal and shear viscosities are set to avoid this kind of numerical instability. Criterion (d) requires that the system be nongeostrophic.

Finally, it must be pointed out that the von Neumann or Fourier analysis yields the necessary conditions only of the global stability for the fully nonlinear 2D shallow-water equations, while the criteria derived from Cheng's theorem are necessary and sufficient conditions for the corresponding linear system.

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APPENDIX A

Cheng's Theory (Cheng 1966): The Distributive Theory of Roots for Polynomials in the Unit Circle

Given an nth-order polynomial with complex coefficients

$$F(Z) = F_0(Z) = b_{0,0}Z^n + b_{1,0}Z^{n-1} + \cdots + b_{n-1,0}Z + b_{n,0} = 0, \quad b_{0,0} \neq 0 \quad \text{(A1)}$$

and setting

$$F^*(Z) = F_0^*(Z) = \overline{b_{n,0}}Z^n + \overline{b_{n-1,0}}Z^{n-1} + \cdots + \overline{b_{1,0}}Z + \overline{b_{0,0}} = 0, \quad \text{(A2)}$$

where Z is a variable in the complex plane and the overbar denotes a complex conjugate, it is easy to prove that the roots of (A1) and (A2) are inverse points to each other on the unit circle.

LEMMA 1. *For any point Z on Γ (where Γ is defined by $|Z| = 1$), it can be proven that*

$$|F(Z)| = |F^*(Z)|. \quad \text{(A3)}$$

LEMMA 2. *Suppose α, β denote any of the variables, and $|\alpha| \neq |\beta|$. If $g(Z) = \alpha F(Z) - \beta F^*(Z)$, then*

$$g^*(Z) = \bar{\alpha}F^*(Z) - \bar{\beta}F(Z),$$
$$F(Z) = \alpha_1 g(Z) - \beta_1 g^*(Z),$$
$$F^*(Z) = \bar{\alpha}_1 g^*(Z) - \bar{\beta}_1 g(Z), \quad \text{(A4)}$$

where

$$\alpha_1 = \frac{\bar{\alpha}}{\alpha\bar{\alpha} - \beta\bar{\beta}}, \quad \beta_1 = \frac{-\beta}{\alpha\bar{\alpha} - \beta\bar{\beta}}.$$

LEMMA 3. *If all roots of (A1) are located within the unit circle, then $|F(Z)| < |F^*(Z)|$ is valid for any point Z in the unit circle.*

In a unit circle, we can choose ξ_k' ($k = 0, 1, \cdots m - 1 < n - 1$) to satisfy

$$|F_k(\xi_k')b_{n-k,k}| \neq |F_k^*(\xi_k')b_{0,k}|, \quad \text{(A5)}$$

$$|F_k(\xi_k')| \neq |F_k^*(\xi_k')|. \quad \text{(A6)}$$

Then, from

$$F_{k+1}(Z) = \frac{F_k^*(\xi_k')F_k(Z) - F_k(\xi_k')F_k^*(Z)}{Z - \xi_k'}, \quad \text{(A7)}$$

one can construct the polynomial series

$$F_0(Z), F_1(Z), \cdots, F_m(Z) \neq 0, \quad m \leq n, \quad \text{(A8)}$$

where

$$F_k(Z) = b_{0,k}Z^{n-k} + b_{1,k}Z^{n-k-1} + \cdots + b_{n-k,k}, \quad \text{(A9)}$$

$$F_k^*(Z) = \overline{b_{n-k,k}}Z^{n-k} + \overline{b_{n-k-1,k}}Z^{n-k-1} + \cdots + \overline{b_{0,k}}. \quad \text{(A10)}$$

Now let us assume that the number of roots for $F_k(Z)$ is p_k inside the unit circle and q_k outside of the unit circle, respectively.

THEOREM 1. *Term $F_k(Z)$ is a polynomial of order $n - k$ ($k = 0, 1, \cdots, m$).*

THEOREM 2. *It is not possible to choose ξ_m' such that $|F_m(\xi_m')b_{n-m,m}| \neq |F_m^*(\xi_m')b_{0,m}|$ and $|F_m(\xi_m')| \neq |F_m^*(\xi_m')|$, in which the necessary and sufficient condition is that the roots for $F(Z)$ and for $F^*(Z)$ are identical.*

THEOREM 3. *The maximum common factors d_k ($k = 0, 1, \cdots, m - 1$) for $F(Z)$ and $F^*(Z)$ are the same.*

Set

$$S_k' = \operatorname{sgn}[|F_k^*(\xi_k')| - |F_k(\xi_k')|] \quad (k = 0, 1, \cdots m - 1), \quad \text{(A11)}$$

where sgn is the sign function. If among S_0', S_1', ..., S_{m-1}', there are j' numbers, such as S_{k1}', S_{k2}', ..., S_{kj}', to be -1 ($0 \leq k_1 < k_2 < \ldots k_j, \leq m - 1$), then we have the following theorem.

THEOREM 4. *Suppose that for a given polynomial (A1), we can find ξ_k' ($k = 0, 1, \cdots, m - 1$) inside the unit circle, which satisfies (A5) and (A6). Then from (A7) we can construct the polynomial series as in (A8).* In this case, we have

$$q_0 = (m - k_1) - (m - k_2) + (m - k_3) - (m - k_4) + \cdots + (-1)^{j'-1}(m - k_{j'}) + (-1)^{j'} q_m. \quad \text{(A12)}$$

THEOREM 5. *Under the same assumptions as in Theorem 4, the necessary and sufficient condition for $p_0 = n$ is $S'_k = 1$ ($k = 0, 1, \cdots, n - 1$).*

If $\xi'_k = 0$ (i.e., at the origin of the unit circle), i.e., all the ξ'_k are chosen equal to zero, then Theorem 5 becomes the following theorem.

THEOREM 5′. *The necessary and sufficient conditions for all roots of the polynomial (A1) to fall within the unit circle are*

$$|b_{n-k,k}| < |b_{0,k}| \ (k = 0, 1, \cdots, n - 1). \quad \text{(A13)}$$

This is due to the fact that when $\xi'_k = 0$, $F_k(0) = b_{n-k,k}$ and $F^*_k(0) = \overline{b_{0,k}}$, then (A6) includes (A5). This can be proven by lemma 3.

COROLLARY. *Suppose that the given polynomial (A1) has real coefficients, and $b_{0,0} > b_{1,0} > \ldots > b_{n,0} > 0$, then all roots of (A1) lie inside the unit circle.*

APPENDIX B

Stability Analysis of the B and C Grids

To verify that the conclusions derived from the A grid are still valid for both B and C grids, in this appendix, the stability analysis of these two widely used grids is carried out. Similar to finite-difference equations (FDEs), (4)–(7), in the A grid, the B-grid FDEs are as follows:

$$\overline{\delta_t U}^t + \hat{U}\overline{\delta_x U}^x + \hat{V}\overline{\delta_y U}^y - fV = -g'\overline{\delta_x \zeta}^y + \frac{\alpha g \hat{D}}{2}\overline{\delta_x T}^y - \hat{\beta} U^{n-1} + A_h(\delta_x^2 U + \delta_y^2 U)^{n-1}, \quad \text{(B1)}$$

$$\overline{\delta_t V}^t + \hat{U}\overline{\delta_x V}^x + \hat{V}\overline{\delta_y V}^y + fU = -g'\overline{\delta_y \zeta}^x + \frac{\alpha g \hat{D}}{2}\overline{\delta_y T}^x - \hat{\beta} V^{n-1} + A_h(\delta_x^2 V + \delta_y^2 V)^{n-1}, \quad \text{(B2)}$$

$$\overline{\delta_t \zeta}^t + \hat{U}\overline{\delta_x \zeta}^x + \hat{V}\overline{\delta_y \zeta}^y + \hat{D}\overline{\delta_x U}^y + \hat{D}\overline{\delta_y V}^x = 0, \quad \text{(B3)}$$

$$\overline{\delta_t T}^t + \hat{U}\overline{\delta_x T}^x + \hat{V}\overline{\delta_y T}^y = K_h(\delta_x^2 T + \delta_y^2 T)^{n-1}. \quad \text{(B4)}$$

The C-grid FDEs can be written in the similar manner as follows:

$$\overline{\delta_t U}^t + \hat{U}\overline{\delta_x U}^x + \hat{V}\overline{\delta_y U}^y - f\overline{V}^{xy} = -g'\delta_x \zeta + \frac{\alpha g \hat{D}}{2}\delta_x T - \hat{\beta} U^{n-1} + A_h(\delta_x^2 U + \delta_y^2 U)^{n-1}, \quad \text{(B5)}$$

$$\overline{\delta_t V}^t + \hat{U}\overline{\delta_x V}^x + \hat{V}\overline{\delta_y V}^y + f\overline{U}^{xy} = -g'\delta_y \zeta + \frac{\alpha g \hat{D}}{2}\delta_y T - \hat{\beta} V^{n-1} + A_h(\delta_x^2 V + \delta_y^2 V)^{n-1}, \quad \text{(B6)}$$

$$\overline{\delta_t \zeta}^t + \hat{U}\overline{\delta_x \zeta}^x + \hat{V}\overline{\delta_y \zeta}^y + \hat{D}\delta_x U + \hat{D}\delta_y V = 0, \quad \text{(B7)}$$

$$\overline{\delta_t T}^t + \hat{U}\overline{\delta_x T}^x + \hat{V}\overline{\delta_y T}^y = K_h(\delta_x^2 T + \delta_y^2 T)^{n-1}, \quad \text{(B8)}$$

where only the superscript $n - 1$ for time is labeled and the other unlabeled superscripts are n. The finite-difference operators are defined as

$$(\delta_x \alpha)_{i,j} = \frac{1}{\Delta x}(\alpha_{i+1/2,j} - \alpha_{i-1/2,j}),$$

$$(\overline{\alpha}^x)_{i,j} = \frac{1}{2}(\alpha_{i+1/2,j} + \alpha_{i-1/2,j}),$$

$$(\overline{\alpha}^{xy})_{i,j} = (\overline{\overline{\alpha}^x}^y)_{i,j}. \quad \text{(B9)}$$

Here $(\delta_y \alpha)_{i,j}$ and $(\overline{\alpha}^y)_{i,j}$ are defined in the same way but with respect to the y axis. The time difference operators are defined in the same manner, that is,

$$(\delta_t \alpha)_{i,j} = \frac{1}{\Delta t}(\alpha_{i,j}^{n+1/2} - \alpha_{i,j}^{n-1/2}),$$

$$(\overline{\alpha}^t)_{i,j} = \frac{1}{2}(\alpha_{i,j}^{n+1/2} + \alpha_{i,j}^{n-1/2}), \quad \text{(B10)}$$

where α denotes any variable.

Similarly, substituting the wave solution (8) to the B-grid FDEs, (B1)–(B4) and the C-grid FDEs, (B5)–(B8), and through the similar rearrangements, we can obtain the exactly same second-order polynomial as (14) with the same coefficients for the diffusion equation and the exactly same sixth-order polynomial as (19) with the same coefficients of (20) for the dynamic system except for the following modifications: M in (18) and (20) is replaced by $M_B = 0.25g'\hat{D}S^2[\sin^2(\alpha_x\Delta s/2)\cos^2(\alpha_y\Delta s/2) + \cos^2(\alpha_x\Delta s/2)\sin^2(\alpha_y\Delta s/2)]$ in the B grid, and M and F in (18) and (20) are replaced by $M_C = 0.25g'\hat{D}S^2[\sin^2(\alpha_x\Delta s/2) + \sin^2(\alpha_y\Delta s/2)]$ and $F_C = F\cos(\alpha_x\Delta s/2)\cos(\alpha_y\Delta s/2)$, respectively, in the C grid. As we expected, these modifications are attributed to the spatial average in the pressure gradient terms in the B grid and in the Coriolis terms in the C grid. It is noted that one weakness of the C-grid model is a change in property of inertial waves from nondispersive waves that are physically correct to the dispersive waves due to the spatial average in the Coriolis terms. However, when Δs becomes very small, F_C tends to be F. Thus, in the C-grid model, using an eddy-resolving grid (i.e., smaller than the baroclinic Rossby radius of deformation) is strongly suggested to avoid the strongly dispersive inertial waves induced by the coarse model grids.

As we can see that the stability criteria derived from the A grid are the same for the B and C grids, except for a difference by a factor of 2 in the gravity wave speed terms for both B and C grids and for the above physical modification of F by F_C in the C grid.

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