Chi-square (χ^2) Goodness-of-fit Tests

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The square of the standard normal variate is known as Chi-Square variate with 1 degree of freedom(d.f.).

If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ and $\chi^2 = \left(\frac{X-\mu}{\sigma}\right)^2$ is a Chi-square variate with 1 d.f.

In general, if $X_i \sim N(\mu_i, \sigma_i^2)$ then

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2, \quad \text{is a Chi-square variate with } n \text{ d.f.}$$

Chi-Square Test

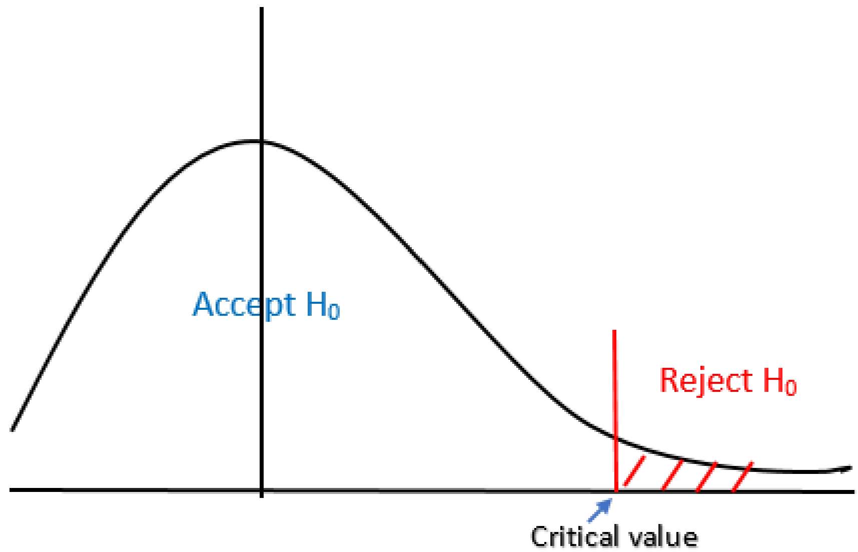
If $O_i (i = 1, 2, \ldots, n)$ be the set of observed (experimental) frequencies and E_i is the corresponding set of expected (theoretical or Hypothetical) frequencies, i.e.,

Events	1	2	$\cdots$	n
Observed frequencies	O_i	O_i	$\cdots$	O_n
Expected frequencies	E_1	E_2	$\cdots$	E_n

then Karl Pearson's chi-square, given by

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

where $\sum O = \sum E$ follows chi-square distribution with $(n-1)$ d.f.



* *Since the* $\chi^2 = \sum \frac{(O-E)^2}{E}$ *depends only on the set of the observed and expected frequencies and d.f. It does not make any assumptions regarding the parent population from which observation are taken. Hence, this test is also known as NON-PARAMETRIC TEST or DISTRIBUTION-FREE TEST.*

Remarks:

- If $\chi^2 = 0$, the observed and theoretical frequencies agree exactly.
- If $\chi^2 > 0$, they do not agree exactly.
- The larger the value of χ^2, the greater is the discrepancy between the observed and the expected frequencies.

Condition for validity of χ^2-test:

- The sample observation should be independent.
- Constraints on the cell frequencies, if any, should be linear. i.e. $\sum O = \sum E$

- The total frequency should be greater than 50.
- No theoretical cell frequency should be less than 5.

Remark:

If any cell frequency is less than 5, then for the applicability of χ^2-test, it is POOLED with the preceding or succeeding frequency so that the pooled frequency is more than 5 and finally adjust for the d.f. lost in pooling.

Illustrative Examples:

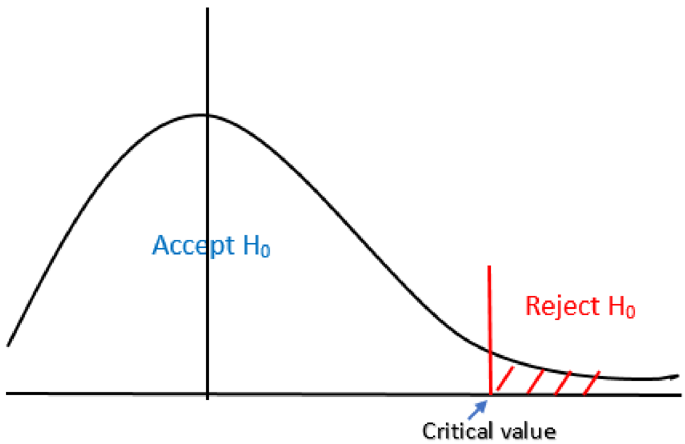
$$\text{3-Steps Approach:} \Rightarrow \begin{cases} \text{Step 1: Define the Null Hypothesis} \\ \text{Step 2: Calculate E and } \chi^2 \\ \text{Step 3: Conclude the result.} \end{cases}$$

Example 1. The number of scooter accidents per month in a certain town where as follow:

$$12 \quad 8 \quad 20 \quad 2 \quad 14 \quad 10 \quad 15 \quad 6 \quad 9 \quad 4$$

Are these frequencies in agreement with the belief that accident conditions were the same during this 10 months period?
Solution:

- Step 1: Null Hypothesis H_0: Given frequencies (no. of accidents per month in a certain town) are CONSISTENT with the belief that accident conditions were same during the 10 months.
- Step 2: Compute "E" and χ^2. Under the Null hypothesis, the accidents are uniformly distributed over the period, so the expected frequency $(E) = 100/10 = 10$.



Month	O	E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
1	12	10	4	0.4
2	8	10	4	0.4
3	20	10	100	10
4	2	10	64	6.4
5	14	10	16	1.6
6	10	10	0	0
7	15	10	25	2.5
8	6	10	16	0.1
9	9	10	1	0.1
10	4	10	36	3.6

$$\chi^2 = \sum \frac{(O-E)^2}{E} = 26.6$$

Step 3: degree of freedom(d.f.)=10-1=9;
Tabulated $\chi^2(0.05)$ for 9 d.f. = 16.92 (critical value)

Since $26.6 > 16.92$, so it is significant and rejects the Null hypothesis at 5% level of significance. Hecnce, we conclude that the accident conditions are CERTAINLY NOT UNIFORM (same) over the 10 month period.

Chi-square Distribution Table

d.f.	.995	.99	.975	.95	.9	.1	.05	.025	.01
1	0.00	0.00	0.00	0.00	0.02	2.71	3.84	5.02	6.63
2	0.01	0.02	0.05	0.10	0.21	4.61	5.99	7.38	9.21
3	0.07	0.11	0.22	0.35	0.58	6.25	7.81	9.35	11.34
4	0.21	0.30	0.48	0.71	1.06	7.78	9.49	11.14	13.28
5	0.41	0.55	0.83	1.15	1.61	9.24	11.07	12.83	15.09
6	0.68	0.87	1.24	1.64	2.20	10.64	12.59	14.45	16.81
7	0.99	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.72
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89
32	15.13	16.36	18.29	20.07	22.27	42.58	46.19	49.48	53.49
34	16.50	17.79	19.81	21.66	23.95	44.90	48.60	51.97	56.06
38	19.29	20.69	22.88	24.88	27.34	49.51	53.38	56.90	61.16
42	22.14	23.65	26.00	28.14	30.77	54.09	58.12	61.78	66.21
46	25.04	26.66	29.16	31.44	34.22	58.64	62.83	66.62	71.20
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15
55	31.73	33.57	36.40	38.96	42.06	68.80	73.31	77.38	82.29
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38
65	39.38	41.44	44.60	47.45	50.88	79.97	84.82	89.18	94.42
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.43
75	47.21	49.48	52.94	56.05	59.79	91.06	96.22	100.84	106.39
80	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33
85	55.17	57.63	61.39	64.75	68.78	102.08	107.52	112.39	118.24
90	59.20	61.75	65.65	69.13	73.29	107.57	113.15	118.14	124.12
95	63.25	65.90	69.92	73.52	77.82	113.04	118.75	123.86	129.97
100	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81