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Hansen's SPA Test

- Consider a decision made under conditions of uncertainty h periods ahead

- Let

$$\{\delta_{k,t-h},\ k = 0, 1, \ldots, m\}$$

be a finite set of decision rules.

- Decisions are evaluated with a real-valued loss function

$$L(\xi_t, \delta_{k,t-h})$$

where ξ_t is a random variable that represents the aspects of the problem that are unknown at the time the decision must be made.

- We evaluate forecasts/predictions in terms of their expected loss,

$$E\left[L\left(\xi_t, \delta_{k,t-h}\right)\right]$$

- We don't need to assume that any of the forecasts are made from a "correctly specified" population model.
    - Indeed, we don't have to believe in or reference any such thing

- Typically, the situation will require the estimation of parameters from a forecasting model as an indirect step in obtaining forecasts

$$\delta_{k,t-h} = \delta_{k,t-h}(\hat{\theta}_{k,t-h})$$

- These $\hat{\theta}_{k,t-h}$ are likely to influence expected loss — typically by increasing loss

- The first model $k=0$ plays a special role and is called the benchmark

- The decision rule $(\delta_{k,t-h})$ can represent a point forecast, an interval forecast, density forecast, trading rule, or other heuristic for decision making under uncertainty

Example (Trading Rule)

- Let $\delta_{k,t-1}$ ($h = 1$ period) be a binary "signal function" that instructs a trader to take either a short ($\delta = -1$) or a long ($\delta = +1$) position in an asset at time $t-1$

- The k^{th} trading rule yields the profit

$$\pi_{k,t} = 1.0 + \delta_{k,t-1}\, r_t$$

where r_t is the return on the asset in period t

$$r_t = \ln P_t - \ln P_{t-1}$$

where P_t is the asset price, index level, or portfolio NAV at time t

- Say we have a forecasting model in the form of a simple linear regression (for simplicity's sake) that takes the form

$$\hat{r}_{k,t} = \hat{\alpha} + \hat{\beta} X_{t-1}$$

- We can generate the signal as

$$\delta_{k,t} = \text{sign}(\hat{r}_{k,t})$$

- Of course we generate many such models
- We wish to know if any of the models $k = 1, \dots, m$ out perform the benchmark model $\delta_{0,t}$ in terms of expected loss
- So we set up a null hypothesis that <u>the benchmark is NOT inferior to any of the alternatives</u>
- Key to this analysis is the relative performance defined as

$$d_{k,t} = L(\xi_t, \delta_{0,t-h}) - L(\xi_t, \delta_{k,t-h}) \quad \text{for } k = 1, \dots, m$$

- So $d_{k,t}$ denotes the performance of model k relative to the benchmark

- We are working with isoelastic utility, so

$$u(\pi_{k,t}) = \frac{(\pi_{k,t})^{1-\gamma}}{1-\gamma} \quad \text{for } \gamma = 2, 3, 4, 5$$

So

$$L(\xi_t, \delta_{k,t}) = -u(\pi_{k,t})$$

- Let's look at an example set of calculations for $\alpha = 2.0$
  - Suppose $r_t = -0.05$ at time t
  - Let $\delta_{0,t} = +1$, that is the benchmark model is a constant "long-only" strategy
  - Suppose that $\delta_{k,t} = -1$ for an arbitrary model k (i.e. the k^{th} model yields a "short" signal)

– Then we have the following results

$$L_{0,t} = -\frac{(1.0 + \underbrace{(+1)}_{\delta_{0,t}}\underbrace{(-0.05)}_{r_t})^{\overbrace{1-2}^{\alpha}}}{\underbrace{1-2}_{\alpha}} = 1.0526$$

$$L_{k,t} = -\frac{(1.0 + \underbrace{(-1)}_{\delta_{k,t}}(-0.05))^{\overbrace{1-2}^{\alpha}}}{\underbrace{1-2}_{\alpha}} = 0.9524$$

$$d_{k,t} = \underbrace{1.0526}_{L_{0,t}} - \underbrace{0.9524}_{L_{k,t}} = \underbrace{0.1002}_{\text{positive loss differential}}$$

NB: Model k outperforms the benchmark

Loss is an "economic bad", so a higher $d_{k,t}$ favors model k more and more as it increases!

- Hypothesis of Interest

  - Stack the $d_{k,t}$ for $1, \ldots, m$ and call it

$$\mathbf{d}_t = (d_{1,t}, d_{2,t}, \ldots, d_{m,t})'$$

bold

which is the vector of relative performances at time t

  - Let $\boldsymbol{\mu} = E[\mathbf{d}_t]$

bold

  - We can formulate our null

$$H_0 : \boldsymbol{\mu} \leq 0$$

- We work under the assumption that model k is better than the benchmark iff $E(d_{k,t}) > 0$

- So, once the d_t have been calculated, we work exclusively from here on with these relative losses

  - Step 1: calculate $\{d_{k,t}\}_{t=1}^{T}$ for $k = 1, \dots, m$
  - Step 2: move on to calculate $T_n^{RC} = \max(n^{1/2}\bar{d}_1, \dots, n^{1/2}\bar{d}_m)$

    where $\bar{d}_k = \frac{1}{n}\sum_{t=1}^{n} d_{k,n}$

- The test statistic becomes

$$T_n^{RC} = \max(n^{1/2}\bar{d}_1, \dots, n^{1/2}\bar{d}_m)$$

- This makes sense b/c we're look the best performing model relative to the benchmark model $(\delta_{0,t} = +1)$

- So far this identical to White's Reality Check (RC)

- Hansen's SPA alters the test statistic as the following

$$T_n^{SPA} = \max\Big[\underbrace{\max_{k=1,\dots,m}\left(\frac{n^{1/2}\bar{d}_k}{\hat{\omega}_k}\right)}_{\text{the best performing model}},\ 0\Big]$$

*can still be negative

↑ None if no model j has $\bar{d}_j > 0$

- Hansen follows White by considering two methods for implementation:

    1. Parametric Monte Carlo
    2. Bootstrap

- We will only consider the bootstrap, as is general practice
    - Hansen uses the Politis & Romano stationary bootstrap

- The stationary bootstrap produces pseudo-time series

$$\{d^*_{b,t}\} \equiv \{d_{\tau_{b,t}}\} \qquad b = 1, \ldots, B$$

where $\{\tau_{b,1}, \ldots, \tau_{b,n}\}$ is constructed by combining blocks of $\{1, \ldots, n\}$ of random lengths

- The typical case is with block length chose from a geometrically distributed parameter $q \in (0,1]$

- B should be chose large enough so as to not be affected by the actual draws of $\tau_{b,t}$

- For us, there is no reason not to set $B \geq 10,000$

- From the pseudo-time series we calculate their sample averages

$$\bar{d}^*_b \equiv \frac{1}{n} \sum_{t=1}^{n} d^*_{b,t} \qquad b = 1, \ldots, B$$

- We seek the distribution of the test statistic under the null hypothesis, so we impose the null by recentering the bootstrap variables around $\hat{\mu}^l, \hat{\mu}^c, \hat{\mu}^u$

$$Z^*_{k,b,t} \equiv d^*_{k,b,t} - g_i(\bar{d}_k) \qquad \text{for}$$

$i = l, c, u$

$b = 1, \ldots, B$

$t = 1, \ldots, n$

- $g_\ell(x) = \max(0, x)$

$$g_c(x) = x \cdot \mathbb{1}\{x \geq -\sqrt{(\hat{\omega}_k^2/n)\, 2 \ln\ln n}\}$$

$$g_u(x) = x$$

- The expected values of $Z^{*}_{k,b,t}$, $i = \ell, c, u$ conditional on $(d_1, \ldots, d_n)$ are given by $\hat{\mu}^\ell, \hat{\mu}^c, \hat{\mu}^u$

- ~~Let $\hat{\mu} = \hat{\mu}^\ell, \hat{\mu}^c, \hat{\mu}^u$ then~~ (skip this)

- The test statistic becomes

$$T^{SPA^*}_{b,n} = \max\left\{0, \max_{k=1,\ldots,m}\left[n^{1/2}\, \bar{Z}^{*}_{k,b} / \hat{\omega}_k\right]\right\}$$

for $b = 1, \ldots, B$

- where

$$\bar{Z}^*_{k,b} = \frac{1}{n}\sum_{t=1}^{n} Z^*_{k,b,t} \quad \text{for} \quad k = 1, \ldots, m$$

$\hat{\omega}_k$ = a variance estimate (I'm really going into this calculation. The SPA function will handle this for you!)

- The Bootstrap p-value is given by

$$\hat{P}_{SPA} = \sum_{b=1}^{B} \frac{\mathbb{1}\{T^{SPA*}_{b,n} > T^{SPA}_{n}\}}{B}$$

- The null should be rejected for small p-values

- There will be three values for each of $i = l, c, u$

(The SPA test spits them out)

- See Table 1 of Hansen (JBES, 2005) to keep things straight