DATA 5610

Time Series Notes III

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Beginning Time Series Topics III

Unit Roots

This basic random walk is

$$x_t = x_{t-1} + \varepsilon_t$$

with

$$E_{t-1}(\varepsilon_t) = 0$$

Note the property:

$$E_t(x_{t+1}) = x_t$$

Because of this property random walks are a popular model for asset prices.

Notice: the random walk is a special case of the AR(1) model in which $\phi = 1$

$$x_t = \phi x_{t-1} + \varepsilon_t$$

Ex:

- x_0, the initial value of the series is a real number denoting the starting value of the process.
- If x_t is the log-price of a stock then x_0 would be the log-price of the stock at its initial public offering (IPO).

If ε_t has a symmetric distribution (normal, t, etc) around zero, then conditional on x_{t-1}, x_t has a 50-50 chance to go up or down, implying that x_t would go up or down at random.

NB: this fits well the economics of the Efficient Markets Hypothesis (EMH). Early tests of the EMH were essentially tests for a random walk.

Within the AR(1) framework $\phi = 1$, which does not satisfy the weak-stationarity assumption. A Random walk is not weakly stationary, and we call it a unit-root nonstationary time series.

Under the random walk model of (log) stock prices the price is not predictable or mean-reverting.

We will take a small detour regarding forecasting to show this.

Forecasting

Forecasting is an important application of time series analysis.

For the AR(p) model:

- Suppose we are at the time index h and interested i n forecasting x_{h+l}, where $l \geq 1$.
- The time index h is called the forecast origin
- The positive integer l is called the forecast horizon.

Let $\hat{x}_h(l)$ be the forecast of x_{h+l} using the minimum squared error loss function and F_n be the collection of information available at the forecast origin h. Then the forecast $\hat{x}_h(l)$ is chosen such that

$$E\{[x_{h+l} - \hat{x}_h(l)]^2 \mid F_h\} \leq \min_g E[(x_{h+l} - g)^2 \mid F_h]$$

where g is a function of the information available at time h (inclusive), that is a function of F_h.

We refer to $\hat{x}_h(l)$ as the l-step ahead forecast of x_t at the forecast origin h.

1-Step Ahead Forecast

$$x_{h+1} = \phi_0 + \phi_1 x_h + \ldots + \phi_p x_{h+1-p} + \varepsilon_{h+1}$$

Under the minimum squared error loss function, the point forecast of x_{h+1} given $F_h = \{x_h, x_{h-1}, \cdots\}$ is –

the conditional expectation

$$\hat{x}_h(1) = E\left(x_{h+1}|x_h\right) = \phi_0 + \sum_{i=1}^{p} \phi_i x_{h+1-i}$$

and the associated forecast error is

$$e_h(1) = x_{h+1} - \hat{x}_h(1) = \varepsilon_{h+1}$$

The variance of the 1-step ahead forecast error is

$$Var[e_h(1)] = Var(\varepsilon_{h+1}) = \sigma_\varepsilon^2$$

If ε_t is normally distributed, then a 95% 1-step ahead interval forecast of x_{h+1} is $\hat{x}_h(1) \pm 1.96\sigma_\varepsilon$

In econometrics ε_{t+1} is often referred to as the shock to the series at time $t+1$

2-Step Ahead

Next consider x_{h+2} at the forecast origin h. From the AR(p) model we have

$$x_{h+2} = \phi_0 + \phi_1 x_{h+1} + \cdots + \phi_p x_{h+2-p} + \varepsilon_{h+2}$$

Taking conditional expectation, we get

$$\hat{x}_h(2) = E\left(x_{h+2}|F_h\right) = \phi_0 + \phi_1\hat{x}_h(l) + \phi_2 x_h + \cdots + \phi_p x_{h+2-p}$$

and forecast error

$$e_h(2) = x_{h+2} - \hat{x}_h(2) - \phi_1[x_{h+1} - \hat{x}_h(1)] + \varepsilon_{h+2} = \varepsilon_{h+2} + \phi_1\varepsilon_{h+1}$$

The variance of the forecast error is

$$\begin{aligned} Var[e_h(2)] &= Var(\varepsilon_{h+2} + \phi\varepsilon_{h+1}) \\ &= (1+\phi_1^2)\sigma_\varepsilon^2 \end{aligned}$$

Note: $Var[e_h(2)] \geq Var[e_h(1)]$

Common sense tells us we are more uncertain about x_{h+2} than about x_{h+1} at time index h.

Multi-Step Ahead

In general

$$x_{h+l} = \phi_0 + \phi_1 x_{h+l-1} + \cdots + \phi_p x_{h+l-p} + \varepsilon_{h+l}$$

and

$$\hat{x}_h(l) = \phi_0 + \sum_{i=1}^{p} \phi_i \hat{x}_h(l-i)$$

The l-step ahead forecast error is

$$e_h(l) = x_{h+l} - \hat{x}_h(l)$$

- It can be shown that for a stationary AR(p) model, $\hat{x}_h(l)$ converges to $E(x_t)$ as $l \to \infty$
- This has the meaning that for such a series the long-term point forecast approaches the unconditional mean.

Forecasting with MA Models

1-Step Ahead (for MA(1))

$$x_{h+1} = \mu + \varepsilon_{h+1} - \theta_1 \varepsilon_h$$

Taking the conditional expectation we have

$$\hat{X}_h(l = 1) = E\left(x_{h+1} | F_h\right) = \mu - \theta_1 \varepsilon_h$$

$$e_h(1) = x_{h+1} - \hat{x}_h(1) = \varepsilon_{h+1}$$

$$Var[e_h(1)] = \sigma_\varepsilon^2$$

2-Step Ahead

$$x_{h+2} = \mu + \varepsilon_{h+2} - \phi\varepsilon_{h+1}$$

We have

$$\hat{X}_h(l = 2) = E\left(x_{h+2}|F_h\right) = \mu$$

$$e_h(2) = x_{h+2} - \hat{x}_h(2) = \varepsilon_{h+2} - \theta\varepsilon_{h+1}$$

$$Var[e_h(2)] = (1 + \theta_1^2)\sigma_\varepsilon^2$$

NB: $\varepsilon_{h+1} = x_{h+1} - \mu + \theta_1\varepsilon_h$

NB: $E(x_{h+1}) = \mu - \theta_1\varepsilon_t - \mu + \theta_1\varepsilon_h = 0$

Note: The 2-step ahead forecast is simply the unconditional mean of the model. This is true for any forecast origin h.

More generally, $\hat{X}_h(l) = \mu$ for $l \geq 2$. Thus for any MA(1) mean reversion take only 1 time period.

Similarly for an MA(2) we have

$$x_{h+l} = \mu + \varepsilon_{h+l} - \theta_1 \varepsilon_{h+l-1} - \theta_2 \varepsilon_{h+l-2}$$

And we obtain

$$\hat{X}_h(1) = \mu - \theta_1 \varepsilon_h - \theta_2 \varepsilon_{h+l-2}$$

$$\hat{X}_h(2) = \mu - \theta_2 \varepsilon_h$$

$$\hat{X}_h(l) = \mu \quad \text{for} \quad l \geq 2$$

Now back to the random walk model:

$$x_t = x_{t-1} + \varepsilon_t$$

1-Step Ahead Forecast

$$\hat{X}_h(1) = E\left[x_{h+1} | x_h, x_{h-1}, \ldots\right] = x_h$$

which is the log stock price at the forecast origin, or in other words the best guess for tomorrow's closing stock price is today's closing stock price. Or the current stock price contains all relevant information regarding the firm.

Note: $F_h = \{x_h\}$

2-Step Forecast

$$\begin{aligned}\hat{X}_h(2) &= E\left[x_{h+2} \mid x_h, x_{h-1}, \ldots\right] = E\left[x_{h+1} + \varepsilon_{h+2} \mid x_h, x_{h-1}, \ldots\right] \\ &= E\left[x_{h+1} \mid x_h, x_{h-1}, \ldots\right] = \hat{X}_h(1) = x_h\end{aligned}$$

This is true for any forecast horizon

$$\hat{X}_h(l) = x_h$$

The MA representation of the random walk model is

$$x_t = \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \cdots$$

The representation has several important implications.

(1.) The l-step ahead forecast error is

$$e_h(l) = \varepsilon_{h+1} + \cdots + \varepsilon_{h+1}$$

so that

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$$Var[e_h(l)] = l\sigma_\varepsilon^2 \quad \text{(which diverges to infinity as } l \to \infty\text{)}$$

(2.) The uncondtional variance of x_t is unbounded because $Var[e_h(l)]$ approaches infinity as l increases.

Random Walk with Drift

$$x_t = \mu + x_{t-1} + \varepsilon_t$$

$$\text{where } \mu = E\left(x_t - x_{t-1}\right)$$

In finance (and macro) μ can be important. It is called the drift.

To see this

$$x_1 = \mu + x_0 + \varepsilon_1$$

$$x_2 = \mu + x_1 + \varepsilon_2 = 2\mu + x_0 + \varepsilon_2 + \varepsilon_1$$

$$\vdots$$

$$x_t = t\mu + x_0 + \varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_1$$