Final Project Part A: Risk-Neutral Monte Carlo Pricing

DATA 5695: Computational Methods in FinTech

Introduction

This final project serves as a capstone that brings together everything we have learned about applying risk-neutral pricing theory to Monte Carlo methods for exotic option payoffs. The main focus is on the logic of control variate sampling. We will also introduce more complex dynamics in the second problem, where we simulate a stochastic volatility model of asset prices.

Control variate sampling marks our departure from the risk-neutral approach and leads us into a truly catallactic approach. Within the risk-neutral paradigm, Clewlow and Carverhill suggest taking inspiration from practitioners' hedging practices to guide the choice of control variates. This is a good starting point, which we will later reinterpret under the catallactic paradigm. From a catallactic perspective, real-world hedges are manifestations of subjective "control variates." Although these two worlds are roughly isomorphic, it can still be somewhat confusing under the risk-neutral paradigm, because the asset price is assumed to have already been hedged. At the very least, it raises the question.

Problem 1: Pricing and Arithmetic Asian Option

This example is taken from Chapter 4 of the book *Implementing Derivatives Models* by Clewlow and Strickland. In this project you will price a European arithmetic Asian call option. This option pays the difference (if positive) between the arithmetic average of the asset price A_T and the strike price K at the maturity date T. The arithmetic average is taken on a set of observations (called fixings) of the asset price S_{t_i} (which we assume follows geometric Brownian motion) at dates $t_i; i = 1 \dots N$.

$$A_T = \frac{1}{N} \sum_{i=1}^{N} S_{t_i}$$

There is no analytical solution for the price of an arithmetic Asian option; however, there is a simple analytical formula for the price of a geometric Asian option. A geometric Asian call option pays the difference (again, only if positive) between the geometric average of the asset price G_T and the strike price K at the maturity date T. The geometric average is defined as

$$G_T = \left(\prod_{i=1}^{N} S_{t_i} \right)^{1/N}$$

Since the geometric average is essentially the product of lognormally distributed variables then it is also lognormally distributed. Therefore the price of the geometric Asian call option is given by a modified Black–Scholes formula:

$$C_{GA} = exp(-rT) \left(exp(a + \frac{1}{2}b)N(x) - KN(x - \sqrt{b}) \right)$$

where

$$\begin{aligned}
a &= \ln(G_t) + \frac{N-m}{N} \left(\ln(S) + \nu(t_{m+1} - t) + \frac{1}{2}\nu(T - t_{m+1}) \right) \\
b &= \frac{(N-m)^2}{N^2}\sigma^2(t_{m+1} - t) + \frac{\sigma^2(T - t_{m+1})}{6N^2}(N-m)(2(N-m)-1) \\
\nu &= r - \delta - \frac{1}{2}\sigma^2 \\
x &= \frac{a - \ln(K) + b}{\sqrt{b}}
\end{aligned}$$

where G_t is the current geometric average and m is the last known fixing. The geometric Asian option makes a good static hedge style control variate for the arithmetic Asian option. To implement the Monte Carlo method we simulate the difference between the arithmetic and geometric Asian options or a hedged portfolio which is long one arithmetic Asian and short one geometric Asian option. This is much faster than using the delta of the geometric Asian option to generate a delta hedge control variate because we do not have to compute the delta at every time step and it is equivalent to a continuous delta hedge.

For this project we will price a 1–year maturity, European Asian call option with a strike price of \$100, a current asset price at \$100 and a volatility of 20%. The continuously compounded interest rate is assumed to be 6% per annum, the asset pays a continuous dividend yield of 3% per annum, and their are 10 equally spaced fixing dates. The simulation has 10 time steps and 10,000 simulations; $K = \$100$, $T = 1$ year, $S = \$100$, $\sigma = 0.2$, $r = 0.06$, $\delta = 0.03$, $N = 10$, $M = 10000$.

The model for asset price dynamics at each time step is Geometric Brownian Motion, as follows

$$S_t = S_{t-1} \times \exp\left(nudt + sigsdt \times \varepsilon\right)$$

where ε is drawn from a standard normal distribution.

A good idea is to precompute the constants as follows:

$$\begin{aligned}
dt &= & \Delta t = \frac{T}{N} = \frac{1}{10} = 0.1 \\
nudt &= & (r - \delta - \frac{1}{2}\sigma^2)\Delta t = (0.06 - 0.03 - 0.5 \times 0.2^2) \times 0.1 = 0.001 \\
sigsdt &= & \sigma\sqrt{\Delta t} = 0.2\sqrt{0.1} = 0.0632
\end{aligned}$$

Then for each simulation $j = 1$ to M where $M = 10,000$, S_t is initialised to $S = 100$, $sumSt = 0$ and $productSt = 1$. Then for each time step $i = 1$ to N, where $N = 10$, S_t is simulated and the sum and product of the asset prices at the fixing times are accumulated.

The estimate of the option price is

$$\hat{C_0} = \frac{1}{M}\sum_{i=1}^{M} C_{0,j}$$

where

$$C_{0,j} = \exp\left(-rT\right)C_{T,j}$$

and

$$C_{T,j} = \max\left(0, A_T - K\right).$$

A measure of the simulation error is the standard deviation of $\hat{C_0}$ which is called the standard error $SE(\,\cdot\,)$ and can be estimated as the sample standard deviation of $C_{0,j}$ divided by the square root of the number of samples.

$$SE(\hat{C_0}) = \frac{SD(C_{0,j})}{\sqrt{M}}$$

where

$$SD(C_{0,j}) = \sqrt{\frac{1}{M-1}\sum_{j=1}^{M}(C_{0,j} - \hat{C_0})^2}$$

The deliverable for this project, in addittion to the project source code, is a table that presents the results of the simulation. In the table present the price, standard error, and relative computation time for the following:

- Simple Monte Carlo
- Antithetic sampling
- Control variate
- Antithetic and control variate

Finally, give a brief statement about your conclusion regarding the trade–off between computation time and variance reduction costs.

Problem 2: Pricing a Lookback Option

This example is taken from Chapter 4 of the book *Implementing Derivatives Models* by Clewlow and Strickland. In this project you will price a European fixed strike lookback call option. This option pays the difference, if positive, between the maximum of a set of observations (called fixings) of the asset price S_{t_i} at dates $t_i; i = 1, \dots, N$ and the strike price. Thus the payoff at the maturity date is

$$\max\left(0, \max\left(S_{t_i}; i = 1, \dots, N\right) - K\right)$$

We will also assume that the asset price and the variance of the asset price returns $V = \sigma^2$ are governed by the following stochastic differential equations:

$$\begin{aligned} dS = & \quad rSdt + \sigma S dz_1 \\ dV = & \quad \alpha(\bar{V} - V)dt + \xi\sqrt{V}dz_2 \end{aligned}$$

and that the Wiener processes dz_1 and dz_2 are uncorrelated, though this can easily be generalized.

There is no analytical solution for the price of a European fixed strike lookback call option with discrete fixings and stochastic volatility. However, there is a simple analytical formula for the price of a continuous fixing fixed strike lookback call with constant volatility.

$$C_{FSLBCall} = G + Se^{\delta T}N(x + \sigma\sqrt{T}) - Ke^{-rT}N(x)$$
$$-\frac{S}{B}\left(e^{-rT}\left(\frac{E}{S}\right)^{B} N(x + (1-B)\sigma\sqrt{T}) - e^{-\delta T}N(x + \sigma\sqrt{T})\right)$$

where

$$\begin{cases} \text{if} \quad K \geq M, \quad \text{then} \quad E = K, G = 0 \\ \text{if} \quad K < M, \quad \text{then} \quad E = M, G = e^{-rT}(M - K) \end{cases}$$

and

$$\begin{aligned} B &= \frac{2(r-\delta)}{\sigma^2} \\ x &= \frac{\ln\left(\frac{S}{E}\right) + \left((r-\delta) - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \end{aligned}$$

and M is the current known maximum. We can therefore use the continuously fixed floating strike lookback call option formula to compute *delta*, *gamma*, and *vega* hedge control variates. Rather than differentiate the above equation with respect to the asset price twice and volatility once which would lead to extremely complex expressions it is more efficient to use finite difference approximations to the partial differentials for *gamma* and *vega*.

A measure of the simulation error is the standard deviation of $\hat{C}_0$ which is called the standard error $SE(\,\cdot\,)$ and can be estimated as the sample standard deviation of $C_{0,j}$ divided by the square root of the number of samples.

$$SE(\hat{C}_0) = \frac{SD(C_{0,j})}{\sqrt{M}}$$

where

$$SD(C_{0,j}) = \sqrt{\frac{1}{M-1}\sum_{j=1}^{M}(C_{0,j} - \hat{C}_0)^2}$$

The deliverable for this project, in addittion to the project source code, is a table that presents the results of the simulation. For example, fill in the missing data for the following table:

	Price	Standard Error	Computation Time
Simple estimate			
Antithetic variate			
Control variates			
Combined variates			

The parameters for the problem are given in the following table.

Strike Price	100
Time to Maturity	1 year
Initial asset price	100
Volatility	20%
Risk–free rate	6%
Continuous Dividend Yield	3%
Mean reversion rate (α)	5.0
Volatility of Volatility (ξ)	0.02
Number of time steps	52
Number of simulations	1000