DATA 5695: Midterm - Part 1

Binomial Option Pricing

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Introduction

This homework assignment is all about the Binomial options pricing model. It is based on Chapters 9, 10, 11 of the McDonald text.

Chapter 9: Parity and Other Option Relationships

9.1 A stock currently sells for $32.00. A 6-month call option with a strike of $35 has a premium of $2.27. Assuming a 4% continuously compounded risk-free rate and a 6% continuous dividend yield, what is the price of the associated put option?

9.2. A stock currently sells for $32.00. A 6-month call option with a strike of $30.00 has a premium of $4.29, and a 6-month put with the same strike has a premium of $2.64. Assume a 4% continuously compounded risk-free rate. What is the present value of the dividends payable over the next 6 months?

9.3 Suppose the S&R index is 800, the continuously compounded risk-free rate is 5%, and the dividend yield is 0%. A 1-year 815-strike European call costs $75 and a 1-year 815-strike European put costs $45. Consider the strategy of buying the stock, selling the 815-strike call, and buying the 815-strike put.

- **a.** What is the rate of return on this position held until the expiration of the options?
- **b.** What is the arbitrage implied by your answer to (a)?
- **c.** What difference between the call and put prices would eliminate arbitrage?
- **d.** What difference between the call and put prices eliminates arbitrage for strike prices of $780, $820, and $840?

9.8 Suppose call and put prices are given by

Strike	50	55
Call Premium	9	10
Put Premium	7	6

What no-arbitrage property is violated? What spread position would you use to effect arbitrage? Demonstrate that the spread position is an arbitrage?

9.9 Suppose call and put prices are given by

Strike	50	55
Call premium	16	10
Put premium	7	14

What no-arbitrage property is violated? What spread position would you use to effect arbitrage? Demonstrate that the spread position is an arbitrage?

9.10 Suppose call and put prices are given by

Strike	50	55	60
Call premium	18	14	9.50
Put premium	7	10.75	14.45

Find the convexity violations. What spread would you use to effect arbitrage? Demonstrate that the spread position is an arbitrage?

9.12 In each case identify the arbitrage and demonstrate how you would make money by creating a table showing your payoff.

- **a.** Consider two European options on the same stock with the same time to expiration. The 90-strike call costs \$10 and the 95-strike call costs \$4.
- **b.** Now suppose these options have 2 years to expiration and the continously compounded interest rate is 10%. The 90-strike call costs \$10 and the 95-strike call costs \$5.25. Show again that there is an arbitrage opportunity. (**Hint:** It is important in this case that the options are European.)
- **c.** Suppose that a 90-strike European call sells for \$15, a 100-strike call sells for \$10, and a 105-strike calls sells for \$6. Show how you could use an asymmetric butterfly to profit from this arbitrage opportunity.

Chapter 10: Binomial Option Pricing – Basic Concepts

10.1 Let $S = \$100$, $K = \$105$, $r = 8\%$, $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 1$.

- **a.** What are the premium, Δ, and B for a European call?
- **b.** What are the premium, Δ, and B for a European put?

10.2 Let $S = \$100$, $K = \$95$, $r = 8\%$, $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 1$.

- **a.** Verify that the price of a European call is \$16.196.
- **b.** Suppose you observe a call price of \$17. What is the arbitrage?
- **c.** Suppose you observe a call price of \$15.50. What is the arbitrage?

10.3 Let $S = \$100$, $K = \$95$, $r = 8\%$, $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, and $d = 0.8$, and $n = 1$.

- **a.** Verify that the price of a European put is \$7.471.
- **b.** Suppose you observe a put price of \$8. What is the arbitrage?
- **c.** Suppose you observe a put price of \$6. What is the arbitrage?

10.4 Obtain at least 5 years' worth of daily stock price data for a stock of your choice.

- **1.** Compute annual volatility using all the data.
- **2.** Compute annual volatility for each calendar year in your data. How does volatility vary over time?
- **3.** Compute annual volatility for the first and second half of each year in your data. How much variation is there in your estimate?

10.5 Obtain at least 10 years of daily data for five stocks. Estimate annual volatility for each year for each asset in your data. What do you observe about the pattern of historical volatility over time? Does historical volatility move in tandem for different assets?

10.6 Let $S = \$100$, $K = \$95$, $\sigma = 30\%$, $r = 8\%$, $T = 1$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 2$. Construct the binomial tree for a call option. At each node provide the premium, Δ, and B.

10.8 Let $S = \$100$, $K = \$95$, $\sigma = 30\%$, $r = 8\%$, $T = 1$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 2$. Construct the binomial tree for a European put option. At each node provide the premium, Δ, and B.

10.10 Let $S = \$100$, $K = \$95$, $\sigma = 30\%$, $r = 8\%$, $T = 1$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 2$. Construct the binomial tree for an American put option. At each node provide the premium, Δ, and B.

10.11 Suppose $S_0 = \$100$, $K = \$50$, $r = 7.696\%$ (continuously compounded), $\delta = 0$, and $T = 1$.

- **a.** Suppose that for $h = 1$, we have $u = 1.2$ and $d = 1.05$. What is the binomial option price for a call option that lives one period? Is there any problem with having $d > 1$?
- **b.** Suppose now that $u = 1.4$ and $d = 0.6$. Before computing the option price, what is your guess about how it will change from your previous answer? Does it change? How do you account for the result? Interpret your answer using put-call parity.
- **c.** Now let $u = 1.4$ and $d = 0.4$. How do you think the call option price will change from (a)? How does it change? How do you account for this? Use put-call parity to explain your answer.

10.12 Let $S = \$100$, $K = \$95$, $r = 8\%$ (continously compounded), $\sigma = 30\%$, $\delta = 0$, $T = 1$ year, and $n = 3$.

- **a.** Verify that the binomial option price for an American call option is \$18.283. Verify that there is never early exercise; hence, a European call would have the same price.
- **b.** Show that the binomial option price for European put option is \$5.979. Verify that put-call parity is satisfied.
- **c.** Verify that the price of an American put is \$6.678.

10.21 For a stock index, $S = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 3\%$, and $T = 3$. Let $n = 3$.

- **a.** What is the price of a European call option with a strike of \$95?
- **b.** What is the price of a European put option with a strike of \$95.
- **c.** Now let $S = \$95$, $K = \$100$, $\sigma = 30\%$, $r = 3\%$, and $\delta = 5\%$. (You have exchanged values for the stock price and strike price and for the interest rate and dividend yield.) Value both options again. What do you notice?

10.22 Repeat the previous problem calculating prices for American options instead of European. What happens?

Chapter 11: Binomial Option Pricing – Selected Topics

For the following problems write three functions in Python: `call_payoff`, `put_payoff`, and `binomial`. The first two functions are to calculate the payoff to plain vanilla call and put options. The third function is to calculate the option premium, Δ, and B for a given pricing problem. Use these functions for any of the problems below for which $n > 3$.

11.1 Consider a one-period binomial model with $h = 1$, where $S = \$100$, $r = 0$, $\sigma = 30\%$, and $\delta = 0.08$. Compute American call options for $K = \$70, \$80, \$90$, and \$100.

- **a.** At which strike(s) does early exercise occur?
- **b.** Use put-call parity to explain why early exercise does not occur at the higher strikes.
- **c.** Use put-call parity to explain why early exercise is sure to occur for all lower strikes than that in your answer to (a).

11.4 Consider a one-period binomial model with $h = 1$, where $S = \$100$, $r = 0.08$, $\sigma = 30\%$, and $\delta = 0$. Compute American put option prices for $K = \$100, \$110, \$120$, and $\$130$.

- **a.** At which strike(s) does early exercise occur?
- **b.** Use put-call parity to explain why early exercise is sure to occur for all strikes greater than that in your answer to (a).

11.5 Repeat Problem 11.4, only set $\delta = 0.08$. What is the lowest strike price at which early exercise will occur? What condition related to put-call parity is satisfied at this strike price?

11.7 Let $S = \$100$, $K = \$100$, $\sigma = 30\%$, $r = 0.08$, $t = 1$, and $\delta = 0$. Let $n = 10$. Suppose the stock has an expected return of 15%.

- **a.** What is the expected return on a European call option? A European put option?
- **b.** What happens to the expected return if you increase the volatility to 50%?

11.8 Let $S = \$100$, $\sigma = 30\%$, $r = 0.08$, $t = 1$, and $\delta = 0$. Suppose the true expected return on the stock is 15%. Set $n = 10$. Compute European call prices, Δ, and B for strikes of $\$70, \$80, \$90, \$100, \$110, \$120$, and $\$130$. For each strike, compute the expected return on the option. What effect does the strike have on the option's expected return?

11.9 Repeat the previous problem, except that for each strike price, compute the expected return on the option for times to expiration of 3 months, 6 months, 1 year, and 2 years. What effect does time to maturity have on the option's expected return?

11.10 Let $S = \$100$, $\sigma = 30\%$, $r = 0.08$, $t = 1$, and $\delta = 0$. Suppose the true expected return on the stock is 15%. Set $n = 10$. Compute European put prices, Δ, and B for strikes of $\$70, \$80, \$90, \$100, \$110, \$120$, and $\$130$. For each strike, compute the expected return on the option. What effect does the strike have on the option's expected return?

11.11 Repeat the previous problem, except that for each strike price, compute the expected return on the option for times to expiration of 3 months, \$6% months, 1 year, and 2 years. What effect does time to maturity have on the option's expected return?

11.12 Let $S = \$100$, $\sigma = 30\%$, $r = 0.08$, $t = 1$, and $\delta = 0$. Using equation (11.12) to compute the probability of reaching a terminal node and $Su^i d^{n-i}$ to compute the price at that node, plot the risk-neutral distribution of year-1 stock prices as in Figures 11.7 and 11.8 for $n = 3$ and $n = 10$.

11.13 Repeat the previous problem for $n = 50$. What is the risk-neutral probability that $S_1 < \$80$? $S_1 < \$120$?

11.14 We saw in Section 10.1 that the undiscounted risk-neutral expected stock price equals the forward price. We will verify this using the binomial tree in Figure 11.4.

- **a.** Using $S = \$100$, $r = 0.08$, and $\$ = 0.08$, what are the 4-month, 8-month, and 1-year forward prices?
- **b** Verify your answers in (a) by computing the risk-neutral expected stock price in the first, second, and third binomial period. Use equation (11.12) to determine the probability of reaching each node.

Bonus Problems

1 Write a Python function to calculate the implied volatility using the binomial options pricing model. Obtain 1-year of historical daily stock price data and compute historical and implied volatilities using your function for at-the-money strike prices. Plot and compare the results.

2 Let $S = \$41$, $\sigma = 30\%$, $r = 0.08$, $t = 1$, and $\delta = 0$. Simulate the delta-hedging process of the options market-maker on a daily basis for a written 40-strike call option. Assume that the true expected rate of return on the stock is $\mu = 15\%$.

- **a.** Using the lognormal expressions for $u = e^{(\mu-\delta-0.5\sigma^2)h+\sigma\sqrt{h}}$ and $d = e^{(\mu-\delta-0.5\sigma^2)h-\sigma\sqrt{h}}$ and the binomial process for the stock price data-generating mechanism.
- **b.** Using geometric Brownian motion for the stock price data-generating mechanism.
- **c.** Using the stationary bootstrap and 1-year of history data as your data-genterating mechanism.

Compare the respective end-of-period hedging payoffs in dollar terms.