Finance 5350 Class Notes

Topic: The Bernoulli and Binomial Distributions

Introduction

This note is a brief review of concepts from discrete probability. We will cover the definition of a *dicrete random variable* and then turn our attention to a discrete probability distribution called the *Bernoulli Distribution* as well as its generalization in the *Binomial Distribution*.

These notes are based on the presentation in Bain and Engelhardt (1992).

Discrete Random Variables

Definition: Discrete Random Variable

If the set of all possible values of a random variable, X, is a countable set, $x_1, x_2, \ldots, x_n$, our $x_1, x_2, \ldots$, then X is called a **discrete random variable**. The function

$$f(x) = P[X = x] \qquad x = x_1, x_2, \ldots$$

that assigns the probability to each possible value x will be called the **discrete probability density function** (discrete pdf).

Definition: The Bernoulli Distribution

A random variable , X, that assumes only the values 0 and 1 is known as a **Bernoulli variable**, and a performance of an experiment with only two types of outcomes is called a **Bernoulli trial**. In particular, if an experiment can result only in "success" (E) or "failure" (E'), then the corresponding Bernoulli variable is

$$X(e) = \begin{cases} 1 & \text{if} \quad e \in E, \\ \\ 0 & \text{if} \quad e \in E' \end{cases}$$

The pdf of X is given by $f(0) = 1 - \theta$ and $f(1) = \theta$. The corresponding distribution is known as a **Bernoulli distribution**, and its pdf can be expressed as

$$f(x) = \theta^x (1 - \theta)^{1-x} \qquad x = 0, 1$$

Example: Drawing Marbles at Random

Consider drawing marbles at random from a collection of 10 black and 20 white marbles. In such a problem, we might regard "black" as success and "white" as failure, or vice versa, in a single draw. If obtaining a black marble is regarded as success, then $\theta = 10/30 = 1/3$ and $1 - \theta = 20/30 = 2/3$.

The Single-Period Binomial Model

Recall that in the single-period binomial option pricing model we could express it in its risk-neutral form as follows

$$f_0 = e^{-rh}[f_u p^* + f_d(1 - p^*)]$$

where f_0 is value of the option (call or put) and f_u is the value of the option in the up state and f_d is the value of the option in the down state. Finally, $p^* = \frac{e^{(rh)} - u}{u - d}$ was called the ***risk-neutral probability***. This is an example of a single Beroulli trial with the probability set to the risk-neutral probability.

A Mathematical Note: The Binomial Coefficient

The number of combinations of n distinct objects chosen r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example:

If five cards are drawn from a deck of cards without replacement, the number of five-card hands is given by

$$\binom{52}{5} = \frac{52!}{5!47!}$$

The Binomial Distribution

An experiment with a sequence of **independent Bernoulli trials** leads to a generalization of the Bernoulli distribution called the ***Binomial Distribution***.

In a sequence of n independent Bernoulli trials with probability of success p on each trial, let X represent the number of successess. The discrete pdf of X is given by

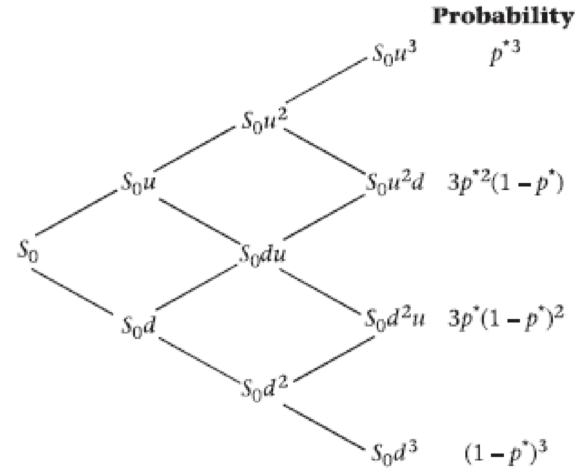
$$b(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n$$

Relation to the Binomial Options Pricing Model

The Binomial distribution discrete pdf can be used to calculate the risk-neutral probability of ending up on a given terminal node of the binomial tree. This can be seen in Figure 11.8 of McDonald (2012) (see page 333).

FIGURE 11.6

Construction of a binomial tree depicting stock price paths, along with risk-neutral probabilities of reaching the various terminal prices.



So, for example, one can find out the probability that you would end up on the second node from the top in this three-period tree using the Binomial pdf as follows:

$$b(2;3,p^*) = \binom{n=3}{x=2}(p^*)^2(1-(p^*)^{3-2} = \frac{3!}{2!1!}(p^*)^2(1-p^*)^1 = 3(p^*)^2(1-p^*)$$

And the probabilities of the other nodes can be found in a like manner for $x = 0, 1, 2, 3$.

Finding Binomial Probabilities in Python

One can use the `binom` object from the `scipy.stats` module in Python as follows:

```
from scipy.stats import binom


## Just some placeholder values - substitute your own for actual problems
n = 3
x = 2
p = 0.45
b = binom.pmf(x, n, p)
```

References

Bain, Lee J., and Max Engelhardt. 1992. *Introduction to Probability and Mathematical Statistics, 2nd Edition.* Duxbury Thomson Learning.

McDonald, Robert L. 2012. *Derivatives Markets, Third Edition.* Pearson.