Binomial Option Pricing: Basic Principles

Financial 5350: Computational Finance

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Introduction to Binomial Option Pricing

- The binomial option pricing model enables us to determine the price of an option, given the characteristics of the stock or other underlying asset
- The binomial option pricing model assumes that the price of the underlying asset follows a binomial distribution - that is, the asset price in each Period can move only up or down by a specified amount
- The binomial model is often referred to as the "Cox-Ross-Rubinstein pricing model"

A One-Period Binomial Tree

- Example
  - Consider a European call option on the stock of XYZ, with a \$40 strike and 1 year to expiration
  - XYZ does not pay dividends (i.e. $\delta = 0$), and its current price is $S_0 = \$41$
  - The continuously compounded risk-free interest rate is 8% (i.e. $r = 0.08$)
  - The following is the corresponding binomial tree:

Computing the Option Price

- Next, consider two portfolios:
  - *Portfolio A:* buy one call option
  - *Portfolio B:* buy 2/3 shares of XYZ and borrow \$18.462 at the risk-free rate, r
- Portfolio Costs:
  - *Portfolio A:* the call premium, **which is unknown and what we are solving for**
  - *Portfolio B:* $2/3 \times \$41 - \$19.462 = \$8.871$

Computing the Option Price Continued

- Payoffs:

Computing the Option Price Continued

- Portfolios A and B have the same payoff. Therefore:
  - Portfolios A and B should have the same cost. Since Portfolio B costs \$8.871, the price of on option must also be \$8.871
  - There is a way to create the payoff to a call by buying shares and borrowing. Portfolio B is a *synthetic call*
  - One option has the risk of $2/3$ shares. The value $2/3$ is the *delta* (Δ) of the option: the number of shares that replicates the option payoff

The Bimonial Solution

- How do we find a replicating portfolio consisting of Δ shares of stock and a dollar amount B in lending, such that the portfolio imitates the option whether the stock or falls?
  - Suppose that the stock has a continuous dividend yield of δ, which is reinvested in the stock. Thus, if you buy one share at time t, at time $t + h$ you will have $e^{\delta h}$ shares
  - If the length of a period is h, the interest factor per period is e^{rh}
  - uS denotes the stock price when the price goes up, and dS denotes the stock price when the price goes down

The Binomial Solution Continued

- The value of the replicating portfolio at time h, with stock price S_h, is

$$\Delta S_h e^{\delta h} + e^{rh} B$$

The Binomial Solution Continued

- At the prices $S_h = uS$ and $S_h = dS$, a successful replicating portfolio will satisfy

$$(\Delta \times uS \times e^{\delta h}) + (B \times e^{rh}) = C_u$$
$$(\Delta \times dS \times e^{\delta h}) + (B \times e^{rh}) = C_d$$

- Solving for Δ and B gives

$$\Delta = e^{-\delta h}\left(\frac{C_u - C_d}{S(u-d)}\right)$$

$$B = e^{-rh}\left(\frac{uC_d - dC_u}{u-d}\right)$$

NB: Please see notes on the full algebraic derivation.

The Bimonial Solution Continued

- The cost of creating the option is the net cash flow required to buy the shares and bonds. Thus, the cost of the option is $\Delta S + B$

$$\Delta S + B = e^{-rh}\left(C_u \frac{e^{(r-\delta)h} - d}{u - d} + C_d \frac{u - e^{(r-\delta)h}}{u - d}\right)$$

- The no-arbitrage condition is

$$d < e^{(r-\delta)h} < u$$

Arbitraging a Mispriced Option

If the observed option price differs from its theoretical price, arbitrage is possible:

- If an option is overpriced, we can sell the option. However, the risk is that the option will be in the money at expiration, and we will be required to deliver the stock. To hedge this risk, we can buy a synthetic option at the same time we sell the actual option

- If an option is underpriced, we buy the option. To hedge the risk associated with the possibility of the stock price falling at expiration, we sell a synthetic option at the same time

Arbitraging a Mispriced Option Continued

Risk-Neutral Pricing

- We can interpret the terms $\frac{(e^{(r-\delta)h}-d)}{(u-d)}$ and $\frac{(u-e^{(r-\delta)h})}{u-d}$ as probabilities
- Let

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

- Then equation (10.3) can then be written as

$$C_0 = e^{-rh}[p^* C_u + (1 - p^*) C_d]$$

- We call p^* the **risk-neutral probability** of an increase in the stock price

Summary

- In order to price an option, we need to know the following:
    - Stock price (S)
    - Strike price(K)
    - The expiration date of the options (T)
    - Standard deviation of returns on the stock (σ)
    - Divided yield (δ)
    - Risk-free rate (r)
- Using the risk-free rate and σ, we can approximate the future distribution of the stock by creating a binomial tree using equation (10.9)
- Once we have the binomial tree, it is possible to price the option using equation (10.3)

Continuously Compounded Returns

Volatility

The Standard Deviation of Continuously Compounded Returns Continued

Constructing u and d

Constructing u and d Continued

Estimating Historical Volatility

Estimating Historical Volatility Continued

One-Period Example with a Forward Tree

One-Period Example with a Forward Tree Continued

A Two-Period European Call

A Two-Period European Call Continued

Pricing the Call Option

Pricing the Call Option Continued

Pricing the Call Option Continued

Many Binomial Periods

Many Bimonial Periods Continued

Many Bimonial Periods Continued

Put Options

Put Options Continued

American Options

American Options Continued

American Options Continued

American Options Continued

Options on Other Assets

Options on a Stock Index

Options on a Stock Index Continued

Options on Futures Contracts

Options on Futures Contracts Continued

Options on Futures Contracts Continued

Options on Futures Contracts Continued

Options on Commodities

Options on Bonds

Summary