

# Optimal delta-hedging under transactions costs

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## Abstract

This paper examines the problem of delta-hedging portfolios of options under transactions costs by maximising expected utility (or minimising a loss function on the replication error). We extend the work of Hodges and Neuberger (1989) to study the optimal strategy under a general cost function with fixed and proportional costs. A computational procedure for solving this problem is described and we develop an efficient computational method for the case of proportional transaction costs. We examine the nature of the solution close to the expiry date and using simulation we compare the performance of the optimal strategies with other common strategies.

**Keywords:** Option pricing; Hedging; Transaction costs; Computational methods; Simulation

**JEL classification:** G13; C61

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## 1. Introduction

This paper describes a stochastic optimal control approach for delta-hedging portfolios of contingent claims in the presence of transaction costs. It extends

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The authors would like to acknowledge helpful discussions with Michael Selby, Chris Strickland, Silio Aparicio; participants in the Financial Options Research Centre's Fifth annual Conference (June 1992), particularly Michael Dempster; and the European Finance Association conference in Copenhagen, August 1993, particularly Constantine Thanassoulas and Raman Uppal; and an anonymous referee. All errors remain our own. Funding for this work was provided by past and present corporate members of the Financial Options Research Centre: Bankers Trust, Banque Indosuez, Credit Suisse First Boston, Kleinwort Benson Investment Management, LIFFE/LTOM, London Clearing House, London Commodity Exchange, HSBC Markets, Tokyo Mitsubishi International, Deutsche Morgan Grenfell, Nomura Bank, SBC Warburgs, UBS Phillips and Drew.

earlier work by Hodges and Neuberger (1989), who analysed the case of writing and hedging a European call option under proportional costs. In this paper we study the optimal strategies under a general cost function with a constant cost per transaction, a cost per unit of asset transacted and a cost proportional to the value transacted. We analyse the nature of the strategies through time including close to the settlement date where the type of settlement, either asset or cash settlement, has a significant effect. The computational method for solving this problem is described in detail and an efficient computational method to solve for the optimal strategies under proportional transaction costs is developed. Then, using simulation, we compare the performance of the optimal strategies against typical sub-optimal strategies such as Black–Scholes and Leland for a variety of portfolios of mixed long and short positions.

The construction of hedging strategies which best replicate the outcomes from portfolios of options (and other contingent claims) in the presence of transactions costs is an important problem. Hedging is central to the theory of option pricing. Arbitrage valuation models, such as that of Black and Scholes (1973), depend on the idea that an option can be perfectly hedged using the underlying asset, so making it possible to create a portfolio which replicates the option exactly. Hedging is also widely used to reduce risk, and the kind of delta-hedging strategies implicit in Black and Scholes are commonly applied, at least approximately, by participants in options markets. Optimal hedging strategies are therefore of direct practical interest. Much of the theory of options assumes that markets are frictionless. This paper considers the impact of transactions costs on delta-hedging and valuation. This is closely related to the valuation issues which arise where the nature of the market dictates that trading is discontinuous, or that the asset process is such that the market is incomplete and contingent claims are not spanned by existing securities.

The first paper to consider the problem of replicating options' payoffs using delta-hedging under transactions costs was Leland (1985). The issue is particularly interesting because under the usual Black–Scholes strategy, implemented as rebalancings at discrete intervals, the expected volume of transactions becomes unbounded as the number of rebalancings is increased. Leland's analysis is set in a continuous-time framework and assumes proportional transactions costs. It describes how by making an adjustment to the variance (which depends on the exogenously specified revision frequency) the Black–Scholes formula can be used to hedge with a zero expected replication error, and with a standard deviation which tends to zero with the length of the rebalancing interval. Neuhaus (1989) contributes some further theoretical insights to this approach.

The method described in this paper follows earlier work by Hodges and Neuberger (1989) and is based on maximising expected utility. Alternatively, we may view it as minimising a loss function defined on the replication error. This approach seems more appropriate than exact replication since it provides much

tighter valuation bounds. Depending on the choice of risk-aversion parameter we can obtain either tight or much looser (but also cheaper) hedging. The approach is in a paradigm similar to that of Davis (1988), Davis and Norman (1988), Taksar et al. (1988) and Dumas and Luciano (1991). These papers describe optimal portfolio strategies to maximise expected utility over an infinite horizon. They extend earlier work by Merton (1971) and Constantinides (1986). However, while these papers are concerned with optimal portfolio strategies, they are not directly concerned with the problems of replicating (or similarly hedging) contingent claims by means of the underlying asset.

Mention should also be made of a number of other recent papers related to ours. The model first proposed by Hodges and Neuberger (1989) has been further studied by Davis and Panas (1991), and Davis et al. (1993). Dixit (1991) and Dumas (1991) provide useful material on smooth pasting conditions which usually apply to problems such as ours. Figlewski (1987) gives some interesting simulation results. Boyle and Vorst (1992) provide an elegant reworking of Leland's analysis within a binomial framework. This has the following interesting feature. Their variance adjustment differs from Leland's, essentially because the binomial assumption distorts the expected absolute price change in any sub-interval even though it provides the correct variance. Wilmott (1993, 1994) further analyses approaches based on hedging at discrete intervals. Edirisinghe, Naik and Uppal (1993) also provide a binomial replication based approach, and apply a technology based on linear programming. Replication can be a dangerous philosophy. Bensaid et al. (1992), also in a discrete-time framework, show that it can be cheaper to dominate a contingent claim than to exactly replicate it. Constantinides and Zariphopoulou (1995) obtain much tighter valuation bounds by maximising expected utility than can be obtained by either replication or dominance objectives. In contrast, Neuberger (1994) shows that under a pure jump process (with fixed jump size), exact replication can provide tight bounds on option values.

Section 2 of the paper introduces the model, based closely on Hodges and Neuberger (1989), describes the general problem of the best replication of a contingent claim (or portfolio of claims) under transactions costs, and develops the solution approach we have adopted. Section 3 describes the computational procedure. In Section 4 we discuss the properties of the optimal strategies for both proportional costs only and a general cost function. Our simulation analysis, in Section 5, compares the hedging performance of a variety of approaches to hedge a small selection of option exposures. Section 6 presents our conclusions.

## **2. The model**

The structure of the model follows that developed in Hodges and Neuberger (1989). Davis and Panas (1991) and Davis et al. (1993) have developed some of its

theoretical properties. We consider an asset whose price  $S_t$  at time  $t$  evolves under the diffusion process described by

$$dS_t = \mu(S_t) dt + \sigma(S_t) dz. \quad (1)$$

The general problem is to replicate the outcomes from a portfolio of contingent claims whose payoffs occur at different dates. For expositional clarity we will consider payoffs at a single future date  $T$ , given by  $C(S_T)$ , but our approach generalises easily. The replication is to be accomplished by holding  $x_t$  units of the asset plus either borrowing or lending at a constant interest rate  $r$ . The holdings in this replicating portfolio are to be actively managed through time, but transactions in the underlying asset involve a transactions cost amounting to  $k(v, S)$  where  $v$  is the volume of asset transacted (either positive or negative) and  $S$  is the (mid) asset price. For example, for the case of costs which are a constant proportion of the value transacted we have

$$k(v, S) = k|v|S. \quad (2)$$

In general it is either impossible or at least undesirable to replicate the contingent claim exactly. For many problems exact replication at finite cost is impossible. For others, while exact replication at finite cost may be possible, it will be too expensive to represent an attractive policy. The replication problem is therefore ill-defined until we have specified a criterion for choosing between alternative replicating strategies. We will assume initially a fairly general expected utility criterion which will later be specialised to a particular function. Thus, we assume that an initial amount of money is invested through time (managed between the risky asset and the risk-free rate). By the terminal date  $T$ , after liquidating the asset holding, an amount of cash  $y_T$  is available to set against the contingent liability  $C(S_T)$ . At this date we have an accumulated surplus of

$$w_T = y_T - C(S_T) \quad (3)$$

net of the option value to be replicated. We define a utility function  $U(w_T)$ , and seek to characterise and calculate replication strategies which maximise the expected value of this utility function. Note that we can allow the horizon date to be later than the expiry date of the contingent claim. Also, since it is definitely possible for  $w_T$  to be negative, we are precluded from using some commonly employed utility functions, such as power or logarithmic utility functions for  $U(\cdot)$ . We shall assume that  $U(w)$  is defined for all real numbers  $w$ , that its first two derivatives exist, are continuous, and satisfy the usual properties for a risk averse utility function, i.e., that  $U_w > 0$  and  $U_{ww} < 0$ . It may seem unrealistic to assume that an options trader for example has an horizon over which they are maximising their expected utility or indeed that of the investment bank which employs them. However, the options trader or institution is likely to have

a number of horizons over which they are effectively maximising their expected utility. For example, end of quarter or year profit and loss reporting. Furthermore, our aim is to characterise the nature of the solutions for this approach to the problem of replication under costs. Whilst the fine details of the solutions will depend on the specific utility function, time horizon, components of the portfolio, and other parameters, the overall characteristics of the solutions will not.

We now describe the structure of the general problem. Using the notation already introduced we define the indirect utility function  $J(\cdot)$  as

$$J(t, S, x, y) = \max \tilde{E} [U(w_T)] \quad (4)$$

as the maximum expected utility possible starting at time  $t$  when the asset price is  $S$ , with initial holdings of  $x$  units of the asset, and an amount  $y$  in cash.  $\tilde{E}[\cdot]$  is the expectation operator under some suitable probability measure, not necessarily the objective one. We also define  $\tilde{\mu}(\cdot)$  as the rate of drift of  $S$  under this measure. The maximum is taken over all feasible transactions policies. At the last date  $T$ , it is clear that by definition  $J(\cdot)$  is obtained trivially as

$$J(T, S, x, y) = U(w_T), \quad (5)$$

where

$$w_T = xS_T + y_T - C(S_T) \quad (6a)$$

corresponding to no costs at termination (i.e. asset settlement permitted), or

$$w_T = xS_T - k(x, S_T) + y_T - C(S_T) \quad (6b)$$

corresponding to cash settlement after transactions costs have been paid.

The indirect utility function  $J(\cdot)$  is solved recursively backwards through time using the Hamilton–Jacobi–Bellman dynamic programming approach of stochastic optimisation.  $J(\cdot)$  evolves backwards as given by

$$J(t, S, x, y) = \max \{E_{dS} [J(t + dt, S + dS, x^*, y(x^*))]\}, \quad (7)$$

where the maximum is taken over the choice of the quantity of the asset  $x^*$  to hold.

This optimal control problem is characterised by the second-order partial differential equation (using subscripts to denote partial derivatives)

$$J_t + \tilde{\mu}(S)J_s + \frac{1}{2}\sigma^2(S)J_{ss} + yrJ_y = 0 \quad (8a)$$

for interior values of  $x \in X$ , subject to the boundary conditions

$$J(T, S, x, y) = U(w_T) \quad (8b)$$

defining  $J$  at the terminal date  $T$ , and

$$J(t, S, x + u, y - uS - k(u, S)) \leq J(t, S, x, y) \quad (8c)$$

which defines the boundary of the region  $X$  on which transactions occur. This expresses the condition that the upper boundary of  $X$  is such that buying more units of the asset does not improve the utility. Similarly, the lower boundary of  $X$  is such that selling some units of the asset does not improve the utility.

The details of the solution to this problem are developed in Appendix A. The solution provides a ‘reservation selling (or buying) price’ for the contingent claim, and also the optimal strategy for hedging it.

For a cost function of the form

$$k(u, S) = k_c + (k_f + k_p S)|u|, \quad (9)$$

where  $k_c$  is a constant cost per transaction,  $k_f$  is a fixed cost per unit of asset transacted and  $k_p$  is a cost proportional to the value transacted, the optimal strategy is characterised by control bands defined by the functions

$$x_-(t, S, y), x_+(t, S, y), \tilde{x}_-(t, S, y), \tilde{x}_+(t, S, y). \quad (10)$$

The discovery that  $x < x_-$  leads to transactions to re-establish  $x$  at the value  $\tilde{x}_-$ , and similarly if  $x > x_+$  it is re-established to  $\tilde{x}_+$ . If transaction costs are simply proportional to the quantity transacted then  $x_- = \tilde{x}_-$  and  $x_+ = \tilde{x}_+$  and if transaction costs have a fixed component only then  $\tilde{x}_- = \tilde{x}_+$ .

The reservation selling (and buying) prices are defined as follows. We define  $J^C(t, S, x, y)$  as the expected utility (under an optimal hedging strategy) of assuming the state contingent liability  $C$ , (e.g.  $C(S_T)$  as before). The individual’s reservation selling price of  $C$ ,  $V_S(C)$  is defined as the price required to provide the same expected utility as not selling the contingent claim. Thus  $V_S$  is defined by the equation

$$J^C(0, S, 0, V_S) = J^0(0, S, 0, 0), \quad (11)$$

where  $J^0$  is defined as  $J^C$ , but with no state contingent liability assumed.

Similarly, we can define the buying price  $V_B$  as the maximum price worth paying to buy the contingent claim, defined by the equation

$$J^{-C}(0, S, 0, -V_B) = J^0(0, S, 0, 0). \quad (12)$$

Under this general formulation, at each date in the calculation, the indirect utility function depends on the three state variables of  $S$ ,  $x$  and  $y$ . We therefore follow Hodges and Neuberger (1989) and specialise the utility function to the negative exponential

$$U(w_T) = -\exp(-\lambda w_T) \quad (13)$$

and also choose the risk-free interest rate  $r$  as the risk adjusted rate of drift,  $\mu^*$ <sup>1</sup>. This reduces the state variables by one and makes the computations relatively straightforward. It also enables us to produce strategies which have the attractively simple properties of not being wealth dependent and not creating risky positions in the absence of any contingent claim to be hedged. The details of the solution to the optimal control problem under this specific set of assumptions are developed in Appendix A. In the following section we develop a computational procedure for solving this problem and a particularly efficient method for the case of proportional transaction costs.

### 3. The computational procedure

In their original work, Hodges and Neuberger used a binomial tree for the asset price, with a large vector for different possible values of  $x$  at each node of the tree. At each stage, the value of the indirect utility function  $H$  was calculated for each element of each vector by the usual binomial averaging scheme. These were then modified to reflect the inequality relationships which stem from undertaking transactions when it improves expected utility to do so (Eqs. (A6)). Although this scheme works reasonably well, it has a number of disadvantages. Firstly, the values of expected utility are exponential in terminal wealth, and therefore computationally liable to 'blow-up' as either underflow or overflow. It is therefore better to work with the logarithm of (minus) the utility, which is simply the reservation price. Secondly, the procedure is computationally expensive and liable to numerical instability. Our improved scheme is as follows.

We use a conventional Cox et al. (1979) binomial scheme for the evolution of  $S$ , with a time step  $\Delta t$ , up and down proportional jumps  $u$  and  $d$ , and the up and down jump probabilities chosen to be equal to  $\frac{1}{2}$  which gives

$$u = \exp((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}), \quad d = \exp((r - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}), \quad (14)$$

Let  $H_{i,j}$  be the indirect utility at node  $i,j$  corresponding to  $i$  time steps and  $j$  upward jumps of the asset price. The reservation price  $V_{i,j}$  at node  $i,j$  is related to the corresponding indirect utility by

$$\exp\{\lambda V_{i,j}\} = -H_{i,j}. \quad (15)$$

<sup>1</sup> See Hodges and Neuberger (1989) for a detailed justification of this approach. Briefly, Dybvig (1992) showed the inseparability of investment and hedging decisions, but individuals or institutions would like to behave as if they were separable. We follow Hodges and Neuberger (1989) and assume that the institution has already optimised their portfolio of assets and liabilities. The institution then has the opportunity to write (or buy) a contingent claim and hedge the risk. We are interested in the optimal incremental transactions the contingent claim generates, the choice of the riskless rate as the drift achieves this objective. See Toft (1996) for an analysis that does not assume a risk-neutral drift in a simulation-based study.

$H$  evolves backwards in the binomial tree as discounted expectations in the usual way, so  $V_{i,j}$  is related to its two successor values  $V_{i+1,j}$ ,  $V_{i+1,j+1}$  by

$$\begin{aligned}\exp\{\lambda V_{i,j}\} &= \exp(-r\Delta t) \frac{\exp\{\lambda V_{i+1,j}\} + \exp\{\lambda V_{i+1,j+1}\}}{2} \\ &= \exp(-r\Delta t) \exp\{\lambda V_{i+1,j}\} \frac{1 + \exp\{(\lambda V_{i+1,j+1} - V_{i+1,j})\}}{2}.\end{aligned}\quad (16)$$

This gives

$$V_{i,j} = \frac{-r\Delta t}{\lambda} + V_{i+1,j} + \frac{1}{\lambda} \ln \left[ \frac{1 + \exp\{\lambda(V_{i+1,j+1} - V_{i+1,j})\}}{2} \right]. \quad (17)$$

In this way we can evolve the reservation price directly backwards through the binomial tree. This method also avoids numerical overflow in the exponential function since it is applied to the difference of adjacent reservation prices.

Once all reservation prices have been evolved backwards at a particular time step they are modified by applying the boundary conditions as described in Appendix A (Eqs. (A.6)). This process is then repeated until we reach the root node of the tree corresponding to the current date.

In the case of proportional transaction costs only we are able to derive a functional approximation to the reservation price, which improves the efficiency of the computational procedure, as follows:

Firstly, we note the power series expansion

$$\ln\left(\frac{1}{2} + \frac{1}{2}e^x\right) = \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots \quad (18)$$

Substituting this into Eq. (17) we obtain

$$V_{i,j} = \frac{-r\Delta t}{\lambda} + \frac{(V_{i+1,j+1} - V_{i+1,j})}{2} + \frac{\lambda(V_{i+1,j+1} - V_{i+1,j})^2}{8} - \dots \quad (19)$$

We can ignore the higher-order terms as long as  $\lambda$  is not too large, and our time steps are also sufficiently small. Note that the simplicity of Eq. (18) relies on the choice of binomial probabilities of one-half.

We can also use this approximation to find the first derivative of  $V_{i,j}$  as

$$V'_{i,j} = \frac{(V'_{i+1,j+1} + V'_{i+1,j})}{2} + \frac{\lambda(V_{i+1,j+1} - V_{i+1,j})(V'_{i+1,j+1} - V'_{i+1,j})}{4} - \dots \quad (20)$$

which is needed to apply the boundary conditions arising from transacting.



Guided by the form of Eq. (18) we make the following functional approximation for  $V$  (dropping the node indices) in the region  $x_- \leq x \leq x_+$ :

$$V = p_* - xS + a_1 z + a_2 z^2 + a_4 z^4, \quad z = x - x_*, \quad (21a)$$

$$V' = -S + a_1 + a_2 z + a_4 z^3, \quad (21b)$$

where  $x_*$  is the optimal value of  $x$  given the opportunity to freely transact to any value.

In the region defined by  $x < x_-$ , the reservation price is simply the linear function defined by the proportional transaction costs

$$V = p_* - x_- S + a_1 z_- + a_2 z_-^2 + a_4 z_-^4 - e^{-r(T-t)}(S - (k_f + k_p S))(x - x_-), \\ z_- = x_- - x_*. \quad (22)$$

and similarly for  $x > x_+$

$$V = p_* - x_+ S + a_1 z_+ + a_2 z_+^2 + a_4 z_+^4 - e^{-r(T-t)}(S + (k_f + k_p S))(x - x_+), \\ z_+ = x_+ - x_*. \quad (23)$$

We also divide the region  $x_- \leq x \leq x_+$  into two sub-regions  $x_- \leq x \leq x_*$  and  $x_* \leq x \leq x_+$  with a different set of coefficients  $a_1, a_2, a_4$  and  $b_1, b_2, b_4$  in the two sub-regions, respectively. This allows the derivatives of  $V$  to be discontinuous at  $x_*$ , which we observed experimentally, and for the control limits to be at different distances from  $x_*$ . So we have a set of ten parameters ( $p_*, a_1, a_2, a_4, b_1, b_2, b_4, x_-, x_*, x_+$ ) defining the reservation price as a function of  $x$  rather than having a large discrete vector approximation.

Given (21a) and (21b) we can obtain  $V$  and  $V'$  from (19) and (20) and using this we must solve for the parameters of (21a) and (21b). We solve for these parameters by fitting the functional form (21a) and its derivative (21b) to the evolved reservation price and its derivative (Eqs. (19) and (20)) as follows: Firstly, we search for the values of  $x$  for which  $V' = -S$ ,  $V' = -e^{-r(T-t)}(S - (k_f + k_p S))$ , and  $V' = -e^{-r(T-t)}(S + (k_f + k_p S))$ , with  $V'$  given by Eq. (20), at which  $x = x_*$ ,  $x = x_-$ , and  $x = x_+$ , respectively. The search is performed by combination of bisection and Newton–Raphson (Press et al. 1992) as in some parts of the binomial tree pure Newton–Raphson does not converge. From these points we can obtain the parameters. For example, for the region  $x_- \leq x \leq x_*$  we have

$$p^* = V(x_*) + x_* S, \quad (24)$$

$$a_1 = V'(x_*) + S, \quad (25)$$

where  $V'(x_*) = V'(x_* + \varepsilon) - V''(x_* + \varepsilon)\varepsilon$  since the discontinuous derivatives invalidates using Eq. (20). Then using Eqs. (21a) and (21b) at  $x = x_-$  we can solve for  $a_2$  and  $a_4$ , noting that  $V'(x_-) = -e^{-r(T-t)}(S - (k_f + k_p S))$  by

construction. We have a similar set of calculations for the coefficients in the region  $x_* \leq x \leq x_+$ .

This procedure gives a large improvement in speed, and an enormous saving in storage requirements, as we now need only store ten parameters at each node of the binomial tree.<sup>2</sup>

#### 4. Properties of optimal hedging strategies

We now provide some general comments regarding the properties of optimal replication strategies. The solution provides a reservation selling price (or buying price) above which it is advantageous to sell (or buy) the contingent claim and hedge the risk using the prescribed strategy. It is worth noting that in this framework the reservation prices depend on the quantity of the claim involved. As we have just seen, the optimal control  $x$  is constrained to evolve between control limits which depend on time, and on the asset price. Under the negative exponential utility assumption the amount of cash accumulated into the replicating portfolio is irrelevant. No controlling action is taken until the control parameter  $x$  attains one of its limits. In the case of transaction costs which are simply proportional to the quantity of asset transacted then immediately  $x$  moves below  $x_-$  we transact to bring it back to  $x_-$ . If transactions costs also have a fixed component then immediately  $x$  moves below  $x_-$  we transact to move it to  $\tilde{x}_-$  which is above  $x_-$ ; and similarly for  $x_+$ . We use some numerical examples to illustrate general features of the solutions, firstly for proportional costs only and then for fixed and proportional costs.

Fig. 1 shows the buy and sell reservation prices computed for a 6-month call option, with Black–Scholes values for comparison. The asset value is  $100p$ , the exercise price is  $100p$ , the volatility is 30% and we have used a zero interest rate. Transactions costs are at 2% (each way) and we have chosen  $\lambda = 1$ . We have used these values throughout the numerical work reported in this paper, except that for our simulations we have used a transactions cost of 1% instead of 2%.

Figs. 2 and 3 plot the control region for fixed time to expiry as a function of the price of the underlying asset. This region evolves through time in a way comparable to the behaviour of the Black–Scholes delta curve for hedging options in the absence of any transactions costs. As  $S$  changes, it is necessary to adjust the delta of the hedge only as much as is required to keep within the region defined by the two curves. The shape of the curves depend on the contingent claim to be hedged, on the level of transactions cost and on the

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<sup>2</sup> Numerical investigation suggests that the errors introduced by the approximation are typically small (1%), and remain bounded as the time step is decreased, for the parameter values used in this work.

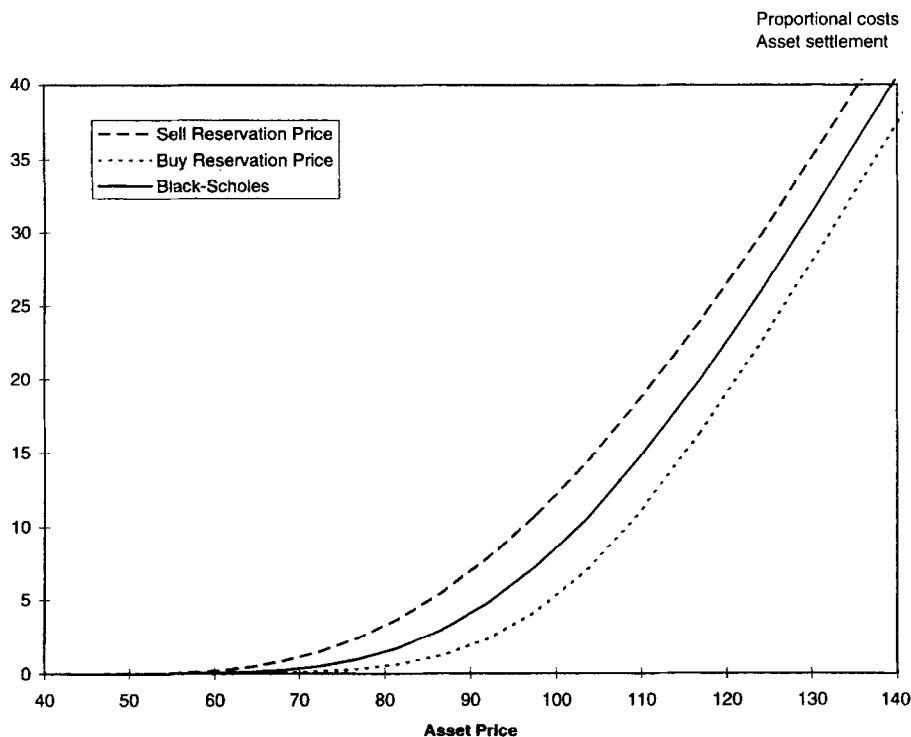


Fig. 1. Black-Scholes and optimal strategy reservation call option prices. Short call option, proportional transaction costs, asset settlement: (solid line) Black-Scholes price; (dashed line) sell reservation price; (dotted line) buy reservation price.

degree of risk aversion. They also reflect the variance intuition of Leland (1985). Leland noted that when there is a short gamma exposure (so that an increase in asset price leads to an increased requirement for the underlying) the transactions cost makes it as if the price movement had been even greater. If we are hedging a short call our control region is flattened, corresponding to an increased variance assumption. Note that for out-of-the-money and in-the-money options, the Black-Scholes delta may be outside the optimal control region. In other words, if we inherit an options book which is currently exactly delta-hedged under Black-Scholes and we face delta-hedging costs, it may nevertheless be optimal to move the hedge away from the Black-Scholes value. If we plot Black-Scholes delta curves on our diagram for different values of volatility we find that they cross the curves we have computed: our curves are not simply 'Leland' ones for simple constant volatility adjustments. We should think of the volatility adjustment as reflecting the expected cost of future hedging transactions, and this depends on the level of the price itself.

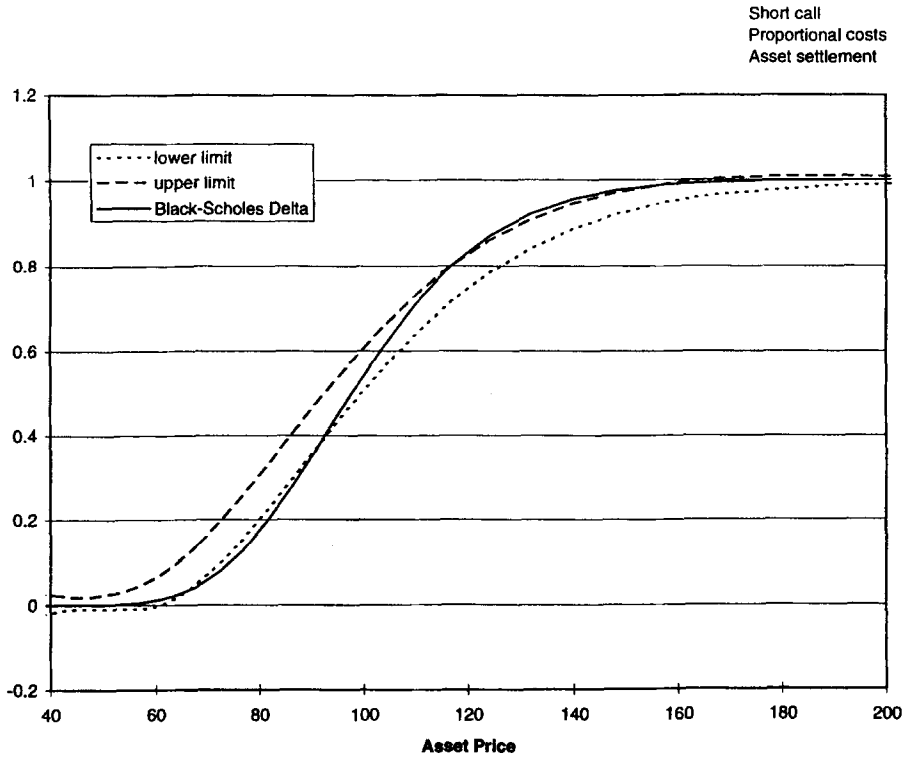


Fig. 2. Optimal strategy control limits and Black-Scholes delta. Short call option, proportional transaction costs, asset settlement: (solid line) Black-Scholes delta; (dashed line) upper limit; (dotted line) lower limit.

Conversely, for the case where an option has been purchased, the positive gamma means we can sell some of the underlying after a price rise, so it is as if the variance were smaller, and hence our hedging region slopes at a steeper angle. In this case the region is wider, which reflects the fact that the convex shaped payoff presents much less risk of large losses. For lower levels of cost or higher levels of risk aversion we should expect the width of the control region to be reduced.

Hedging more complex positions is especially interesting, since with a combination of both long and short positions Leland's method may find it hard to know whether to increase or decrease the variance. The optimal-control approach has no such problems. Conditional on any pre-specified contingent payoffs, which can even occur at differing dates, it gives the best hedge for a given degree of risk aversion. The risk aversion reflects the trade-off to be made between the expected cost of managing the hedge and its variance. We

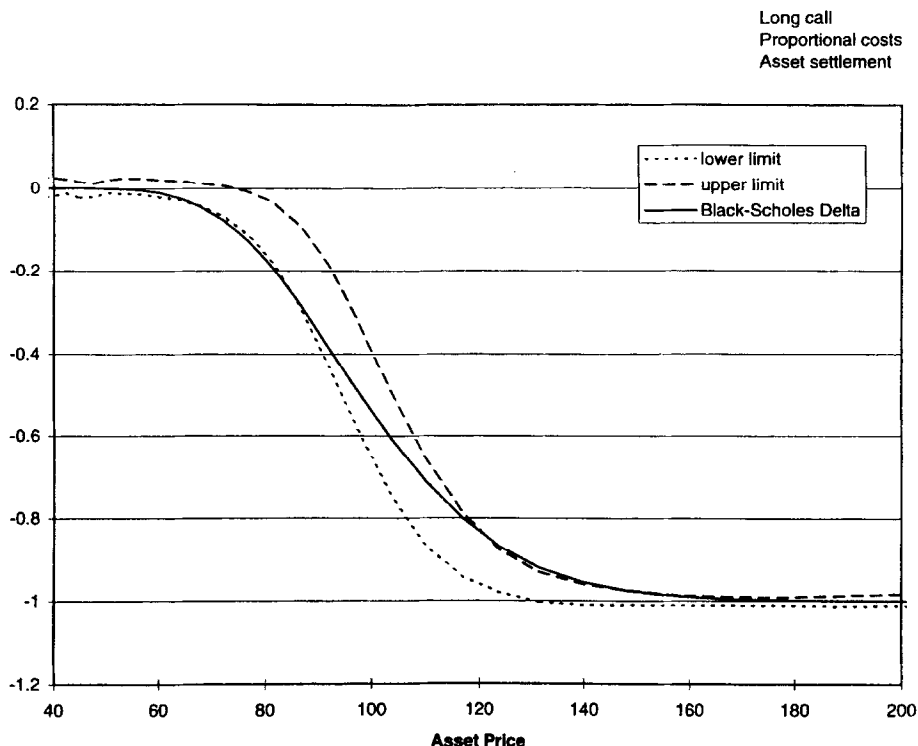


Fig. 3. Optimal strategy control limits and Black-Scholes delta. Long call option, proportional transaction costs, asset settlement: (solid line) Black-Scholes delta; (dashed line) upper limit; (dotted line) lower limit.

have chosen to simulate hedging the payoffs from a bull spread and a butterfly spread in order to illustrate hedging a portfolio with mixed positions. The results are described in the next section.

We next turn to the behaviour of the boundary as a function of time. Close to expiry, the shape of the control boundary depends critically on the assumptions about settlement. Where asset settlement is permitted (Fig. 4) it is not worth paying transaction costs immediately prior to expiry just to avoid minor replication errors. In this case the limits  $x_-$  and  $x_+$  flare outwards near to the expiry date. However, if settlement must be made in cash (Fig. 5), any excess position in the underlying had better be liquidated sooner rather than later. Thus if the asset price is below the exercise price then the option will expire worthless and so the control limits both tend to zero. If the asset price is above the exercise price then the upper limit tends to one (the amount of asset we will need) and the lower limit tends to zero so that we defer buying the asset if we are

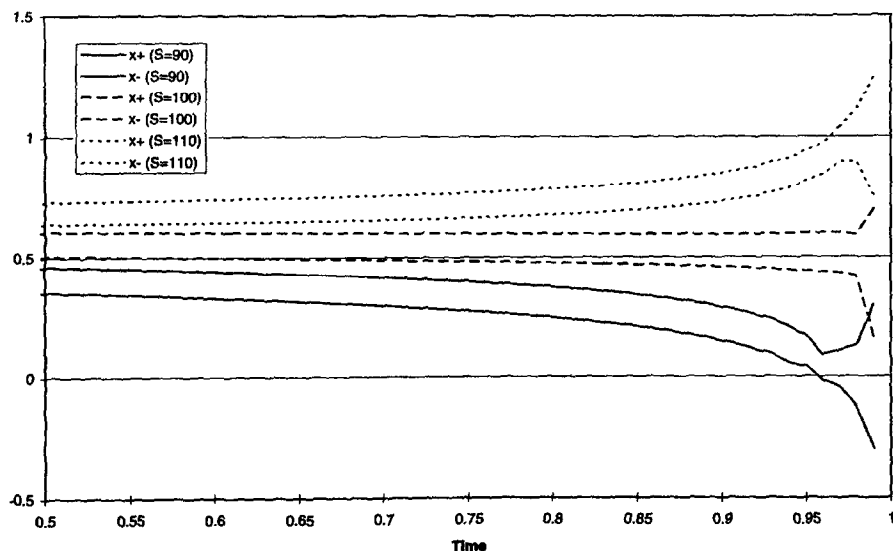


Fig. 4. Optimal strategy control limits for a short call with proportional transaction costs and asset settlement: (solid line) asset price equal to 90; (dashed line) asset price equal to 100; (dotted line) asset price equal to 110.

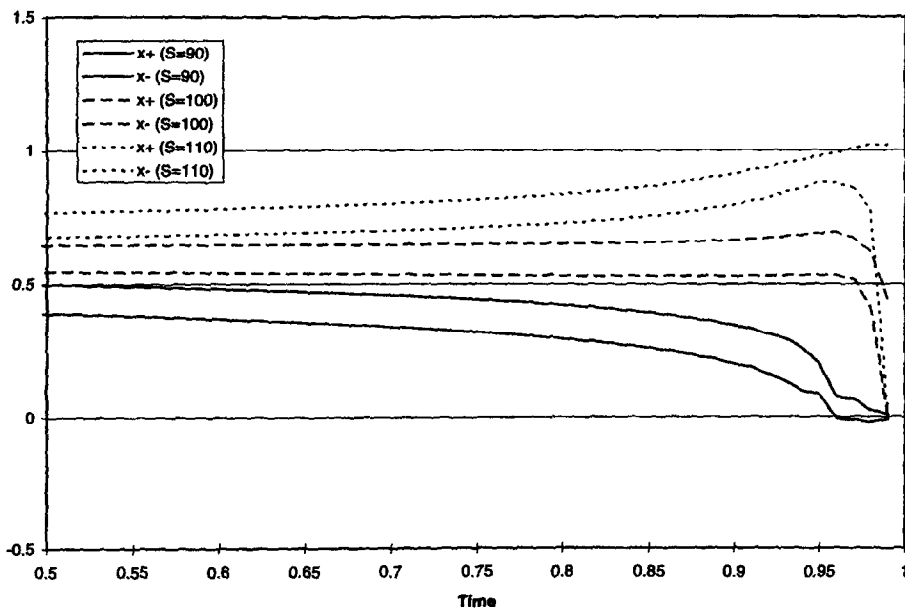


Fig. 5. Optimal strategy control limits for a short call with proportional transaction costs and cash settlement: (solid line) asset price equal to 90; (dashed line) asset price equal to 100; (dotted line) asset price equal to 110.

short. These features are evident from Figs. 4 to 7 which show how the control limits evolve through time for in, at and out of the money values of the asset price.

It is worth noting how the control limits we compute have fairly constant separation once we are back from the immediate pre-expiration transient. They are not obviously related to the gamma of the claim, though that is clearly one aspect of their determinants.

We now turn to the general cost function case. Figs. 6 and 7 show the control limits for a short call with a fixed cost component ( $k_c$ ) of 0.2 and a proportional cost component of 2%, 3 days before expiration of the option. Fig. 6 is for asset settlement and Fig. 7 is for cash settlement. The inner limits are basically the same as for the proportional cost case through time, for example in the cash settlement case (Fig. 7), for out-of-the-money values of the asset price the inner limits tend to zero. The outer limits essentially follow the inner limits but their separation is determined by the relative sizes of the fixed and proportional components of the transaction costs. As the asset price increases the proportional cost increases relative to the fixed cost and so the outer limits tend to the inner limits. Conversely as the asset price decreases the proportional cost decrease relative to the fixed cost and so the outer limits flare out away from the inner limits.

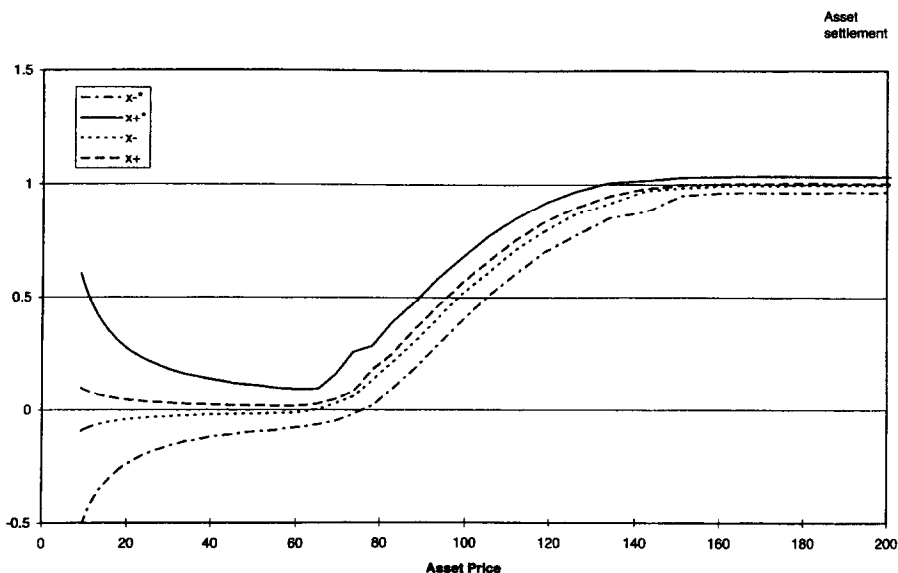


Fig. 6. Optimal strategy control limits for a short call with fixed and proportional transaction costs and asset settlement: (dashed and dotted line) lower outer control limit; (solid line) upper outer control limit; (dotted line) lower inner control limit; (dashed line) upper inner control limit.

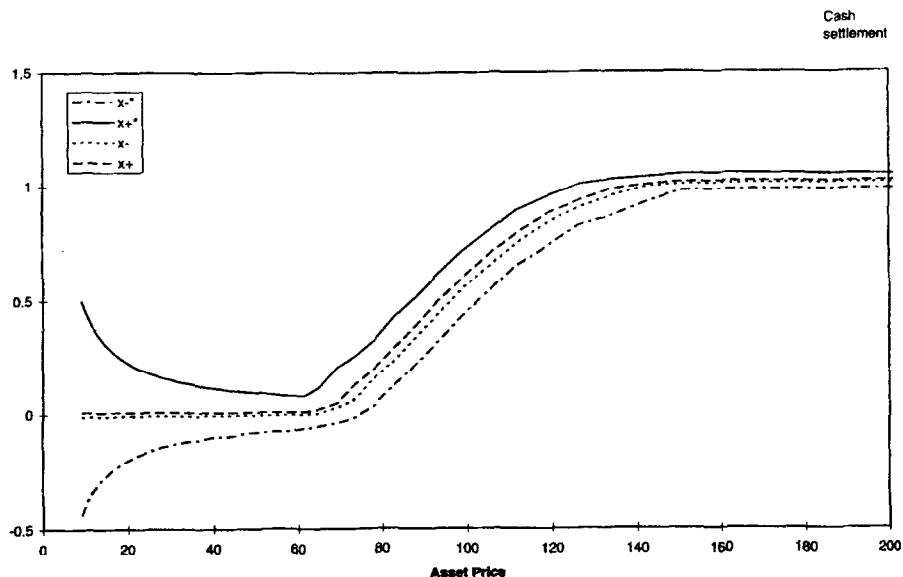


Fig. 7. Optimal strategy control limits for a short call with fixed and proportional transaction costs and cash settlement: (dashed and dotted line) lower outer control limit; (solid line) upper outer control limit; (dotted line) lower inner control limit; (dashed line) upper inner control limit.

## 5. A simulation analysis

In this section we compare the performance of the optimal delta-hedging strategy against other common strategies: Black–Scholes, Leland and a heuristic strategy based on the optimal strategy. For the Black–Scholes and Leland strategies it is necessary to choose a replication interval which reflects the investor's risk aversion. We compute the relevant delta at the start of each interval and adjust the holding in the underlying asset accordingly. The heuristic strategy is designed to highlight the relative importance of the two facets of the optimal strategy: the optimal delta and the width of the control region around this delta. The heuristic strategy therefore centres the optimal control region on the Black–Scholes delta rather than the optimal delta. For the optimal and heuristic strategies we must choose the risk-aversion parameter  $\lambda$ , we then wait until the holding in the underlying asset moves outside the outer control limits at which point we adjust the holding to the nearest inner control limit.

In order to compare the four strategies we require a suitable metric which does not favour a priori any one strategy. For example, if we chose to compare the strategies in terms of expected utility then the optimal strategy would dominate. We simply adopt a mean-variance framework and plot the expected hedging error relative to the Black–Scholes 'fair' value against the standard



deviation of the hedging error. This metric has the advantage of allowing intuitive interpretation of the trade-off of hedging error or cost against hedge standard deviation or effectiveness. The hedging error is defined as follows: we sell or buy the contingent claim for the Black–Scholes (no transactions cost) fair value and use the proceeds to replicate the claim under the transactions costs. At maturity we compute the cash value of the portfolio (the value of the underlying asset held less the borrowing and the liability). This value discounted back to the present is the hedging error of the strategy relative to the no transactions costs case.

Our simulations are based on replication over a year where the minimum revision interval is one day. We simulate replication of selling and buying a European call (with an exercise price of 100), a bull spread<sup>3</sup> (with a lower exercise price of 100 and an upper exercise price of 110) and buying a butterfly spread (with exercise prices at 95, 100 and 105). The price of these options is set, as stated above, at the no transactions costs fair value. For all the simulations the initial underlying asset price is 100, the annualised volatility is 30% and the riskless rate is 0. We set the proportional transactions costs to be 1% of the value of any single trade in the underlying asset.

The simulation proceeds as follows: Firstly, we compute the binomial lattice solution to the optimal strategy. We save a table of the control limits for each binomial lattice value of the underlying asset and each time step (we arrange for the binomial lattice to have the same number of time steps as the simulation for simplicity). The initial hedge portfolios are then set up and we begin simulation of the underlying price path. After each daily time step we check the optimal and heuristic hedges to see if the holding in the underlying asset is outside the outer control limits. This is done by interpolating for the limits at the current underlying price from the saved table. Note that it is possible for the underlying price to move outside the range of asset prices in the table. If this occurs we must recompute the binomial lattice solution. However, by arranging for the binomial lattice to have five nodes rather than one at the current time, we almost never have to recompute the solution<sup>4</sup>. If the holding is outside the outer control limits we rebalance the hedge back to the nearest inner control limit. At each Black–Scholes/Leland replication interval we rebalance these hedges. Finally, at maturity we compute the cost of each strategy and collect the statistics necessary to compute the mean and variance of the costs. This path simulation is repeated 1000 times.

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<sup>3</sup> Note that the Leland strategy is strictly only applicable to globally convex or concave payoff functions. In our implementation of the Leland strategy for the bull and butterfly spread we use the sign of the Black–Scholes gamma to determine the direction in which the volatility is adjusted in an attempt to account for this problem.

<sup>4</sup> For certain asset price processes we could of course quantify this probability.

Figs. 8–12 summarise the results of the simulations. The curves correspond to rebalancing intervals of 1–12 days for the Black–Scholes and Leland strategies and to  $\lambda$  from 0.2 to 10.0 for the optimal and heuristic strategies.

Consider first the replication of a European call (Figs. 8 and 9). For the optimal strategy and the Leland strategy the expected hedging error is strictly monotonically increasing with decreasing standard deviation of the hedging error (increasing risk aversion). But for the Black–Scholes and heuristic strategies this is not the case. As we rebalance more and more frequently the variance of the total cost begins to be dominated by the variance of the transaction costs. As the risk aversion increases the heuristic strategy tends towards the Black–Scholes strategy. This is because the control band becomes very narrow and since it is centred on the Black–Scholes delta we obtain the Black–Scholes strategy in the limit. The heuristic strategy helps us to understand the relationship between the Black–Scholes, Leland and optimal strategies. At low risk aversion the control band is wide and the optimal delta tends towards the Black–Scholes delta. As the risk aversion increases the band becomes narrower and the optimal delta deviates from Black–Scholes.

The curves for the Black–Scholes and Leland strategies exhibit more variability than those for the optimal and heuristic strategies because for low levels of

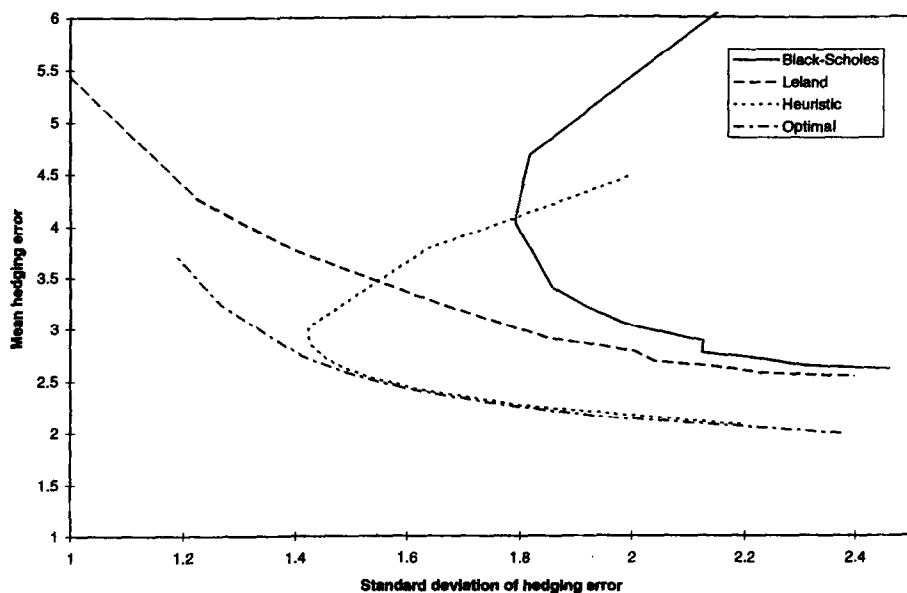


Fig. 8. Comparison of hedging strategies for a short European call: (solid line) is the Black–Scholes strategy with discrete time rebalancing; (dashed line) Leland strategy; (dotted line) heuristic strategy; (dash and dot line) optimal strategy.

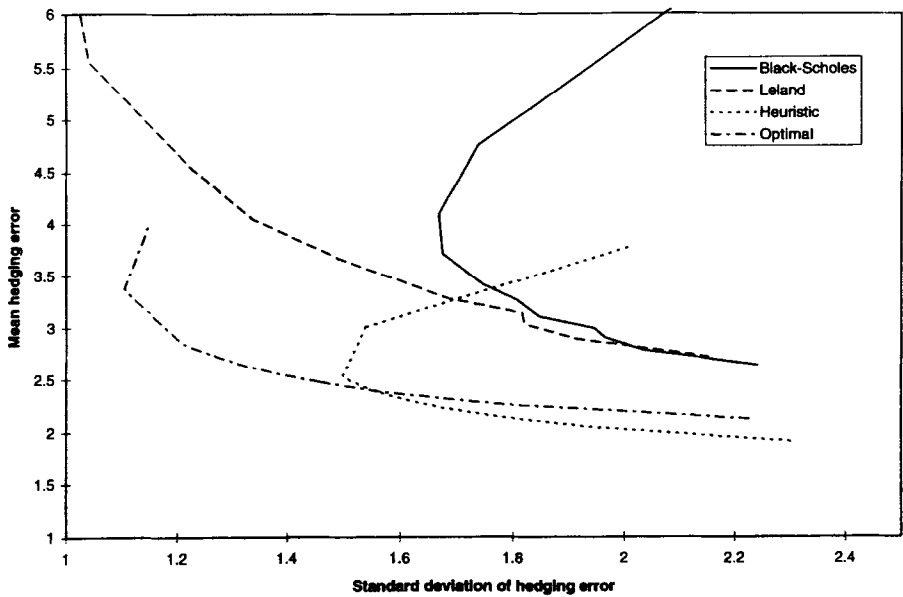


Fig. 9. Comparison of hedging strategies for a long European call: (solid line) is the Black–Scholes strategy with discrete time rebalancing, (dashed line) Leland strategy; (dotted line) heuristic strategy; (dash and dot line) optimal strategy.

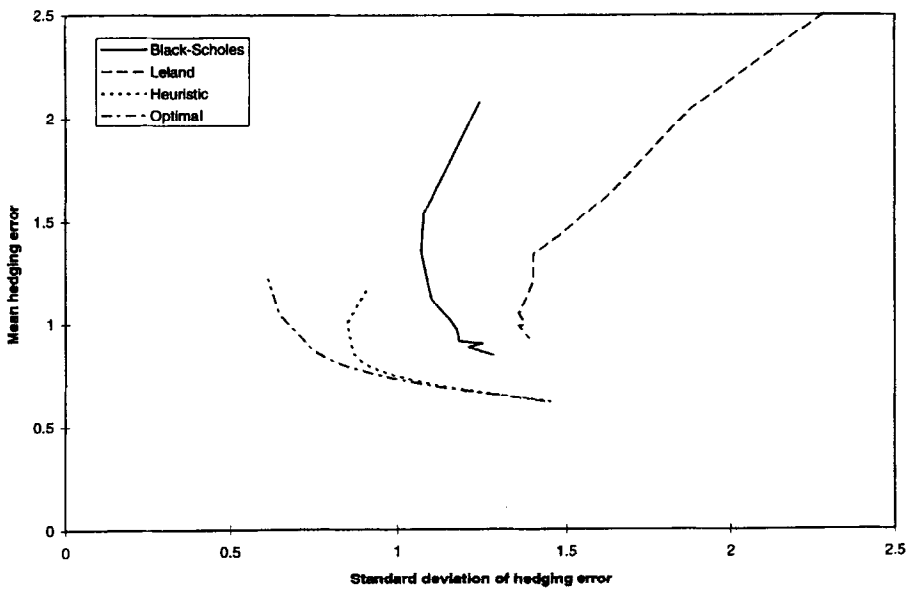


Fig. 10. Comparison of hedging strategies for a short bull spread: (solid line) is the Black–Scholes strategy with discrete time rebalancing, (dashed line) Leland strategy; (dotted line) heuristic strategy; (dash and dot line) optimal strategy.

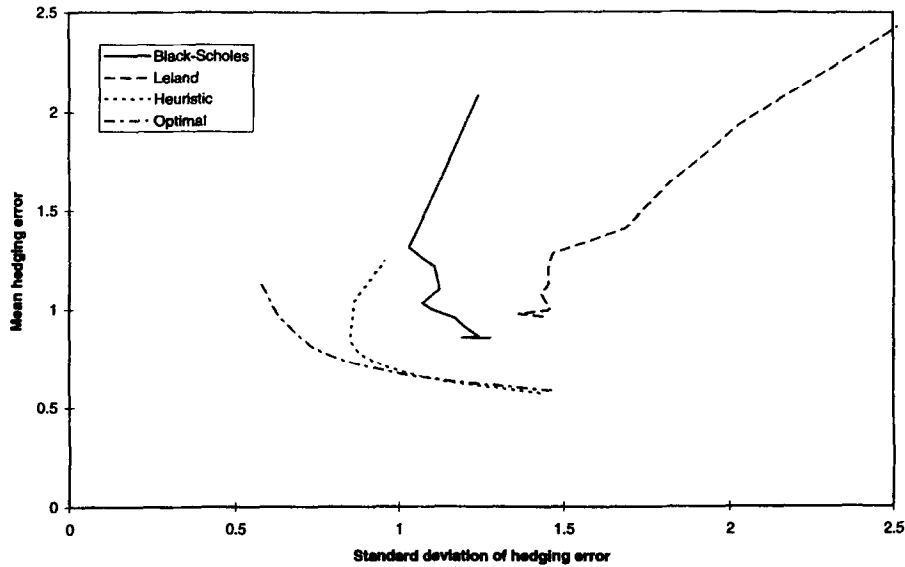


Fig. 11. Comparison of hedging strategies for a long bull spread: (solid line) Black-Scholes strategy with discrete time rebalancing; (dashed line) Leland strategy; (dotted line) heuristic strategy; (dash and dot line) optimal strategy.

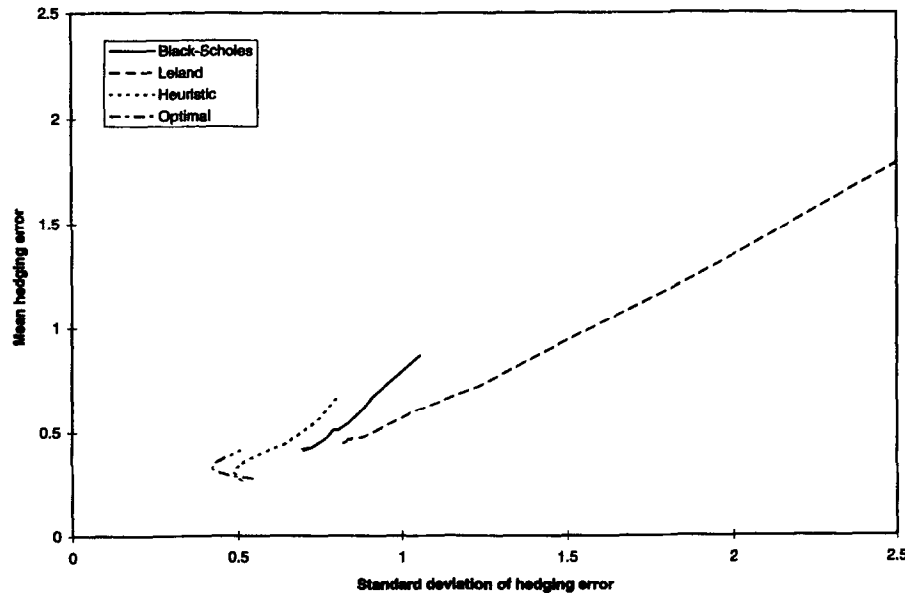


Fig. 12. Comparison of hedging strategies for a long butterfly spread: (solid line) Black-Scholes strategy with discrete time rebalancing; (dashed line) Leland strategy; (dotted line) heuristic strategy; (dash and dot line) optimal strategy.

risk aversion they rebalance very infrequently and the times when they do rebalance can be severely sub-optimal. In contrast, the optimal strategy only rebalances and incurs costs when it is optimal to do so thus making best use of the costs it is incurring.

For the bull spread (Figs. 10 and 11) we obtain the same qualitative results but the details are different. The Leland strategy suffers a similar problem to the Black–Scholes and heuristic strategies for high levels of risk aversion in that the standard deviation is not monotonically decreasing with expected cost. In fact, the Leland strategy now performs worse than Black–Scholes. This is because (as we noted earlier) the Leland strategy is only strictly applicable for globally concave or convex payoff functions where the volatility is adjusted upwards or downwards, respectively.

The expected costs under the optimal strategy are less for long positions because it is able to trade-off the reduced risk of the position with less accurate hedging. This is most striking for the case of the long butterfly position which has very limited risk of loss. The optimal strategy totally dominates the other strategies.

## **6. Summary**

In this paper we have studied the problem of delta-hedging under transactions costs, using the stochastic optimal control approach first described by Hodges and Neuberger (1989). Rather than seeking a strategy for exact replication, which is liable to be expensive and may be dominated by other strategies, this approach obtains the optimal hedging strategy to maximise expected utility (or to minimise a loss function defined on the replication error). Under proportional transactions costs this results in strategies characterised by a control band within which delta must be maintained. With a fixed cost component the strategies consist of outer control limits which trigger rebalancing to the nearest inner limit.

This method has the advantage over Leland's approach in that it works just as well for hedging mixed portfolios of long and short positions, and also mixed maturity dates. The paper describes the solution technique for a general cost function with fixed and proportional components and develops an efficient computational method for the proportional cost case which substantially increases the speed and reduces the storage required for the calculation. The characteristics of the optimal strategies are discussed, and a simulation study is completed to compare the hedging characteristics of some alternative strategies. The strategies we tested were chosen so that we could examine which features of a hedging strategy are most important: hedging to the 'correct' delta or hedging only to within a band in order to conserve transactions costs. The simulations show that the optimal control approach is substantially more effective than

discrete rebalancing strategies such as the standard implementation of Black–Scholes or Leland's method. Furthermore, while the target delta and the band around it are both important, surprisingly good hedges can be obtained (at least for low levels of risk aversion) by hedging using a control region of the right width but based around an incorrect central delta.

## Appendix A

Here we present the derivation of the optimal control problem under our specific choice of negative exponential utility.

Since from (4) and (13):

$$J(t, S, x, y) = E[-\exp\{-\lambda w_T\}] \quad (\text{A.1})$$

and the management of  $x$  through time is independent of  $y$ ,

$$J(t, S, x, y) = J(t, S, x, 0) \exp\{-\lambda y e^{r(T-t)}\}. \quad (\text{A.2})$$

If we define a new indirect utility function

$$H(t, S, x) = J(t, S, x, 0) \quad (\text{A.3})$$

then we may derive the following new equations and boundary conditions for  $H$ , which correspond to our previous Eqs. (8):

$$H_t + rS H_s + \frac{1}{2} \sigma^2(S) H_{ss} = 0, \quad (\text{A.4a})$$

$$H(T, S, x) = -\exp\{-\lambda w_T\}, \quad (\text{A.4b})$$

$$H(t, S, x + u) \geq H(t, S, x) \exp\{-\lambda(uS - k(u, S))e^{r(T-t)}\} \quad (\text{A.4c})$$

In terms of the reservation price this becomes

$$V(t, S, x + u) \geq V(t, S, x) - (uS - k(u, S))e^{r(T-t)}. \quad (\text{A.5})$$

Substituting out general cost function (9) we obtain

$$V(t, S, x + u) \geq V(t, S, x) - (uS - k_c - (k_f + k_p S)u)e^{r(T-t)}. \quad (\text{A.6})$$

For the special case of proportional transactions cost,  $k_c = 0$ , this translates to the linear slope condition

$$V_x = -e^{r(T-t)}(S - (k_f + k_p S)) \quad (\text{A.6a})$$

for  $x = \tilde{x}_-$ ,

$$V_x = -e^{r(T-t)}(S + (k_f + k_p S)) \quad (\text{A.6b})$$

for  $x = \tilde{x}_+$ .

We then obtain the following boundary conditions for  $x_-$  and  $x_+$

$$V(t, S, \tilde{x}_- + u) \geq V(t, S, \tilde{x}_-) - u(S - (k_f + k_p S)) e^{r(T-t)} + k_c e^{r(T-t)} \quad (\text{A.6c})$$

and

$$V(t, S, \tilde{x}_+ + u) \geq V(t, S, \tilde{x}_+) - u(S - (k_f + k_p S)) e^{r(T-t)} + k_c e^{r(T-t)}, \quad (\text{A.6d})$$

so that the outer limits are defined by the values of  $x$  at which, by extending the curve from  $V(t, S, \tilde{x}_\pm)$  with the straight line of slope  $S + (k_f + k_p S)e^{r(T-t)}$ , the difference between the curve and the reservation price is  $k_c e^{r(T-t)}$ .

Our valuation formulae for selling and buying values  $V_S$  and  $V_B$  simplify as follows. As before,  $V_S$  is defined by the equation,

$$J^C(0, S, 0, V_S) = J^0(0, S, 0, 0) \quad (\text{A.7})$$

which now can be expressed as

$$H^C(0, S, 0) \exp\{-\lambda V_S e^{rT}\} = H^0(0, S, 0) = -1,$$

so

$$V_S = \frac{1}{\lambda} e^{-rT} \ln(-H^C). \quad (\text{A.8})$$

Similarly, for the buying price  $V_B$ , we have

$$V_B = -\frac{1}{\lambda} e^{-rT} \ln(-H^{-C}). \quad (\text{A.9})$$

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