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The Crash of '87: Was It Expected? The Evidence from Options Markets

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ABSTRACT

Transactions prices of S&P 500 futures options over 1985–1987 are examined for evidence of expectations ***prior*** to October 1987 of an impending stock market crash. First, it is shown that out-of-the-money puts became unusually expensive during the year preceding the crash. Second, a model is derived for pricing American options on jump-diffusion processes with systematic jump risk. The jump-diffusion parameters implicit in options prices indicate that a crash was expected and that implicit distributions were negatively skewed during October 1986 to August 1987. Both approaches indicate no strong crash fears during the 2 months immediately preceding the crash.

Attempts to explain what caused the stock market crashes around the world in October 1987 have suffered from the paucity of major economic developments occurring around that time that could have triggered the crashes. Shifting expectations regarding monetary policy, foreign investors' fears of a dollar decline, increasing riskiness of assets—none of these appeared major enough, if present at all, to explain the magnitude of the crashes. And although portfolio insurance strategies could conceivably magnify the effects of a jump in fundamentals, the initiating jump in fundamentals appears to be lacking.

This paper examines the alternative hypothesis that the U.S. stock market crashed because it was expected to crash. It is conceivable that the crash was a self-fulfilling prophesy—a "rational bubble." *Ex post*, the behavior of the stock market strongly resembles a bubble. A dramatic 42% gain in the stock market over most of 1987 was reversed by an even more dramatic 23% decline on October 19 and 20 (see Figure 1), leaving the stock market at year-end essentially unchanged from its level in January. The Blanchard and Watson (1982) model of a rational bubble, with an explosive divergence away from the fundamentals-determined level that is sustained by an expected jump return to that level, appears consistent with the actual behavior of the stock market.

One must of course be wary of explanations offered after the fact. Eyeballing the behavior of the S&P 500 index is an inadequate test of the rational bubble hypothesis, conditioned as it is upon the knowledge that a

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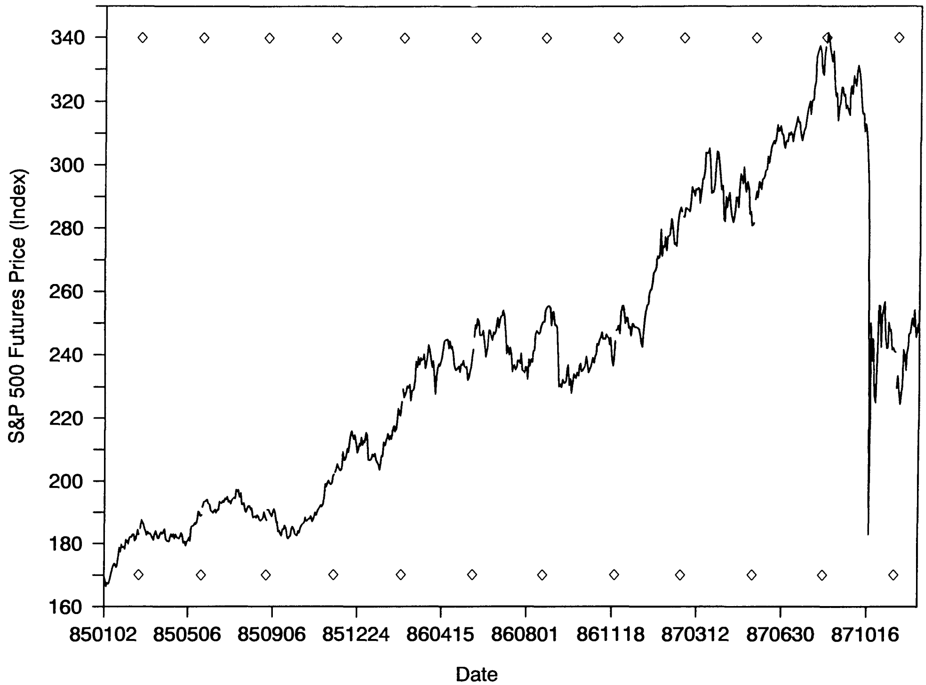


Figure 1. S&P 500 futures prices, 1985–1987. Futures contracts with 1 to 4 months maturity, noon quotes. Diamonds (◇) indicate maturity shifts.

crash actually occurred. A more relevant test is whether there were reliable harbingers foretelling the crash, since it is the expected crash that sustains the explosive bubble. One would expect, for instance, pronounced unanimity that the market is "overvalued." In Shiller's (1989) survey taken immediately after the crash, most individual and institutional investors said that they had thought prior to the crash that the market was overpriced. But, as Shiller acknowledges, the survey results may be biased by 20-20 hindsight.

This paper takes the alternative approach of using options prices to examine whether a crash was expected. Options prices offer direct insights into the climate of expectations prior to the crash. Given that call options will pay off only when the underlying asset prices is in excess of the exercise price (i.e., the call finishes "in the money") and puts only when the reverse is true, the instantaneous set of call and put option prices across all exercise prices gives a very direct indication of market participants' aggregate subjective distributions. For instance, an assessed risk of a market crash will lead to put options on S&P 500 futures with exercise prices well below the current futures price ("out-of-the-money" puts) being priced higher than calls with exercise prices well *above* the futures price (out-of-the-money calls); the chance of large downward movements in the market makes the puts more likely to finish in the money than the calls. Out-of-the-money (OTM) put options on S&P 500 futures were in fact unusually expensive relative to OTM calls during the year preceding the crash, especially during the periods October 1986–February 1987 and June–August 1987. Ironically, relative OTM put prices subsided when the market peaked in August 1987 and were back to "normal" levels during the 2 months immediately preceding the crash.

The theoretical foundations and the empirical evidence for the assertion that OTM puts were "unusually" expensive are given in Section I. Earlier theoretical work is summarized concerning the relationship between OTM call and put option prices for standard distributional hypotheses:[1] constant elasticity of variance processes (including arithmetic and geometric Brownian motion); stochastic volatility processes; and jump-diffusions. Most of the standard specifications of these models (e.g., the continuous-time versions of ARCH and GARCH stochastic volatility models; jump-diffusions with mean-zero jumps) can be ruled out *a priori* as inconsistent with observed option prices. They imply that OTM puts should trade at a slight discount relative to OTM calls, contrary to fact. Within these classes of distributional hypotheses, therefore, transactions prices for American options of S&P 500 futures indicate either that market participants expected substantial *negative* jumps (i.e., crashes) in the market during the year preceding the crash or that they thought market volatility would rise rapidly if the market fell.

Section II examines the former hypothesis. An option pricing model for American options on jump-diffusion processes when jump risk is systematic and nondiversifiable is derived under the hypothesis of time-separable power utility. The parameters of the "risk-neutral" process *implicit* in transactions prices of call and put options on S&P 500 futures for a given day are estimated via nonlinear least squares:

1) the volatility conditional on no jumps,
2) the probability of a jump,
3) the mean jump size (positive or negative) conditional on a jump occurring, and
4) the standard deviation of jump sizes conditional on a jump occurring.

The estimation is repeated for all days over 1985–1987, yielding a chronology of crash fears. Using the restrictions on preferences, estimates of the market's perception of actual jump-diffusion parameters can be derived from the estimates of implicit parameters. Option prices turn out to be insensitive to the specification of preferences, so the two sets of parameters are virtually identical.

The resulting parameter estimates indicate that negative jumps began to be expected starting in October 1986, with the distributions implicit in options prices especially negatively skewed during October 1986–February 1987 and June–August 1987. The implicit crash fears subsided as the market peaked in August 1987, only to resurge after the stock market crash actually

[1]See Bates (1988a).

occurred. The evidence from options markets is, therefore, that the rise in the market until August 1987 may have been a bubble, but that if so it burst in August—not on October 19.

I. Measures of Asymmetry Under Standard Distributional Hypotheses

A. Theoretical Foundations

Fundamental to the pricing of European and American options is the derivation from the *actual* distribution of the asset price of an *equivalent* "risk-neutral" distribution that summarizes the prices of relevant Arrow-Debreu state-contingent claims. Options are then priced at the discounted expected value of their future payoffs, using this risk-neutral distribution. For processes such as geometric Brownian motion and constant elasticity of variance for which options are redundant assets that can be replicated by a dynamic trading strategy in the underlying asset and a riskless bond, the equivalent risk-neutral distribution can be derived via no-arbitage conditions. For more complicated processes such as stochastic volatility and jump-diffusion processes, such replication is not feasible. Deriving the appropriate risk-neutral probability measure in those cases requires pricing volatility risk or jump risk, which in turn typically requires additional restrictions on distributions and/or on preferences. The two standard approaches are: 1) assume the additional risk is nonsystematic and therefore has price zero; or 2) assume the representative investor has time-separable power utility, and preferably log utility, so that Cox, Ingersoll, and Ross (1985b) separability results can be invoked to price the additional risk.[2]

Once the risk-neutral distribution has been derived, pricing European call and put options is relatively straightforward. The terminal payoff of a European call option maturing T periods from now given terminal asset price realization S_{t+T} and strike price X_c is $\max(S_{t+T} - X_c, 0)$. Under the standard assumption that the short-term interest rate r is constant over the lifetime of the option, the price of the call conditional upon a current underlying asset price of S_t will be

$$\begin{aligned} c(S_t, T; X_c) &= e^{-rT} E_t^* \max(S_{t+T} - X_c, 0) \\ &= e^{-rT} E_t^* [S_{t+T} - X_c \mid S_{t+T} \geq X_c] \operatorname{Prob}^* [S_{t+T} \geq X_c], \quad (1) \end{aligned}$$

the discounted expected payoff conditional upon finishing "in the money" times the probability of finishing in the money. Expectations and probabilities are calculated using the risk-neutral distribution, conditional on all information currently available at time t. Similarly, the terminal payoff of a

[2] Examples of the former approach include Hull and White (1987), Johnson and Shanno (1987), and Scott (1987) for pricing options under stochastic volatility and Merton (1976a) for pricing options under jump-diffusions. Examples of the latter include Wiggins (1987) and Melino and Turnbull (1990) for stochastic volatility and Bates (1988b), Naik and Lee (1990), and the discussion below for jump-diffusions.

European put with strike price X_p that matures T periods from now is $\max(X_p - S_{t+T}, 0)$; its current price will be

$$p(S_t, T; X_p) = e^{-rT}E_t^* \max[X_p - S_{t+T}, 0]$$
$$= e^{-rT}E_t^*[X_p - S_{t+T} \mid S_{t+T} \le X_p]\,\text{Prob}^*[S_{t+T} \le X_p]. \quad (2)$$

Under the risk-neutral distribution, $E_t^*(S_{t+T}) = F_{t,t+T}$, the current forward price on the asset for delivery T periods from now. By arbitrage, $F_{t,t+T} = S_t e^{bT}$, where b is the proportional "cost of carry" to maintaining a position in the underlying asset. For nondividend paying stocks, $b = r$, the opportunity cost of not holding the risk-free asset. For foreign exchange, $b = r - r^*$, the domestic/foreign interest differential. An important institutional feature of futures contracts is that the cost of carry b is zero. The margin requirements for large investors of taking futures positions can be met partly by posting Treasury bills, partly by posting cash margins (on which brokers pay money market rates), so there is no opportunity cost to such positions.

A key and somewhat obvious insight from equations (1) and (2) is that out-of-the-money (OTM) European calls reflect conditions in the upper tail of the risk-neutral distribution, while OTM European puts reflect conditions in the lower tail. If the strike prices of the put and call are spaced symmetrically around $E^*(S_{t+T}) = F$ (see Figure 2), the symmetry or asymmetry of the risk-neutral distribution will be directly reflected in the relative prices of these out-of-the-money calls and puts. Symmetric risk-neutral distributions imply equal prices for OTM European calls and puts; skewed distributions create systematic divergences.

Consequently, one could use the observed relative prices of European calls and puts where such exist to judge whether these are consistent with the skewness of the risk-neutral distribution derived from any specific distributional hypothesis—an exercise roughly comparable to looking at "moneyness biases." Unfortunately, the major exchange-traded options on market indexes are *American* options, which can be exercised at any time up until maturity. The feasibility of early exercise obscures the influence of the risk-neutral distribution's skewness upon relative prices of out-of-the-money calls and puts. The problem is that the optimality of early exercise, which determines the "early-exercise premium" markup of American over European option prices, depends primarily upon the cash flows of the underlying asset and only secondarily upon the skewness of its distribution. The influence of the cash flows upon early exercise is captured in the cost of carry parameter b. When b is significantly positive, American puts have a greater probability of early exercise than American calls and therefore will command a greater markup over European prices.[3] The reverse is true for negative

[3]An extreme but well-known example is the case of options that do not pay dividends ($b = r$): it will never be optimal to exercise an American call early on such stocks regardless of what the distribution is, whereas it may be optimal to exercise an American put early at some point in the future.

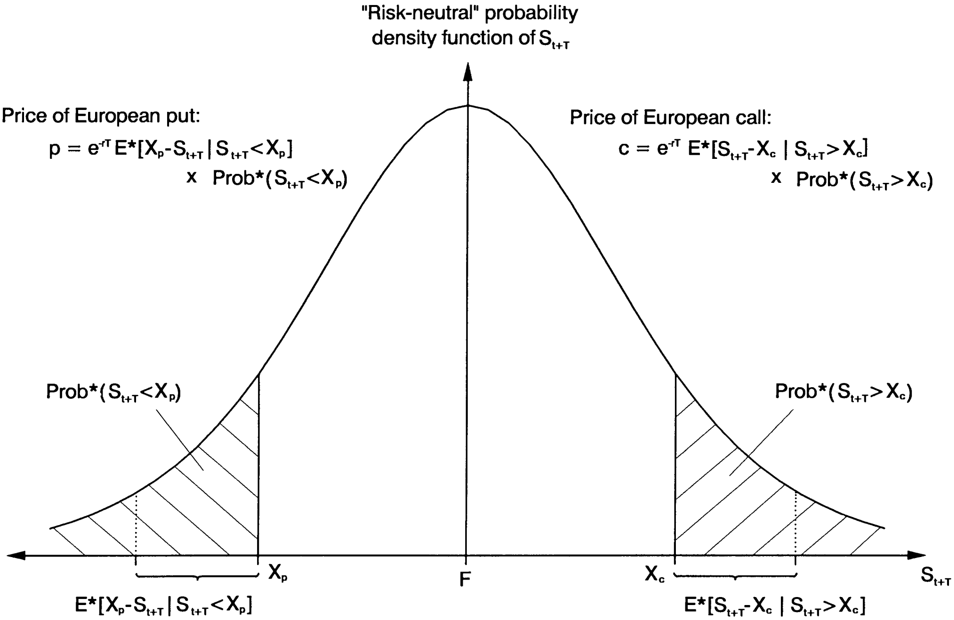


Figure 2. Determination of European call and put prices. European call and put prices are determined by the relevant discounted conditional expectation times the tail probability, using the "risk-neutral" probability measure. A symmetric distribution implies call and put options symmetrically out-of-the-money (OTM) will be priced identically. Skewed distributions induce systematic divergences between OTM call and put prices.

values of b.[4] Consequently, judging the merits of any distributional hypothesis generally requires explicit computation of American option prices and comparing actual and model prices—a laborious process that has been done extensively for geometric Brownian motion but hardly at all for other distributions.[5]

In the special case of American options on futures contracts, however, the fact that the cost of carry is zero creates a knife-edge case in which the symmetry or asymmetry of the risk-neutral distribution is mirrored in the symmetry or asymmetry of the early-exercise decision for calls and puts and also in the early-exercise premia. For these options, relative prices of out-of-the-money calls and puts *can* be used as a quick diagnostic of the symmetry

[4]Shastri and Tandon (1986) examine the difference between American and European call and put prices under geometric Brownian motion for foreign currency options, for which $b = r - r^*$ can be positive or negative.

[5]Whaley (1986) looked at the American options on S&P 500 futures considered in this paper. He found that during 1983, an American option pricing model predicated on geometric Brownian motion underpriced in-the-money calls and out-of-the-money puts relative to observed market prices while overpricing out-of-the-money calls and in-the-money puts.

or skewness of the risk-neutral distribution, and thereby as a diagnostic of the merits of the underlying distributional hypothesis. I call this diagnostic a "skewness premium."

Definition: The *x% skewness premium* is defined as the percentage deviation of $x\%$ out-of-the-money call prices from $x\%$ out-of-the-money put prices:

$$SK(x) \equiv c(S, T; X_c)/p(S, T; X_p) - 1 \text{ for European options in general,} \tag{3a}$$

$$SK(x) \equiv C(F, T; X_c)/P(F, T; X_p) - 1 \text{ for American futures options,} \tag{3b}$$

where

$$X_p = F/(1 + x) < F < X_c = F(1 + x),\ x > 0,$$

and F is the forward price on the underlying asset when options are European, or the underlying futures price for American futures options.[6]

Intuitively from Figure 2, the skewness premium should be directly related to the skewness of the risk-neutral distribution.[7] The deviation between OTM call and put prices, as measured by the skewness premium, was examined in earlier work for three major classes of stochastic processes:[8]

1) *constant elasticity of variance (CEV) processes*,[9] special cases of which include arithmetic and geometric Brownian motion;
2) *stochastic volatility processes*,[10] the benchmark models being those for which volatility evolves independently of the asset price,[11] and
3) *jump-diffusion processes*, the benchmark model being Merton's (1976a,b) specification of log-symmetric jumps with zero mean.[12]

The relationships for these classes between the *actual* stochastic process, the "risk-neutral" stochastic processes used in pricing options, and the

[6]Cox, Ingersoll, and Ross (1981) noted that interest rate volatility can cause the forward and futures prices to diverge. Cornell and Reinganum (1981), however, found the divergences empirically negligible.

[7]Note that the strike prices of the out-of-the-money put and call are spaced *geometrically* around F, rather than arithmetically. This turns out to be convenient given the prevalence of postulated "log-symmetric" distributions such as geometric Brownian motion. For the small values of out-of-the-money parameter x typically observed in options markets, however, the difference between arithmetic and geometric spacing is small.

[8]See Bates (1988a).

[9]See Cox and Ross (1976) and Cox and Rubinstein (1985).

[10]Major papers on pricing options under stochastic volatility include Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Wiggens (1987), and Melino and Turnbull (1990).

[11]Nelson (1990) has shown that the continuous-time limit of standard ARCH and GARCH models is a stochastic volatility model with increments to volatility *independent* of those to the asset price. Within the option pricing papers cited above, the independent increments special case is the most tractable and, consequently, the most studied. Nelson's (1991) exponential ARCH model allows nonzero correlation between asset price and volatility innovations.

[12]The Merton (1976a) model actually allows a broader specification of the distribution of jump amplitudes. However, implementation of the model (Merton (1976b) and Ball and Torous (1983, 1985)) has focussed upon this special case.

skewness premium are given in Table I. Most of the processes imply that the risk-neutral distribution is roughly symmetric and slightly positively skewed. The benchmark stochastic volatility and jump-diffusion models, for instance, generate "fat-tailed" distributions that are essentially symmetrically fat-tailed. As a result, call options $x\%$ out-of-the-money should trade at a slight 0% to $x\%$ *markup* over the correspondingly out-of-the-money puts, if any of the standard distributional hypotheses is correct. To get OTM call and put prices deviating by more than this narrow range within these classes of distributional hypotheses requires nonstandard parameter values.

PROPOSITION 1: (from Bates (1988a)): *For European options in general and for American options on futures, the skewness premium has the following properties for the distributions listed above regardless of the maturity of the options*:

1) $0\% \le SK(x) \le x\%$ *for*
   - *i) arithmetic and geometric Brownian motion*
   - *ii) "standard" CEV processes*
   - *iii) benchmark stochastic volatility and jump-diffusion processes*
2) $SK(x) < 0\%$ *only if*
   - *i) volatility of returns increases as the market falls,*[13] *or*
   - *ii) negative jumps are expected under the risk-neutral distribution*
3) $jSK(x) > x\%$ *if and only if*
   - *i) volatility of returns increases as the market rises, or*
   - *ii) positive jumps are expected under the risk-neutral distribution.*

In consequence, prices of American options on S&P 500 futures can be used quite directly to assess the skewness of the implicit distribution, and thereby to judge the merits of alternative distributional hypotheses.

B. Empirical Evidence

Transactions data for call and put options on S&P 500 futures, as well as transactions data for the underlying futures contracts, were obtained from the Chicago Mercantile Exchange for the years 1985–1987. The data, known as the "Quote Capture" report, consist of the time and price of every

[13] CEV processes with $\rho < 0$, or stochastic volatility processes with $\mathrm{Cov}(dS/S, d\sigma) < 0$. Time series studies of individual stocks and of the market (e.g., Christie (1982) and Gibbons and Jacklin (1988)) have found that volatility of returns dS/S does rise as prices fall—an observation that appears to hold across most categories of stocks. Explicit estimates by Gibbons and Jacklin over 1962–1985 (and subsamples thereof) found CEV parameter values for individual stocks almost invariably between 0 and 1. Based on historical data, therefore, stochastic volatility models suggest a skewness premium between 0% and $x\%$ should be observed. Of course, it is quite possible that historical estimates are badly out of date.

Table 1

Relative Prices of OTM Calls and Puts (SK) under Standard Distributional Hypotheses

Actual Stochastic Process	"Risk-neutral" Process[a]	Skewness Premium[b]
1. CEV processes		
$dS = \mu S\,dt + \sigma S^{\rho}\,dZ$	$dS = bS\,dt + \sigma S^{\rho}\,dZ$	$SK < 0$ only if $\rho < 0$ $0 < SK < x\%$ for $0 < \rho < 1$ $SK > x\%$ for $\rho > 1$
Special cases $\rho = 0$: arithmetic Brownian motion $\rho = 1$: geometric Brownian motion $0 < \rho < 1$: standard leverage effects		
2. Stochastic volatility processes		
$\begin{cases} dS/S = \mu\,dt + \sigma_t\,dZ \\ d\sigma_t/ = \alpha(\sigma_t)\,dt + \upsilon(\sigma_t)\,dZ_\sigma \end{cases}$	$\begin{cases} dS/S = b\,dt + \sigma_t\,dZ \\ d\sigma_t = [\alpha(\sigma_t) + \Phi_\sigma]\,dt + \upsilon(\sigma_t)\,dZ_\sigma \end{cases}$	
$\mathrm{Cov}(dZ, dZ_\sigma) = \rho_{s\sigma}\,dt$	$\mathrm{Cov}(dZ, dZ_\sigma) = \rho_{s\sigma}\,dt$ $\Phi_\sigma \equiv \mathrm{Cov}(dJ_w / J_w, d\sigma)$	$SK = x\%$ for $\rho_{s\sigma} = 0$
	Standard Assumptions[c] a) nonsystematic volatility risk ($\Rightarrow \Phi_\sigma = 0$); ***or*** b) isoelastic utility, CRTS technologies ($\Rightarrow \Phi_\sigma = \Phi_\sigma(\sigma)$)	$SK \gtreqless x\%$ as $\rho_{s\sigma} \gtreqless 0$

Table 1—*Continued*

Actual Stochastic Process	"Risk-neutral" Process[a]	Skewness Premium[b]
3. Jump-diffusion processes		
$dS/S = (\mu - \lambda\bar{k})\,dt + \sigma\,dZ + k\,dq$	$dS/S = (b - \lambda^*\bar{k}^*)\,dt + \sigma\,dZ + k^*\,dq^*$	
Prob$(dq = 1) = \lambda\,dt$	Prob$(dq^* = 1) = \lambda^*\,dt$	
$1 + k$ log-normally distributed	$\lambda^* = \lambda E[1 + \Delta J_w / J_w]$	$SK = x\%$ for $\bar{k}^* = 0$
	$\bar{k}^* = \bar{k} + \mathrm{Cov}(k, \Delta J_w / J_w)/E(1 + \Delta J_w / J_w)$	$SK \gtreqless x\%$ as $\bar{k}^* \gtreqless 0$
	Standard Assumptions[c]	
	a) nonsystematic jump risk	
	$(\Rightarrow \lambda^* = \lambda,\ \bar{k}^* = \bar{k})$; ***or***	
	b) isoelastic utility, CRTS technologies	
	$(\Rightarrow 1 + k^*$ log-normal; $\lambda^*, \bar{k}^*$ constant)	

Notation:

Z and Z_σ are Wiener processes. J_w is the indirect marginal utility of optimally invested wealth for the representative investor. ΔJ_w is the jump-contingent random change in J_w.

[a] For derivation of the "risk-neutral" processes from the actual process, see the references cited above.

[b] SK is defined as the percentage deviation of call prices from put prices for options $x\%$ out-of-the-money relative to the forward (or futures) price for European options in general (b unconstrained) and for American options on futures $(b = 0)$.

[c] This highlights only a few of the assumptions typically imposed. Additional restrictions are also required, including frictionless markets, restrictions on interest rate processes, etc.

transaction in which the price changed from the previous transaction. In addition, bid and ask prices are also recorded if the bid price is above or the ask price is below the price of the previous transaction. There is no information given regarding the volume of transactions at a particular price.

Options on S&P 500 futures over the period were available for March, June, September, and December expiration dates. Intraquarterly options were introduced in 1987; those were ignored in this study. The last trading date for the options was initially the third Friday of the month, the expiration date of the underlying futures contracts, but was subsequently changed to the day before in the second quarter of 1986 because of "triple witching hour" problems. The options are on a cash settlement basis, with the underlying contract being $500 times the S&P 500 index. Contracts are available for exercise prices at 5-point intervals ranging in- and out-of-the-money relative to the current level of the futures price on the S&P 500 index, plus the contracts opened previously. The set of options contracts available for a given maturity therefore depends upon the past movements of the stock market during the history of that maturity of option. Option maturities range up to 9 months, but the options most actively traded are those with maturities under five months. Roughly two-thirds of the transactions during 1985–1987 were in calls, one-third in puts.

Three exclusionary restrictions were applied to the data. First, only contracts of a single maturity were considered for any day: namely, contracts with maturities between 1 and 4 months (28–118 days). Longer maturities were too thinly traded, and shorter maturities were too near maturity to contain much information about implicit distributions. Second, to avoid days with thin trading, at least 20 transactions in calls and at least 20 in puts were required for a day's data to be retained. Finally, transactions in at least 4 strike classes for calls and 4 for puts were required, to ensure a "moneyness" range sufficient to distinguish amongst alternative distributional hypotheses. Only 4 days out of 735 were eliminated on the basis of these latter two restrictions. The resulting data set contains from 100 to 1800 quotes per day for calls and puts of all strike classes, with an average of about 400 per day. The futures price underlying the option price was taken to be the nearest preceding transactions price in the futures market. Lapsed time between the futures transaction and the options transaction averaged 5 1/2 seconds.

Since options exist only for specific exercise prices, the skewness premium measure of asymmetry cannot be implemented directly. For each OTM call with exercise price $x\%$ above the futures price, there will not in general exist a corresponding OTM put with exercise price exactly $x\%$ below the futures price. However, theoretical distributions and no-arbitrage conditions imply that options prices are continuous, monotone, and convex functions of the exercise price. Options prices for desired exercise prices were therefore interpolated from a constrained cubic spline fitted through the *ratio* of options prices to futures prices, as a function of the exercise price/futures price ratio X/F.[14] An example of the fits for a typical day is given in Figure 3. The fits were excellent prior to the crash (see Figure 4), with typical standard errors for options prices of about 0.04% of the futures price—roughly 2 price ticks. Standard errors were enormous after the crash, indicating a lack of consensus about appropriate options prices.

The resulting skewness premia from January 2, 1985 to December 31, 1987 for options at-the-money and 4% out-of-the-money are given in Figure 5. The at-the-money graph is another measure of the accuracy of interpolation. For the theoretical distributions listed above, at-the-money calls and puts should

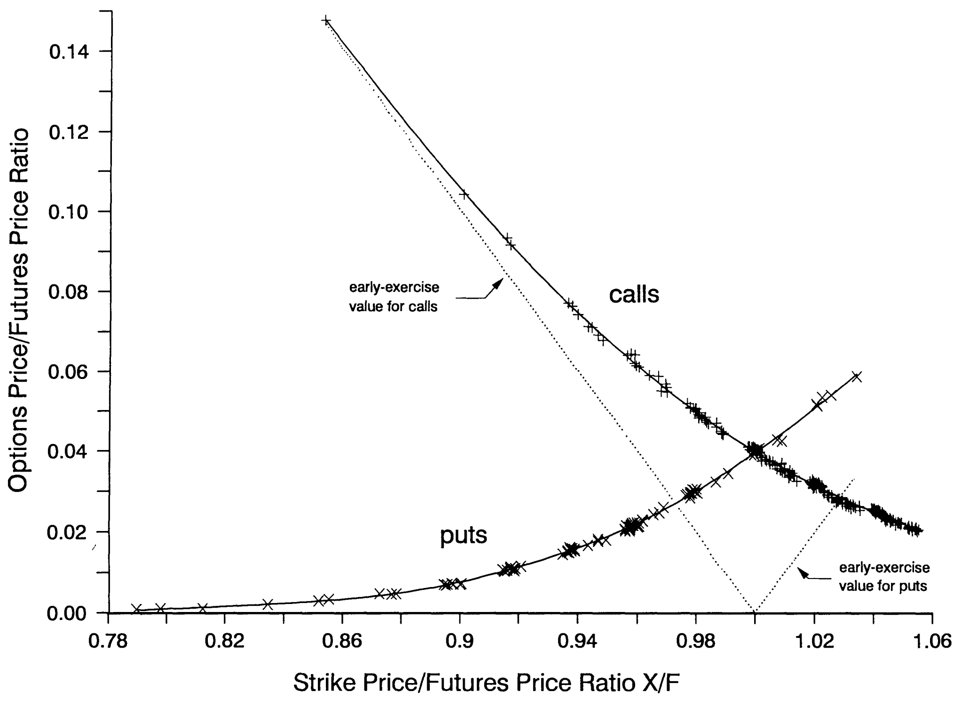


Figure 3. March 21, 1986 prices of calls and puts on S&P 500 futures: transactions prices, constrained cubic splines and early-exercise values, as a fraction of the underlying futures price. Cubic splines were fitted through the transactions data subject to convexity, monotonicity, and level constraints.

[14] The breakpoints for the cubic polynomials were the average X/F ratio for each strike class, with the deepest in- and out-of-the-money strike classes excluded. Thus, for $N+1$ strike classes there were $N-1$ breakpoints and N cubic polynomials. $N+5$ linear Kuhn-Tucker constraints on the $4N$ cubic coefficients sufficed to ensure that the cubic spline was 1) convex, 2) monotone with slope (in absolute value) between 0 and 1, and 3) greater than or equal to the immediate-exercise value. The constrained cubic spline that minimized root mean squared error was estimated using the constrained optimization subroutine CONOPT in GQOPT. Calls and puts had separate splines. For 62% of the splines, no constraint was binding.

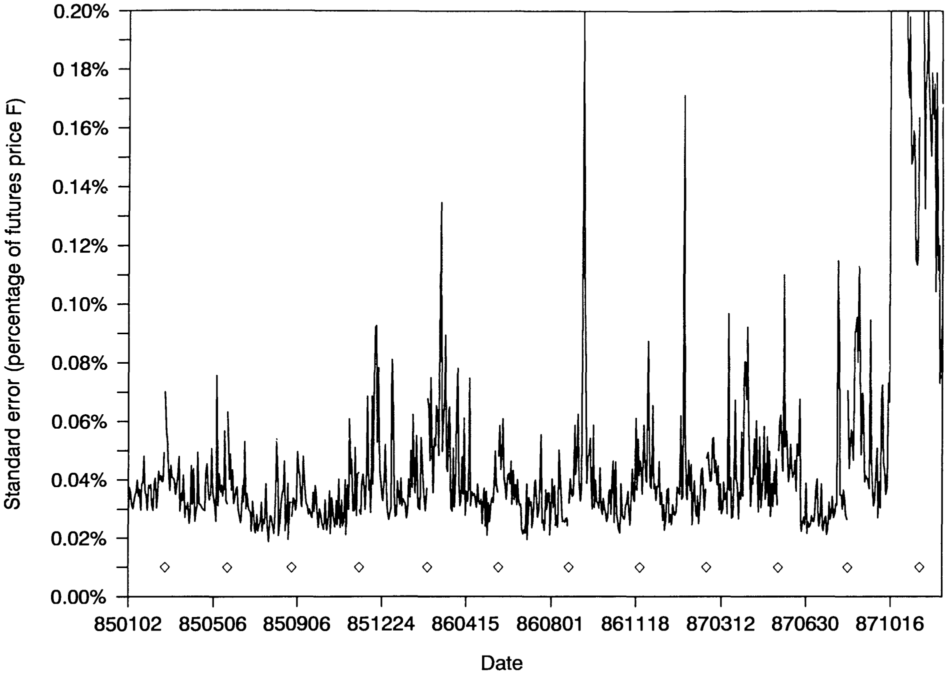


Figure 4. Standard errors from fitted cubic splines, 1985-1987. Graph shows the daily overall root mean squared error from constrained cubic splines fitted separately to call data and to put data, adjusted for foregone degrees of freedom. Standard errors are in percentages of the futures price F. With a futures price around 250, 0.04% of F represents a standard error of about 2 price ticks.

be priced identically,[15] yielding a skewness premium value of 0%—which is in fact observed, except after the crash in October 1987.

The 4% skewness premium shown in Figure 5 indicates fairly gradual shifts over time in implicit assessments of skewness. In 1985, the premium was typically in excess of the 4% benchmark, suggesting an assessment of considerable upside potential. Over most of 1986 the premium was roughly in the 0-4% range of standard distributional hypotheses. Starting late in 1986, however, strong assessments of downside risk (negative skewness premia) began emerging, growing especially pronounced during the periods October 1986-February 1987 and June-August 1987. The 6% plunge in the stock market on September 11-12, 1986 was presumably a major contributor to this perception of downside risk, although it was not until a month later that the skewness premium became markedly negative. In the first week of August, 1987, 4% OTM puts were about *25% more expensive* than corre-

[15]Unlike for European options, there are no arbitrage-based restrictions equating prices of at-the-money *American* calls and puts—only inequality restrictions limiting how far they can deviate.

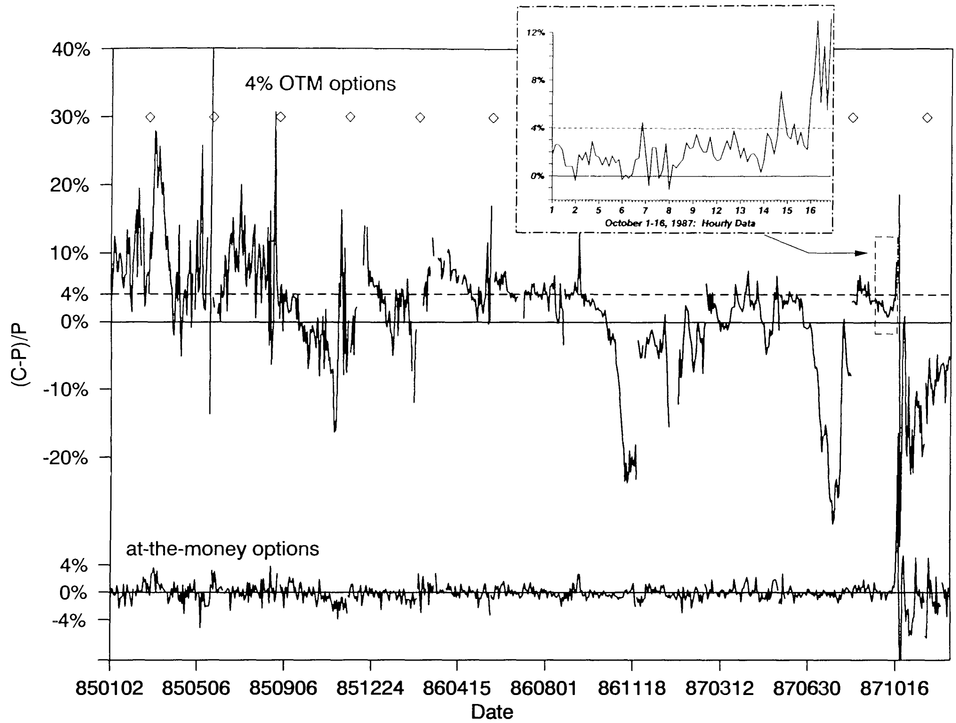


Figure 5. Percentage deviation of call from put prices (skewness premia) for options at-the-money and 4% out-of-the-money, 1985–1987. The small deviations from the theoretical value of 0% shown in the ATM graph prior to October 19, 1987 suggest that the interpolation method used is reliable over this period. The 4% skewness premium calculated from prices of 4% OTM options is a measure of implicit skewness. Deviations above (below) the standard [0%, 4%] range indicate greater positive (negative) skewness than that implied by standard distributional hypotheses. Inset shows 4% skewness premia calculated *hourly* during October 1987 prior to the crash. Diamonds (◇) indicate shifts in the maturity date of the options.

spondingly OTM calls, whereas standard distributional hypotheses imply the puts should have been 0–4% *cheaper*. The fears subsided concurrently with the stock market reaching its peak in August, with OTM put prices returning to levels comparable with OTM calls up until the stock market crash in October. Judging from this metric, there may have been fears of a crash in the year prior to its actual occurrence, but the decline in the market in August alleviated those fears.[16]

Figure 5 also shows the 4% skewness premium calculated from *hourly* data during October, 1987. The graphs confirm the above observation that the crash came as a surprise. Even as late as Friday afternoon, October 16, there

[16]The 2% and 6% skewness premia show a similar picture. Indeed, premia typically move in proportion: the 2% (6%) skewness premium is usually about one-half ($1\,{}^1\!/_2$) times the size of the 4% premium.

was no negative skewness implicit in options prices. Once the crash had occurred, however, options prices imply that market participants started perceiving considerable downside risk and continued to do so through the end of 1987.

II. Option Pricing Under Asymmetric Jump-Diffusion Processes

A. The Model

The above measure of asymmetry indicates that distributions more skewed than those hitherto generally considered would better explain observed option prices. For this reason, and because of what transpired on October 19–20, 1987, it is posited that the S&P 500 index follows a stochastic differential equation with possibly asymmetric, random jumps:

$$dS/S = \left[\mu - \lambda\bar{k} - d_t\right] dt + \sigma\, dZ + k\, dq, \tag{4}$$

where

μ is the instantaneous cum-dividend expected return on the asset;
d_t is the (flow rate) dividend yield;[17]
σ is the instantaneous variance conditional on no jumps;
Z is a standard Wiener process;
k is the random percentage jump conditional upon a Poisson-distributed event occurring, where $1 + k$ is log-normally distributed: $\ln(1 + k) \sim N(\gamma - {}^1/_2\, \delta^2, \delta^2) \equiv N(\gamma', \delta^z)$, $E(k) \equiv \bar{k} = e^{\gamma} - 1$;
λ is the frequency of Poisson events;
and q is a Poisson counter with intensity λ: $\text{Prob}(dq = 1) = \lambda\, dt$, $\text{Prob}(dq = 0) = 1 - \lambda\, dt$.

The process resembles geometric Brownian motion most of the time, but on average λ times per year the price jumps discretely by a random amount. For constant $\mu - d_t$, the variance, coefficient of skewness, and coefficient of kurtosis of $\ln(S_{t+T}/S_t)$ are given by

$$v^2 T = \left\{\sigma^2 + \lambda\left[(\gamma')^2 + \delta^2\right]\right\} T \tag{5a}$$

$$SKEW = \lambda\gamma'\left[(\gamma')^2 + 3\,\delta^2\right] T^{-1/2}/v^3 \tag{5b}$$

$$KURT = 3 + \lambda\left[(\gamma')^4 + 6(\gamma')^2\,\delta^2 + 3\,\delta^4\right] T^{-1}/v^4. \tag{5c}$$

The distribution is leptokurtic ($KURT \geq 3$), with skewness depending upon the sign of γ'. As T increases (from daily to monthly to annual holding periods), the moments of $\ln(S_{t+T}/S_t)$ converge towards those of a normal distribution. However, the skewness premium measures the distribution of

[17]The focus of this paper will be on options on S&P 500 futures. Since the futures price is ex-dividend, the results below will also be valid for discrete dividend payments made at period's end, provided the end-of-period dividend yield is nonstochastic during the lifetime of the option.

S_{t+T}/S_t, rather than that of $\ln(S_{t+T}/S_t)$. The noncentral moments of S_{t+T}/S_t for constant $\mu - d$ are given by the function

$$M_n \equiv E\left[\left(S_{t+T}/S_t\right)^n\right] = \exp\left\{n\left(\mu - d - \lambda\bar{k}\right)T \right. \\ \left. + {}^1/_2\left(n^2 - n\right)\sigma^2 T + \lambda T\left[e^{n\gamma + {}^1/_2(n^2-n)\delta^2} - 1\right]\right\}. \quad (6)$$

The variance, coefficient of skewness, and coefficient of kurtosis for S_{t+T}/S_t are given by

$$\mathrm{Var}\left(S_{t+T}/S_t\right) = M_2 - \left(M_1\right)^2 \quad (7a)$$

$$SKEW = \left[M_3 - 3M_1M_2 + 2\left(M_1\right)^3\right]/(\mathrm{Var})^{3/2} \quad (7b)$$

$$KURT = \left[M_4 - 4M_1M_3 + 6\left(M_1\right)^2M_2 - 3\left(M_1\right)^4\right]/(\mathrm{Var})^2. \quad (7c)$$

Skewness and kurtosis do not depend upon the drift term $\mu - d$.

The postulated process differs from previous work on option pricing under jump-diffusion processes (Merton (1976a,b) and Ball and Torous (1983,1985)) in certain important directions. First, the jumps are allowed to be *asymmetric*, i.e., with nonzero mean. Values of the expected percentage jump size $\bar{k}$ greater (less) than zero imply that the distribution is positively (negatively) skewed relative to geometric Brownian motion.

Second, since the underlying asset is a futures contract on the S&P 500 index, it is hardly plausible to maintain Merton's simplifying assumption that jump risk is nonsystematic and diversifiable. Instead, the appropriate risk-neutral jump-diffusion must be derived via restrictions on technologies and preferences. The following restrictions are imposed:

A1) Markets are frictionless: there are no transactions costs or differential taxes, trading takes place continuously, and there are no restrictions on borrowing or selling short.

A2) Optimally invested wealth W_t follows a jump-diffusion,[18]

$$dW/W = \left(\mu_w - \lambda\bar{k}_w - C/W\right)dt + \sigma_w\,dZ_w + k_w\,dq, \quad (8)$$

where μ_w is constant and k_w is the random percentage jump in wealth conditional on the Poisson event occurring. $1 + k_w$ is log-normally distributed: $\ln(1 + k_w) \sim N(\gamma_w - {}^1/_2\,\delta_w^2, \delta_w^2)$, $E(k_w) \equiv \bar{k}_w = \exp(\gamma_w) - 1$, and $\mathrm{Cov}[\ln(1 + k), \ln(1 + k_w)] \equiv \delta_{sw}$.

A3) The representative consumer has time-separable power utility

$$E_\tau \int_\tau^\infty e^{-\rho\tau} U(C_\tau)\,d\tau,\ U(C) = \left(C^{1-R} - 1\right)/(1 - R). \quad (9)$$

[18]The process for optimally invested wealth can be derived from an extension of the Cox, Ingersoll, and Ross (1985b) production economy, with multiple investment opportunities that are represented by constant returns to scale jump-diffusion processes with simultaneous jumps. See Bates (1988b).

By construction, jump risk is systematic: all asset prices and wealth jump simultaneously, albeit by possibly different amounts. The standard assumption of a constant instantaneous riskless rate r is not imposed but rather follows from assumptions A1–A3. These assumptions yield a tractable option pricing model.

PROPOSITION 2: *Contingent claims are priced as if investors were risk-neutral and the asset price followed the jump-diffusion*

$$dS/S = \left(b - \lambda^* \bar{k}^*\right) dt + \sigma \, dZ + k^* \, dq^*, \tag{10}$$

where

b is the cost of carry coefficient ($r - d_t$ for stock options, 0 for futures options)
σ and δ are as before,
$\lambda^* = \lambda E[1 + \Delta J_w / J_w] = \lambda \exp[-R\gamma_w + {}^1\!/_2 R(1 + R)\, \delta_w^2]$
q^* is a Poisson counter with intensity λ^*,
$1 + k^*$ is a lognormal random variable:

$$E\left(k^*\right) \equiv \bar{k}^* = \bar{k} + \mathrm{Cov}\left(k, \Delta J_w / J_w\right) / E\left(1 + \Delta J_w / J_w\right)$$
$$= \exp\left(\gamma - R\, \delta_{sw}\right) - 1 \equiv e^{\gamma^*} - 1; \ln\left(1 + k^*\right) \sim N\left(\gamma^* - {}^1\!/_2\, \delta^2, \delta^2\right).$$

Proof: See Appendix I.

The process (10), expressed as a jump-diffusion, is actually the specification of Arrow-Debreu contingent claims prices given aggregate relative risk aversion R. λ^*, for instance, is the cost per unit time of jump insurance; i.e., $\lambda^* \Delta t$ is, to order $o(\Delta t)$, the price at time t of an Arrow-Debreu security that pays off \$1 in the event of a jump occurring within the interval $(t, t + \Delta t]$, and \$0 otherwise. Under risk neutrality, such insurance is priced at the actuarially fair rate: $\lambda^* = \lambda$. When jumps tend to be negative ($\bar{k}_w < 0$) and investors are risk-averse, the cost of jump insurance λ^* exceeds λ. Under the assumptions of the model, the price of such insurance is state-independent.[19] Note that the *implicit* mean jump size $\bar{k}^*$ will typically be downwardly biased relative to the true mean jump size $\bar{k}$. The extent of this bias depends upon the covariance of asset price jumps with jumps in the marginal utility of optimally invested wealth, and will be examined below. As noted in Table I, the sign of $\bar{k}^*$ determines whether the $x\%$ skewness premium is greater than or less than $x\%$. In the event of "firm-specific" jump risk ($\bar{k}_w = \delta_w = \delta_{sw} = 0$), the model reduces to the Merton model: $\bar{k}^* = \bar{k}$, $\lambda^* = \lambda$.

Pricing European options from process (10) is straightforward; see Merton (1976a). European calls are priced at what would be the discounted expected

[19] A special case of this model is in Jones (1984), who looks at option pricing under deterministic jump amplitudes ($\delta = 0$). Jones assumes λ^* is constant which, along with the deterministic-jumps assumption, implies an arbitrage-based option pricing formula similar to equation (11) below.

value of their terminal payoffs if the terminal distribution were determined by (10):[20]

$$
\begin{aligned}
c(F,T;X) &= e^{-rT}\sum_{n=0}^{\infty} \text{Prob}^*(n\,jumps)\,E_t^*\left[\max(F_{t+T}-X,0)\mid n\,jumps\right]\\
&= e^{-rT}\sum_{n=0}^{\infty}\left[e^{-\lambda^* T}(\lambda^* T)^n/n!\right]\left[Fe^{b(n)T}N(d_{1n}) - XN(d_{2n})\right], \quad (11)
\end{aligned}
$$

where

$$
\begin{aligned}
b(n) &= (b-\lambda^*\bar{k}^*) + n\gamma^*/T = -\lambda^*\bar{k}^* + n\gamma^*/T,\\
d_{1n} &= \left[\ln(F/X) + b(n)T + {}^1/_2(\sigma^2 T + n\,\delta^2)\right]/(\sigma^2 T + n\,\delta^2)^{1/2}, \text{ and}\\
d_{2n} &= d_{1n} - (\sigma^2 T + n\,\delta^2)^{1/2},
\end{aligned}
$$

since the cost of carry b is zero for futures. European puts have an analogous formula:

$$
\begin{aligned}
p(F,T;X) &= e^{-rT}\sum_{n=0}^{\infty} \text{Prob}^*(n\,jumps)\,E_t^*\left[\max(X-F_{t+T},0)\mid n\,jumps\right]\\
&= e^{-rT}\sum_{n=0}^{\infty}\left[e^{-\lambda^* T}(\lambda^* T)^n/n!\right]\left[XN(-d_{2n}) - Fe^{b(n)T}N(-d_{1n})\right]. \quad (12)
\end{aligned}
$$

There are no known analytic solutions for American calls. Finite-difference methods can be used to evaluate option prices accurately for given jump-diffusion parameters but at a prohibitive cost in computer time. To estimate the parameters implicit in observed option prices using such methods would require a Cray. Consequently, I have developed an accurate and inexpensive quadratic approximation for evaluating American options written on jump-diffusion processes. The approximation is an extension of the one developed by MacMillan (1987) and extended by Barone-Adesi and Whaley (1987) for evaluating American options written on geometric Brownian motion processes; details are in Appendix II. The resulting formula for American calls on future contracts is

$$
C(F,T;X) = \begin{cases} c(F,T;X) + XA_2\left[(F/X)/y_c^*\right]^{q_2} & \text{for } F/X < y_c^*,\\ F - X & \text{for } F/X \geq y_c^* \end{cases} \quad (13)
$$

[20] Numerical evaluation of European calls and puts requires truncation of the infinite sum. The method used calculates a unimodal "cap" on the summation terms $(\kappa(\lambda T e^{\gamma^*})]^n/n!$ for calls, $\kappa'(\lambda T)^n/n!$ for puts, for constant κ and κ') and extends the summation in two directions from the peak term ($n \approx \lambda T e^{\gamma^*}$ for calls) until either additional terms would not increase accuracy or 1000 terms were reached.

where $A_2 = (y_c^*/q_2)[1 - c_F(y_c^*, T; 1)]$, q_2 is the positive root to

$$\tfrac{1}{2}\sigma^2 q^2 + \left(-\lambda^*\bar{k}^* - \tfrac{1}{2}\sigma^2\right)q - r/\left(1 - e^{-rT}\right) + \lambda^*\left[e^{\gamma^* q + \frac{1}{2}q(q-1)\delta^2} - 1\right] = 0, \tag{14}$$

and the critical futures price/exercise price ratio $y_c^* \geq 1$ above which the call is exercised immediately is given implicitly by

$$y_c^* - 1 = c\left(y_c^*, T; 1\right) + \left(y_c^*/q_2\right)\left[1 - c_F\left(y_c^*, T; 1\right)\right]. \tag{15}$$

Similarly, American puts have values approximated by

$$P(F, T; X) = \begin{matrix} p(F, T; X) + XA_1\left[(F/X)/y_p^*\right]^{q_1} & \text{for } F/X > y_p^*, \\ X - F & \text{for } F/X \leq y_p^* \end{matrix}, \tag{16}$$

where $A_1 = (y_p^*/-q_1)[1 + p_F(y_p^*, T; 1)]$,
q_1 *is the negative root to equation* (14),

and the critical futures price/exercise price ratio $y_p^* \leq 1$ below which the put is exercised immediately is given implicitly by

$$1 - y_p^* = p\left(y_p^*, T; 1\right) + \left(y_p^*/-q_1\right)\left[1 + p_F\left(y_p^*, T; 1\right)\right]. \tag{17}$$

The parameters q_1, q_2, y_c^* and y_p^* can be evaluated rapidly via Newton's method for given parameters r, σ, λ^*, $\bar{k}^*$, and δ and for given time to maturity T. The approximations are quite accurate; as shown in Table II, the approximation error is typically substantially less than 0.05, the size of one price tick.

B. Estimation

The procedure discussed above gives an American option pricing formula as a function of state variables S and T and parameters X, r, σ, λ^*, $\bar{k}^*$, and δ. The first four are known. The instantaneous risk-free rate can be proxied by Treasury bill rates: I use rates derived from the average of bid and ask discounts on Treasury bills maturing close to the maturity of the option.[21] The average jump size $\bar{k}^*$, jump frequency λ^*, jump dispersion δ, and standard deviation σ (conditional on no jumps) are not known. This paper takes the approach of estimating the jump parameters *implicit* in option prices. The parameters estimated are of course those of the "risk-neutral" jump-diffusion process. Inferring the true parameters requires additional assumptions about the degree of relative risk aversion and about the degree to which jumps in the S&P 500 are related to jumps in wealth.

Option prices within a given day as a fraction of the corresponding futures price were assumed to be the corresponding model prices plus a random

[21] Interestingly, the bid/ask spread on Treasury bills jumped substantially in the summer before the crash. The spread came down again in September.

Table II

Theoretical Futures Options Values under Asymmetric Jump-Diffusion Processes Futures Price $F = 250$. Parameters: $r = 0.10$, $T = 0.25$.

Jump-Diffusion Parameters	Exercise Price X	Call Options: European $c(F, T; X)$[a]	Call Options: American $C(F, T; X)$ Finite Difference Method[b]	Call Options: American $C(F, T; X)$ Quadratic Approximation Method[c]	Put Options: European $p(F, T; X)$[a]	Put Options: American $P(F, T; X)$ Finite Difference Method[b]	Put Options: American $P(F, T; X)$ Quadratic Approximation Method[c]
1) $\sigma = 0.1414$	220	29.49	30.03	30.01	0.23	0.23	0.23
$\lambda = 0$	235	16.39	16.54	16.53	1.76	1.76	1.77
$\gamma = 0$	250	6.88	6.91	6.92	6.88	6.91	6.92
$\delta = 0$	265	2.04	2.05	2.06	16.67	16.82	16.82
	280	0.42	0.42	0.43	29.68	30.15	30.12
2) $\sigma = 0.10$	220	29.45	30.00	30.01	0.19	0.19	0.19
$\lambda = 10$	235	16.25	16.41	16.42	1.62	1.63	1.63
$\gamma = 0.01$	250	6.81	6.84	6.86	6.81	6.82	6.85
$\delta = 0.03$	265	2.17	2.18	2.18	16.79	16.90	16.91
	280	0.56	0.56	0.56	29.82	30.21	30.19
3) $\sigma = 0.10$	220	29.58	30.05	30.04	0.33	0.33	0.33
$\lambda = 10$	235	16.49	16.61	16.61	1.86	1.87	1.88
$\gamma = -0.01$	250	6.79	6.80	6.83	6.79	6.82	6.85
$\delta = 0.03$	265	1.88	1.89	1.89	16.51	16.67	16.68
	280	0.35	0.35	0.35	29.61	30.09	30.09

Table II

Table II—*Continued*

		Call Options			Put Options		
			American $C(F, T; X)$			American $P(F, T; X)$	
Jump-Diffusion Parameters	Exercise Price X	European $c(F, T; X)$[a]	Finite Difference Method[b]	Quadratic Approximation Method[c]	European $p(F, T; X)$[a]	Finite Difference Method[b]	Quadratic Approximation Method[c]
4) $\sigma = 0.10$	220	29.30	30.00	30.00	0.04	0.04	0.04
$\lambda = 0.25$	235	15.62	15.80	15.92	0.99	0.99	0.99
$\gamma = 0.20$	250	6.28	6.34	6.42	6.28	6.28	6.29
$\delta = 0$	265	2.65	2.68	2.72	17.28	17.29	17.32
	280	1.42	1.44	1.45	30.68	30.76	30.81
5) $\sigma = 0.10$	220	30.14	30.28	30.32	0.88	0.89	0.90
$\lambda = 0.25$	235	16.71	16.72	16.75	2.08	2.10	2.13
$\gamma = -0.20$	250	6.02	6.01	6.02	6.02	6.06	6.14
$\delta = 0$	265	1.11	1.11	1.11	15.74	15.91	16.03
	280	0.09	0.09	0.09	29.35	30.00	30.01

[a]European options prices were computed using equations (11) and (12).

[b]Finite-difference American options prices were generated recursively using the Cox and Rubinstein (1985) binomial option pricing methodology for the diffusion part, augmented by probability-weighted numerical integration for jump-contingent expected values. The time interval was about 1/5 day for runs 1, 4, 5 and about 1/10 of a day for runs 2 and 3. A comparison of the finite-difference and Merton model prices for European options indicated an accuracy of the former of ± 0.003 for runs 1, 4, and 5 and ± 0.012 for runs 2 and 3.

[c]The quadratic approximation American options prices were computed using equations (13) and (16).

additive disturbance term:[22]

$$\frac{V_j}{F_j} = \frac{V(F_j, T; X_j, \sigma, \lambda^*, \bar{k}^*, \delta)}{F_j} + \epsilon_j, j = 1, \ldots, \text{NOBS}_t. \tag{18}$$

Given the homogeneity of the model in F and X, this is equivalent to the nonlinear regression

$$(V/F)_j = V(1, T; (X/F)_j, \sigma, \lambda^*, \bar{k}^*, \delta) + \epsilon_j. \tag{19}$$

A cross-sectional data sample of pooled calls and puts with identical maturities in the 1–4 month range was used, and the implicit parameters σ_t, λ_t^*, $\bar{k}_t^*$, and δ_t for that day were estimated via nonlinear least squares. Similar regressions were run for all days in the 1985–1987 data sample. The implicit parameters were not constrained to be constant over time. While reestimating the parameters daily is admittedly potentially inconsistent with the assumption of constant or slow-changing parameters used in deriving the option pricing model, such estimation was felt to be valuable, for two reasons:

1) a chronology of parameter estimates and of implicit moments over time—skewness and kurtosis as well as volatility—could thereby be generated, indicating market sentiment on a daily basis over the 1985–1987 period, and
2) stylized facts for the future specification of more complicated dynamic models could thereby be generated.

Several tricks were used to speed optimization. First, rather than optimizing over $(\sigma, \lambda^*, \bar{k}^*, \delta)$, the optimization was over the transformed parameter space

$$\langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle = \langle \ln(v), N^{-1}(f), \gamma^*, \ln(\delta) \rangle \in \mathbb{R}^4,$$

where $v = \{\sigma^2 + \lambda^*[(\gamma^*)^2 + \delta^2]\}^{1/2}$ is the implicit volatility, $f = \lambda^*[(\gamma^*)^2 + \delta^2]/v^2$ is the fraction of variance attributable to jumps, and $N^{-1}(\cdot)$ is the inverse of the cumulative normal. Optimizing over the implicit volatility effectively reduced the dimensions of the optimization from 4 to 3, because of a ridge in the likelihood surface essentially perpendicular to v. Option prices for a given day had about the same implicit volatility regardless of specific jump-diffusion parameters.

Second, there are problems with the overabundance of data. Using transactions data, there are up to 2000 data per day; estimating the parameters for a single day was slow even on a mainframe. Consequently, a *representative* set of 20 call and 20 put prices, expressed as a fraction of the futures price F and

[22]A drawback of additive errors is that option prices cannot be negative. The alternative approach of multiplicative error terms has the drawback that given a minimum tick size for option prices, errors do not in fact decline proportionately for lower-priced, farther out-of-the-money options. Such regressions consequently weight the far OTM options too heavily while virtually ignoring other options prices.

evenly spaced over the X/F domain, was constructed for each day using the atheoretic constrained cubic splines of the first section. Given the lack of noise in option prices around this cubic spline fit (Figures 3 and 4), plus the fact that the constrained cubic spline was estimated using the same loss function as in regression (19) above, this representative data set was felt to summarize accurately the information contained in a day's worth of data. Parameters were then estimated via volume-weighted nonlinear least squares on the representative data,[23] using the quadratic hillclimbing software of Goldfeld and Quandt, GQOPT method GRADX, six starting values. Extensive runs on individual days confirmed that this representative data set yielded parameter estimates identical or observationally equivalent to those estimated directly from the actual data set. All standard errors and all tests of overidentifying restrictions reported below were of course calculated using the actual data set.

C. Results

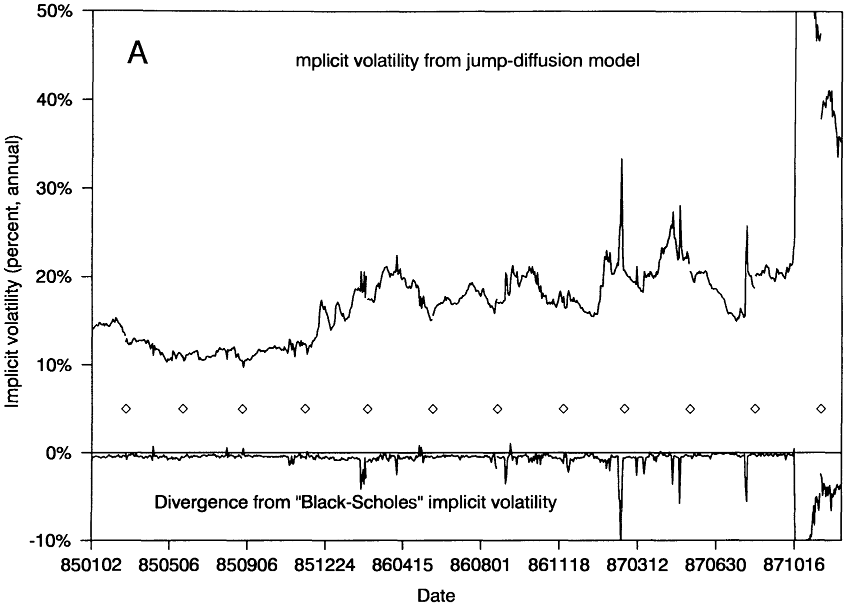
Figures 6A–6D chronicle the estimated implicit volatilities $v = (\sigma^2 + \lambda^*[(\gamma^*)^2 + \delta^2]\}^{1/2}$ per year, the implicit jump frequencies λ_t^* per year, the implicit average jump size $\bar{k}_t^*$, and the implicit jump dispersion δ_t for all days in the period 1985–1987. The graphs evince the marked change in implicit distributions that started in October 1986. Prior to that time, the parameters (with some exceptions) indicate an essentially log-symmetric, fat-tailed distribution, with jumps of zero mean and 2%–8% standard deviations expected on monthly to annual frequencies. After October 1986, expectations of predominantly negative jumps ($\bar{k}^* < 0$) are evident in option prices, especially during October 1986–February 1987 and June–August 1987. Figure 7, which shows expected jumps per year ($\lambda^*\bar{k}^*$), highlights the strong crash fears that accompanied the stock market reaching its peak in August 1987. Crash fears subsided markedly when the stock market peaked, although expectations of negative jumps continued up until shortly before the stock market crash. Crash fears returned in full force once the stock market crash occurred.

Figures 8 and 9 chronicle the implicit coefficients of skewness and kurtosis for F_{t+T}/F_t, using a standardized one-month holding period.[24] The graphs indicate that implicit distributions became *negatively* skewed starting in October 1986, particularly during October 1986–February 1987 and June–August 1987, as well as after the crash. The negative jumps estimated in the two months immediately preceding the crash were not large or frequent enough for the implicit distribution of F_{t+T}/F_t to be negatively skewed.[25] Kurtosis was of necessity greater than three, since jump-diffusions are leptokurtic, and typically ranged from 3 to 10 for monthly returns, with

[23] Weights were calculated by apportioning the incremental weight of each actual datum to the two representative data flanking it, with the apportionment depending linearly on proximity.

[24] The actual holding period varies between 4 and 1 months with the varying maturity of the options used.

[25] Concurrently, the $x\%$ skewness premia were less than $x\%$ but still positive.



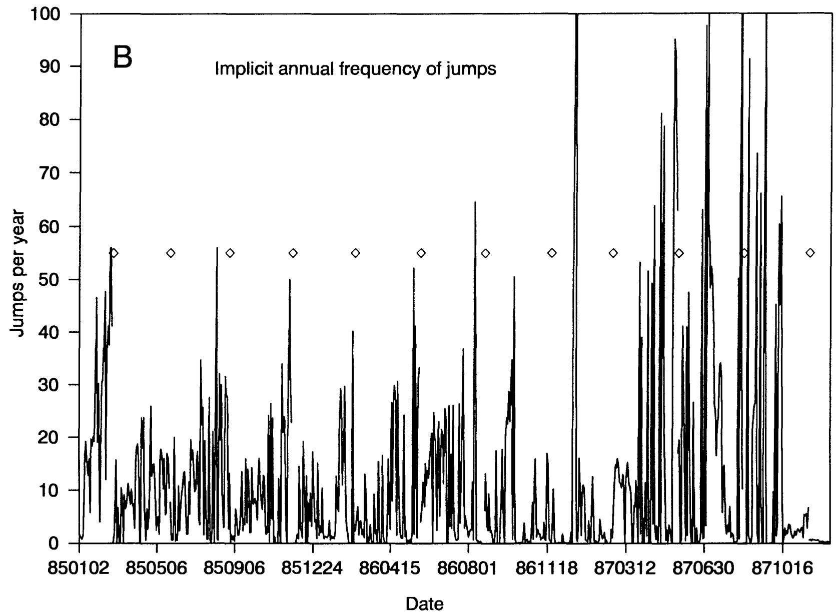


Figure 6. Jump-diffusion parameters implicit in call and put options on S&P 500 futures, 1985–1987. Parameters are: A) the implicit annual volatility $\upsilon = \{\sigma^2 + \lambda^*[(\gamma^*)^2 + \delta^2]\}^{1/2}$ and its deviation from the volatility σ estimated from the no-jump model; B) the implicit annual frequency of jumps λ^*; C) the implicit expected percentage jump size $\bar{k}^*$ conditional on a jump occurring; D) the implicit standard deviation of percentage jump sizes δ. Diamonds (◇) indicate shifts in the maturity date of the options.

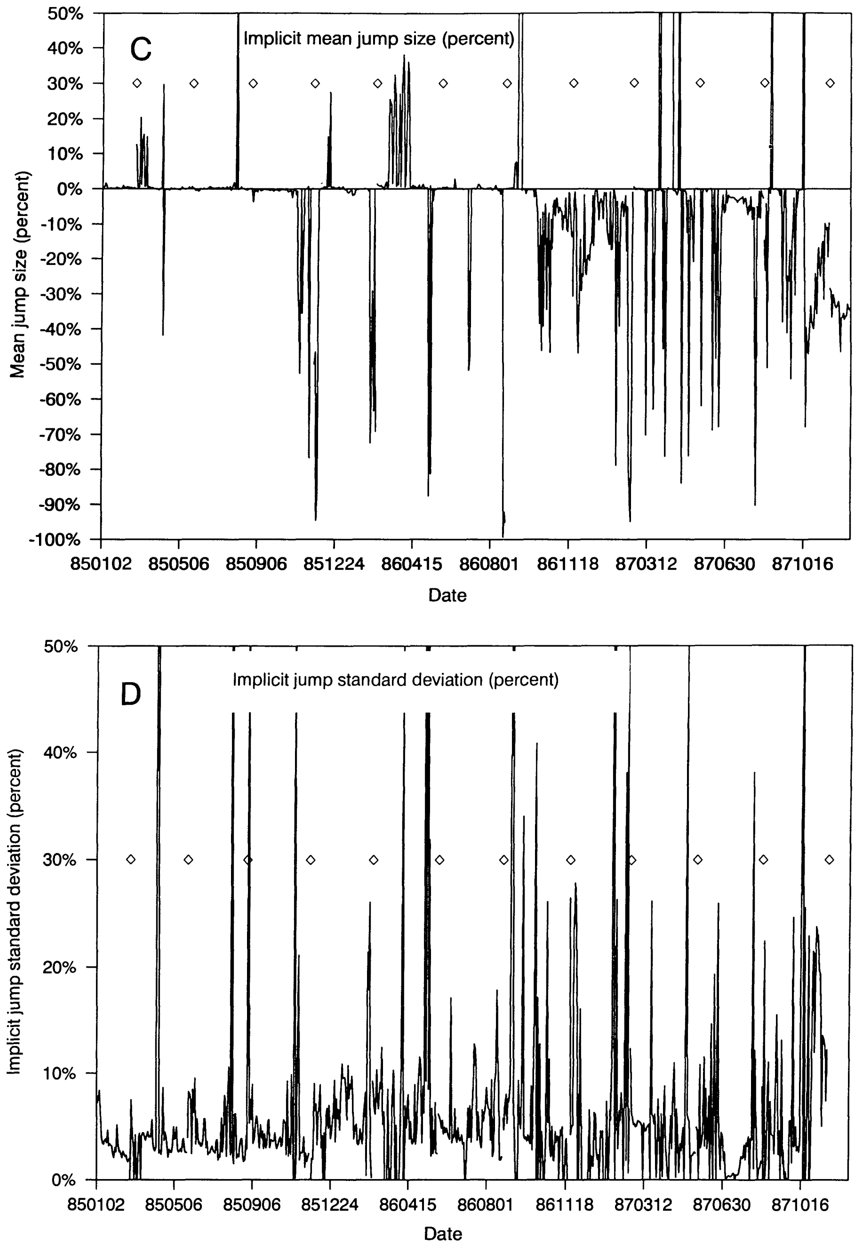


Figure 6.—*Continued*

some extreme outliers. The graphs of the moments (Figures 6A, 8, and 9) also make the point that although the daily estimated parameters may bounce around, the associated implicit distributions evolve rather more smoothly. Figure 10 shows the evolution of implicit probability density functions over time.

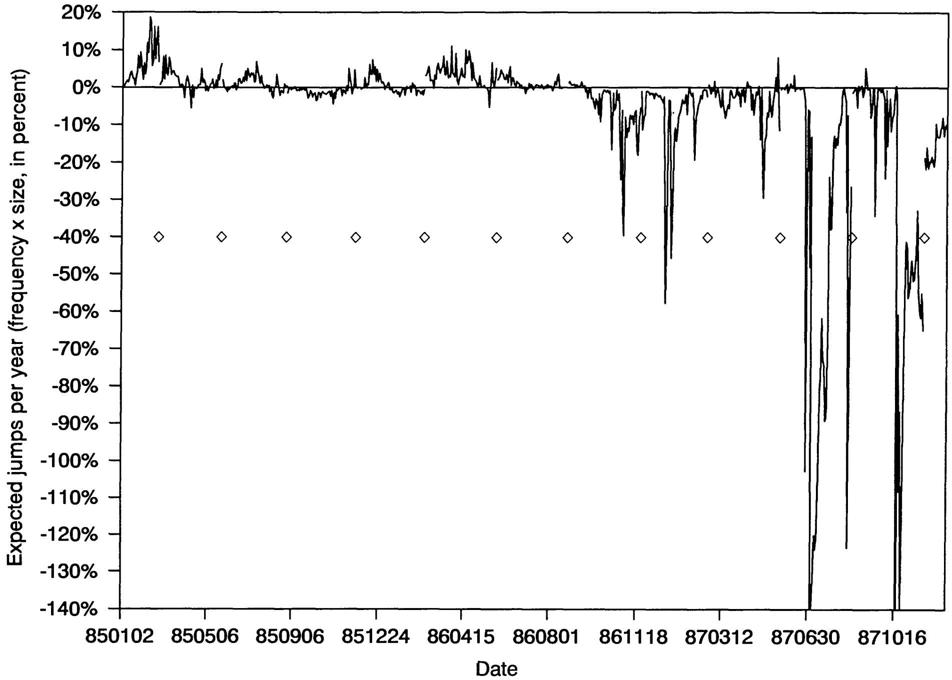


Figure 7. Expected jumps per year $\lambda^*\bar{k}^*$ implicit in call and put options on S&P 500 futures, 1985–1987. Diamonds (◇) indicate shifts in the maturity date of the options.

The above inferences are of course based upon the estimated parameters of the risk-neutral jump-diffusion rather than those of the actual distribution. However, the true parameters λ and γ do not differ qualitatively or quantitatively from the risk-neutral parameters $\lambda^* = \lambda\exp[-R\gamma_w + {}^1/_2 R(1 + R)\,\delta_w^2]$ and $\gamma^* = \gamma - R\,\delta_{sw}$, given plausible assumptions about relative risk aversion R and about the degree to which total wealth is affected by jumps in the S&P 500 index. For instance, if $R = 2$ (the "Samuelson presumption"), equity comprises half of wealth, and jumps occur only in stock prices, then, approximately,

$$\begin{aligned}
\delta_{sw} &\approx (\delta)\left({}^1/_2\delta\right) = {}^1/_2\,\delta^2 \\
\lambda^* &= \lambda\exp\left[-2\left({}^1/_2\gamma\right) + {}^1/_2 R(1 + R)\left({}^1/_2\,\delta^2\right)\right] \approx \lambda\exp(-\gamma) \\
\gamma^* &= \gamma - 2\left[\delta\left({}^1/_2\,\delta\right)\right] \approx \gamma,
\end{aligned}$$

given that estimates of δ^2 are small (≤ 0.005). Inferences from the risk-neutral parameters about the distribution of jumps apply equally to the true parameters, although the implicit jump frequency slightly overstates the true jump frequency when jumps have a negative mean. Correspondingly, implicit skewness and kurtosis will be slightly upwardly biased in magnitude relative to the true values when jumps have a negative mean.

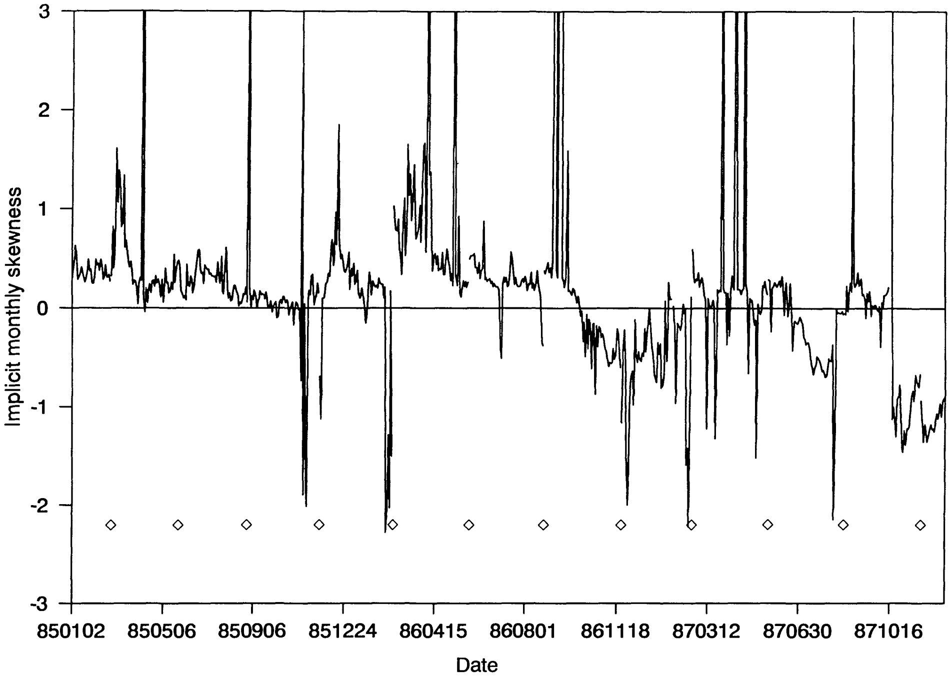


Figure 8. Skewness of $F_{t+1\ \text{month}}/F_t$ implicit in call and put options on S&P 500 futures, 1985–1987. Skewness is estimated using the jump-diffusion model discussed in the text, and the jump-diffusion parameters estimated above. The reported skewness is for a standardized one-month holding period. Diamonds (◇) indicate shifts in the maturity date of the options; option maturity varies between 1 and 4 months.

The jump-diffusion model fits the actual data quite well prior to the crash—as well in fact as the atheoretic cubic spline fits shown in Figure 4.[26] The fit was markedly better than the American option pricing model premised on geometric Brownian motion (GBM) during the periods of pronounced implicit skewness prior to the crash (Figure 11), with standard errors up to 0.06% of F, or 3 ticks lower.[27] GBM did not do badly at other times, with standard errors only $0-1$ tick higher. Nevertheless, the no-jumps restriction of geometric Brownian motion was rejected at 10^{-6} significance levels for 90% of the days prior to the stock market crash.[28] The hypothesis of identical

[26]Prior to the crash, the average difference in fits between the cubic splines and the jump-diffusion model was 0.004% of the underlying futures price—roughly 1/5 of a price tick.

[27]Ball and Torous (1985, p. 155) asserted that there were no "operationally significant differences between the Black-Scholes and the Merton model prices," in the context of pricing options on NYSE stocks. However, their jump-diffusion model restricts jump sizes to having a zero mean and is therefore incapable *a priori* of eliminating "moneyness bias" pricing errors of the American version of the Black-Scholes model such as those being observed here.

[28]The no-jumps restriction was rejected at 10^{-16} significance for 70% of the days prior to October 19, 1987.

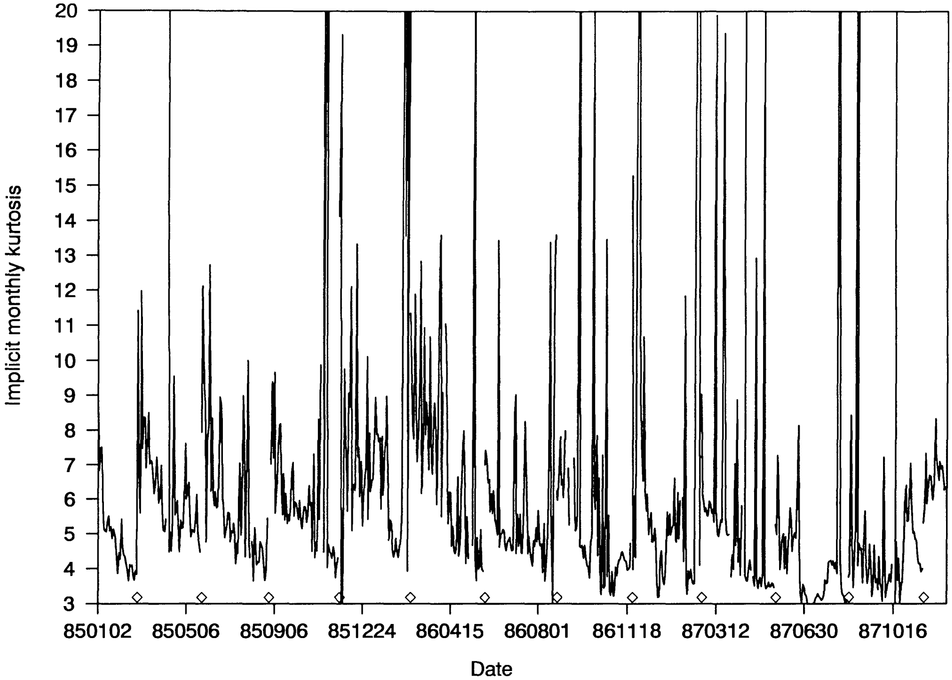


Figure 9. Kurtosis of $F_{t+1\ \text{month}}/F_t$ implicit in call and put options on S&P 500 futures, 1985–1987. See notes to Figure 8.

jump-diffusion parameters for calls and for puts did not appreciably worsen the fit relative to fitting parameters to calls and to puts independently. Nevertheless, these overidentifying restrictions were rejected at 10^{-6} significance for 50% of the days prior to the stock market crash.[29] The hypothesis of constant or slow-changing jump-diffusion parameters implicit in the option pricing model was not tested. Given the nonstationarity of implicit volatilities evident in Figure 6A, however, that hypothesis would be overwhelmingly rejected.

III. Summary and Conclusions

This paper has shown that there was a strong perception of downside risk on the market during the year preceding the stock market crash. The crash fears first emerged one month after the stock market plunged 6% on September 11–12, 1986 and were especially pronounced during October 1986–February 1987 and June–August 1987 as well as after the crash. Two

[29] Given the absence of noise in options prices evident in Figures 3 and 4, F-tests are extremely powerful. A constrained model tends almost invariably to be rejected at standard significance levels in favor of any more general alternative. It is, consequently, unclear what the appropriate significance level is—hence the reporting above of the *absolute* improvement in standard errors from relaxing constraints.

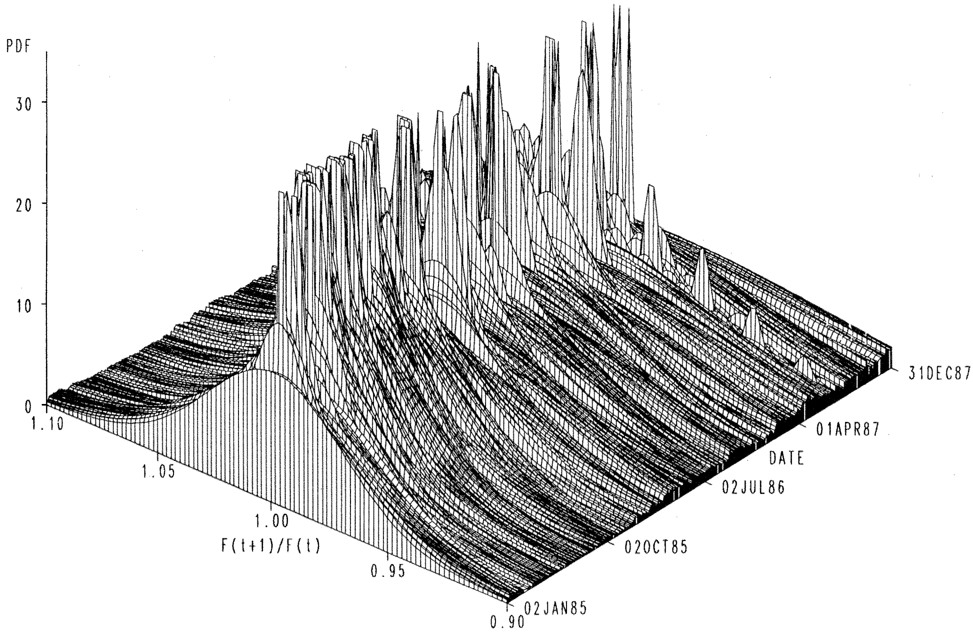


Figure 10. Probability density function of $F_{t+1\ \mathrm{month}}/F_t$ implicit in call and put options on S&P 500 futures, 1985–1987. See notes to Figure 8.

methods were used to establish this "stylized fact:" first, that out-of-the-money puts, which provide crash insurance, were unusually expensive relative to out-of-the-money calls. This high price for crash insurance cannot be explained by standard options pricing models with positively skewed distributions, such as Black-Scholes, constant elasticity of variance, or GARCH. Second, a jump-diffusion model was fitted to daily options prices during 1987, and expected negative jumps were invariably found starting a year prior to the crash. Again, downside risk, as indicated by negative implicit coefficients of skewness, was most pronounced in October 1986–February 1987 and in June–August 1987, as well as after the crash.

By either measure, there were no strong fears of a crash in the 2 months immediately preceding October 19, 1987—not even late on Friday afternoon, October 16. If there was a rational bubble in the stock market, one would have to conclude that it burst in mid-August—not in mid-October.

Appendix I
Option Pricing Under Systematic Jump Risk

A.1. Capital Asset Pricing Model

Derivation of an options pricing model under assumptions A1–A3 is as follows. Define

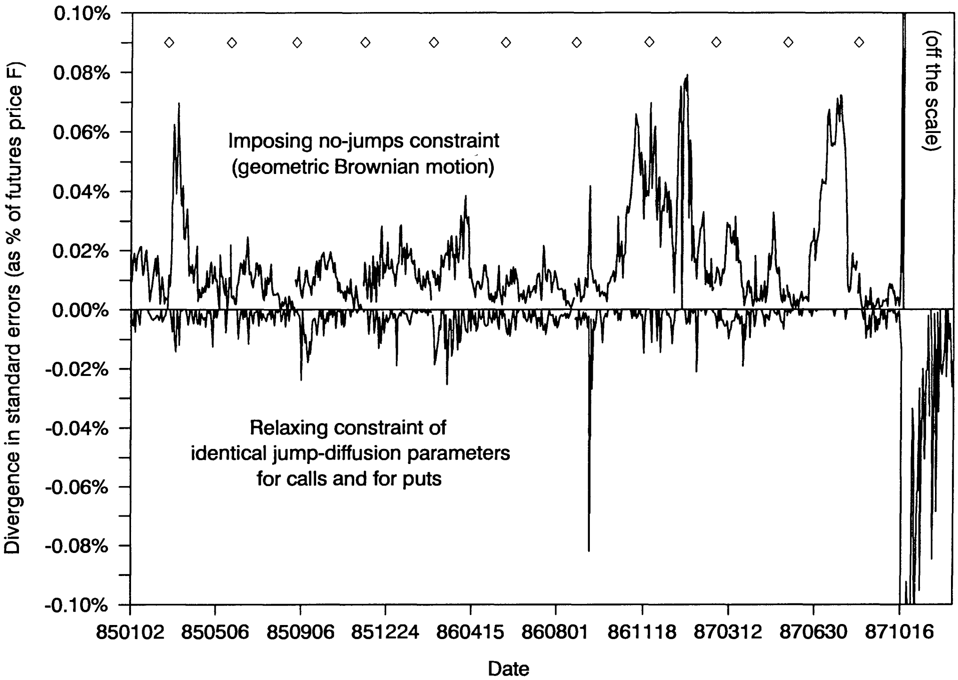


Figure 11. Changes in standard errors resulting from imposition/relaxation of overidentifying constraints. The graph shows: 1) the increase in daily standard errors over 1985–1987 from imposing the no-jumps restriction $\lambda^* = 0$ (geometric Brownian motion) *relative* to the standard errors of the jump-diffusion model estimated on call and put data pooled daily; 2) the decrease in daily standard errors over 1985–1987 from relaxing the constraint of identical jump-diffusion parameters for calls and for puts. Changes in standard errors are percentages of the futures price F. With a futures price around 250, a change of 0.02% of F represents a change in standard errors of about 1 price tick.

$$J(W, t) = \max_{\{C_\tau\}} E_t \int_t^\infty e^{-\rho\tau} U(C_\tau)\, d\tau \tag{A1}$$

as the indirect utility of wealth at time t under the optimal consumption plan. By standard perturbation arguments (borrow risklessly and invest in the risky asset), the instantaneous excess return on a risky asset with instantaneous dividend yield d_t per unit time must satisfy

$$E_t\{J_w(W_{t+dt}, t + dt)[S_{t+dt}/S_t + d_t\, dt - e^{r\, dt}]\} = 0; \tag{A2}$$

dividing by $J_w(W, t)$ and rearranging yields the standard result

$$E_t(dS/S) + d_t\, dt - r\, dt = -E_t[(dJ_w/J_w)(dS/S)] + o(dt), \tag{A3}$$

where $\lim_{dt \to 0} o(dt)/dt = 0$. That is, required cum-dividend excess returns are higher for assets that tend to pay off when marginal utility of wealth is low. The Ito expansion of dJ_w for jump-diffusion processes is

$$dJ_w = J_t\, dt + J_w\, dW_{dq=0} + {}^1\!/_2 J_{ww} \sigma_w^2 W^2\, dt + \Delta J_w\, dq + o(dt), \tag{A4}$$

where

$$\Delta J_w = J_w(W(1+k_w),t) - J_w \text{ is the (random) jump in marginal utility of wealth conditional on a jump occurring, and}$$
$$dW_{dq=0} = (\mu_w - \lambda\bar{k}_w - C/W)W\,dt + \sigma_w W\,dZ_w \text{ comprises the diffusion components of } dW.$$

Plugging expansion (A4) into equation (A3) and ignoring terms of order $o(dt)$ yields the fundamental capital asset pricing model for jump-diffusion processes with systematic jump risk.

PROPOSITION A1: *When asset prices follow jump-diffusion processes with jumps occurring simultaneously, the equilibrium cum-dividend excess return is*

$$\mu - r = R(W,t)\sigma_{sw} - \lambda E_{dq=1}\left[(\Delta J_w/J_w)(\Delta S/S)\right], \tag{A5}$$

where

$R(W,t) = (-WJ_{ww}/J_w)$ *is the coefficient of relative risk aversion, and*
$\sigma_{sw} = E_{dq=0}[(dS/S)(dW/W)]/dt$ *is the instantaneous covariance per unit time between asset and market returns conditional on no jumps.*

The instantaneous riskless rate for borrowing and lending is given by the fact that equation (A5) must also hold for the market's expected return μ_w:

$$r(W,t) = -E(dJ_w/J_w)/dt = \mu_w - R(W,t)\sigma_w^2 + \lambda E_{dq=1}\left[(\Delta J_w/J_w)k_w\right]. \tag{A6}$$

Equation (A5) holds in general. Assumptions (A2) and (A3) imply that $J_w(W,t) = Ae^{-\rho t}W^{-R}$ for some constant A, with the following implications:

1) the coefficient of relative risk aversion $R(W,t) = R$ is constant,
2) $\Delta J_w/J_w = (1+k_w)^{-R} - 1$ does not depend on wealth, and
3) the instantaneous riskfree rate r is constant.[30]

The CAPM for jump-diffusions is then

$$\mu - r = R\sigma_{sw} - \lambda E\left\{\left[(1+k_w)^{-R} - 1\right]k\right\}, \tag{A7}$$

where

$$r = \mu_w - R\sigma_w^2 + \lambda E\left\{\left[(1+k_w)^{-R} - 1\right]k_w\right\}. \tag{A8}$$

A.2. Option pricing

Equation (A7) applies to any asset and consequently also to contingent claims written upon the asset. Writing the price of the claim as $V(S,t)$,[31] the

[30] It is standard in the option pricing literature to impose the constant interest rate restriction directly, justifying it as empirically plausible for short-term options. In the above derivation, constant interest rates follow from the assumption that μ_w is constant.

[31] Assumptions (A2) and (A3), in yielding a state-independent CAPM (A7), imply that the price of a contingent claim written on the asset will not in general depend on W.

equilibrium expected return β on the claim is

$$(\beta - r)V = RSV_s\sigma_{sw} + \lambda E\left[(\Delta J_w / J_w)\,\Delta V\right] \tag{A9}$$

where $\Delta V = V(S(1+k), t) - V$ is the (random) jump in the price of the claim conditional on a jump occurring. By Ito's lemma, the expected return also satisfies

$$\beta V = V_t + SV_s\left(\mu - \lambda\bar{k} - d_t\right) + {}^1\!/_2\sigma^2 S^2 V_{ss} + \lambda E(\Delta V). \tag{A10}$$

Combining (A9) and (A10), and using (A5) for μ yields the fundamental differential equation for contingent claims under jump-diffusions.

PROPOSITION A2: *The price of any contingent claim $V(S, t)$ satisfies the partial differential equation*

$$V_t + \left\{b - \lambda E\left[(J_w^*/J_w)k\right]\right\}SV_s + {}^1\!/_2\sigma^2 S^2 V_{ss} + \lambda E\left[(J_w^*/J_w)\,\Delta V\right] = rV, \tag{A11}$$

subject to claim-specific boundary conditions, where

$b = r - d_t$ *is the cost of carry on the underlying asset,*
$J_w^*/J_w = J_w(W(1+k), t)/J_w = (1 + k_w)^{-R}$, *and*
$\Delta V = V(S(1+k), t) - V.$

Expression (A11) can *almost* be interpreted as risk-neutral pricing of the contingent claim conditional on S following a jump-diffusion with expected return b and proportional jumps k, except that the jump-contingent expectations over k and over ΔV (which depends on k) are weighted by the jump-contingent marginal utility of wealth $J_w^*/J_w = (1 + k_w)^{-R}$. A transformation of the probability measure makes the risk-neutral pricing statement exact and facilitates computation of contingent claims prices.

PROPOSITION A3:[32] *Under assumptions (A2) and (A3), the price of any contingent claim $V(S, t)$ satisfies the partial differential equation*

$$V_t + \{b - \lambda^*\bar{k}^*\}SV_s + {}^1\!/_2\sigma^2 S^2 V_{ss} + \lambda^* E\left[V\left(S(1+k^*), t\right) - V\right] = rV \tag{A12}$$

[32] The random variables $z = \ln(1+k)$ and $y = \ln(1+k_w)$ have a bivariate normal distribution with p.d.f. $f(z, y)$. For $h(z)$, an arbitrary function of z,

$$\begin{aligned}\lambda E\left[(J_w^*/J_w)h(z)\right] &= \lambda E\left[e^{-Ry}h(z)\right]\\ &= \lambda E\left(e^{-Ry}\right)\int_z\left\{\int_y\left[e^{-Ry}/E\left(e^{-Ry}\right)\right]f(z,y)\,dy\right\}h(z)\,dz\\ &= \lambda E\left(e^{-Ry}\right)\int_z h(z)f^*(z)\,dz = \lambda^* E\left[h(z^*)\right].\end{aligned}$$

where $\lambda^* \equiv \lambda E(e^{-Ry}) = \lambda E[(1+k_w)^{-R}]$. Since $f(z, y)$ is the p.d.f. of a bivariate normal, $f^*(z)$ is the p.d.f. of a normal distribution, with the moments given below.

subject to claim-specific boundary conditions, where

$b = r - d_t$ *is the cost of carry on the underlying asset,*
$\lambda^* = \lambda E(J_w^*/J_w) = \lambda \exp[-R\gamma_w + {}^1/_2 R(1+R)\,\delta_w^2]$, *and*
$1 + k^*$ *is a lognormal random variable*: $E(1+k^*) = 1 + \bar{k}^*$
$= \exp(\gamma - \mathrm{R}\,\delta_{sw}) = \exp(\gamma^*)$; $\ln(1+k^*) \sim N(\gamma^* - {}^1/_2\,\delta^2, \delta^2)$.

That is, options are priced *as if* investors were risk neutral and the asset price followed the jump-diffusion

$$dS^*/S^* = \left(b - \lambda^*\bar{k}^*\right) dt + \sigma\, dZ + k^*\, dq^*. \qquad \text{(A13)}$$

Using the transformation of variable $F_{t,T} = Se^{b(T-t)}$, options on futures can be shown to solve a similar partial differential equation, with cost of carry b_F for futures appropriately set equal to zero:

$$V_t - \lambda^*\bar{k}^* F V_F + {}^1/_2 \sigma^2 F^2 V_{FF} + \lambda^* E\left[V\left(F(1+k^*), t\right) - V\right] = rV. \qquad \text{(A14)}$$

Appendix II

Quadratic Approximation to American Option Values for Jump-Diffusion Processes

The American call option price $C(S, T; X)$ must meet the boundary conditions

$$C(S, 0; X) = \max(S - X, 0), \qquad \text{(B1)}$$
$$C(S_c^*, T; X) = S_c^* - X, \text{ and} \qquad \text{(B2)}$$
$$C_s(S_c^*, T; X) = 1, \qquad \text{(B3)}$$

where S_c^* is the critical early-exercise price on the underlying asset (relative to X) above which the option will always be exercised immediately; determination of S_c^* is part of the problem. The option must satisfy the Bellman equation in the interior of the no-stopping region:

$$-V_T + \left(b - \lambda^*\bar{k}^*\right) SV_s + {}^1/_2 \sigma^2 S^2 V_{ss} + \lambda^* E\left[V\left(Se^J, T; X\right) - V\right] = rV, \qquad \text{(B4)}$$

where $V(S, T; X) = C(S, T; X)$, J is a random normal variable distributed $N(\gamma^* - {}^1/_2\,\delta^2, \delta^2)$, and $\bar{k}^* = e^{\gamma^*} - 1$. European call options ($V = c$) solve equation (B4) subject only to terminal condition (B1).

Given there exists a series solution for the European option price, the problem is to find a good approximation for the early-exercise premium

$$\epsilon_c(S, T; X) \equiv C(S, T; X) - c(S, T; X). \qquad \text{(B5)}$$

Given the linearity of equation (B4) in V and its partials, ϵ_c also must solve (B4) in the no-stopping interior. The premium is homogeneous in S and X: $\epsilon_c(S, T; X) = X\epsilon_c(S/X, T; 1)$. Furthermore, without loss of generality, the premium can be written as

$$\epsilon_c(S, T; X) = XK(T) f(S/X, K(T)) \equiv XK(T) f(y, K), \qquad \text{(B6)}$$

where $y \equiv S/X$ and $K(T)$ is an arbitrary function of time to maturity T. The partial derivatives of ϵ are $\epsilon_s = Kf_y$, $\epsilon_{ss} = K/Xf_{yy}$, and $\epsilon_T = XK(T)f + XKK_T f_K$. From (B4), the function $f(y, K)$ must satisfy

$$\begin{aligned} &{}^1\!/_2 \sigma^2 y^2 f_{yy} + (b - \lambda^* \bar{k}^*) y f_y - rf\left[1 + (K_T/rK)(1 + Kf_K)/f\right] \\ &\quad + \lambda^* E\left[f(ye^J, K) - f\right] = 0, \end{aligned} \tag{B7}$$

Choosing $K(T) = 1 - e^{-rT}$ simplifies the expression to

$$\begin{aligned} &{}^1\!/_2\, \sigma^2 y^2 f_{yy} + (b - \lambda^* \bar{k}^*) y f_y - (r/K) f \\ &\quad - r(1 - K) f_K + \lambda^* E\left[f(ye^J, K) - f\right] = 0, \end{aligned} \tag{B8}$$

which, for calls, is solved subject to boundary conditions

$$f(y, 0) = 0, \tag{B9a}$$
$$f(0, K) = 0, \tag{B9b}$$
$$f(y_c^*, K) = (y_c^* - 1) - c(y_c^*, T; 1), \text{ and} \tag{B9c}$$
$$f_y(y_c^*, K) = 1 - c_s(y_c^*, T; 1). \tag{B9d}$$

Conditions (B9c) and (B9d) require that the early-exercise premium *smoothly* approach the stopped or early exercise value

$$f(y, K) = (y - 1) - c(y, T; 1)$$

as y approaches the critical spot price/strike price ratio y_c^* above which the call is always exercised early.

Apart from the term $r[1 - K(T)]f_K$, expression (B8) is an ordinary differential equation in y. The quadratic approximation for the early exercise premium originally proposed by MacMillan (1987) is generated by ignoring this term and solving the ordinary differential equation subject to the boundary conditions (B9). The choice of $K(T) = 1 - e^{-rT}$ ensures that the approximation becomes exact as T approaches 0 or ∞.

Under the approximation, (B8) becomes

$$ {}^1\!/_2\, \sigma^2 y^2 f_{yy} + (b - \lambda^* \bar{k}^*) y f_y - (r/K) f + \lambda^* E\left[f(ye^J, K) - f\right] = 0. \tag{B10}$$

The general solution to this is of the form

$$f(y) = A_1 y^{q_1} + A_2 y^{q_2}, \tag{B11}$$

where q_1 and q_2 are the roots to

$$ {}^1\!/_2\, \sigma^2 q^2 + \left(b - \lambda^* \bar{k}^* - {}^1\!/_2\, \sigma^2\right) q - r/K(T) + \lambda^*\left[e^{\gamma^* q + {}^1\!/_2\, q(q-1)\delta^2} - 1\right] = 0. \tag{B12}$$

One root (q_1) is negative, the other (q_2) is positive. For given values of the parameters r, σ, λ^*, $\bar{k}^*$, and δ, accurate values of the roots q_1 and q_2 can be rapidly determined from (B12) via Newton's method. Starting values are obtained from the quadratic equation given by expanding $\exp[\gamma^* q + {}^1\!/_2\, \delta^2 q(q - 1)]$ in (B12) in a second-order Taylor expansion, ignoring powers of γ^* and δ higher than 2 and using the approximation $\bar{k}^* \approx \gamma^* + {}^1\!/_2(\delta^*)^2$:

$$^1\!/_2\, v^2 q^2 + \left[b - {}^1\!/_2\, v^2\right] q - r/K(T) \approx 0, \tag{B13}$$

where $v^2 \equiv \sigma^2 + \lambda^*[(\gamma^*)^2 + \delta^2]$ is the implicit variance per unit time. The approximation (B13) indicates that for jumps with plausible amplitudes ($|\gamma^*|$ and δ substantially less than 1), the parameter of curvature q essentially depends upon jump parameters λ^*, γ^*, and δ only insofar as they contribute to the average variance per unit time of the underlying process. Furthermore, since $r/K(T) = r/(1 - e^{-rT}) \approx 1/T$, the parameter of curvature q is insensitive to the interest rate.

Boundary condition (B9a) rules out the negative root; $A_1 = 0$ for calls. Conditions (B9c) and (B9d) pin down the critical early-exercise ratio y_c^* and the coefficient A_2. Since $f(y) = A_2 y^{q_2}$ and $f'(y) = A_2(q_2/y)y^{q_2}$, (B9c) and (B9d) imply that y_c^* is the *implicit* solution to

$$y_c^* - 1 = c\left(y_c^*, T; 1\right) + \left(y_c^*/q_2\right)\left[1 - c_s\left(y_c^*, T; 1\right)\right], \tag{B14}$$

and A_2 is given by

$$A_2 = \left(y_c^*/q_2\right)\left[1 - c_s\left(y_c^*, T; 1\right)\right]. \tag{B15}$$

For given parameters, y_c^* can be solved from (B15) via Newton's method, using Merton's series solution for the European call option values and partials $c(\cdot)$, $c_s(\cdot)$, and $c_{ss}(\cdot)$.

A similar expression holds for the quadratic approximation to the early exercise premium on American puts. The positive root is ruled out; $f(y) = A_1 y^{q_1}$. Solving this subject to the boundary conditions

$$f\left(y_p^*\right) = \left(1 - y_p^*\right) - p\left(y_p^*, T; 1\right) \quad \text{and} \tag{B16c}$$

$$f_y\left(y_p^*\right) = -1 - p_s\left(y_p^*, T; 1\right) \tag{B16d}$$

yields the critical early-exercise spot price/exercise price ratio y_p^* as the implicit solution to

$$1 - y_p^* = p\left(y_p^*, T; 1\right) + \left(y_p^*/-q_1\right)\left[1 + p_s\left(y_p^*, T; 1\right)\right], \tag{B17}$$

with

$$A_1 = \left(y_p^*/-q_1\right)\left[1 + p_s\left(y_p^*, T; 1\right)\right]. \tag{B18}$$

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