Financial Forwards and Futures

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Section 5.1 Alternative Ways to Buy a Stock

Introduction

- Financial futures and forwards
  - On stocks and indexes
  - On currencies
  - On interest rates
- How are they used?
- How are they priced?
- How are they hedged?

Alternative Ways to Buy a Stock

- Four different payment and receipt timing combinations
  - Outright purchase: ordinary transaction
  - Fully leveraged purchase: investor borrows the full amount
  - Prepaid forward contract: pay today, receive the share later
  - Forward contract: agree on price now, pay/receive later
- Payments, receipts, and their timing

TABLE 5.1 Four different ways to buy a share of stock that has price S_0 at time 0. At time 0 you agree to a price, which is paid either today or at time T. The shares are received either at 0 or T. The interest rate is r.

Description	Pay at Time	Receive Security at Time	Payment
Outright purchase	0	0	S_0 at time 0
Fully leveraged purchase	T	0	S_0e^{rT} at time T
Prepaid forward contract	0	T	?
Forward contract	T	T	$? \times e^{rT}$

Pricing Prepaid Forwards

- If we can price the *prepaid* forward (F^P), then we can calculate the price for a forward contract

$$F = \text{Future Value of} \quad F^P$$

- Three possible methods to price prepaid forwards
  - Pricing by analogy
  - Pricing by discounted cash flows
  - Pricing by arbitrage
- For now, assume that there are no dividends

Pricing Prepaid Forwards (cont'd)

- Pricing by analogy
  - In the absence of dividends, the timing of delivery is irrelevant
  - Price of the prepaid forward contract same as current stock price
  - $F^P = S_0$ (where the asset is bought at $t = 0$, delivered at $t = T$)
- Pricing by discounted present value (α: risk-adjusted discount rate)
  - If expected $t = T$ stock price at $t = 0$ is $E_0(S_T)$, then $F^P = E_0(S_T)e^{-\alpha T}$
  - Since $t = 0$ expected value of price at $t = T$ is $E_0(S_T) = S_0 e^{\alpha T}$
  - Combining the two, $F^P_{0,T} = S_0 e^{\alpha T} = S_0$

Pricing Prepaid Forwards (cont'd)

- Pricing by arbitrage
  - ***Arbitrage***: a situation in which one can generate positive cash flow by simultaneously buying and selling related assets, with no net investment and with no risk. Free money!
  - If at time $t = 0$, the prepaid forward price somehow exceeded the stock price, i.e., $F^P_{0,T} > S_0$, an arbitrageur could do the following

TABLE 5.2 Cash flows and transactions to undertake arbitrage when the prepaid forward price, $F^P_{0,T}$, exceeds the stock price, S_0.

Transaction	Cash Flows	
	Time 0	Time T (expiration)
Buy stock @ S_0	$-S_0$	$+S_T$
Sell prepaid forward @ $F^P_{0,T}$	$+F^P_{0,T}$	$-S_T$
Total	$F^P_{0,T} - S_0$	0

- The price mechanism will ensure that these sort of arbitrage opportunities cannot persist, at equilibrium we can expect: $F^P_{0,T} = S_0$

Pricing Prepaid Forwards (cont'd)

- What if there are dividends? Is $F^P_{0,T} = S_0$ still valid?
  - No, because the holder of the forward will not receive dividends that will be paid to the holder of the stock, $F^P_{0,T} > S_0$
  - $F^P_{0,T} = S_0$ - PV(all dividends paid from $t = 0$ to $t = T$)
- For discrete dividends D_{t_i} at times $t_i, i = 1, \ldots, n$
  - The prepaid forward price: $F^P_{0,T} = S_0 - \sum\limits_{i=1}^{n} PV_0(D_{t_i})$
  - For continuous dividends with an annualized yield δ, the prepaid forward price is $F^P_{0,T} = S_0 e^{-\delta T}$

Pricing Prepaid Forwards (cont'd)

- Example 5.1
  - XYZ stock costs \$100 today and is expected to pay a quarterly dividend of \$1.25. If the risk-free rate is 10% compounded continuously, how much does a 1-year prepaid forward cost?
  - $F^P_{0,1} = \$100 - \sum\limits_{i=1}^{4} \$1.25 e^{-0.025 i} = \$95.30$

Pricing Prepaid Forwards (cont'd)

- Example 5.2

- The index is \$125 and the dividend yield is 3% continously compounded. How much does a 1-year prepaid forward cost?
- $F_{0,1}^{P} = \$125e^{-0.03} = \$121.31$

Section 5.3 Forward Contracts on Stock

Pricing Forwards on Stock

- Forward price is the future value of the *prepaid* forward price
  - No dividends
  - $F_{0,T} = FV(F_{0,T}^{P}) = FV(S_0) = S_0e^{rT}$
  - Continuous dividends

$$F_{0,T} = S_0e^{(r-\delta)T}$$

Pricing Forwards on Stock (cont'd)

- Forward premium
  - The difference between current forward price and stock price
  - Can be used to infer the current stock price from forward price
  - Definition:
    - Forward premium: $F_{0,T}/S_0$
    - Annualized forward premium $= (1/T)\ln(F_{0,T}/S_0)$

Creating a Synthetic Forward

- One can offset the risk of a forward by creating a *synthetic* forward to offset a position in the actual forward contract
- How can one do this? (assume continuous dividends at rate δ)
  - Recall the long forward payoff at expiration $= S_T - F_{0,T}$
  - Borrow and purchase shares as follows

TABLE 5.3 Demonstration that borrowing $S_0e^{-\delta T}$ to buy $e^{-\delta T}$ shares of the index replicates the payoff to a forward contract, $S_T - F_{0,T}$.

	Cash Flows	
Transaction	**Time 0**	**Time T (expiration)**
Buy $e^{-\delta T}$ units of the index	$-S_0e^{-\delta T}$	$+S_T$
Borrow $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
Total	0	$S_T - S_0e^{(r-\delta)T}$

- Note that the total payoff at expiration is same as forward premium

Creating a Synthetic Forward (cont'd)

- The idea of creating synthetic forward leads to following
  - Forward = Stock - zero-coupon bond
  - Stock = Forward - zero-coupon bond
  - Zero-coupon bond = Stock - forward
- Cash-and-Carry arbitrage: Buy the index, short the forward

TABLE 5.6 Transactions and cash flows for a cash-and-carry: A market-maker is short a forward contract and long a synthetic forward contract.

	Cash Flows	
Transaction	**Time 0**	**Time T (expiration)**
Buy tailed position in stock, paying $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_T$
Borrow $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
Short forward	0	$F_{0,T} - S_T$
Total	0	$F_{0,T} - S_0e^{(r-\delta)T}$

Creating a Synthetic Forward (cont'd)

- Cash-and-carry arbitrage with transaction costs
  - Trading fees, bid-ask spreads, different borrowing/lending rates, the price effect of trading in large quantities, make arbitrage harder
  - Suppose
    - Bid-ask spreads: for stock $S^b < S^a$, and for forward $F^b < F^a$
    - Cost k of transacting forward
    - Interest rate for borrowing and lending are $r^b < r^l$
    - No dividends and no time T transaction costs for simplicity
  - Arbitrage possible if
    - $F^b > F^+ = (S_0^a + 2k)e^{r^b T}$
    - $F^a < F^- = (S_0^b - 2k)e^{r^l T}$

Other Issues in Forward Pricing

- Does the forward price predict the future price?
  - According to the formula $F_{0,T} = S_0e^{-(r-\delta)T}$ the forward price conveys no additional information beyond what S_0, r, and δ provides
  - Moreover, the forward price underestimates the future stock price
- Forward pricing formula and cost of carry

$$\text{Forward Price} = \text{Spot Price} + \underbrace{\text{Interest to carry the asset} - \text{asset lease rate}}_{\text{Cost of carry}, \quad (r-\delta)S}$$