

Finance 6470 - Derivatives Markets

Binomial Model Derivation

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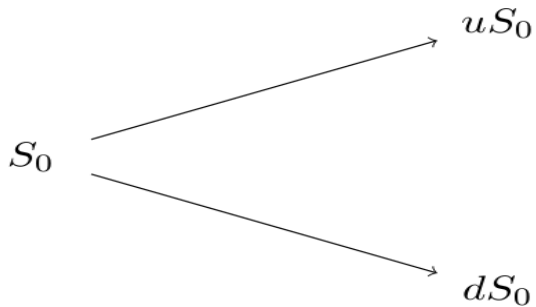
Introduction

What follows is a derivation of the single-period Binomial option pricing formula. This derivation is slightly different than the one found in your textbook. I use different variable names than the text in order to be more consistent with the Black–Scholes model.

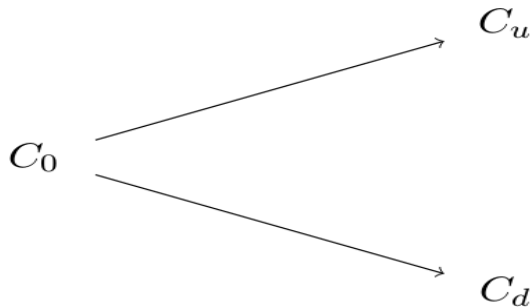
One-Period Trees

To fix ideas, recall that our simple assumption of binomial prices leads to two binomial trees: one for the stock price, and one for the option price:

Stock Price Tree



Option Price Tree



The Replicating Portfolio Concept

The basic idea of the Binomial Option Pricing Model is to set up a replicating portfolio to synthetically replicate the European call option payoff. This leads to a simple equation:

$$C_0 = \Delta S + B$$

where Δ and B are chosen with care so as to perfectly replicate the call option¹. This begs the question: just how are Δ and B chosen? We can solve for these parameters by noting that the following must hold:

$$C_u = \Delta uS + Be^{rh}$$

$$C_d = \Delta dS + Be^{rh}$$

¹The same logic applies for put options, so we can talk only about call options without loss of generality.

Solving for B

We can now see how to solve for these parameters. First we will solve for Be^{rh} in the second equation as follows:

$$Be^{rh} = C_d - \Delta dS$$

and plug it into the first for Be^{rh} as follows:

$$C_u = \Delta uS + C_d - \Delta dS$$

Solving for Δ

We notice that B has now disappeared from the first equation and we can solve for Δ as follows:

$$\Delta S(u - d) = C_u - C_d$$

which leads to:

$$\Delta = \frac{C_u - C_d}{S(u - d)}$$

So now we have solved for the correct value of Δ that gives us the number of shares we need to hold in our portfolio to synthetically replicate the call option. We can now plug this Δ into $Be^{rh} = C_d - \Delta dS$ to get an equation, for which the only unknown is B and solve for it. We do this as follows:

$$Be^{rh} = C_d - \left(\frac{C_u - C_d}{S(u - d)} \right) dS$$

This we can rearrange as:

$$\begin{aligned} Be^{rh} &= C_d \frac{(u-d)}{(u-d)} - \left(\frac{dC_u - dC_d}{u-d} \right) \\ &= \frac{uC_d - dC_d - dC_u + dC_d}{u-d} \\ &= \frac{uC_d - dC_u}{u-d} \end{aligned}$$

Finally, we can multiply both sides of the equation by e^{-rh} to get the following:

$$B = e^{-rh} \left(\frac{uC_d - dC_u}{u-d} \right)$$

The No-Arbitrage Solution

We now know what the values of Δ and B need to be to perfectly replicate the call option. Since we can observe these quantities, we can figure out by applying the **law of one price** (or in other words by assuming no arbitrage opportunities exist) the equilibrium price of the call option now, or C_0 .

We simply plug in for Δ and B in the following:

$$\begin{aligned} C_0 &= \Delta S + B \\ &= \left(\frac{C_u - C_d}{S(u - d)} \right) S + e^{-rh} \left(\frac{uC_d - dC_u}{u - d} \right) \end{aligned}$$

The Risk-Neutral Representation

Essentially we could stop here. We are done. We have derived the single-period Binomial Option Pricing Model. But we will keep working to rearrange this equation to express it in such a manner to get even more deep intuition from it.

We can rewrite the model as follows:

$$\begin{aligned}C_0 &= \left(\frac{C_u - C_d}{S(u - d)} \right) S + e^{-rh} \left(\frac{uC_d - dC_u}{u - d} \right) \\&= \left(\frac{C_u - C_d}{(u - d)} \right) + e^{-rh} \left(\frac{uC_d - dC_u}{u - d} \right) \\&= e^{-rh} \left(\frac{e^{rh}C_u - e^{rh}C_d + uC_d - dC_u}{u - d} \right) \\&= e^{-rh} \left(C_u \frac{e^{rh} - d}{u - d} + C_d \frac{u - e^{rh}}{u - d} \right)\end{aligned}$$

Finally, we can let $p_u^* = \frac{e^{rh}-d}{u-d}$ and $p_d^* = \frac{u-e^{rh}}{u-d}$. Now we can write the model simply as:

$$C_0 = e^{-rh} [C_u p_u^* + C_d p_d^*]$$